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Multiple positive and sign-changing solutions of an elliptic equation with fast increasing weight and critical growth

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ABSTRACT

We consider the following equation

$$-\mathrm{div}\big(K(x)\nabla u\big) = \lambda K(x)|x|^{\beta}|u|^{q-2}u + Q(x)K(x)|u|^{2^*-2}u, \quad x \in \mathbb{R}^N,$$

where $N \geq 3$, $2 < q < 2^* = 2N/(N-2)$, $\lambda > 0$ is a parameter, $K(x) = \exp(|x|^{\alpha}/4)$, $\alpha \geq 2$, $\beta = (\alpha - 2)\frac{(2^*-q)}{(2^*-2)}$ and $0 \leq Q(x) \in C(\mathbb{R}^N)$. Using variational methods and delicate estimates, we establish some existence and multiplicity of positive and sign-changing solutions for the problem, provided that the maximum of $Q(x)$ is achieved at different points.

1. Introduction

In this paper, we are concerned with the existence and multiplicity of positive and sign-changing solutions for the following problem:

$$-\mathrm{div}\big(K(x)\nabla u\big) = \lambda K(x)|x|^{\beta}|u|^{q-2}u + Q(x)K(x)|u|^{2^*-2}u, \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $N \geq 3$, $2 < q < 2^* = 2N/(N-2)$, $\lambda > 0$ is a parameter, $K(x) = \exp(|x|^{\alpha}/4)$, $\alpha \geq 2$, $\beta = (\alpha-2)\frac{(2^*-q)}{(2^*-2)}$ and $0 \leq Q(x) \in C(\mathbb{R}^N)$ is assumed to satisfy the following condition:

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$(Q1)$ There exist k different points $a^1, a^2, \ldots, a^k$ in $\mathbb{R}^N$ such that $Q(a^j)$ are strict maximums and satisfy

$$Q(a^j) = Q_M = \max\{Q(x) : x \in \mathbb{R}^N\} > 0, \quad j = 1, 2, \ldots, k;$$

$(Q2)$ None of the points $a^1, a^2, \ldots, a^k$ is an origin;
$(Q2')$ One of the points $a^1, a^2, \ldots, a^k$ is an origin;
$(Q3)$ $Q_M - Q(x) = \begin{cases} o\big(|x - a^j|^{N-(N-2)q/2}\big), & \text{if } a^j \neq 0, \\ o\big(|x - a^j|^{N+\beta-(N-2)q/2}\big), & \text{if } a^j = 0, \end{cases}$ for x near a^j, $j = 1, 2, \ldots, k$.

For $\alpha = q = 2, \lambda \equiv (N-2)/(N+2)$ and $Q(x) \equiv 1$, equation (1.1) is originated from finding self-similar solutions of the form

$$w(t, x) = t^{\frac{2-N}{N+2}} u(xt^{-1/2})$$

for the evolution equation

$$w_t - \Delta w = |w|^{4/(N-2)} w \quad on \quad (0, \infty) \times \mathbb{R}^N.$$

See [6,9] for a detailed description.

Equation (1.1) with $\alpha = q = 2, Q(x) \equiv 1$ has been treated by many authors. See [10,13,14,16] and reference therein. When $q = 2, Q(x) \equiv 1$, Catrina et al. [4] have obtained some existence results of the Brezis–Nirenberg type and have showed that the critical dimension of the problem depends on the value of α. Later on, when $Q(x) \equiv 1$, by using Mountain Pass Theorem and Linking Theorem, Furtado et al. [7] have proved that there are a positive solution if $2 < q < 2^*$ and a sign-changing solution if $q = 2$. Recently, Furtado et al. [8] have considered the following equation

$$-\mathrm{div}\big(K(x)\nabla u\big) = K(x) f(u) + \lambda K(x)|u|^{2^*-2} u, \quad x \in \mathbb{R}^N, \tag{1.2}$$

where $f(u)$ is superlinear and subcritical. In that article, for any given $k \in \mathbb{N}$, they have shown that there exists $\lambda^* = \lambda^*(k) > 0$ such that (1.2) has at least k pairs of solutions for $\lambda \in \big(0, \lambda^*(k)\big)$. But they can not give any information about the sign of these solutions. We also refer the interested reader to [2,3,15,20] for various existence results in the case $K(x) \equiv 1$ and $\alpha = 2$. As far as we know, we have not seen any multiplicity of positive and sign-changing solutions for problem (1.1) with the fast increasing weights $K(x)$ and $2 < q < 2^*$ in the literature.

The aim of this paper is to use the shape of the graph of $Q(x)$ to prove the existence and multiplicity of both positive and sign-changing solutions for problem (1.1), this property has been firstly observed by Cao and Noussair [2,3]. For the problem considered here, some different phenomena may appear since we have an additional weighted function $K(x)$. We will combine the effect of $K(x)$ and the shape of $Q(x)$ to study (1.1). Our main results are:

Theorem 1.1. *Assume conditions $(Q1)$, $(Q2)$ and $(Q3)$. If $N \geq 3$ and $\frac{2N-2}{N-2} < q < 2^*$, then there exists $\lambda_0 > 0$, such that (1.1) has at least k positive solutions for $\lambda \in (0, \lambda_0)$.*

Theorem 1.2. *Assume conditions $(Q1)$, $(Q2)$ and $(Q3)$. Then there exists $\lambda_0 > 0$, such that (1.1) has at least k sign-changing solutions, if one of the following statements holds:*
(i) $N \geq 4$, $\frac{2N-2}{N-2} < q < 2^*, \lambda \in (0, \lambda_0)$;
(ii) $N = 3, 5 < q < 2^*, \lambda \in (0, \lambda_0)$.

The following two Theorems consider a different case from the above two Theorems, in which one of the points $a^1, a^2, \ldots, a^k$ is an origin.

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Theorem 1.3. *Assume conditions (Q1), (Q2′) and (Q3). Then there exists $\lambda_0 > 0$, such that (1.1) has at least k positive solutions, if one of the following statements holds:*
(i) $N > 2 + \alpha/2, \frac{2N-2}{N-2} < q < 2^*, \lambda \in (0, \lambda_0)$;
(ii) $3 \le N \le 2 + \alpha/2, 2^* - 4/\alpha < q < 2^*, \lambda \in (0, \lambda_0)$.

Theorem 1.4. *Assume conditions (Q1), (Q2′) and (Q3). Then there exists $\lambda_0 > 0$, such that (1.1) has at least k sign-changing solutions, if one of the following statements holds:*
(i) $N \ge \alpha + 2, \frac{2N-2}{N-2} < q < 2^*, \lambda \in (0, \lambda_0)$;
(ii) $3 \le N < \alpha + 2, 2^* - 2/\alpha < q < 2^*, \lambda \in (0, \lambda_0)$.

Problem (1.1) is variational in nature. Indeed, for any $\alpha \ge 2$, let us denote by $H(\alpha)$ the Hilbert space obtained as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\| = \left(\int\limits_{\mathbb{R}^N} K(x)|\nabla u|^2 dx \right)^{1/2}.$$

We also define the weighted Lebesgue spaces

$$L^q_K(\alpha) = \left\{ u \text{ measurable in } \mathbb{R}^N \ : \ \|u\|^q_{q,K} = \int\limits_{\mathbb{R}^N} K(x)|x|^\beta |u|^q dx < \infty \right\}.$$

From [7], we know that the embedding $H(\alpha) \hookrightarrow L^q_K(\alpha)$ is continuous for $2 \le q \le 2^*$ and compact for $2 \le q < 2^*$. It then follows that the following functional:

$$I_\lambda(u) = \frac{1}{2} \int\limits_{\mathbb{R}^N} K(x)|\nabla u|^2 dx - \frac{\lambda}{q} \int\limits_{\mathbb{R}^N} K(x)|x|^\beta |u|^q dx - \frac{1}{2^*} \int\limits_{\mathbb{R}^N} Q(x)K(x)|u|^{2^*} dx$$

is well defined on $H(\alpha)$ and there exists a one to one correspondence between the critical points and the weak solutions of (1.1). Here, we say that $u \in H(\alpha)$ is a weak solution of (1.1), if for any $v \in H(\alpha)$, there holds

$$\int\limits_{\mathbb{R}^N} K(x) \left[\nabla u \nabla v - \lambda |x|^\beta |u|^{q-2} uv - Q(x)|u|^{2^*-2} uv \right] dx = 0.$$

In the proofs of Theorems 1.1–1.4, we will employ the methods which has been introduced previously by Tarantello [17,18] and later refined by Cao and Noussair [2] (see also [5]). The main difficulties lie in two aspects. Firstly, since the embedding $H(\alpha) \hookrightarrow L^{2^*}_K(\mathbb{R}^N)$ is not compact, the functional I_λ satisfies (PS) condition only locally. Different from the test functions used in [4,2], we use the following test function

$$z_{\varepsilon,a^j} = K(x)^{-1/2} \varphi(x - a^j) \frac{C_N \varepsilon^{(N-2)/2}}{(\varepsilon^2 + |x - a^j|^2)^{(N-2)/2}},$$

where $C_N = \left(N(N-2)/Q_M\right)^{(N-2)/4}$, $\varphi(x - a^j)$ is a cut off function, and then succeed to prove that the energy level belongs to the range where (PS) condition hold. Secondly, in comparison with [2], the calculations here are more delicate due to the presence of $|x|^\beta$. To overcome this difficulty, we consider two distinct cases whether or not the condition $(Q2)$ holds. The calculations will be done under $(Q2)$ and $(Q2')$ respectively (see Appendix A and B below).

This paper is organized as follows. In the next section, we give some notations and preliminaries. We prove Theorem 1.1 in section 3, and Theorem 1.2 in section 4. Section 5 is devoted to the proofs of Theorems 1.3 and 1.4. In Appendix A and B, some important estimates are shown.

2. Notations and preliminaries

Throughout this paper, we write $\int u$ instead of $\int_{\mathbb{R}^N} u(x)dx$. $B_r(x)$ is a ball centered at x with radius r. Set $\overline{B}_r(x), S_r(x)$ denote the closure and the boundary of $B_r(x)$, respectively. $\to$ denotes strong convergence. $\rightharpoonup$ denotes weak convergence. All limitations hold as $n \to \infty$ unless otherwise stated. C and C_i denote various positive constants whose values can vary from line to line. What's more, the following minimization problem will be useful in what follows:

$$S = \inf\left\{ \int K(x)|\nabla u|^2 \quad ; \quad u \in H(\alpha), \int K(x)|u|^{2^*} = 1 \right\}. \tag{2.1}$$

It is worth mentioning that this constant S is equal to the best Sobolev constant, see [4].

For $j = 1, 2, \dots, k$, by the condition $(Q1)$, we can choose $\eta > 0$ small enough such that $B_{2\eta}(a^j)$ are disjoint, $0 < C_1 < |x+a^j| < C_2$ for $x \in \overline{B}_{2\eta}(a^j)$ with $a^j \neq 0$, and $Q(x) < Q(a^j)$ for $x \in \overline{B}_{2\eta}(a^j) \setminus \{a^j\}$. Let $g : H(\alpha) \to \mathbb{R}^N$ be defined by

$$g(u) = \frac{\displaystyle\int xK(x)|u|^{2^*}}{\displaystyle\int K(x)|u|^{2^*}}.$$

Set

$$\mathcal{M}_\lambda = \{u \in H(\alpha) : u \not\equiv 0, \langle I'_\lambda(u), u\rangle = 0\},$$

and then define, for $j = 1, 2, \dots, k$,

$$\begin{aligned}
\mathcal{M}^j_\lambda &= \{u \in \mathcal{M}_\lambda : g(u) \in B_\eta(a^j)\},\\
\mathcal{U}^j_\lambda &= \{u \in \mathcal{M}_\lambda : g(u) \in S_\eta(a^j)\},\\
\mathcal{V}^j_\lambda &= \{u : u^\pm \in \mathcal{M}^j_\lambda\},\\
\mathcal{W}^j_\lambda &= \{u : u \in \mathcal{M}_\lambda, g(u^\pm) \in \overline{B}_\eta(a^j) \text{ and either } g(u^+) \in S_\eta(a^j) \text{ or } g(u^-) \in S_\eta(a^j)\},
\end{aligned}$$

where $u^+ = \max\{u, 0\}$ and $u^- = \min\{u, 0\}$. Correspondingly we also define

$$m^j_\lambda = \inf_{u\in\mathcal{M}^j_\lambda} I_\lambda(u), \qquad \overline{m}^j_\lambda = \inf_{u\in\mathcal{U}^j_\lambda} I_\lambda(u),$$

$$M^j_\lambda = \inf_{u\in\mathcal{V}^j_\lambda} I_\lambda(u), \qquad \overline{M}^j_\lambda = \inf_{u\in\mathcal{W}^j_\lambda} I_\lambda(u),$$

then we have the following result.

Lemma 2.1. *Assume $\lambda > 0$ and conditions $(Q1)$, $(Q2)$ and $(Q3)$. If $N \geq 3$ and $\frac{2N-2}{N-2} < q < 2^*$, then*

$$m^j_\lambda < \frac{S^{N/2}}{NQ_M^{(N-2)/2}}, \tag{2.2}$$

for $j = 1, 2, \dots, k$.

Proof. We take $z_\varepsilon = z_{\varepsilon,a^j} = K(x)^{-1/2}\varphi(x-a^j)U_{\varepsilon,a^j}$ from Appendix A. Define the function $h(t) = I_\lambda(tz_\varepsilon), t \geq 0$. Clearly, $h(t)$ has a unique maximum point $t_\varepsilon > 0$ which follows that $t_\varepsilon z_\varepsilon \in \mathcal{M}_\lambda$. This and the fact that z_ε has symmetric support about a^j imply that $t_\varepsilon z_\varepsilon \in \mathcal{M}_\lambda^j$. Thus, (2.2) will follow if we can show

$$\sup_{t>0} I_\lambda(tz_\varepsilon) < \frac{S^{N/2}}{NQ_M^{(N-2)/2}}.$$

In order to prove that, we set $\psi(t) = \frac{t^2}{2}\int K(x)|\nabla z_\varepsilon|^2 - \frac{t^{2^*}}{2^*}\int Q(x)K(x)|z_\varepsilon|^{2^*}$. By using $\psi'(t) = 0$, we deduce that $\psi(t)$ achieves its maximum at

$$T_{Max} = \left(\frac{\int K(x)|\nabla z_\varepsilon|^2}{\int Q(x)K(x)|z_\varepsilon|^{2^*}}\right)^{(N-2)/4}.$$

Consequently,

$$\begin{aligned}
\sup_{t>0} I_\lambda(tz_\varepsilon) &= I_\lambda(t_\varepsilon z_\varepsilon) \\
&= \psi(t_\varepsilon) - \frac{\lambda}{q}t_\varepsilon^q \int K(x)|x|^\beta |z_\varepsilon|^q \\
&\leq \psi(T_{Max}) - \frac{\lambda}{q}t_\varepsilon^q \int K(x)|x|^\beta |z_\varepsilon|^q \\
&= \frac{1}{N}\left[\frac{\int K(x)|\nabla z_\varepsilon|^2}{(\int Q(x)K(x)|z_\varepsilon|^{2^*})^{(N-2)/N}}\right]^{N/2} - \frac{\lambda}{q}t_\varepsilon^q \int K(x)|x|^\beta |z_\varepsilon|^q.
\end{aligned}$$

Next, we can assume that $t_\varepsilon \geq C_1 > 0$. If not, there exists a sequence $\varepsilon_n \to 0^+$ such that $t_{\varepsilon_n} \to 0$. It then follows from the continuity of I_λ and the boundedness of $\{z_{\varepsilon_n}\}$ that

$$\sup_{t>0} I_\lambda(tz_{\varepsilon_n}) = I_\lambda(t_{\varepsilon_n} z_{\varepsilon_n}) \to 0 < \frac{S^{N/2}}{NQ_M^{(N-2)/2}},$$

namely, the proof is finished. So, we can assume $t_\varepsilon \geq C_1 > 0$.

Thus, if $N \geq 3$ and $\frac{2N-2}{N-2} < q < 2^*$, we get from Appendix (A.1a)–(A.1c), (A.7a) and (A.12) that for $\varepsilon > 0$ sufficiently small

$$\begin{aligned}
\sup_{t>0} I_\lambda(tz_\varepsilon) &= \frac{1}{N}\left[\frac{A_1 + \tau(\varepsilon)}{\left(Q_M A_2 + O(\varepsilon^N) + o(\varepsilon^{N-(N-2)q/2})\right)^{(N-2)/N}}\right]^{N/2} - \frac{\lambda}{q}t_\varepsilon^q \int K(x)|x|^\beta |z_\varepsilon|^q \\
&\leq \frac{S^{N/2}}{NQ_M^{(N-2)/2}} + \tau(\varepsilon) + O(\varepsilon^N) + o(\varepsilon^{N-(N-2)q/2}) - C\varepsilon^{N-(N-2)q/2} \\
&< \frac{S^{N/2}}{NQ_M^{(N-2)/2}},
\end{aligned}$$

where

$$\tau(\varepsilon) = \begin{cases} O(\varepsilon^{N-2}) + O(\varepsilon) + O(\varepsilon^2), & \text{if} \quad N > 4, \\ O(\varepsilon^{N-2}) + O(\varepsilon) + O(\varepsilon^2|\ln\varepsilon|), & \text{if} \quad N = 4, \\ O(\varepsilon^{N-2}) + O(\varepsilon|\ln\varepsilon|) + O(\varepsilon), & \text{if} \quad N = 3. \end{cases}$$

This completes the proof of Lemma 2.1. □

Lemma 2.2. *Assume condition* $(Q1)$. *Then there exists* $\lambda_0 > 0$ *such that*

(1) $\overline{m}_\lambda^j > \dfrac{S^{N/2}}{NQ_M^{(N-2)/2}}$,

(2) $\overline{M}_\lambda^j > \dfrac{2S^{N/2}}{NQ_M^{(N-2)/2}}$,

for $j = 1, 2, \ldots, k$, *and* $\lambda \in (0, \lambda_0)$.

Proof. To prove part(1), suppose to the contrary that there are sequences $\lambda_n \to 0$, and $\{u_n\} \subset \mathcal{U}_{\lambda_n}^j$ such that

$$I_{\lambda_n}(u_n) \to c \leq \frac{S^{N/2}}{NQ_M^{(N-2)/2}},$$
$$\int K(x)|\nabla u_n|^2 - \lambda_n \int K(x)|x|^\beta |u_n|^q = \int Q(x)K(x)|u_n|^{2^*}.$$

Then it is easy to check that $\{u_n\}$ is bounded in $H(\alpha)$ and then $\lambda_n \int K(x)|x|^\beta |u_n|^q \to 0$. Moreover, since $\int Q(x)K(x)|u_n|^{2^*} \leq Q_M S^{-2^*/2}(\int K(x)|\nabla u_n|^2)^{2^*/2}$ and $\int K(x)|x|^\beta |u_n|^q \leq C(\int K(x)|\nabla u_n|^2)^{q/2}$ (see [7, p. 1038, Proposition 2.1]), we can deduce from the last equality that there exist constants C_1 and C_2 such that

$$\int K(x)|\nabla u_n|^2 \geq C_1 > 0 \quad \text{and} \quad \int Q(x)K(x)|u_n|^{2^*} \geq C_2 > 0,$$

for all $n = 1, 2, \cdots$. Thus, we can choose $t_n > 0$ such that $v_n = t_n u_n$ satisfies

$$\int K(x)|\nabla v_n|^2 = \int Q_M K(x)|v_n|^{2^*},$$

and

$$t_n = \left(\frac{\int Q(x)K(x)|u_n|^{2^*} + \lambda_n \int K(x)|x|^\beta |u_n|^q}{\int Q_M K(x)|u_n|^{2^*}} \right)^{(N-2)/4}$$

are bounded. Since $Q(x) \leq Q_M$ and $\lambda_n \displaystyle\int K(x)|x|^\beta |u_n|^q \to 0$, we have $t_n \to t_0 \leq 1$. Indeed, $t_0 = 1$. This follows easily from

$$\begin{aligned}
\frac{S^{N/2}}{NQ_M^{(N-2)/2}} &\leq \lim_{n\to\infty} \frac{1}{N} \int K(x)|\nabla v_n|^2 \\
&= \lim_{n\to\infty} \frac{1}{N} t_n^2 \int K(x)|\nabla u_n|^2 \\
&= \lim_{n\to\infty} t_n^2 \Bigg[\frac{1}{N} \left(\int K(x)|\nabla u_n|^2 - \lambda_n \int K(x)|x|^\beta |u_n|^q \right) \\
&\qquad + \left(\frac{1}{2} - \frac{1}{q} \right) \lambda_n \int K(x)|x|^\beta |u_n|^q \Bigg]
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} t_n^2 I_{\lambda_n}(u_n) \\
&= t_0^2 c \\
&\le t_0^2 \frac{S^{N/2}}{NQ_M^{(N-2)/2}}.
\end{aligned}
$$

The inequalities above also show that

$$
c = \frac{S^{N/2}}{NQ_M^{(N-2)/2}} \quad \text{and} \quad \lim_{n\to\infty} \int K(x)|\nabla v_n|^2 = \frac{S^{N/2}}{Q_M^{(N-2)/2}}. \tag{2.3}
$$

Let $w_n = v_n/(\int K(x)|v_n|^{2^*})^{1/2^*}$. We first notice that

$$
\int K(x)|\nabla w_n|^2 \to S,
$$

which means that $\{w_n\}$ is a minimizing sequence for the problem (2.1). Next, by using a result of Lions [11, p. 158, Lemma I.1], we can find a point $x_0 \in \mathbb{R}^N$ and a subsequence (still denoted by $\{w_n\}$) satisfying

$$
\lim_{n\to\infty} \int \phi K(x)|w_n|^{2^*} = \phi(x_0), \qquad \text{for any } \phi \in C(\mathbb{R}^N). \tag{2.4}
$$

In particular, we have

$$
g^i(u_n) = \frac{\displaystyle\int x_i K(x)|u_n|^{2^*}}{\displaystyle\int K(x)|u_n|^{2^*}} = \frac{\displaystyle\int x_i K(x)|w_n|^{2^*}}{\displaystyle\int K(x)|w_n|^{2^*}} \to (x_0)_i,
$$

and so, from $g(u_n) \in S_\eta(a^j)$, we have $x_0 \in S_\eta(a^j)$. Then, we can apply (2.3) and (2.4) to obtain that

$$
\begin{aligned}
\lim_{n\to\infty} I_{\lambda_n}(u_n) &= \lim_{n\to\infty} \left[\frac{1}{N}\left(\int K(x)|\nabla u_n|^2 - \lambda_n \int K(x)|x|^\beta |u_n|^q \right) \right. \\
&\qquad \left. + \left(\frac{1}{2} - \frac{1}{q}\right)\lambda_n \int K(x)|x|^\beta |u_n|^q \right] \\
&= \lim_{n\to\infty} \frac{1}{N} \int Q(x)K(x)|u_n|^{2^*} \\
&= \lim_{n\to\infty} \frac{1}{N} \int Q(x)K(x)|v_n|^{2^*} \\
&= \frac{Q(x_0)}{NQ_M} \lim_{n\to\infty} \int Q_M K(x)|v_n|^{2^*} \\
&= \frac{Q(x_0)}{NQ_M} \lim_{n\to\infty} \int K(x)|\nabla v_n|^2 \\
&= \frac{Q(x_0)}{NQ_M} \frac{S^{N/2}}{Q_M^{(N-2)/2}} \\
&< \frac{S^{N/2}}{NQ_M^{(N-2)/2}},
\end{aligned}
$$

a contradiction to (2.3). Hence part (1) holds.

For part (2), we argue by contradiction once more. If, to the contrary, there are $\lambda_n \to 0$ and $\{u_n\} \subset \mathcal{W}^j_\lambda$ such that

$$I_{\lambda_n}(u_n) \to c \le \frac{2S^{N/2}}{NQ_M^{(N-2)/2}},$$

$$\int K(x)|\nabla u_n^\pm|^2 - \lambda_n \int K(x)|x|^\beta |u_n^\pm|^q = \int Q(x)K(x)|u_n^\pm|^{2^*}.$$

In view of $I_{\lambda_n}(u_n) = I_{\lambda_n}(u_n^+) + I_{\lambda_n}(u_n^-)$, we must have

$$\lim_{n\to\infty} I_{\lambda_n}(u_n^+) \le \frac{S^{N/2}}{NQ_M^{(N-2)/2}} \quad \text{or} \quad \lim_{n\to\infty} I_{\lambda_n}(u_n^-) \le \frac{S^{N/2}}{NQ_M^{(N-2)/2}}.$$

But, either of the two inequalities above will lead to a contradiction, as in the proof of (1). □

3. Multiple positive solutions under the condition $(Q2)$

In this section, we will prove Theorem 1.1. First, we have

Lemma 3.1. *Assume conditions* $(Q1)$, $(Q2)$ *and* $(Q3)$. *There exist* $\lambda_0 > 0$ *and a sequence* $\{u_n^j\} \subset \mathcal{M}^j_\lambda$ *such that*

$$u_n^j \ge 0, \quad I_\lambda(u_n^j) \to m_\lambda^j, \quad I'_\lambda(u_n^j) \to 0,$$

for $j = 1, 2, \dots, k$, *and* $\lambda \in (0, \lambda_0)$.

Proof. Following the same ideas presented in lemma 2.4 of [2], we have that $\overline{\mathcal{M}}^j_\lambda = \mathcal{M}^j_\lambda \cup \mathcal{U}^j_\lambda$ and $\mathcal{U}^j_\lambda$ is the boundary of $\overline{\mathcal{M}}^j_\lambda$. By Lemmas 2.1 and 2.2, we have there exists $\lambda_0 > 0$ such that $m^j_\lambda < \overline{m}^j_\lambda$ for $\lambda \in (0, \lambda_0)$. Therefore, we get

$$m^j_\lambda = \inf\{I_\lambda(u) : u \in \overline{\mathcal{M}}^j_\lambda\}.$$

Using Ekeland's variational principle, we obtain a minimizing sequence $\{u_n^j\} \subset \mathcal{M}^j_\lambda$ with the following properties:

(*a*) $I_\lambda(u_n^j) < m^j_\lambda + \frac{1}{n}$,

(*b*) $I_\lambda(u_n^j) \le I_\lambda(w) + \frac{1}{n}\|w - u_n^j\|$, for any $w \in \overline{\mathcal{M}}^j_\lambda$.

From $I_\lambda(|u|) = I_\lambda(u)$, we may assume $u_n^j \ge 0$. For each u_n^j, by the same argument as in Lemma 2.4 of Tarantello [17], there are $\varepsilon_n > 0$ and a differential function t_n defined for $w \in H(\alpha)$, $w \in B_{\varepsilon_n}(0)$ such that $t_n(w)(u_n^j - w) \in \mathcal{M}^j_\lambda$, and

$$\langle t'_n(0), v\rangle = \frac{2\int K(x)\nabla u_n^j \nabla v - q\lambda \int K(x)|x|^\beta |u_n^j|^{q-2}u_n^j v - 2^* \int Q(x)K(x)|u_n^j|^{2^*-2}u_n^j v}{\int K(x)|\nabla u_n^j|^2 - (q-1)\lambda \int K(x)|x|^\beta |u_n^j|^q - (2^*-1)\int Q(x)K(x)|u_n^j|^{2^*}} \tag{3.1}$$

Choose $0 < \rho < \varepsilon_n$ and let $w_\rho = \rho u$ with $\|u\| = 1$. Fix n and set $z_\rho = t_n(w_\rho)(u_n^j - w_\rho)$. Since $z_\rho \in \mathcal{M}^j_\lambda$, it then follows from (b) that

$$I_\lambda(z_\rho) - I_\lambda(u_n^j) \ge -\frac{1}{n}\|z_\rho - u_n^j\|.$$

From the definition of Fréchet derivative, we also have

$$\langle I'_\lambda(u_n^j), z_\rho - u_n^j\rangle + o(\|z_\rho - u_n^j\|) \geq -\frac{1}{n}\|z_\rho - u_n^j\|.$$

Hence,

$$\langle I'_\lambda(u_n^j), (u_n^j - w_\rho) + \big(t_n(w_\rho)-1\big)(u_n^j - w_\rho) - u_n^j\rangle \geq -\frac{1}{n}\|z_\rho - u_n^j\| + o(\|z_\rho - u_n^j\|)$$

which implies that

$$-\langle I'_\lambda(u_n^j), w_\rho\rangle + \big(t_n(w_\rho)-1\big)\langle I'_\lambda(u_n^j), u_n^j - w_\rho\rangle \geq -\frac{1}{n}\|z_\rho - u_n^j\| + o(\|z_\rho - u_n^j\|).$$

From which, we have

$$\begin{aligned} & -\rho\langle I'_\lambda(u_n^j), u\rangle + \frac{t_n(w_\rho)-1}{t_n(w_\rho)}\langle I'_\lambda(z_\rho), t_n(w_\rho)(u_n^j - w_\rho)\rangle \\ & \quad + \big(t_n(w_\rho)-1\big)\langle I'_\lambda(u_n^j) - I'_\lambda(z_\rho), u_n^j - w_\rho\rangle \\ & \geq -\frac{1}{n}\|z_\rho - u_n^j\| + o(\|z_\rho - u_n^j\|) \end{aligned} \tag{3.2}$$

Due to $t_n(w_\rho)(u_n^j - w_\rho) \in \mathcal{M}_\lambda^j$, we also have $\langle I'_\lambda(z_\rho), t_n(w_\rho)(u_n^j - w_\rho)\rangle = 0$. This and (3.2) yield

$$\begin{aligned} \langle I'_\lambda(u_n^j), u\rangle \leq & \frac{1}{n}\frac{\|z_\rho - u_n^j\|}{\rho} + \frac{o(\|z_\rho - u_n^j\|)}{\rho} \\ & + \frac{t_n(w_\rho)-1}{\rho}\langle I'_\lambda(u_n^j) - I'_\lambda(z_\rho), u_n^j - w_\rho\rangle \end{aligned} \tag{3.3}$$

From $\|u\| = 1$, (3.1) and the boundedness of $\{u_n^j\}$, it follows that

$$\begin{aligned} \lim_{\rho\to 0}\frac{|t_n(w_\rho)-1|}{\rho} &= \lim_{\rho\to 0}\frac{|t_n(0+\rho u) - t_n(0)|}{\rho} \\ &= \langle t'_n(0), u\rangle \\ &\leq \|t'_n(0)\| \\ &\leq C_1. \end{aligned}$$

Note that

$$\begin{aligned} \|z_\rho - u_n^j\| &= \|t_n(w_\rho)(u_n^j - w_\rho) - (u_n^j - w_\rho) - w_\rho\| \\ &= \|\left(t_n(w_\rho) - 1\right)\left(u_n^j - w_\rho\right) - w_\rho\| \\ &\leq |t_n(w_\rho) - 1|\cdot\|u_n^j - w_\rho\| + \|w_\rho\| \\ &= |t_n(w_\rho)-1|C_2 + \rho. \end{aligned}$$

Furthermore, for fixed n, since $\langle I'(u_n^j), u_n^j\rangle = 0$ and $(u_n^j - w_\rho) \to u_n^j$ as $\rho \to 0$, we obtain by letting $\rho \to 0$ in (3.3) that

$$\langle I'(u_n^j), u\rangle \leq \frac{C}{n}$$

from which it follows that $I'_\lambda(u_n^j) \to 0$. Thus, the proof is complete. $\square$

Lemma 3.2. *Assume condition* $(Q1)$ *and* $\lambda > 0$.

(1) *If* $\{u_n\} \subset \mathcal{M}_\lambda$ *be a sequence satisfying*

$$I_\lambda(u_n) \to c < \frac{S^{N/2}}{NQ_M^{(N-2)/2}} \quad \textit{and} \quad I'_\lambda(u_n) \to 0,$$

then $\{u_n\}$ *is relatively compact in* $H(\alpha)$.

(2) *If* $\{u_n\} \subset \mathcal{V}_\lambda^j$ *be a sequence satisfying*

$$I_\lambda(u_n) \to c < m_\lambda^j + \frac{S^{N/2}}{NQ_M^{(N-2)/2}} \quad \textit{and} \quad I'_\lambda(u_n) \to 0,$$

then $\{u_n\}$ *is relatively compact in* $H(\alpha)$.

Proof. (**1**). It is easy to check that $\{u_n\}$ is bounded in $H(\alpha)$. Thus, we can assume that

$$\begin{gathered} u_n \rightharpoonup u_0 \text{ in } H(\alpha), \\ u_n \to u_0 \text{ in } L_K^q(\alpha), \quad 2 \le q < 2^*, \\ u_n \to u_0 \text{ a.e. on } \mathbb{R}^N. \end{gathered}$$

We claim that $I'_\lambda(u_0) = 0$. Indeed, let us consider $v \in C_0^\infty(\mathbb{R}^N)$ and set $\Omega = \operatorname{supp}(v)$. By Young's inequality, we have

$$\left|K(x)|x|^\beta |u_n|^{q-2} u_n v\right| \le C\Big(K(x)|u_n|^q + K(x)|v|^q\Big) \text{ a.e. in } \Omega.$$

Since $\lim_{n\to\infty} \int K(x)|u_n|^q = \int K(x)|u_0|^q$ and $v \in C_0^\infty(\mathbb{R}^N)$, we can apply Lebesgue Theorem to obtain

$$\lim_{n\to\infty} \int K(x)|x|^\beta |u_n|^{q-2} u_n v = \int K(x)|x|^\beta |u_0|^{q-2} u_0 v. \tag{3.4}$$

Since $|u_n|^{2^*-2} u_n \to |u_0|^{2^*-2} u_0$ in $L_K^1(\Omega)$, a similar argument as above shows that

$$\lim_{n\to\infty} \int Q(x)K(x)|u_n|^{2^*-2} u_n v = \int Q(x)K(x)|u_0|^{2^*-2} u_0 v. \tag{3.5}$$

From (3.4), (3.5) and the weak convergence of $\{u_n\}$, we conclude that

$$0 = \lim_{n\to\infty} \langle I'_\lambda(u_n), v\rangle = \langle I'_\lambda(u_0), v\rangle, \quad \text{for any} \quad v \in C_0^\infty(\mathbb{R}^N).$$

Hence, the claim follows by density and then

$$\int K(x)|\nabla u_0|^2 - \lambda \int K(x)|x|^\beta |u_0|^q = \int Q(x)K(x)|u_0|^{2^*},$$

$$I_\lambda(u_0) = \left(\frac{1}{2} - \frac{1}{q}\right) \int K(x)|\nabla u_0|^2 + \left(\frac{1}{q} - \frac{1}{2^*}\right) \int Q(x)K(x)|u_0|^{2^*} \ge 0.$$

Set $w_n = u_n - u_0$. According to Brezis–Lieb Lemma [1], we obtain that

$$\int Q(x)K(x)|u_n|^{2^*} = \int Q(x)K(x)|u_0|^{2^*} + \int Q(x)K(x)|w_n|^{2^*} + o(1)$$

and

$$\int K(x)|\nabla u_n|^2 = \int K(x)|\nabla u_0|^2 + \int K(x)|\nabla w_n|^2 + o(1).$$

One has that

$$\begin{aligned}\int K(x)|\nabla w_n|^2 &= \int K(x)|\nabla u_n|^2 - \int K(x)|\nabla u_0|^2 + o(1)\\ &= \lambda \int K(x)|x|^\beta |u_n|^q + \int Q(x)K(x)|u_n|^{2^*} - \int K(x)|\nabla u_0|^2 + o(1)\\ &= \lambda \int K(x)|x|^\beta |u_n|^q + \int Q(x)K(x)|u_0|^{2^*} + \int Q(x)K(x)|w_n|^{2^*}\\ &\quad - \int K(x)|\nabla u_0|^2 + o(1)\\ &= \int Q(x)K(x)|w_n|^{2^*} + o(1).\end{aligned}$$

Now assume

$$\int K(x)|\nabla w_n|^2 \to l \geq 0, \qquad \int Q(x)K(x)|w_n|^{2^*} \to l \geq 0.$$

If $l \neq 0$, it then follows from $\int Q(x)K(x)|w_n|^{2^*} \leq Q_M \int K(x)|w_n|^{2^*}$ and (2.1) that $l \geq S^{N/2}/Q_M^{(N-2)/2}$. Thus,

$$\begin{aligned}\lim_{n\to\infty} I_\lambda(u_n) &= I_\lambda(u_0) + \lim_{n\to\infty} I_\lambda(w_n)\\ &\geq I_\lambda(u_0) + \frac{1}{N}\int K(x)|\nabla w_n|^2\\ &\geq I_\lambda(u_0) + \frac{S^{N/2}}{NQ_M^{(N-2)/2}}\\ &\geq \frac{S^{N/2}}{NQ_M^{(N-2)/2}}\end{aligned}$$

which contradicts to the hypothesis in (1). Hence $l = 0$, i.e. $u_n \to u_0$ in $H(\alpha)$.

(**2**). From $\{u_n\} \subset \mathcal{V}_\lambda^j$, we get $I_\lambda(u_n^\pm) \geq m_\lambda^j$ and

$$\int K(x)|\nabla u_n^\pm|^2 = \lambda \int K(x)|x|^\beta |u_n^\pm|^q + \int Q(x)K(x)|u_n^\pm|^{2^*},$$

$$I_\lambda(u_n) = I_\lambda(u_n^+) + I_\lambda(u_n^-).$$

Since $I_\lambda(u_n) \to c < m_\lambda^j + S^{N/2}/NQ_M^{(N-2)/2}$ and $I_\lambda(u_n^\pm) \geq m_\lambda^j$, one must have

$$\lim_{n\to\infty} I_\lambda(u_n^\pm) < \frac{S^{N/2}}{NQ_M^{(N-2)/2}}.$$

It is trivial to show $\{u_n\}$ is bounded in $H(\alpha)$, so we may assume

$$u_n \rightharpoonup u_0 \text{ in } H(\alpha),$$

$$u_n \to u_0 \text{ a.e. on } \mathbb{R}^N,$$

$$u_n^\pm \rightharpoonup u_0^\pm \text{ in } H(\alpha).$$

Set $w_n^\pm = u_n^\pm - u_0^\pm$. We claim that $\|w_n^\pm\| \to 0$. Otherwise, we can establish the contradiction as in the proof of part (1). Thus we complete the proof. □

Proof of Theorem 1.1. For $j = 1, 2, \dots, k$, Lemma 3.1 implies that there are $\lambda_0 > 0$ and $\{u_n^j\} \subset \mathcal{M}_\lambda^j$ satisfying $u_n^j \geq 0$, $I_\lambda(u_n^j) \to m_\lambda^j$ and $I'_\lambda(u_n^j) \to 0$ for $\lambda \in (0, \lambda_0)$. It then follows from Lemmas 2.1 and 3.2 that $u_n^j \to u^j$ and $u^j \not\equiv 0$ is a weak solution of (1.1). By a standard elliptic regularity argument then $u^j \in C^2(\mathbb{R}^N)$ and the strong maximum principle imply that u^j is a positive solution of (1.1). Finally, since $g(u^j) \in B_\eta(a^j)$ and $B_\eta(a^j)$ are disjoint, we conclude that u^i and u^j are distinct if $i \neq j$. This completes the proof. □

4. Multiple sign-changing solutions under the condition $(Q2)$

Now, we are in a position to prove Theorem 1.2, following Cao and Noussair [2], we first show the following two lemmas.

Lemma 4.1. *Assume $\lambda > 0$ and conditions $(Q1)$, $(Q2)$ and $(Q3)$. For $j = 1, 2, \dots, k$, we have*

$$M_\lambda^j < m_\lambda^j + \frac{S^{N/2}}{N Q_M^{(N-2)/2}}, \tag{4.1}$$

if one of the following statements holds:
(i) $N \geq 4$, $\frac{2N-2}{N-2} < q < 2^*$;
(ii) $N = 3, 5 < q < 2^*$.

Proof. Let $z_\varepsilon = z_{\varepsilon,a^j} = K(x)^{-1/2}\varphi(x - a^j)U_{\varepsilon,a^j}$ be as Appendix A. Clearly, the proof will be completed if we can prove that:

(1) For each $\lambda > 0$, $j = 1, 2, \dots, k$ and $\varepsilon > 0$ small enough, there are $s_\varepsilon, t_\varepsilon \in [1/2, 2]$ such that $(s_\varepsilon u^j - t_\varepsilon z_\varepsilon)^\pm \in \mathcal{M}_\lambda^j$, where u^j is the positive solution obtained in Theorem 1.1.

(2) $\displaystyle\sup_{s,t\in[1/2,2]} I_\lambda(su^j - tz_\varepsilon) < m_\lambda^j + \frac{S^{N/2}}{N Q_M^{(N-2)/2}}$.

Proof of (1). Let δ be as Appendix A and set

$$d_1 = \min\{u^j(x) : |x - a^j| \leq 2\delta\},$$
$$d_2 = \max\{u^j(x) : |x - a^j| \leq 2\delta\},$$
$$A = 2\left(\frac{4C_N}{d_1}\right)^{1/(N-2)},$$
$$B = \frac{1}{2}\left(\frac{C_N K(a^j)^{-1/2}}{4d_2}\right)^{1/(N-2)}.$$

From the definition of z_ε, we know that $|x - a^j| \geq \sqrt{\varepsilon}A$ implies that $\frac{1}{2}u^j(x) \geq 2z_\varepsilon(x)$, and $|x - a^j| \leq \sqrt{\varepsilon}B$ implies that $2u^j(x) \leq \frac{1}{2}z_\varepsilon(x)$. Now define

$$\Sigma_\varepsilon^+ = \left\{x \in \mathbb{R}^N : su^j(x) \geq tz_\varepsilon(x), \quad t, s \in [1/2, 2]\right\},$$
$$\Sigma_\varepsilon^2 = \{x \in \mathbb{R}^N : |x - a^j| \geq \sqrt{\varepsilon}A\},$$
$$\Sigma_\varepsilon^3 = \{x \in \mathbb{R}^N : |x - a^j| \geq \sqrt{\varepsilon}B\}.$$

It is easy to see that

$$\Sigma_\varepsilon^2 \subset \Sigma_\varepsilon^+ \subset \Sigma_\varepsilon^3.$$

Next, we estimate the following integrals

$$\int K(x)|\nabla(su^j - tz_\varepsilon)^\pm|^2, \quad \int K(x)|x|^\beta|(su^j - tz_\varepsilon)^\pm|^q, \quad \int Q(x)K(x)|(su^j - tz_\varepsilon)^\pm|^{2^*}.$$

One has

$$\begin{aligned}
\int K(x)|\nabla(su^j - tz_\varepsilon)^+|^2 &= \int_{\Sigma_\varepsilon^+} K(x)|\nabla(su^j - tz_\varepsilon)|^2 \\
&\geq \int_{\Sigma_\varepsilon^2} K(x)|\nabla(su^j - tz_\varepsilon)|^2 \\
&\geq \int_{\Sigma_\varepsilon^2} K(x)|\nabla(su^j)|^2 - 2st\int_{\Sigma_\varepsilon^2} K(x)|\nabla u^j||\nabla z_\varepsilon| \\
&\geq \int K(x)|\nabla(su^j)|^2 - \int_{\mathbb{R}^N\setminus\Sigma_\varepsilon^2} K(x)|\nabla(su^j)|^2 \\
&\quad -8\left(\int K(x)|\nabla u^j|^2\right)^{1/2}\left(\int_{\Sigma_\varepsilon^2} K(x)|\nabla z_\varepsilon|^2\right)^{1/2}.
\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}
&\int K(x)|\nabla(su^j - tz_\varepsilon)^+|^2 \\
\leq &\int_{\Sigma_\varepsilon^3} K(x)|\nabla(su^j - tz_\varepsilon)^+|^2 \\
\leq &\int_{\Sigma_\varepsilon^3} K(x)|\nabla(su^j)|^2 + \int_{\Sigma_\varepsilon^3} K(x)|\nabla(tz_\varepsilon)|^2 + 8\int_{\Sigma_\varepsilon^3} K(x)|\nabla u^j||\nabla z_\varepsilon| \\
\leq &\int K(x)|\nabla(su^j)|^2 + 4\int_{\Sigma_\varepsilon^3} K(x)|\nabla z_\varepsilon|^2 \\
&+8\left(\int_{\Sigma_\varepsilon^3} K(x)|\nabla u^j|^2\right)^{1/2}\left(\int_{\Sigma_\varepsilon^3} K(x)|\nabla z_\varepsilon|^2\right)^{1/2}.
\end{aligned}$$

By easy calculation, we have $\int_{\Sigma_\varepsilon^3} K(x)|\nabla z_\varepsilon|^2 \leq C\varepsilon^{(N-2)/2}$ for some positive constant C. Then it follows from the inequalities above that

$$\left|\int K(x)|\nabla(su^j - tz_\varepsilon)^+|^2 - \int K(x)|\nabla(su^j)^+|^2\right| \leq C\varepsilon^{(N-2)/4}. \tag{4.2}$$

For the second integral, from an elementary inequality

$$(a-b)^r \geq a^r - ra^{r-1}b, \quad \text{for } a \geq b \geq 0 \text{ and } r > 2,$$

we have that

$$\begin{aligned}\int K(x)|x|^\beta|(su^j-tz_\varepsilon)^+|^q &= \int\limits_{\Sigma_\varepsilon^+} K(x)|x|^\beta|su^j-tz_\varepsilon|^q\\ &\geq \int\limits_{\Sigma_\varepsilon^2} K(x)|x|^\beta|su^j-tz_\varepsilon|^q\\ &\geq \int\limits_{\Sigma_\varepsilon^2} K(x)|x|^\beta|su^j|^q - qs^{q-1}t\int\limits_{\Sigma_\varepsilon^2} K(x)|x|^\beta|su^j|^{q-1}z_\varepsilon\\ &\geq \int K(x)|x|^\beta|su^j|^q - \int\limits_{\mathbb{R}^N\setminus\Sigma_\varepsilon^2} K(x)|x|^\beta|su^j|^q\\ &\quad -qs^{q-1}t\int\limits_{\Sigma_\varepsilon^2} K(x)|x|^\beta|su^j|^{q-1}z_\varepsilon.\end{aligned}$$

Moreover, it is easy to verify that

$$\int K(x)|x|^\beta|(su^j-tz_\varepsilon)^+|^q \leq \int K(x)|x|^\beta|su^j|^q,$$

$$\int\limits_{\mathbb{R}^N\setminus\Sigma_\varepsilon^2} K(x)|x|^\beta|su^j|^q \leq C_1\varepsilon^{N/2},$$

$$\int\limits_{\Sigma_\varepsilon^2} K(x)|x|^\beta|su^j|^{q-1}z_\varepsilon \leq C_2\varepsilon^{(N-2)/2},$$

for some positive constants C_1, C_2. From the inequalities above we obtain

$$\left|\int K(x)|x|^\beta|(su^j-tz_\varepsilon)^+|^q - \int K(x)|x|^\beta|su^j|^q\right| \leq C\varepsilon^{(N-2)/2}. \tag{4.3}$$

Similarly, we also have

$$\left|\int Q(x)K(x)|(su^j-tz_\varepsilon)^+|^{2^*} - \int Q(x)K(x)|su^j|^{2^*}\right| \leq C\varepsilon^{(N-2)/2}. \tag{4.4}$$

Define

$$h^+(s,t) = \int K(x)|\nabla(su^j-tz_\varepsilon)^+|^2 - \lambda\int K(x)|x|^\beta|(su^j-tz_\varepsilon)^+|^q - \int Q(x)K(x)|(su^j-tz_\varepsilon)^+|^{2^*}.$$

Since $\int K(x)|\nabla u^j|^2 - \lambda\int K(x)|x|^\beta|u^j|^q - \int Q(x)K(x)|u^j|^{2^*} = 0$, we can use (4.2)–(4.4) to obtain

$$h^+(s,t) = \int K(x)|\nabla su^j|^2 - \lambda\int K(x)|x|^\beta|su^j|^q - \int Q(x)K(x)|su^j|^{2^*} + O(\varepsilon^{(N-2)/4}).$$

By $I_\lambda(u^j) > 0$ and $\langle I_\lambda'(u^j), u^j\rangle = 0$, we can further obtain that, for $\varepsilon > 0$ small enough

$$h^+\left(\frac{1}{2},t\right) > 0, \quad h^+(2,t) < 0, \quad \text{for any } t \in \left[\frac{1}{2},2\right]. \tag{4.5}$$

Next, define:

$$h^-(s,t) = \int K(x)|\nabla(su^j-tz_\varepsilon)^-|^2 - \lambda\int K(x)|x|^\beta|(su^j-tz_\varepsilon)^-|^q - \int Q(x)K(x)|(su^j-tz_\varepsilon)^-|^{2^*}$$

Obviously,

$$h^-(s,t) = \int K(x)|\nabla(tz_\varepsilon - su^j)^+|^2 - \lambda \int K(x)|x|^\beta |(tz_\varepsilon - su^j)^+|^q - \int Q(x)K(x)|(tz_\varepsilon - su^j)^+|^{2^*}.$$

Let $\Sigma_\varepsilon^- = \{x \in \mathbb{R}^N : tz_\varepsilon(x) > su^j(x)\}$, it then follows that $\Sigma_\varepsilon^- = \mathbb{R}^N \setminus \Sigma_\varepsilon^+$ and $\mathbb{R}^N \setminus \Sigma_\varepsilon^3 \subset \Sigma_\varepsilon^- \subset \mathbb{R}^N \setminus \Sigma_\varepsilon^3$.

By using similar calculations, as above, we have

$$\left| \int K(x)|\nabla(tz_\varepsilon - su^j)^+|^2 - \int K(x)|\nabla(tz_\varepsilon)|^2 \right| \le C\varepsilon^{N/4},$$
$$\left| \int K(x)|x|^\beta |(tz_\varepsilon - su^j)^+|^q - \int K(x)|x|^\beta |tz_\varepsilon|^q \right| \le C\varepsilon^{N/2}, \quad \text{if} \quad q > \frac{2N-2}{N-2},$$
$$\left| \int Q(x)K(x)|(tz_\varepsilon - su^j)^+|^{2^*} - \int Q(x)K(x)|tz_\varepsilon|^{2^*} \right| \le C\varepsilon^{N/2},$$

and therefore

$$\begin{aligned} h^-(s,t) &= \int K(x)|\nabla(tz_\varepsilon)|^2 - \lambda \int K(x)|x|^\beta |tz_\varepsilon|^q - \int Q(x)K(x)|tz_\varepsilon|^{2^*} + O(\varepsilon^{N/4}) \\ &= t^2 \int K(x)|\nabla z_\varepsilon|^2 - \lambda t^q \int K(x)|x|^\beta |z_\varepsilon|^q - t^{2^*} \int Q(x)K(x)|z_\varepsilon|^{2^*} + O(\varepsilon^{N/4}) \\ &= t^2 A_1 + \tau(\varepsilon) - \lambda C\varepsilon^{N-(N-2)q/2} - t^{2^*} Q_M A_2 + o(\varepsilon^{N-(N-2)q/2}) + O(\varepsilon^{N/4}) \end{aligned} \tag{4.6}$$

where the last equality follows from Appendix (A.1a)–(A.1c), (A.7a) and (A.12).

Noting $A_1 = Q_M A_2$, then as in (4.5) we get, for $\varepsilon > 0$ small enough

$$h^-\left(t, \frac{1}{2}\right) > 0, \quad h^-(t,2) < 0, \quad \text{for any } t \in \left[\frac{1}{2}, 2\right]. \tag{4.7}$$

Using (4.5), (4.7) and a theorem by Miranda [12] (see also [19, p. 701, Theorem 1]), we derive that there are $s_\varepsilon, t_\varepsilon \in [1/2, 2]$ satisfying

$$h^+(s_\varepsilon, t_\varepsilon) = h^-(s_\varepsilon, t_\varepsilon) = 0$$

which implies $(s_\varepsilon u^j - t_\varepsilon z_\varepsilon)^\pm \in \mathcal{M}_\lambda$.

To complete the proof of part (1), it suffices to show that, for $\varepsilon > 0$ small enough

$$g\big((s_\varepsilon u^j - t_\varepsilon z_\varepsilon)^\pm\big) \in B_\eta(a^j).$$

First, we consider $g\big((s_\varepsilon u^j - t_\varepsilon z_\varepsilon)^+\big)$. One has

$$|(s_\varepsilon u^j - t_\varepsilon z_\varepsilon)^+|^{2^*} \le |s_\varepsilon u^j|^{2^*}$$

and

$$|(s_\varepsilon u^j - t_\varepsilon z_\varepsilon)^+|^{2^*} \ge |s_\varepsilon u^j|^{2^*} - 2^* s_\varepsilon^{2^*-1} t_\varepsilon |u^j|^{2^*-1} z_\varepsilon$$

on Σ_ε^+. Based on the two above inequalities we further obtain

$$0 \ge |(s_\varepsilon u^j - t_\varepsilon z_\varepsilon)^+|^{2^*} - |s_\varepsilon u^j|^{2^*} \ge -2^* s_\varepsilon^{2^*-1} t_\varepsilon |u^j|^{2^*-1} z_\varepsilon$$

and so

$$\left| \int K(x)\left[|(s_\varepsilon u^j - t_\varepsilon z_\varepsilon)^+|^{2^*} - |s_\varepsilon u^j|^{2^*}\right] \right| \le 2^* s_\varepsilon^{2^*-1} t_\varepsilon \int K(x)|u^j|^{2^*-1} z_\varepsilon, \tag{4.8}$$

$$\left| \int x_i K(x)\left[|(s_\varepsilon u^j - t_\varepsilon z_\varepsilon)^+|^{2^*} - |s_\varepsilon u^j|^{2^*}\right] \right| \le 2^* s_\varepsilon^{2^*-1} t_\varepsilon \int K(x)|x_i||u^j|^{2^*-1} z_\varepsilon, \tag{4.9}$$

for any x_i.

From (4.8) and (4.9), we get

$$g^i\big((s_\varepsilon u^j - t_\varepsilon z_\varepsilon)^+\big) \le \frac{\displaystyle\int x_i K(x)|s_\varepsilon u^j|^{2^*} + 2^* 2^{2^*} \int K(x)|x_i||u^j|^{2^*-1} z_\varepsilon}{\displaystyle\int K(x)|s_\varepsilon u^j|^{2^*} - 2^* 2^{2^*} \int K(x)|u^j|^{2^*-1} z_\varepsilon} \tag{4.10}$$

and

$$g^i\big((s_\varepsilon u^j - t_\varepsilon z_\varepsilon)^+\big) \ge \frac{\displaystyle\int x_i K(x)|s_\varepsilon u^j|^{2^*} - 2^* 2^{2^*} \int K(x)|x_i||u^j|^{2^*-1} z_\varepsilon}{\displaystyle\int K(x)|s_\varepsilon u^j|^{2^*} + 2^* 2^{2^*} \int K(x)|u^j|^{2^*-1} z_\varepsilon}. \tag{4.11}$$

Together with the estimate $\int K(x)|u^j|^{2^*-1} z_\varepsilon \le C\varepsilon^{(N-2)/2}$, (4.10) and (4.11) imply that

$$\begin{aligned}
g^i(u^j) - C\varepsilon^{(N-2)/2} &= \frac{\displaystyle\int x_i K(x)|u^j|^{2^*} - C\varepsilon^{(N-2)/2}}{\displaystyle\int K(x)|u^j|^{2^*} + C\varepsilon^{(N-2)/2}} \\
&\le g^i\big((s_\varepsilon u^j - t_\varepsilon z_\varepsilon)^+\big) \\
&\le \frac{\displaystyle\int x_i K(x)|u^j|^{2^*} + C\varepsilon^{(N-2)/2}}{\displaystyle\int K(x)|u^j|^{2^*} - C\varepsilon^{(N-2)/2}} \\
&= g^i(u^j) + C\varepsilon^{(N-2)/2},
\end{aligned}$$

for some positive constant C, independent of ε, which implies that $g\big((s_\varepsilon u^j - t_\varepsilon z_\varepsilon)^+\big) \in B_\eta(a^j)$, since $u^j \in \mathcal{M}_\lambda^j$ and $\mathcal{M}_\lambda^j$ is an open set. The proof for $g\big((s_\varepsilon u^j - t_\varepsilon z_\varepsilon)^-\big) \in B_\eta(a^j)$ is very similar. Thus, we complete the proof of (1).

Proof of (2).

$$\begin{aligned}
& I_\lambda(su^j - tz_\varepsilon) \\
= {} & I_\lambda(su^j) + \frac{t^2}{2}\int K(x)|\nabla z_\varepsilon|^2 + \frac{\lambda}{q}\int K(x)|x|^\beta |su^j|^q + \frac{1}{2^*}\int Q(x)K(x)|su^j|^{2^*} \\
& - st\int Q(x)K(x)|u^j|^{2^*-1} z_\varepsilon - st\lambda \int K(x)|x|^\beta |u^j|^{q-1} z_\varepsilon \\
& - \frac{\lambda}{q}\int K(x)|x|^\beta |su^j - tz_\varepsilon|^q - \frac{1}{2^*}\int Q(x)K(x)|su^j - tz_\varepsilon|^{2^*} \\
= {} & I_\lambda(su^j) + \frac{t^2}{2}\int K(x)|\nabla z_\varepsilon|^2 - \lambda\frac{t^q}{q}\int K(x)|x|^\beta |z_\varepsilon|^q \\
& - \frac{t^{2^*}}{2^*}\int Q_M K(x)|z_\varepsilon|^{2^*} - \frac{t^{2^*}}{2^*}\int \big(Q(x) - Q_M\big)K(x)|z_\varepsilon|^{2^*} \\
& - \frac{\lambda}{q}\int K(x)|x|^\beta \left[|su^j - tz_\varepsilon|^q - (su^j)^q - (tz_\varepsilon)^q + qst(u^j)^{q-1} z_\varepsilon\right] \\
& - \frac{1}{2^*}\int Q(x)K(x)\left[|su^j - tz_\varepsilon|^{2^*} - (su^j)^{2^*} - (tz_\varepsilon)^{2^*} + 2^* st (u^j)^{2^*-1} z_\varepsilon\right].
\end{aligned}$$

Using condition $(Q1)$ and Lemma A.3, if $q > \frac{2N-2}{N-2}$, we have from the following elementary inequality

$$|a+b|^r \geq |a|^r + |b|^r - C(|a|^{r-1}|b| + |a||b|^{r-1}), \qquad \text{for any } a,\, b \in \mathbb{R},\, r > 1$$

that

$$\begin{aligned}
&\left| \int K(x)|x|^\beta \left[|su^j - tz_\varepsilon|^q - (su^j)^q - (tz_\varepsilon)^q + qst(u^j)^{q-1} z_\varepsilon \right] \right| \\
&\leq C \Big(\int K(x)|x|^\beta \left[s^{q-1}t(u^j)^{q-1} z_\varepsilon + st^{q-1}(u^j) z_\varepsilon^{q-1} \right] \Big) \\
&\leq C_1 \Big(\int \left(z_\varepsilon^{q-1} + z_\varepsilon \right) \Big) \\
&= O\left(\varepsilon^{N-(N-2)(q-1)/2} \right) + O\left(\varepsilon^{(N-2)/2} \right),
\end{aligned}$$

and

$$\begin{aligned}
&\left| \int Q(x)K(x) \left[|su^j - tz_\varepsilon|^{2^*} - (su^j)^{2^*} - (tz_\varepsilon)^{2^*} + 2^* st(u^j)^{2^*-1} z_\varepsilon \right] \right| \\
&\leq C \left[\int K(x) \left(s^{2^*-1} t(u^j)^{2^*-1} z_\varepsilon + st^{2^*-1}(u^j) z_\varepsilon^{2^*-1} \right) \right] \\
&\leq C_1 \left(\int K(x) \left(z_\varepsilon^{2^*-1} + z_\varepsilon \right) \right) \\
&= O\left(\varepsilon^{(N-2)/2} \right).
\end{aligned}$$

Thus, we have from Lemma A.3 that

$$\begin{aligned}
&\sup_{s,t \in [1/2,2]} I_\lambda(su^j - tz_\varepsilon) \\
&\leq m_\lambda^j + \sup_{t \geq 0} \left[\frac{t^2}{2} \int K(x)|\nabla z_\varepsilon|^2 - \frac{t^{2^*}}{2^*} \int Q_M K(x)|z_\varepsilon|^{2^*} - \lambda \frac{t^q}{q} \int K(x)|x|^\beta |z_\varepsilon|^q \right] \\
&\quad + o\big(\varepsilon^{N-(N-2)q/2} \big) + O\big(\varepsilon^{N-(N-2)(q-1)/2} \big) + O\big(\varepsilon^{(N-2)/2} \big).
\end{aligned}$$

Moreover, as in the proof of Lemma 2.1, we obtain

$$\begin{aligned}
&\sup_{t \geq 0} \left[\frac{t^2}{2} \int K(x)|\nabla z_\varepsilon|^2 - \frac{t^{2^*}}{2^*} \int Q_M K(x)|z_\varepsilon|^{2^*} - \lambda \frac{t^q}{q} \int K(x)|x|^\beta |z_\varepsilon|^q \right] \\
&\leq \frac{S^{N/2}}{N Q_M^{(N-2)/2}} + \tau(\varepsilon) - C\varepsilon^{N-(N-2)q/2}.
\end{aligned}$$

Therefore, we can derive for $\varepsilon > 0$ small enough

$$\begin{aligned}
\sup_{s,t \in [1/2,2]} I_\lambda(su^j - tz_\varepsilon) &\leq m_\lambda^j + \frac{S^{N/2}}{N Q_M^{(N-2)/2}} + \tau(\varepsilon) - C\varepsilon^{N-(N-2)q/2} \\
&\quad + o\big(\varepsilon^{N-(N-2)q/2} \big) + O\big(\varepsilon^{N-(N-2)(q-1)/2} \big) + O\big(\varepsilon^{(N-2)/2} \big) \\
&< m_\lambda^j + \frac{S^{N/2}}{N Q_M^{(N-2)/2}},
\end{aligned}$$

if $N \geq 4$ and $\frac{2N-2}{N-2} < q < 2^*$. Clearly, the above inequality also holds for the case $N = 3$ and $5 < q < 2^*$. This completes the proof of (2). □

By Lemmas 2.1, 2.2 and 4.1, we have that there is $\lambda_0 > 0$ such that

$$\overline{M}_\lambda^j > M_\lambda^j,$$

for $\lambda \in (0, \lambda_0)$. Furthermore, similar to the proof of Lemma 3.1, we can establish the following lemma.

Lemma 4.2. *Assume conditions (Q1), (Q2) and (Q3). There exist $\lambda_0 > 0$ and a sequence $\{u_n^j\} \subset \mathcal{V}_\lambda^j$ such that*

$$I_\lambda(u_n^j) \to m_\lambda^j, \quad I'_\lambda(u_n^j) \to 0, \tag{4.12}$$

for $\lambda \in (0, \lambda_0)$.

Proof of Theorem 1.2. For $j = 1, 2, \ldots, k$, Lemma 4.2 implies that there are $\lambda_0 > 0$ and $\{u_n^j\} \subset \mathcal{V}_\lambda^j$ satisfying $I_\lambda(u_n^j) \to m_\lambda^j$ and $I'_\lambda(u_n^j) \to 0$ for $\lambda \in (0, \lambda_0)$. It then follows from Lemmas 3.2 and 4.1 that $u_n^j \to u^j$ and u^j is a weak solution of (1.1). Clearly, there exists a positive constant C such that $\|u^\pm\| \geq C > 0$, for all $u^\pm \in \mathcal{M}_\lambda^j$, and so $(u^j)^\pm \not\equiv 0$. Finally, since $g(u^j) \in B_\eta(a^j)$ and $B_\eta(a^j)$ are disjoint, we conclude that u^i and u^j are distinct if $i \neq j$. The proof of Theorem 1.2 is complete. □

5. Proofs of Theorems 1.3 and 1.4

In order to prove Theorem 1.3, we need the following Lemma, which can be proved from Lemmas B.1, B.2 and B.3, by an argument similar to that of Lemma 2.1.

Lemma 5.1. *Assume $\lambda > 0$ and conditions (Q1), (Q2′) and (Q3). For $a^1 = 0$, we have*

$$m_\lambda^1 < \frac{S^{N/2}}{NQ_M^{(N-2)/2}},$$

if one of the following statements holds:
(i) $N \geq \alpha + 2, 2 < q < 2^*$;
(ii) $3 \leq N < \alpha + 2, 2^* - 4/\alpha < q < 2^*$.

Proof of Theorem 1.3. Similar to the proof of Theorem 1.1, by Lemmas 5.1, 2.2 and 3.2, we can obtain that there are $\lambda_0 > 0$ and $\{u_n^1\} \subset \mathcal{M}_\lambda^1$ satisfying $u_n^1 \geq 0$, $u_n^1 \to u^1$ and $\lambda \in (0, \lambda_0)$. It then follows that $u^1 \not\equiv 0$ is a weak solution of (1.1) and $u^1 \geq 0$. Furthermore, by a standard elliptic regularity argument then $u^1 \in C^2(\mathbb{R}^N)$ and the strong maximum principle imply that u^1 is a positive solution of (1.1). From this and Theorem 1.1, it is easy to see that Theorem 1.3 holds. □

On the other hand, to prove Theorem 1.4, applying a similar argument of Lemma 4.1, we also can get the next result from Lemmas B.1, B.2 and B.3.

Lemma 5.2. *Assume $\lambda > 0$ and conditions (Q1), (Q2′) and (Q3). For $a^1 = 0$, we have*

$$M_\lambda^1 < m_\lambda^1 + \frac{S^{N/2}}{NQ_M^{(N-2)/2}},$$

if one of the following statements holds:
(i) $N \geq 2\alpha + 2, 2 < q < 2^*$;
(ii) $3 \leq N < 2\alpha + 2, 2^* - 2/\alpha < q < 2^*$.

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Proof of Theorem 1.4. Similar to the proof of Theorem 1.2, by Lemmas 5.2, 2.2 and 3.2, we conclude that there are $\lambda_0 > 0$ and $\{u_n^1\} \subset \mathcal{V}_\lambda^1$ satisfying $u_n^1 \to u^1$ and $\lambda \in (0, \lambda_0)$. It then follows that u^1 is a weak solution of (1.1) and $(u^1)^\pm \not\equiv 0$. Combining this with Theorem 1.2, it is easy to see that Theorem 1.4 holds. □

Appendix A

In this Appendix, we prove some estimates which will be used in the proofs of Theorem 1.1 and 1.2. Define a cutoff function $\varphi(x) \in C_0^2(\mathbb{R}^N)$ such that $\varphi(x) \equiv 1$ in $B_\delta(x_0)$, $\varphi(x) \equiv 0$ outside $B_{2\delta}(x_0)$, $0 \le \varphi \le 1$ and $0 < \delta < \eta/2$. Define

$$z_{\varepsilon,a^j} = K(x)^{-1/2}\varphi(x-a^j)\frac{C_N\varepsilon^{(N-2)/2}}{(\varepsilon^2 + |x-a^j|^2)^{(N-2)/2}},$$

where $C_N = \big(N(N-2)/Q_M\big)^{(N-2)/4}$, and set

$$U_{\varepsilon,a^j} = \frac{C_N\varepsilon^{(N-2)/2}}{(\varepsilon^2 + |x-a^j|^2)^{(N-2)/2}}.$$

For simplicity, we write z_ε instead of z_{ε,a^j} when there is no confusion.

Lemma A.1. *Assume condition (Q2) holds. Then for $\varepsilon > 0$ small,*

$$\begin{cases} \int K(x)|\nabla z_\varepsilon|^2 = A_1 + O(\varepsilon^{N-2}) + O(\varepsilon) + O(\varepsilon^2), & \text{if } N > 4, \quad \text{(a)} \\ A_1 + O(\varepsilon^{N-2}) + O(\varepsilon) + O(\varepsilon^2|\ln\varepsilon|), & \text{if } N = 4, \quad \text{(b)} \\ A_1 + O(\varepsilon^{N-2}) + O(\varepsilon|\ln\varepsilon|) + O(\varepsilon), & \text{if } N = 3, \quad \text{(c)} \end{cases} \tag{A.1}$$

where

$$A_1 = C_N^2(N-2)^2\int\frac{|x|^2}{(1+|x|^2)^N}.$$

Proof. By easy calculation, one has

$$\begin{aligned} &\int K(x)|\nabla z_\varepsilon|^2 \\ &= \int \varphi^2(x-a^j)\left[|\nabla U_{\varepsilon,a^j}| - \frac{\alpha}{4}w_\varepsilon(x\cdot\nabla U_{\varepsilon,a^j})|x|^{\alpha-2} + \frac{\alpha^2}{64}U_{\varepsilon,a^j}^2|x|^{2(\alpha-1)}\right] \\ &\quad +2\int \varphi(x-a^j)U_{\varepsilon,a^j}\nabla\big(\varphi^2(x-a^j)\big)\cdot\Big(\nabla U_{\varepsilon,a^j} - \frac{\alpha}{8}U_{\varepsilon,a^j}|x|^{\alpha-2}x\Big) \\ &\quad + \int U_{\varepsilon,a^j}^2|\nabla\big(\varphi(x-a^j)\big)|^2. \end{aligned}$$

First of all, if $N > 2$, we have

$$\begin{aligned} \int \varphi^2(x-a^j)|\nabla U_{\varepsilon,a^j}| &= C_N^2(N-2)^2\varepsilon^{N-2}\int\frac{|x-a^j|^2\varphi^2(x-a^j)}{(\varepsilon^2+|x-a^j|^2)^N} \\ &= C_N^2(N-2)^2\varepsilon^{N-2}\left(\int\frac{|x|^2}{(\varepsilon^2+|x|^2)^N} + \int\frac{\big(\varphi^2(x)-1\big)|x|^2}{(\varepsilon^2+|x|^2)^N}\right) \\ &= C_N^2(N-2)^2\varepsilon^{N-2}\left(\varepsilon^{2-N}\int\frac{|x|^2}{(1+|x|^2)^N} + O(1)\right) \\ &= C_N^2(N-2)^2\int\frac{|x|^2}{(1+|x|^2)^N} + O(\varepsilon^{N-2}). \end{aligned} \tag{A.2}$$

Secondly, we claim that

$$\begin{cases} \int \varphi^2(x-a^j)\frac{\alpha}{4}U_{\varepsilon,a^j}(x\cdot\nabla U_{\varepsilon,a^j})|x|^{\alpha-2} = O(\varepsilon), & \text{if } N>3, \quad \text{(a)} \\ O(\varepsilon|\ln\varepsilon|), & \text{if } N=3, \quad \text{(b)} \end{cases} \tag{A.3}$$

for $\varepsilon > 0$ small enough. Indeed, we have

$$\begin{aligned} &\int \varphi^2(x-a^j)\frac{\alpha}{4}U_{\varepsilon,a^j}(x\cdot\nabla U_{\varepsilon,a^j})|x|^{\alpha-2} \\ &= \frac{\alpha(N-2)}{4}\int_{B_{2\delta}(a^j)} \frac{\varepsilon^{N-2}\varphi^2(x-a^j)\big(x\cdot(x-a^j)\big)|x|^{\alpha-2}}{(\varepsilon^2+|x-a^j|^2)^{N-1}} \\ &\le C\varepsilon^{N-2}\int_{B_{2\delta}(0)} \frac{\varphi^2(x)\big((x+a^j)\cdot x\big)|x+a^j|^{\alpha-2}}{(\varepsilon^2+|x|^2)^{N-1}} \\ &\le C\varepsilon^{N-2}\int_{B_{2\delta}(0)} \frac{|x|}{(\varepsilon^2+|x|^2)^{N-1}} \\ &= C\varepsilon^{N-2}\varepsilon^{3-N}\int_0^{2\delta/\varepsilon} \frac{rdr}{(1+r^2)^{N-1}} \\ &= O(\varepsilon), \end{aligned}$$

whenever $N > 3$. Similar calculations can prove that (A.3b) holds.

Then, by using similar arguments to the above, we also have for $\varepsilon > 0$ small enough

$$\begin{cases} \int \varphi^2(x-a^j)\frac{\alpha^2}{64}U^2_{\varepsilon,a^j}|x|^{2(\alpha-1)} = O(\varepsilon^2), & \text{if } N>4, \quad \text{(a)} \\ O\left(\varepsilon^2|\ln\varepsilon|\right), & \text{if } N=4, \quad \text{(b)} \\ O(\varepsilon), & \text{if } N=3, \quad \text{(c)} \end{cases} \tag{A.4}$$

$$\int \varphi(x-a^j)U_{\varepsilon,a^j}\nabla\big(\varphi^2(x-a^j)\big)\cdot\Big(\nabla U_{\varepsilon,a^j} - \frac{\alpha}{8}U_{\varepsilon,a^j}|x|^{\alpha-2}x\Big) = O(\varepsilon^{N-2}). \tag{A.5}$$

$$\int U^2_{\varepsilon,a^j}|\nabla\big(\varphi(x-a^j)\big)|^2 = O(\varepsilon^{N-2}). \tag{A.6}$$

Combining (A.2)–(A.6), it is easy to see that Lemma A.1 holds. □

Lemma A.2. *Assume condition (Q2) holds. Then for $\varepsilon > 0$ small,*

$$\begin{cases} \int K(x)|x|^\beta|z_\varepsilon|^q \ge C\varepsilon^{N-(N-2)q/2}, & \textit{if } N\ge 4, \quad \text{(a)} \\ C\varepsilon^{3-q/2}, & \textit{if } N=3,\ 3<q<6, \quad \text{(b)} \\ C\varepsilon^{3-q/2}|\ln\varepsilon|, & \textit{if } N=3,\ q=3, \quad \text{(c)} \\ C\varepsilon^{q/2}, & \textit{if } N=3,\ 2<q<3. \quad \text{(d)} \end{cases} \tag{A.7}$$

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Proof. We have

$$\begin{aligned}
&\int K(x)|x|^\beta |z_\varepsilon|^q \\
&= \int_{B_{2\delta}(a^j)} \frac{K(x)|x|^\beta K(x)^{-q/2} C_N^q \varepsilon^{(N-2)q/2} \varphi^q(x-a^j)}{(\varepsilon^2+|x-a^j|^2)^{q(N-2)/2}} \\
&= C_N^q \varepsilon^{(N-2)q/2} \int_{B_{2\delta}(0)} \frac{K(x+a^j)^{1-q/2}|x+a^j|^\beta \varphi^q(x)}{(\varepsilon^2+|x|^2)^{q(N-2)/2}} \\
&\geq C\varepsilon^{(N-2)q/2} \int_{B_{2\delta}(0)} \frac{\varphi^q(x)}{(\varepsilon^2+|x|^2)^{q(N-2)/2}} \\
&= C\varepsilon^{(N-2)q/2} \left(\int_{B_{2\delta}(0)} \frac{1}{(\varepsilon^2+|x|^2)^{q(N-2)/2}} + \int_{B_{2\delta}(0)} \frac{\varphi^q(x)-1}{(\varepsilon^2+|x|^2)^{q(N-2)/2}} \right) \\
&= C\varepsilon^{(N-2)q/2} \left(\varepsilon^{N-(N-2)q} \int_{B_{2\delta/\varepsilon}(0)} \frac{1}{(1+|x|^2)^{q(N-2)/2}} + \int_{B_{2\delta}(0)} \frac{\varphi^q(x)-1}{(\varepsilon^2+|x|^2)^{q(N-2)/2}} \right) \\
&= C\varepsilon^{N-q(N-2)/2} + O(\varepsilon^{q(N-2)/2})
\end{aligned}$$

whenever $q > N/(N-2)$. Then, (A.7a) follows because the fact $N \geq 4$ implies $q > N/(N-2)$. The proofs of (A.7b)–(A.7d) are similar. □

Lemma A.3. *Assume conditions* (Q2) *and* (Q3) *hold. Then for* $\varepsilon > 0$ *small,*

$$(i) \int K(x)|z_\varepsilon|^{2^*} = A_2 + O(\varepsilon^N), \quad \textit{with} \quad A_2 = C_N^{2^*} \int \frac{1}{(1+|x|^2)^N}, \tag{A.8}$$

$$(ii) \int K(x)|z_\varepsilon|^{2^*-1} = O(\varepsilon^{(N-2)/2}), \tag{A.9}$$

$$(iii) \int K(x)|z_\varepsilon| = O(\varepsilon^{(N-2)/2}), \tag{A.10}$$

$$(iv) \int K(x)|z_\varepsilon|^{q-1} = O(\varepsilon^{N-(N-2)(q-1)/2}), \quad \textit{if } \frac{2N-2}{N-2} < q < 2^* \tag{A.11}$$

$$(v) \int Q(x)K(x)|z_\varepsilon|^{2^*} = Q_M A_2 + O(\varepsilon^N) + o(\varepsilon^{N-(N-2)q/2}). \tag{A.12}$$

Proof. We only prove part (*i*). Parts (*ii*)–(*v*) of the Lemma can be proved by a similar argument.

One has

$$\begin{aligned}
&\int K(x)|z_\varepsilon|^{2^*} \\
&= \int \frac{K(x)K(x)^{-N/(N-2)} C_N^{2^*} \varepsilon^N \varphi^{2^*}(x-a^j)}{(\varepsilon^2+|x-a^j|^2)^N} \\
&= C_N^{2^*} \varepsilon^N \int \frac{K(x+a^j)^{-2/(N-2)} \varphi^{2^*}(x)}{(\varepsilon^2+|x|^2)^N} \\
&= C_N^{2^*} \varepsilon^N \int \frac{1}{(\varepsilon^2+|x|^2)^N} + C_N^{2^*} \varepsilon^N \int \frac{K(x+a^j)^{-2/(N-2)} \varphi^{2^*}(x) - 1}{(\varepsilon^2+|x|^2)^N}.
\end{aligned}$$

Noting that $K(x+a^j)^{-2/(N-2)} \leq 1$ for $N > 2$, we have

$$C_N^{2^*} \varepsilon^N \int \frac{K(x+a^j)^{-2/(N-2)} \varphi^{2^*}(x) - 1}{(\varepsilon^2+|x|^2)^N} \leq C_N^{2^*} \varepsilon^N \int \frac{\varphi^{2^*}(x)-1}{(\varepsilon^2+|x|^2)^N} = O(\varepsilon^N),$$

and therefore

$$\int K(x)|z_\varepsilon|^{2^*} = C_N^{2^*}\varepsilon^N \int \frac{1}{(\varepsilon^2+|x|^2)^N} + O(\varepsilon^N) = C_N^{2^*}\int \frac{1}{(1+|x|^2)^N} + O(\varepsilon^N).$$

Hence, (i) holds. The proof is complete. $\square$

Appendix B

Being different from Appendix A, we consider the case one of the points $a^1, a^2, \dots, a^k$ is an origin in this Appendix. Without loss of generality, we can assume $a^1 = 0$. Let z_{ε,a^j} be as Appendix A. By the same argument of the proofs of Lemmas A.1, A.2 and A.3, the following three Lemmas can be proved.

Lemma B.1. *For $\varepsilon > 0$ small and $a^1 = 0$,*

$$\int K(x)|\nabla z_{\varepsilon,a^1}|^2 = \begin{cases} A_1 + O(\varepsilon^{N-2}) + O(\varepsilon^\alpha) + O(\varepsilon^{2\alpha}), & \text{if } N > 2\alpha+2,\\ A_1 + O(\varepsilon^{N-2}) + O(\varepsilon^\alpha) + O(\varepsilon^{2\alpha}|\ln\varepsilon|), & \text{if } N = 2\alpha+2,\\ A_1 + O(\varepsilon^{N-2}) + O(\varepsilon^\alpha) + O(\varepsilon^{N-2}), & \text{if } \alpha+2 < N < 2\alpha+2,\\ A_1 + O(\varepsilon^{N-2}) + O(\varepsilon^\alpha|\ln\varepsilon|) + O(\varepsilon^{N-2}), & \text{if } N = \alpha+2,\\ A_1 + O(\varepsilon^{N-2}) + O(\varepsilon^{N-2}) + O(\varepsilon^{N-2}), & \text{if } 3 \le N < \alpha+2,\end{cases}$$

where

$$A_1 = C_N^2 (N-2)^2 \int \frac{|x|^2}{(1+|x|^2)^N}.$$

Lemma B.2. *For $\varepsilon > 0$ small and $a^1 = 0$,*

$$(i)\quad \int K(x)|x|^\beta |z_{\varepsilon,a^1}|^q \ge \begin{cases} C\varepsilon^{N+\beta-(N-2)q/2}, & \text{if } \dfrac{2^*\alpha}{\alpha+2} < N \le 2^*,\\ C\varepsilon^{N+\beta-(N-2)q/2}|\ln\varepsilon|, & \text{if } N = \dfrac{2^*\alpha}{\alpha+2},\\ C\varepsilon^{(N-2)q/2}, & \text{if } 2 \le N \le \dfrac{2^*\alpha}{\alpha+2},\end{cases}$$

$$(ii)\quad \int K(x)|x|^\beta |z_{\varepsilon,a^1}| = O(\varepsilon^{(N-2)/2}).$$

Lemma B.3. *Assume condition $(Q3)$ holds. Then for $\varepsilon > 0$ small and $a^1 = 0$,*

$$(i)\quad \int K(x)|z_{\varepsilon,a^1}|^{2^*} = A_2 + O(\varepsilon^N), \quad \text{with} \quad A_2 = C_N^{2^*}\int \frac{1}{(1+|x|^2)^N},$$

$$(ii)\quad \int K(x)|z_{\varepsilon,a^1}|^{2^*-1} = O(\varepsilon^{(N-2)/2}),$$

$$(iii)\quad \int K(x)|z_{\varepsilon,a^1}| = O(\varepsilon^{(N-2)/2}),$$

$$(iv)\quad \int Q(x)K(x)|z_{\varepsilon,a^1}|^{2^*} = Q_M A_2 + O(\varepsilon^N) + o(\varepsilon^{N+\beta-(N-2)q/2}).$$

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