

Regularization of saddle-fold singularity for nonsmooth differential systems

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ABSTRACT

This paper is concerned with the saddle-fold singularity that is a codimension two singularity for piecewise smooth planar differential systems. The approach consists in apply the Regularization's Method to the unfolding of the saddle-fold singularity in order to obtain a 2-parameter family of smooth vector fields. Our main result says that the regularized family undergoes thought a classical Bogdanov–Takens bifurcation.

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1. Introduction

The classical theory of dynamical systems for smooth vector fields on the plane and on surfaces is very well established. Stability theories for such systems were done mainly by Andronov–Pontryagin [1] and Peixoto [8]. A vector field is structurally stable if its phase portrait does not change under small perturbations. Andronov–Pontryagin–Peixoto provide necessary and sufficient conditions to a vector field be structurally stable. After the understanding of structurally stable systems, importance was given to the study of bifurcations. We say that a bifurcation occurs in a family of vector fields when we observe an abrupt change at the phase portrait of the elements of the family when the parameter that governs the family crosses a certain value, called bifurcation point. It is clear that a bifurcation point is not structurally stable. In some sense it is possible to classify a bifurcation according to its level of degeneracy. Roughly speaking, the minimum number of parameters in a family that explains the behaviors of all vector fields in a neighborhood of a fixed element of the family is called codimension of the element. The greater the codimension of the bifurcation, the higher the level of degeneracy of the element.

The study of piecewise smooth vector fields is more recent, and the theory of discontinuous dynamical systems has had advance since the pioneering works of Andronov [1] and Filippov [3]. In the article [6], for example, is contained a list which classifies the local bifurcations of codimension 1 and some global

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bifurcations of discontinuous systems in the plane. And [4] complements this study by giving a complete classification for the local bifurcations of codimension 2 for Filippov Planar Systems. There are other works about the study of bifurcations in piecewise smooth vector fields in 3D, see for instance [2].

The Regularization's Method, introduced by Sotomayor–Teixeira in [10], provides a way to approximate piecewise smooth systems by smooth ones. The advantage of this method is that all the classic results established for smooth systems can be applied in the regularizations of the discontinuous systems. In the study of bifurcations a natural question that arises is: What happens to a particular bifurcation of codimension k in the world of piecewise smooth systems when being regularized? Does its codimension hold, increase, or decrease?

The answers of the previous questions, for $k = 0$ and $k = 1$, were given in the works [9] and [7], respectively. The Sotomayor–Teixeira Regularization's Method was used in [7] to explain the local bifurcations of codimension 1 listed at [6], by comparison with the bifurcations of the respective regularized families. Precisely, consider $Z_\lambda = (X_\lambda, Y_\lambda)$ a 1-parameter family of discontinuous vector fields having a straight line Σ as its discontinuity region. In each half-plane determined by Σ is set a vector field, X_λ or Y_λ . The cases considered in [7] consist to take Σ as the x -axis, the vector field Y_λ constant and transversal to the Σ , and X_λ a vector field with a hyperbolic equilibrium (focus, saddle, node), which may still be visible or invisible. For each type of hyperbolic equilibrium of X_λ there exist bifurcations for the discontinuous vector field Z_λ , arising the collision of these equilibrium points with the discontinuity region Σ , under the λ parameter variation. Given $\varepsilon > 0$, the function $\varphi_\varepsilon(y) = \frac{1}{2} + \frac{y}{2\sqrt{y^2 + \varepsilon^2}}$ was used in [7] to obtain the family of regularized vector fields

$$Z_{\lambda,\varepsilon}(x, y) = (1 - \varphi_\varepsilon(y))Y_\lambda(x, y) + \varphi_\varepsilon(y)X_\lambda(x, y).$$

In [7] it was confirmed that codimension one bifurcations, in the world of piecewise smooth vector fields, presented in [6], after the regularization process, keep the codimension in the world of smooth systems.

As far as we know, there is no result on this direction for the case $k = 2$. We recall that the local bifurcations for piecewise smooth systems of codimension 2 were classified in [4]. An important codimension 2 singularity presented in [4] is the *saddle-fold singularity*.

The aim of this paper is to prove that the codimension of the regularization of the saddle-fold singularity, is also a codimension 2 singularity in the world of smooth systems. The main result in the paper is the following.

Theorem 1. *The regularization of the unfolding of the saddle-fold singularity is given by a 2-parameter family $Z_{\eta,\delta}^R$ and there exist parameter values $\eta = \eta_0$ and $\delta = \delta_0$ such that the regularized family $Z_{\eta,\delta}^R$ undergoes through a classical Bogdanov–Takens bifurcation at (η_0, δ_0) .*

A more precise statement of this theorem can be found in Section 4. In the practice if we have some result that says that the codimension is maintained by regularizations then our task would be to regularize, to a smooth system, and to study its codimension in the smooth world. Unfortunately, we still do not have a complete answer to the question if the codimension of the singularity hold, increase, or decrease. But our result, in addition to the works [7] and [9], exhibits some singularities where the codimension of the singularity is maintained after a regularization process.

The paper is organized as follows. In Section 2 we give the preliminaries concepts that will be used along the paper including the classical Bogdanov–Takens Theorem. In Section 3 we present the saddle-fold singularity and its unfolding. In Section 4 we present a general definition of regularization of a discontinuous vector field and also define when the regularization is of the transition type; we apply it to the unfolding of the saddle-fold singularity and we give the precise statement of our main result. We finish the paper with Section 5 where we prove the main result.

2. Preliminaries

We consider a smooth submanifold $\Sigma \subset \mathbb{R}^2$ given by $\Sigma = f^{-1}(0)$, where the smooth function $f : \mathcal{D} \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ have $0 \in \mathbb{R}$ as a regular value and \mathcal{D} as an open neighborhood of the origin.

The curve Σ , called the *discontinuity set*, separates the open set \mathcal{D} in two open regions $\Sigma^+ = \{(x, y) \in \mathcal{D} : f(x, y) > 0\}$ and $\Sigma^- = \{(x, y) \in \mathcal{D} : f(x, y) < 0\}$. In this paper we assume without loss of generality that the function $f(x, y) = y$ defines Σ in neighborhood of $(0, 0)$.

Let us denote by $\Omega^r(\mathcal{D})$ the set of C^r planar vector fields ($r > 1$), and by $\mathcal{X}^r(\mathcal{D})$ the set of planar discontinuous vector fields $Z = (X, Y)$ given by

$$Z(x, y) = \begin{cases} X(x, y), & \text{if } (x, y) \in \Sigma^+, \\ Y(x, y), & \text{if } (x, y) \in \Sigma^-, \end{cases} \quad (1)$$

where $X, Y \in \Omega^r(\mathcal{D})$.

We can identify the following regions in Σ :

- (1) *Crossing region*: $\Sigma_c = \{p \in \Sigma : Xf(p) \cdot Yf(p) > 0\}$,
- (2) *Sliding region*: $\Sigma_s = \{p \in \Sigma : Xf(p) < 0, Yf(p) > 0\}$,
- (3) *Escaping region*: $\Sigma_e = \{p \in \Sigma : Xf(p) > 0, Yf(p) < 0\}$,

where $Xf(p) = \langle \nabla f(p), X(p) \rangle$.

In the regions Σ_s and Σ_e , the Filippov convention gives us a sliding vector field F_Z , which is the convex combination of vector fields X and Y , see details in [3]

$$F_Z(p) = (1 - l)Y(p) + lX(p), \quad l = \frac{Yf(p)}{Yf(p) - Xf(p)}.$$

A point $p \in \mathcal{D}$ is a *singularity* of (1) if: $p \in \Sigma^+ \cup \Sigma^-$ is an equilibrium of X or Y (that is, $X(p) = 0$ or $Y(p) = 0$), $p \in \Sigma_s \cup \Sigma_e$ is a pseudo-equilibrium (that is, $F_Z(p) = 0$), or p is a tangency point ($Xf(p) = 0$ or $Yf(p) = 0$).

A planar vector field $X \in \Omega^r(\mathcal{D})$ has a *quadratic tangency* with Σ at $T \in \Sigma$ when $Xf(T) = 0$ and $X^2f(T) = X(Xf)(T) = \langle \nabla(Xf)(T), X(T) \rangle \neq 0$.

Now we present the classical Bogdanov–Takens theorem and the bifurcation diagram (see Fig. 1) that can be found in [5].

Theorem 2 (Bogdanov–Takens). *Suppose that a planar system*

$$\dot{x} = F(x, \beta), \quad x \in \mathbb{R}^2, \quad \beta \in \mathbb{R}^2,$$

has a singularity $x_0 = (0, 0)$ where a linear part have a double zero eigenvalue at $\beta_0 = (0, 0)$. Assume that following generic conditions are satisfied:

(BT.0) *the Jacobian matrix $A_0 = JF(x_0, \beta_0) \neq 0$;*

(BT.1) *$a_{20}(\beta_0) + b_{11}(\beta_0) \neq 0$;*

(BT.2) *$b_{20}(\beta_0) \neq 0$; and*

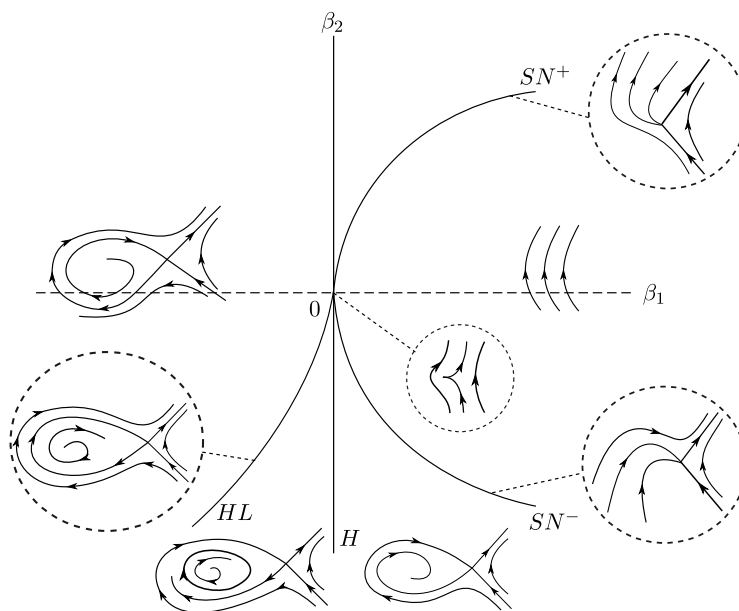


Fig. 1. Bogdanov–Takens bifurcation.

(BT.3) the map

$$(x, \beta) \mapsto \left(F(x, \beta), \operatorname{tr}(JF(x, \beta)), \det(JF(x, \beta)) \right)$$

is regular at (x_0, β_0) . Then, the planar system admits a Bogdanov–Takens bifurcation at (x_0, β_0) .

Remark 1. In above theorem,

$$\begin{aligned} a_{20}(\beta) &= \frac{\partial^2}{\partial y_1^2} \langle F(y_1 v_0 + y_2 v_1, \beta), w_0 \rangle \Big|_{x=0}, \\ b_{20}(\beta) &= \frac{\partial^2}{\partial y_1^2} \langle F(y_1 v_0 + y_2 v_1, \beta), w_1 \rangle \Big|_{x=0}, \\ b_{11}(\beta) &= \frac{\partial^2}{\partial y_1 \partial y_2} \langle F(y_1 v_0 + y_2 v_1, \beta), w_1 \rangle \Big|_{x=0}, \end{aligned}$$

where $y_1, y_2 \in \mathbb{R}$ and v_0, v_1 (resp. w_0, w_1) are the eigenvector and generalized eigenvector of A_0 (resp. A_0^T),

$$A_0 v_0 = 0, \quad A_0 v_1 = v_0, \quad A_0^T w_1 = 0, \quad A_0^T w_0 = w_1,$$

satisfying $\langle v_0, w_0 \rangle = \langle v_1, w_1 \rangle = 1$ and $\langle v_1, w_0 \rangle = \langle v_0, w_1 \rangle = 0$.

3. The saddle-fold singularity

Consider a planar discontinuous vector field $Z = (X, Y) \in \mathcal{X}^r(\mathcal{D})$. When the planar vector fields X and Y have, respectively, a hyperbolic saddle and a quadratic tangency (or fold) at the same point $p \in \Sigma$, we say that p is a *saddle-fold singularity* (see Fig. 2).

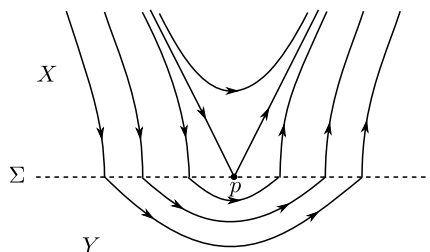


Fig. 2. Saddle-fold singularity.

There are two types of saddle-fold singularity, which are distinguished by $p \in \partial\Sigma_c$ (boundary of Σ_c) or $p \in \partial\Sigma_s \cap \partial\Sigma_e$. In [4] are given generic conditions on a discontinuous vector field with a saddle-fold singularity $p \in \partial\Sigma_c$ in order to obtain a local bifurcation of codimension 2.

We take the saddle-fold singularity $p = (0, 0)$ with the normal form

$$Z(x, y) = \begin{cases} X(x, y) = (x + y, 4x + y), & \text{if } y > 0, \\ Y(x, y) = (\alpha, x), & \text{if } y < 0, \end{cases} \quad (2)$$

where $\alpha > 0$ is a fixed constant.

The normal form (2) satisfies the generic conditions given in [4] since the modulus of the eigenvalues of the saddle are distinct and the associated eigenspaces are transversal to Σ . In addition, $p \in \partial\Sigma_c = \Sigma \setminus \{(0, 0)\}$.

The unfolding is given by

$$Z_{\lambda, \mu}(x, y) = \begin{cases} X_{\mu}(x, y) = (x + y - \mu, 4x + y), & \text{if } y > 0, \\ Y_{\lambda}(x, y) = (\alpha, x - \lambda), & \text{if } y < 0, \end{cases} \quad (3)$$

with $\alpha > 0$ fixed and $\lambda, \mu \in \mathbb{R}$. The phase portraits of (3) can be seen in Fig. 3. These phase portraits were known before, see Figs. 25, 26 and 27 in [4].

Thus, the saddle point $S_{\mu} = (-\mu/3, 4\mu/3)$ of vector field X_{μ} is visible ($\mu > 0$), invisible ($\mu < 0$), or is on the boundary Σ ($\mu = 0$), depending on the sign of μ . And the parameter λ in the system (3) acts by moving the fold of the vector field Y_{λ} , changing its position in relation to the fold of the vector field X_{μ} . For values $\mu > 0$ have the birth of a periodic orbit for $\lambda < 0$, which is repelling and persists throughout Region 2 of the parameter space (λ, μ) . This periodic orbit breaks down when it reaches the invariant manifolds of the saddle (curve $\gamma = \{(\lambda, \mu); \mu = -3\lambda/2\}$ in Fig. 3), becoming a homoclinic loop.

4. Regularization

In some sense, we want to smooth the 2-parameter family (3) of piecewise smooth differential system and see what happens to the codimension of the bifurcation. In order to perform this smoothing we present a general definition of regularization of a discontinuous vector field and also define when the regularization is of transition type.

Definition 1. Let $Z = (X, Y) \in \mathcal{X}^r(\mathcal{D})$ be a planar discontinuous vector field, with discontinuity set Σ . A regularization of Z is a 1-parameter family of planar vector fields $Z_{\varepsilon} \in \Omega^r(\mathcal{D})$ satisfying the property that Z_{ε} converges pointwise to Z in $\mathcal{D} \setminus \Sigma$.

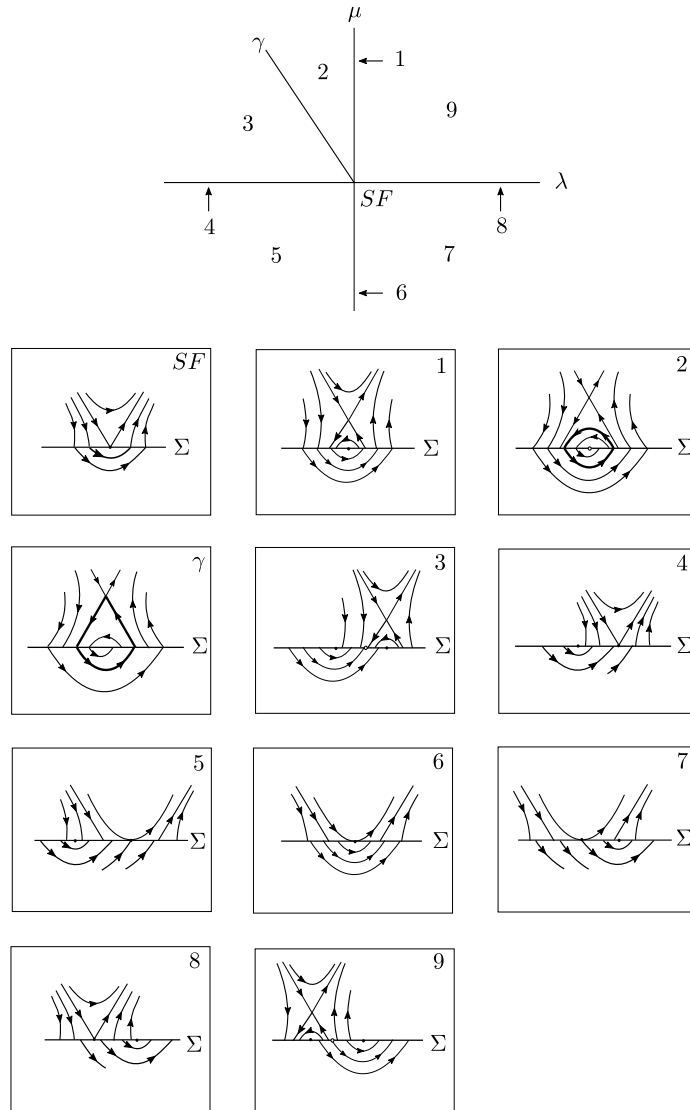


Fig. 3. Bifurcation diagram: unfolding of the saddle-fold singularity.

Definition 2. A C^∞ monotonous function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\lim_{x \rightarrow -\infty} \varphi(x) = 0 \quad \text{and} \quad \lim_{x \rightarrow +\infty} \varphi(x) = 1,$$

is called a *transition function*.

Definition 3. Let $Z = (X, Y) \in \mathcal{X}^r(\mathcal{D})$ be a planar discontinuous vector field, where $\Sigma = f^{-1}(0)$ is defined by a smooth function $f : \mathcal{D} \subset \mathbb{R}^2 \rightarrow \mathbb{R}$. Given $\varepsilon > 0$ and φ a transition function, we say that

$$Z_\varepsilon(p) = (1 - \varphi_\varepsilon(f(p)))Y(p) + \varphi_\varepsilon(f(p))X(p), \quad (4)$$

is a *regularization of transition type*, where $\varphi_\varepsilon(x) = \varphi(x/\varepsilon)$. In this case, ε is said to be a *regularization parameter*.

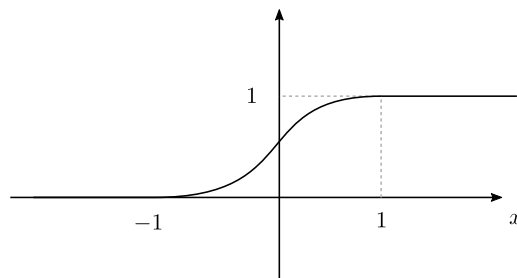


Fig. 4. Transition function of Sotomayor–Teixeira.

Example 1. Consider $Z = (X, Y)$ a planar discontinuous vector field, $\Sigma = f^{-1}(0)$ given by $f(x, y) = y$, and φ the C^∞ function

$$\varphi(x) = \frac{1}{2} + \frac{x}{2\sqrt{x^2 + 1}}.$$

Thus, given $\varepsilon > 0$, the regularization

$$Z_\varepsilon(x, y) = \left(\frac{1}{2} - \frac{y}{2\sqrt{y^2 + \varepsilon^2}} \right) Y(x, y) + \left(\frac{1}{2} + \frac{y}{2\sqrt{y^2 + \varepsilon^2}} \right) X(x, y),$$

is a regularization of transition type.

Example 2 (*Sotomayor–Teixeira Regularization's Method*). Consider a C^∞ function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ satisfying the properties

$$\varphi(x) = \begin{cases} 0, & \text{if } x \leq -1, \\ 1, & \text{if } x \geq 1, \end{cases} \quad \text{and } \varphi'(x) > 0 \text{ if } x \in (-1, 1). \quad (5)$$

In this case, the regularization is called *Sotomayor–Teixeira Regularization*, and the φ function satisfying the properties (5) is called a *transition function of Sotomayor–Teixeira* (Fig. 4). Given a discontinuous vector field $Z = (X, Y)$, the method consists of considering the convex combination of the vector fields X and Y described in (4) by using transition functions (see details in [10]).

In this paper we use the function

$$\varphi(x) = \frac{1}{2} + \frac{x}{2\sqrt{x^2 + 1}} \quad (6)$$

to compute the regularizations of transition type of discontinuous vector fields. Thus, given $\varepsilon > 0$, the regularization of a discontinuous vector field will be described through the function family

$$\varphi_\varepsilon(x) = \varphi(x/\varepsilon) = \frac{1}{2} + \frac{x}{2\sqrt{x^2 + \varepsilon^2}}. \quad (7)$$

Although not being exactly a transition function of Sotomayor–Teixeira, this function has been used in recent studies in [7] and has shown good results in that work. First, the function (7) is so close to any transition function of Sotomayor–Teixeira as we want, by taking positive values of decreasing ε on the function family φ_ε (see Fig. 5). The second reason lies in the simplicity of the expression of (6), which allows greater ease in algebraic manipulation and calculation of regularizations.

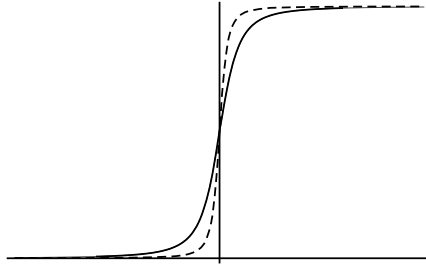


Fig. 5. Transition function (7), for $\varepsilon = 0.5$ (solid) and $\varepsilon = 0.2$ (dashed).

Thus, using the function (6) to regularize the family of discontinuous vector fields (3), we obtain the family of regularized vector fields

$$Z_{\lambda,\mu,\varepsilon}(x,y) = (1 - \varphi_\varepsilon(y))Y_\lambda(x,y) + \varphi_\varepsilon(y)X_\mu(x,y).$$

Denote $Z_{\lambda,\mu,\varepsilon} = (Z_{\lambda,\mu,\varepsilon}^1, Z_{\lambda,\mu,\varepsilon}^2)$. So

$$\begin{aligned} Z_{\lambda,\mu,\varepsilon}^1(x,y) &= \left(\frac{1}{2} - \frac{y}{2\sqrt{y^2+\varepsilon^2}}\right)\alpha + \left(\frac{1}{2} + \frac{y}{2\sqrt{y^2+\varepsilon^2}}\right)(x+y-\mu), \\ Z_{\lambda,\mu,\varepsilon}^2(x,y) &= \left(\frac{1}{2} - \frac{y}{2\sqrt{y^2+\varepsilon^2}}\right)(x-\lambda) + \left(\frac{1}{2} + \frac{y}{2\sqrt{y^2+\varepsilon^2}}\right)(4x+y). \end{aligned}$$

Through the following change of variables and rescaling of parameters: $x = \varepsilon\bar{x}$, $y = \varepsilon\bar{y}$, $\alpha = \varepsilon\bar{\alpha}$, $\lambda = \varepsilon\eta$, $\mu = \varepsilon\delta$, we obtain the family of regularized vector fields $Z_{\eta,\delta}^R = (Z_1, Z_2)$, where

$$\begin{aligned} Z_1(x,y) &= \left(\frac{1}{2} - \frac{y}{2\sqrt{y^2+1}}\right)\alpha + \left(\frac{1}{2} + \frac{y}{2\sqrt{y^2+1}}\right)(x+y-\delta), \\ Z_2(x,y) &= \left(\frac{1}{2} - \frac{y}{2\sqrt{y^2+1}}\right)(x-\eta) + \left(\frac{1}{2} + \frac{y}{2\sqrt{y^2+1}}\right)(4x+y), \end{aligned} \quad (8)$$

and we keep to using the (x,y,α) notation instead of $(\bar{x},\bar{y},\bar{\alpha})$.

Our main goal is to find conditions on α so that the system $Z_{\eta,\delta}^R$ passes through a Bogdanov–Takens bifurcation, at some specific value of the parameters $\eta = \eta_0$ and $\delta = \delta_0$. It will be seen in the proof of Theorem 1 that this is possible if α is not root of three polynomials R_1 , R_2 and R_3 , given below:

$$\begin{aligned} R_1(t) &= -32768t^{14}(9 + 512t^2)(245 + 16t^2(199 + 688t^2 + 256t^4)) \\ &\quad (-1125 + 16t^2(-130 + 1279t^2 + 14656t^4 + 256t^6)), \\ R_2(t) &= 2592t^{10}(361 + 129074t^2 + 7793725t^4 + 191102976t^6) \\ &\quad (-33768075 - 194462570t^2 + 7722010973t^4 + \\ &\quad 113346222160t^6 + 537224695680t^8 + 883362299904t^{10} + \\ &\quad 154241335296t^{12} + 14495514624t^{14}), \\ R_3(t) &= 324t^5(16 + 5816t^2 + 393649t^4 + 8847360t^6) \\ &\quad (2178000 + 13491800t^2 + 26866125t^4 + 32019984t^6 + \\ &\quad 682112t^8). \end{aligned} \quad (9)$$

Define the set $S = S_1 \cup S_2 \cup S_3$, where $S_k = \{t \in \mathbb{R} : R_k(t) = 0\}$ for $k = 1, 2, 3$. We will prove in the next section the following theorem.

Theorem 1 (Main theorem). Let $Z_{\lambda,\mu}$ be the 2-parameter family of discontinuous vector fields given by (3), with $\alpha > 0$ and $\alpha \notin S$. There exist parameter values $\eta = \eta_0$ and $\delta = \delta_0$ such that the respective regularized family $Z_{\eta,\delta}^R$ undergoes through a classical Bogdanov–Takens bifurcation at (η_0, δ_0) .

5. Proof of Theorem 1

The first step in the proof Theorem 1 is the following Proposition.

Proposition 1. There exist an equilibrium point $x_0 = (x_0, y_0)$ of system (8) and parameter values $\beta_0 = (\eta_0, \delta_0)$ for which $\text{tr}_S(A)(x_0, \beta_0) = \det_S(A)(x_0, \beta_0) = 0$, where $\text{tr}_S(A)$ and $\det_S(A)$ are trace and determinant of the Jacobian matrix of (8) at (x_0, β_0) .

Before starting the proof of Proposition 1 we would like to remark the following. Determine the singularities of system (8) is the first step to detect possible local bifurcations of smooth vector fields with one or more parameters. So, it is natural that the equilibrium points depend on system parameters, i.e., $x(\eta, \delta)$ and $y(\eta, \delta)$. However, due to the expression $\sqrt{y^2 + 1}$ in (8), determine the singularities of the system by solving $(Z_1, Z_2) = (0, 0)$ in the variables x and y is impractical. In order to prove Proposition 1, we will use the fact that $\pm y + \sqrt{y^2 + 1} \neq 0$ for all $y \in \mathbb{R}$, to express the parameter values η and δ , in terms of x and y .

Proof of Proposition 1. For each $(x, y) \in \mathbb{R}^2$ system (8) has an equilibrium point for the following parameter values

$$\begin{aligned}\eta &= \frac{y(y + \sqrt{y^2 + 1}) + x(3y + 5\sqrt{y^2 + 1})}{-y + \sqrt{y^2 + 1}}, \\ \delta &= \frac{y^2 + x(y + \sqrt{y^2 + 1}) + y(\sqrt{y^2 + 1} - \alpha) + \alpha\sqrt{y^2 + 1}}{y + \sqrt{y^2 + 1}}.\end{aligned}\tag{10}$$

Denoting the Jacobian matrix $JZ_{\eta,\delta}^R(x, y) = A = (a_{ij})$, $1 \leq i, j \leq 2$, we have

$$\begin{aligned}a_{11} &= \frac{1}{2} + \frac{y}{2\sqrt{y^2 + 1}}, & a_{12} &= \frac{x + 2y + y^3 + (y^2 + 1)^{3/2} - (\alpha + \delta)}{2(y^2 + 1)^{3/2}}, \\ a_{21} &= \frac{5}{2} + \frac{3y}{2\sqrt{y^2 + 1}}, & a_{22} &= \frac{3x + 2y + y^3 + (y^2 + 1)^{3/2} + \eta}{2(y^2 + 1)^{3/2}},\end{aligned}$$

and

$$\begin{aligned}\text{tr}(A) &= \frac{3(x + y) + 2y^3 + 2(y^2 + 1)^{3/2} + \eta}{2(y^2 + 1)^{3/2}}, \\ \det(A) &= \frac{p(y) + \hat{y}q(x, y)}{4(y^2 + 1)^2},\end{aligned}$$

where $p(y) = -6y^4 - 12y^2 + (3\alpha + \eta + 3\delta)y - 4$, $q(x, y) = -6y^3 - 10y - 2x + 5\alpha + 5\delta + \eta$, and $\hat{y} = \sqrt{y^2 + 1}$. In order to investigate local bifurcations of codimension 2 under the variation of the parameters (η, δ) , we will substitute the values of η and δ , given in (10), in the expressions of $\text{tr}(A)$ and $\det(A)$ obtaining new expressions $\det_S(A)(x, y)$ and $\text{tr}_S(A)(x, y)$ given below, and in the following compute the intersection of the curves $\det_S(A)(x, y) = 0$ and $\text{tr}_S(A)(x, y) = 0$ in the xy -plane. So,

$$\begin{aligned} tr_S(A)(x, y) &= \frac{1 + y^2 + (4x + y)\widehat{y}}{(y^2 + 1)^{3/2}(-y + \widehat{y})}, \\ det_S(A)(x, y) &= \frac{r(x, y) + \widehat{y}s(x, y)}{2(y^2 + 1)^{3/2}}, \end{aligned}$$

where $r(x, y) = -y^3 + 8xy^2 + 2\alpha y^2 + 4x - 2y + 5\alpha$, and $s(x, y) = -y^2 + 8xy - 2\alpha y - 2$.

The equation $tr_S(A)(x, y) = 0$ gives us $x = -\frac{1}{4}(y + \sqrt{y^2 + 1})$. Replacing in equation $det_S(A)(x, y) = 0$, we get

$$\frac{-5y^3 - 3\sqrt{1 + y^2} + 5\alpha + y^2(-5\sqrt{1 + y^2} + 2\alpha) - y(5 + 2\sqrt{1 + y^2}\alpha)}{2(1 + y^2)^{3/2}} = 0.$$

By simplifying the numerator we get

$$-5y - 5y^3 + 5\alpha + 2y^2\alpha + \sqrt{1 + y^2}(-3 - 5y^2 - 2y\alpha) = 0,$$

or equivalently $-5y - 5y^3 + 5\alpha + 2y^2\alpha = -\sqrt{1 + y^2}(-3 - 5y^2 - 2y\alpha)$. When applying square on both sides we get the polynomial equation

$$40\alpha y^5 + 5y^4 + 102\alpha y^3 + (14 - 16\alpha^2)y^2 + 62\alpha y + (9 - 25\alpha^2) = 0.$$

For each α fixed, $P(y, \alpha) = 40\alpha y^5 + 5y^4 + 102\alpha y^3 + (14 - 16\alpha^2)y^2 + 62\alpha y + (9 - 25\alpha^2)$ is a polynomial of degree 5 in the variable y . Therefore, the solution of the non-linear system

$$\begin{cases} tr_S(A)(x, y) = 0 \\ det_S(A)(x, y) = 0 \end{cases}$$

is given by a pair (x, y) satisfying $P(y, \alpha) = 0$ and $x = -\frac{1}{4}(y + \sqrt{y^2 + 1})$, where P is the polynomial with real coefficients $P(t) = 40\alpha t^5 + 5t^4 + 102\alpha t^3 + (14 - 16\alpha^2)t^2 + 62\alpha t + (9 - 25\alpha^2)$. Since $\alpha > 0$ and P is odd degree, we have $\lim_{t \rightarrow +\infty} P(t) = +\infty$ and $\lim_{t \rightarrow -\infty} P(t) = -\infty$. Thus, by Bolzano's Theorem, there is $y_0 \in \mathbb{R}$ such that $P(y_0) = 0$. The proof follows by considering $x_0 = -\frac{1}{4}(y_0 + \sqrt{y_0^2 + 1})$ and (η_0, δ_0) given in (10) by $\eta_0 = \eta(x_0, y_0)$ and $\delta_0 = \delta(x_0, y_0)$. \square

Now we prove that the generic conditions of Theorem 2 are satisfied for the system given by the family of regularized vector fields (8). First, the Proposition 1 ensures that in equilibrium point $x_0 = (x_0, y_0)$, the system $\dot{x} = Z_{\eta, \delta}^R(x, \beta)$ has, at $\beta_0 = (\eta_0, \delta_0)$, the Jacobian matrix $A_0 = JZ_{\eta, \delta}^R(x_0, \beta_0)$ with double zero eigenvalue. Furthermore, the matrix A_0 is non-zero since

$$a_{11}(x_0, \beta_0) = \frac{1}{2} + \frac{y_0}{2\sqrt{y_0^2 + 1}} \neq 0.$$

Thus, it remains to prove the generic conditions (BT.1), (BT.2) and (BT.3). The validity of these items is ensured by lemmas below.

Lemma 1. Consider $Z_{\eta, \delta}^R$ the family of regularized vector fields given by (8) where $\alpha > 0$, and α is not a root of the polynomial R_1 , where R_1 is given in (9). Then, the Bogdanov–Takens generic condition (BT.1) is satisfied at $\eta_0 = \eta(x_0, y_0)$, $\delta_0 = \delta(x_0, y_0)$.

Proof. Denote A_0 simply by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Since $\text{tr}(A_0) = \det(A_0) = 0$ we can rewrite $A_0 = \begin{pmatrix} a & b \\ -a^2/b & -a \end{pmatrix}$. The computation of eigenvectors and generalized eigenvectors gives us

$$v_0 = (1, -a/b), \quad v_1 = (0, 1/b), \quad w_0 = (1, 0), \quad w_1 = (a, b).$$

Thus, considering the translation

$$x \mapsto x - x_0, \quad y \mapsto y - y_0$$

for the system (8), we get

$$\begin{aligned} a_{20}(\beta_0) &= \frac{\partial^2}{\partial y_1^2} \left\langle Z_{\eta, \delta}^R(y_1 v_0 + y_2 v_1, \beta_0), w_0 \right\rangle \Big|_{x_0} \\ &= \frac{(y_0 + \hat{y}_0)^2 (3y_0 + 3y_0^3 + 2\hat{y}_0 + 3y_0^2 \hat{y}_0) \alpha}{(\hat{y}_0 (1 + 3y_0^2 + 2y_0^4 + 2y_0 \hat{y}_0 + 2y_0^3 \hat{y}_0 - 2\hat{y}_0 \alpha)^2)}, \end{aligned}$$

and

$$\begin{aligned} b_{11}(\beta_0) &= \frac{\partial^2}{\partial y_1 \partial y_2} \left\langle Z_{\eta, \delta}^R(y_1 v_0 + y_2 v_1, \beta_0), w_1 \right\rangle \Big|_{x_0} \\ &= \frac{B_1 + \hat{y}_0 B_2}{\hat{y}_0 (1 + 3y_0^2 + 2y_0^4 + 2y_0 \hat{y}_0 + 2y_0^3 \hat{y}_0 - 2\hat{y}_0 \alpha)^2}, \end{aligned}$$

where $\hat{y}_0 = \sqrt{1 + y_0^2}$ and

$$\begin{aligned} B_1 &= -15y_0^2 - 75y_0^4 - 108y_0^6 - 48y_0^8 - 7y_0\alpha + 5y_0^3\alpha + 12y_0^5\alpha + 6\alpha^2, \\ B_2 &= -3y_0 - 39y_0^3 - 84y_0^5 - 48y_0^7 - 5\alpha - y_0^2\alpha + 12y_0^4\alpha. \end{aligned}$$

Therefore, computing the Bogdanov–Takens generic condition (BT.1):

$$a_{20}(\beta_0) + b_{11}(\beta_0) = -3 \frac{N_1 + N_2}{(y_0^2 + 1)^{3/2} (-4\alpha^2 + 8\alpha(y_0^2 + 1)y_0 + y_0^2 + 1)},$$

where

$$\begin{aligned} N_1 &= Y_0(y_0 + y_0^3 + \alpha + 8y_0^2\alpha + 8y_0^4\alpha), \\ N_2 &= y_0^2 + y_0^4 + 8y_0^3\alpha + 8y_0^5\alpha + 2\alpha^2. \end{aligned}$$

Suppose that $N_1 + N_2 = 0$. So, $N_1 = -N_2$, and applying square on both sides we obtain $G(y_0, \alpha) = 0$ where G is the polynomial of degree 8 in variable y :

$$\begin{aligned} G(y, \alpha) &= y^2 + 2y^4 + y^6 + 2y\alpha + 20y^3\alpha + 34y^5\alpha + 16y^7\alpha \\ &\quad + \alpha^2 + 13y^2\alpha^2 + 92y^4\alpha^2 + 144y^6\alpha^2 + 64y^8\alpha^2 \\ &\quad - 32y^3\alpha^3 - 32y^5\alpha^3 - 4\alpha^4. \end{aligned}$$

Thus, together with the polynomial $P(y, \alpha) = 40\alpha y^5 + 5y^4 + 102\alpha y^3 + (14 - 16\alpha^2)y^2 + 62\alpha y + (9 - 25\alpha^2)$ of Proposition 1, the system

$$\begin{cases} G(y_0, \alpha) = 0 \\ P(y_0, \alpha) = 0 \end{cases}$$

has a solution. Since y_0 is the root of the polynomials P and G , then the resultant $\text{Res}(G, P)_{y_0}(\alpha) = -32768\alpha^{14}(9 + 512\alpha^2)(245 + 16\alpha^2(199 + 688\alpha^2 + 256\alpha^4))(-1125 + 16\alpha^2(-130 + 1279\alpha^2 + 14656\alpha^4 + 256\alpha^6)) = 0$. But, $\text{Res}(G, P)_{y_0}(\alpha) = R_1(\alpha)$, given by (9), contradicting the initial hypothesis. This concludes the proof of Lemma 1. \square

Lemma 2. Consider $Z_{\eta, \delta}^R$ the family of regularized vector fields given by (8) where $\alpha > 0$ and α is not a root of the polynomial R_2 , where R_2 is given in (9). Then, the Bogdanov–Takens generic condition (BT.2) is satisfied at $\eta_0 = \eta(x_0, y_0)$, $\delta_0 = \delta(x_0, y_0)$.

Proof. Similarly to Lemma 1, we compute

$$\begin{aligned} b_{20}(\beta_0) &= \frac{\partial^2}{\partial y_1^2} \left\langle Z_{\eta, \delta}^R(y_1 v_0 + y_2 v_1, \beta_0), w_1 \right\rangle \Big|_{x_0} \\ &= \frac{M_0 + M_1 \hat{y}_0 + M_2 \hat{y}_0^2}{2\hat{y}_0(1 + 3y_0^2 + 2y_0^4 + 2y_0\hat{y}_0 + 2y_0^3\hat{y}_0 - 2\hat{y}_0\alpha)^2}, \end{aligned}$$

where $\hat{y}_0 = \sqrt{1 + y_0^2}$ and

$$\begin{aligned} M_0 &= y_0(-2 - 3y_0^2 + 43y_0^4 + 92y_0^6 + 48y_0^8 + 21y_0\alpha + 9y_0^3\alpha - 12y_0^5\alpha - 12\alpha^2), \\ M_1 &= -2 - 8y_0^2 + 58y_0^4 + 160y_0^6 + 96y_0^8 + 33y_0\alpha + 24y_0^3\alpha - 24y_0^5\alpha - 12\alpha^2, \\ M_2 &= -5y_0 + 15y_0^3 + 68y_0^5 + 48y_0^7 + 12\alpha + 15y_0^2\alpha - 12y_0^4\alpha. \end{aligned}$$

Supposing $M_0 + M_1\hat{y}_0 + M_2\hat{y}_0^2 = 0$ and manipulating this equation, we obtain $H(y_0, \alpha) = 0$, where H is the polynomial:

$$\begin{aligned} H(y, \alpha) &= 4 - 13y^2 - 38y^4 - 21y^6 + 36y\alpha - 252y^3\alpha - 612y^5\alpha \\ &\quad - 324y^7\alpha - 96\alpha^2 + 9y^2\alpha^2 - 951y^4\alpha^2 - 2208y^6\alpha^2 - 1152y^8\alpha^2 \\ &\quad - 504y\alpha^3 - 216y^3\alpha^3 + 288y^5\alpha^3 + 144\alpha^4. \end{aligned}$$

Therefore, $\text{Res}(H, P)_{y_0}(\alpha) = -2592\alpha^{10}(361 + 129074\alpha^2 + 7793725\alpha^4 + 191102976\alpha^6)(-33768075 - 194462570\alpha^2 + 7722010973\alpha^4 + 113346222160\alpha^6 + 537224695680\alpha^8 + 883362299904\alpha^{10} + 154241335296\alpha^{12} + 14495514624\alpha^{14}) = 0$.

But, $\text{Res}(H, P)_{y_0}(\alpha) = R_2(\alpha)$, given by (9), contradicting the initial hypothesis. Therefore, $b_{20}(\beta_0) \neq 0$. \square

Lemma 3. Consider $Z_{\eta, \delta}^R$ the family of regularized vector fields given by (8) where $\alpha > 0$ and α is not a root of the polynomial R_3 , where R_3 is given in (9). Then, the Bogdanov–Takens generic condition (BT.3) is satisfied at $\eta_0 = \eta(x_0, y_0)$, $\delta_0 = \delta(x_0, y_0)$.

Proof. In fact, the \mathbb{R}^4 -application

$$\begin{aligned} T(x, y, \eta, \delta) &= (T_1, T_2, T_3, T_4) \\ &= \left(Z_1(x, \beta), Z_2(x, \beta), \text{tr}(JZ_{\eta, \delta}^R(x, \beta)), \det(JZ_{\eta, \delta}^R(x, \beta)) \right) \end{aligned}$$

has Jacobian matrix:

$$JT(x, y, \eta, \delta) = \begin{pmatrix} \frac{\partial T_1}{\partial x} & \frac{\partial T_1}{\partial y} & \frac{\partial T_1}{\partial \eta} & \frac{\partial T_1}{\partial \delta} \\ \frac{\partial T_2}{\partial x} & \frac{\partial T_2}{\partial y} & \frac{\partial T_2}{\partial \eta} & \frac{\partial T_2}{\partial \delta} \\ \frac{\partial T_3}{\partial x} & \frac{\partial T_3}{\partial y} & \frac{\partial T_3}{\partial \eta} & \frac{\partial T_3}{\partial \delta} \\ \frac{\partial T_4}{\partial x} & \frac{\partial T_4}{\partial y} & \frac{\partial T_4}{\partial \eta} & \frac{\partial T_4}{\partial \delta} \end{pmatrix},$$

and determinant:

$$\det(JT(x_0, \beta_0)) = \frac{(5\hat{y}_0 + 2\alpha + y_0(9 + 9y_0^2 + 9y_0\hat{y}_0 + 11y_0\alpha + 13\hat{y}_0\alpha))}{(2\hat{y}_0^3)}.$$

Thus, T is regular in (x_0, β_0) if the determinant of the linear part is not zero. Supposing $\det(JT(x_0, \beta_0)) = 0$ and manipulating this equation, we obtain $K(y_0, \alpha) = 0$, where K is the polynomial:

$$K(y, \alpha) = 25 + 34y^2 + 9y^4 + 94y\alpha + 130y^3\alpha + 36y^5\alpha - 4\alpha^2 + 125y^2\alpha^2 + 48y^4\alpha^2.$$

Therefore,

$$\begin{aligned} \text{Res}(K, P)_{y_0}(\alpha) &= 324\alpha^5(16 + 5816\alpha^2 + 393649\alpha^4 + 8847360\alpha^6)(2178000 + 13491800\alpha^2 + 26866125\alpha^4 \\ &\quad + 32019984\alpha^6 + 682112\alpha^8) = 0. \end{aligned}$$

But, $\text{Res}(K, P)_{y_0}(\alpha) = R_3(\alpha)$, given by (9), contradicting the initial hypothesis. Therefore, $T(x, y, \eta, \delta)$ is regular at (x_0, β_0) . \square

The previous lemmas are not a great restriction for the vector field (8) to satisfy the items (BT.1), (BT.2) and (BT.3) of Theorem 2, because all vector fields (8) such that $\alpha > 0$ and $\alpha \notin S$ admit a Bogdanov–Takens bifurcation, where $S = S_1 \cup S_2 \cup S_3$ and $S_k = \{t \in \mathbb{R} : R_k(t) = 0\}$ for $k = 1, 2, 3$. Note that S is a finite subset of \mathbb{R} .

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