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Laplace transformation of vector-valued distributions and applications to Cauchy-Dirichlet problems

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ABSTRACT

We present two new proofs of the exchange theorem for the Laplace transformation of vector-valued distributions. We then derive an explicit solution to the Dirichlet problem of the polyharmonic operator in a half-space. Finally, we obtain explicit solutions to Cauchy-Dirichlet problems of iterated wave- and Klein-Gordon-operators in half-spaces.

1. Introduction

The exchange theorem for the Laplace transformation $\mathscr{L}$ states that

$$\mathscr{L}(S * T) = \mathscr{L}S \cdot \mathscr{L}T \qquad (S \in \mathscr{S}'(\Gamma), T \in \mathscr{O}'_C(\Gamma))$$

with notation as explained in Section 2 below (cf. [29, Prop. 7, p. 308]). A first task of this study is to present two new proofs for the exchange theorem for vector-valued distributions S and T. The original proof was given by L. Schwartz in his theory of vector-valued distributions. Our first proof follows an idea indicated at the beginning of L. Schwartz' proof, namely to apply a proposition on the convolution of two vector-valued distributions $S \in \mathscr{H}(E)$, $T \in \mathscr{K}(F)$, in which both the space $\mathscr{H}$ and its strong dual $\mathscr{H}'_b$ are assumed to be nuclear. Our second proof relies on a theorem of R. Shiraishi on the convolution of vector-valued distributions that supposes only $\mathscr{H}$ (and not necessarily $\mathscr{H}'_b$) to be nuclear.

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Our second goal is to use the Laplace transform of vector-valued distributions to deduce explicit solution formulae for the Cauchy-Dirichlet problem of the operators

$$(\Delta_n + \partial_y^2 - \partial_t^2)^m \qquad \text{(iterated wave operator)}$$
$$(\Delta_{2n+1} + \partial_y^2 - \partial_t^2 - \xi^2)^m \qquad \text{(iterated Klein-Gordon operator)}$$

in the half-space $y > 0$ (Propositions 14 and 16) by starting with the solutions of the Dirichlet problem of the elliptic operator

$$(\Delta_n + \partial_y^2 - p^2)^m \qquad \text{(iterated metaharmonic operator)}$$

(Propositions 9 and 11). We assume the Cauchy data $u|_{t=0}$, $\partial_t u|_{t=0}$, $\ldots$, $\partial_t^{2m-1} u|_{t=0}$ to vanish. In the terminology of R. Courant and D. Hilbert such problems are called "transient response problems" [7, p. 224]. Compare also [14, p. 85]. The expression "metaharmonic" is borrowed from [11] and from [34].

In Proposition 7 we recall the distributionally formulated solution to the Dirichlet problem of the iterated Laplace (i.e., polyharmonic) operator $(\Delta_n + \partial_y^2)^m$ in the half-space $y > 0$, which was presented for the first time in [8].

We note that our method could also be used, e.g., to derive explicit formulae for the solution to the Cauchy-Dirichlet problem of the operator

$$(\Delta_n + \partial_y^2 - \partial_t)^m \qquad \text{(iterated heat operator)}$$

in the half-space $y > 0$.

For $m = 1$, the solution of the Cauchy-Dirichlet problem can be found by an odd extension of the sought solution in the half-space, application of the distributional differentiation formula and convolution with the fundamental solution (in classical terms, by application of a representation theorem by means of Green's function). If $m > 2$ the solution by extension is not known to us.

Our notation is standard, mostly following [26,28,29].

2. New proofs of L. Schwartz' exchange theorem for the Laplace transform of the convolution of vector-valued distributions

Let us first recall L. Schwartz' version [28, Prop. 43, p. 186]:

Theorem 1. *Let Γ be a non-void open convex subset of $\mathbb{R}^n$. Let E and F be separated locally convex topological vector spaces. Then there is a hypocontinuous (with respect to bounded sets) convolution mapping*

$$\overset{*}{\otimes} : \mathscr{S}'(\Gamma)(E) \times \mathscr{S}'(\Gamma)(F) \to \mathscr{S}'(\Gamma)(E \widehat{\otimes}_\pi F).$$

For two Laplace-transformable distributions $S \in \mathscr{S}'(\Gamma)(E)$, $T \in \mathscr{S}'(\Gamma)(F)$ with values in E and F, respectively, and their Laplace images $\mathscr{L}S, \mathscr{L}T$ we have

$$\mathscr{L}(S \overset{*}{\otimes} T) = (\mathscr{L}S) \overset{\cdot}{\otimes} (\mathscr{L}T).$$

We will now explain the notions appearing in this theorem. First, the mappings $\overset{*}{\otimes}$ and $\overset{\cdot}{\otimes}$ in Theorem 1 are defined as in [28, Proposition 3, p. 37] and would be denoted by $*_\pi$ and $\cdot_\pi$ there, respectively. Also, $E \widehat{\otimes}_\pi F$ denotes the quasi-completion of $E \otimes_\pi F$.

Definition 2 *([26, p. 58], [29, p. 303]).* Let $\Gamma \subseteq \mathbb{R}^n$ be non-void and convex. The space of *Laplace transformable distributions* $\mathscr{S}'(\Gamma)$ is defined as

$$\mathscr{S}'(\Gamma) = \bigcap_{\xi \in \Gamma} e^{\xi x} \mathscr{S}'_x = \{S \in \mathscr{D}' \mid \forall \xi \in \Gamma : e^{-\xi x} S(x) \in \mathscr{S}'_x\},$$

where $\mathscr{S}'$ is the space of temperate distributions on $\mathbb{R}^n$. $\mathscr{S}'(\Gamma)$ is endowed with the projective topology with respect to the linear maps $\mathscr{S}'(\Gamma) \to \mathscr{S}'$, $S(x) \mapsto e^{-\xi x} S(x)$ for $\xi \in \Gamma$.

As usual, we denote by $\mathscr{O}'_C$ the space of rapidly decreasing distributions on $\mathbb{R}^n$. By defining

$$\mathscr{O}'_C(\Gamma) = \bigcap_{\xi \in \Gamma} e^{\xi x} \mathscr{O}'_{C,x}$$

analogously to $\mathscr{S}'(\Gamma)$ we have $\mathscr{O}'_C(\Gamma) = \mathscr{S}'(\Gamma)$ if $\Gamma \neq \emptyset$ is convex and open [29, p. 303]. Also, for such Γ the space $\mathscr{O}'_C(\Gamma)$ is a commutative algebra with respect to convolution, which in turn is continuous [29, Corollaire, p. 304].

Let us recall L. Schwartz' definition of the Laplace transformation of *scalar valued* distributions and the Paley-Wiener-Schwartz theorem:

Definition and Theorem 3 *([26, Prop. 22, p. 76], [29, Prop. 6, p. 306]). Let $\emptyset \neq \Gamma \subseteq \mathbb{R}^n$ be open and convex and $T^\Gamma := \Gamma + i\mathbb{R}^n$ the tube domain over Γ. The* Laplace transformation *$\mathscr{L}$ is the mapping*

$$\mathscr{L} \colon \mathscr{O}'_C(\Gamma) \to \mathscr{H}(T^\Gamma), \quad S \mapsto \mathscr{L}S(p) = \langle 1(x), e^{-px} S(x)\rangle, \quad p \in T^\Gamma.$$

The vector-valued scalar product $\langle\ ,\ \rangle$ is defined on $\mathscr{O}_C \times \mathscr{O}'_C(\mathscr{H}(T^\Gamma))$ due to $e^{-px}S(x) \in \mathscr{O}'_{C,x}(\mathscr{H}(T^\Gamma_p))$ [29, Cor., p. 302].

$\mathscr{L}$ is an isomorphism if $\mathscr{H}(T^\Gamma)$ is endowed with the projective topology (with respect to the compact subsets K of Γ) of the inductive limits

$$\mathscr{H}(T^K) = \{f \colon T^K \to \mathbb{C} \textit{ holomorphic} \mid \exists m \in \mathbb{N}_0 : (1 + |p|^2)^{-m} f(p) \in L^\infty(T^K)\}.$$

Concerning the proof of Theorem 1, L. Schwartz remarks that it could be realized by applying Proposition 3 in [28, p. 37] to the spaces $\mathscr{S}'(\Gamma)(E)$ and $\mathscr{S}'(\Gamma)(F)$. But this procedure would require the proof of the nuclearity of the space $\mathscr{S}'(\Gamma)$ "which is easy" and of the nuclearity of its strong dual "which is not so easy". Thus, he proceeds differently [28, p. 187]. However, we aim at performing the proof in such a manner as L. Schwartz remarked. As a byproduct we sharpen Theorem 1 slightly.

Theorem 4. *Let $\emptyset \neq \Gamma \subseteq \mathbb{R}^n$ be an open and convex set and E and F separated locally convex topological vector spaces. There exists a* unique *bilinear,* continuous *mapping*

$$\overset{*}{\otimes} \colon \mathscr{O}'_C(\Gamma)(E) \times \mathscr{O}'_C(\Gamma)(F) \to \mathscr{O}'_C(\Gamma)(E \widehat{\otimes}_\pi F)$$

which extends the mapping

$$\begin{aligned} \overset{*}{\otimes} \colon (\mathscr{O}'_C(\Gamma) \otimes E) \times (\mathscr{O}'_C(\Gamma) \otimes F) &\to \mathscr{O}'_C(\Gamma)(E \otimes_\pi F) \\ (S \otimes e, T \otimes f) &\mapsto (S * T) e \otimes f, \end{aligned}$$

wherein $* \colon \mathscr{O}'_C(\Gamma) \times \mathscr{O}'_C(\Gamma) \to \mathscr{O}'_C(\Gamma)$ is the continuous convolution and $\otimes \colon E \times F \to E \otimes_\pi F$ the canonical bilinear and continuous mapping.

If $\mathscr{L}\colon \mathscr{O}'_C(\Gamma)(E) \to \mathscr{H}(T^\Gamma)(E)$ *is the Laplace transformation of* E*-valued distributions then we have*

$$\mathscr{L}(S \overset{*}{\otimes} T) = \mathscr{L}S \overset{\cdot}{\otimes} \mathscr{L}T$$

for $S \in \mathscr{O}'_C(\Gamma)(E)$, $T \in \mathscr{O}'_C(\Gamma)(F)$. *Summarizing, we have the commutative diagram*

$$\begin{tikzcd}
\mathscr{O}'_C(\Gamma)(E) \times \mathscr{O}'_C(\Gamma)(F) \arrow[r, "\overset{*}{\otimes}"] \arrow[d, "\mathscr{L}\times\mathscr{L}", "\wr\|"'] & \mathscr{O}'_C(\Gamma)(E\widehat{\otimes}_\pi F) \arrow[d, "\mathscr{L}", "\wr\|"'] \\
\mathscr{H}(T^\Gamma)(E) \times \mathscr{H}(T^\Gamma)(F) \arrow[r, "\overset{\cdot}{\otimes}"] & \mathscr{H}(T^\Gamma)(E\widehat{\otimes}_\pi F).
\end{tikzcd}$$

Proof. The claim about the map $\overset{*}{\otimes}$ follows from Proposition 3 in [28, p. 37] because the space $\mathscr{O}'_C(\Gamma)$ has the strict approximation property [26, Proposition 16, p. 59], is nuclear and its strong dual is nuclear (see Lemma 5 below), and because the convolution $\mathscr{O}'_C(\Gamma) \times \mathscr{O}'_C(\Gamma) \xrightarrow{*} \mathscr{O}'_C(\Gamma)$ is continuous. Concerning $\overset{\cdot}{\otimes}$, we note that the multiplication $\cdot : \mathscr{H}(T^\Gamma) \times \mathscr{H}(T^\Gamma) \to \mathscr{H}(T^\Gamma)$ is continuous, and since $\mathscr{H}(T^\Gamma) \cong \mathscr{O}'_C(\Gamma)$, we may again apply Proposition 3 in [28, p. 37]. Finally, commutativity of the diagram follows from that of its scalar variant, which holds due to [29, Proposition 7, p. 308]. □

Lemma 5. *Let* $\emptyset \neq \Gamma \subseteq \mathbb{R}^n$ *be open and convex.*

(i) *The space* $\mathscr{O}'_C(\Gamma)$ *is nuclear and complete.*
(ii) *The strong dual* $(\mathscr{O}'_C(\Gamma))'_b$ *of* $\mathscr{O}'_C(\Gamma)$ *is nuclear.*

Proof. (i) The nuclearity of $\mathscr{O}'_C(\Gamma)$ is an immediate consequence of [13, Corollaire 2, p. 48] and the nuclearity of $\mathscr{O}'_C$ ([13, Théorème 16, p. 131]).

By [25, Proposition 5.3, p. 52] the space $\mathscr{O}'_C(\Gamma)$ is complete.

(ii) The projective limit $\mathscr{O}'_C(\Gamma) = \mathscr{S}'(\Gamma)$ is countable due to

$$\bigcap_{\xi\in\Gamma} e^{\xi x}\mathscr{S}'_x = \bigcap_{\xi\in\Gamma\cap\mathbb{Q}^n} e^{\xi x}\mathscr{S}'_x.$$

We only have to show the inclusion "$\supseteq$", the rest being elementary. For this, given $T \in \bigcap_{\xi\in\Gamma\cap\mathbb{Q}^n} e^{\xi x}\mathscr{S}'_x$ and $\xi \in \Gamma$, choose $\xi_1, \ldots, \xi_k \in \Gamma \cap \mathbb{Q}^n$ such that ξ is in the convex hull of $\{\xi_1, \ldots, \xi_k\}$. By [29, p. 301],

$$e^{-\xi x} = \alpha(x, \xi) \sum_{j=1}^{k} e^{-\xi_j x}$$

with $\alpha(.,\xi) \in \mathscr{B}$, so we have

$$e^{-\xi x}T(x) = \alpha(x,\xi) \sum_{j=1}^{k} \underbrace{e^{-\xi_j x}T(x)}_{\in \mathscr{S}'_x} \in \mathscr{S}'_x.$$

It follows that $\mathscr{S}'(\Gamma)$ is given by the projective limit

$$\mathscr{S}'(\Gamma) = \varprojlim_{\Gamma_f\subseteq\Gamma \text{ finite}} \mathscr{S}'(\Gamma_f).$$

This limit is reduced because the inclusions $\mathscr{D} \subseteq \mathscr{S}'(\Gamma) \subseteq \mathscr{S}'(\Gamma_f)$ and density of $\mathscr{D}$ in $\mathscr{S}'(\Gamma_f)$ imply that $\mathscr{S}'(\Gamma)$ also is dense.

By [25, 4.4, p. 139] the dual $(\mathscr{S}'(\Gamma))'$ endowed with the Mackey topology $\tau((\mathscr{S}'(\Gamma))', \mathscr{S}'(\Gamma))$ can be identified with the inductive limit of the spaces

$$((\mathscr{S}'(\Gamma_f))', \tau((\mathscr{S}'(\Gamma_f))', \mathscr{S}'(\Gamma_f))).$$

Because $\mathscr{S}'(\Gamma)$ is nuclear and complete it is semireflexive, hence the Mackey topology on its dual equals the strong topology [17, Prop. 4, p. 228 and Prop. 8, p. 218]. Hence, we have

$$(\mathscr{S}'(\Gamma))'_b = \varinjlim_{\Gamma_f \subseteq \Gamma \cap \mathbb{Q}^n} (\mathscr{S}'(\Gamma_f))'_b$$

and the nuclearity of $(\mathscr{O}'_C(\Gamma))'_b = (\mathscr{S}'(\Gamma))'_b$ follows by [13, Cor. 1, p. 48] from the nuclearity of $(\mathscr{S}'(\Gamma_f))'_b$. To see that the latter space is nuclear we note that $\mathscr{S}'(\Gamma_f)$, as a finite projective limit of (DFS)-spaces, is itself a (DFS)-space because this class of spaces is stable under the formation of finite products and closed subspaces [20, Theorem A.5.13, p. 253]. Furthermore, $\mathscr{S}'(\Gamma_f)$ is nuclear by [13, Cor. 1, p. 48], hence its strong dual is nuclear by [13, Théorème 7, p. 40]. $\square$

One can even show that $\mathscr{O}'_C(\Gamma)$ is ultrabornological – however, this is quite intricate to prove and will therefore be published separately.

Remark 6. A third proof of Theorem 4 can be given by means of [32, Theorem 2, p. 196], see also [4, Theorem 5, p. 18]. Compared to [29, Prop. 3, p. 37] it has the advantage that the nuclearity of $\mathscr{O}'_C(\Gamma) = \mathscr{S}'(\Gamma)$ is sufficient (Lemma 5 (i)), while nuclearity of its strong dual $\mathscr{O}'_C(\Gamma)'_b$ need not be established. Instead, one has to show that $\mathscr{O}'_C(\Gamma)$ is $\dot{B}$-normal (which is straightforward) and that $\mathscr{O}'_C(\Gamma) \otimes E$ is strictly dense in $\mathscr{O}'_C(\Gamma)(E)$, which in turn is implied by the strict approximation property of $\mathscr{S}'(\Gamma)$ ([29, Proposition 16, p. 59]). In fact, by using [4, Prop. 1, p. 19] we can even dispense with showing $\dot{B}$-normality of $\mathscr{O}'_C(\Gamma)$.

3. Poisson kernels for Dirichlet problems

Our next aim is to reformulate a known result on the Poisson kernels of the Dirichlet problems of polyharmonic operators in half-spaces and to apply the partial Fourier transformation in order to deduce the Poisson kernels of the Dirichlet problems of the iterated metaharmonic operators in half-spaces. Then the theory of vector-valued distributions is applied in order to continue the results analytically. This method goes back to H.G. Garnir [11].

We follow the terminology of [2, p. 635] and [31, p. 140]: The *Poisson kernel* of the j-th Dirichlet problem for the operator

$$(\Delta_n + \partial_y^2 - \xi^2)^m, \quad \Delta_n = \partial_1^2 + \cdots + \partial_n^2, \quad m, n \in \mathbb{N},\ \xi \in \mathbb{R}$$

in the half-space $\mathbb{H} = \{(x, y) \in \mathbb{R}^{n+1} : x \in \mathbb{R}^n,\ y > 0\}$, $j = 0, \ldots, m-1$, is the distribution $E_j \in \mathcal{D}'(\mathbb{H})$ for which

$$\begin{aligned} &(\Delta_n + \partial_y^2 - \xi^2)^m E_j = 0 \qquad E_j \in \mathcal{O}_M(\mathbb{H}) = \{\varphi \in \mathcal{E}(\mathbb{H}) \mid \forall \alpha \in \mathbb{N}_0^n\ \exists k \in \mathbb{N}_0 : \\ &\qquad\qquad\qquad\qquad\qquad\qquad (1 + |x|^2)^{-k/2} \partial^\alpha \varphi \in \mathcal{C}_0(\mathbb{H})\} \\ &\partial_y^k E_j|_{y=0} = \delta(x)\delta_{jk}, \quad k = 0, \ldots, m-1 \ \text{ in } \mathcal{D}'(\mathbb{R}^n). \end{aligned}$$

Here, $\mathcal{C}_0(\mathbb{H}) = \{\psi \in \mathcal{C}(\mathbb{H}) : \lim_{|(x,y)|\to\infty} \psi(x,y) = 0\}$. The existence of the restrictions $\partial_y^k E_j|_{y=0}$ will follow from the explicit form of E_j in Proposition 7 below (see also [16, Theorem 4.4.8, p. 115]).

For a more general notion of Poisson kernel we refer to [33, Section 4.5, p. 137].

To begin with, we use [8, Satz 3] to derive the following result:

Proposition 7. *The Poisson kernel of the j-th Dirichlet problem for the polyharmonic (i.e., iterated Laplace) operator $(\Delta_n + \partial_y^2)^m$ is given by*

$$E_j = \frac{2}{\omega_{n+1}} \frac{y^m}{j!(m-1-j)!} (-\partial_y)^{m-j-1} \left(\frac{1}{(|x|^2+y^2)^{\frac{n+1}{2}}} \right),$$

where $\omega_{n+1} = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}$ denotes the surface measure of the unit sphere in $\mathbb{R}^{n+1}$.

Proof. Let $\varphi \in \mathcal{D}(\mathbb{R}^n_x)$, $\check{\varphi}(x) = \varphi(-x)$, and denote by $*$ the convolution with respect to the x-variables. Then it follows from [8, Satz 3] that $E_j * \check{\varphi} \in \mathcal{E}_{xy}(\mathbb{H})$ is the unique solution to

$$\begin{aligned} &(\Delta_n + \partial_y^2)^m (E_j * \check{\varphi}) = 0 \\ &\lim_{y\searrow 0} \partial_y^k (E_j * \check{\varphi})(x) = \check{\varphi}(x)\delta_{jk}, \quad k = 0, 1, \ldots, m-1. \end{aligned}$$

Consequently, $(\Delta_n + \partial_y^2)^m E_j = 0$ and $\lim_{y\searrow 0} \partial_y^k \langle E_j, \varphi\rangle = \varphi(0)\delta_{jk}$, $k = 0, 1, \ldots, m-1$, i.e., $\lim_{y\searrow 0} \partial_y^k E_j = \delta(x)\delta_{jk}$, $k = 0, 1, \ldots, m-1$. □

Remark 8. We point out the following particular cases of Proposition 7:

(a) For $m = 1$, $j = 0$ we obtain the well-known Poisson kernel of the Dirichlet-problem for $\Delta_n + \partial_y^2$ in the half-space $y > 0$ to be

$$\frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \frac{y}{(|x|^2+y^2)^{\frac{n+1}{2}}},$$

cf., e.g., [31, (1.2), p. 163] or [9, p. 37, Th. 14].

(b) The choice $m = 2$, $j = 0$, $j = 1$ gives the Poisson kernels of the Dirichlet problem for the *biharmonic operator* $(\Delta_n + \partial_y^2)^2$ in the half-space $y > 0$:

$$E_0 = \frac{2\Gamma\left(\frac{n+3}{2}\right)}{\pi^{\frac{n+1}{2}}} \frac{y^3}{(|x|^2+y^2)^{\frac{n+3}{2}}}, \quad E_1 = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \frac{y^2}{(|x|^2+y^2)^{\frac{n+1}{2}}}.$$

In [8], J. Edenhofer cites [3] for this result.

The solution u to

$$\begin{aligned} &(\partial_x^2 + \partial_y^2)^2 u = 0 \quad \text{in } y > 0 \\ &u|_{y=0} = g_0, \ \partial_y u|_{y=0} = g_1 \quad (n = 1, \ m = 2, \ j = 0, 1), \end{aligned}$$

in the form

$$u(x,y) = \frac{2y^3}{\pi} \int_{\mathbb{R}} g_0(x-\xi) \frac{d\xi}{(\xi^2+y^2)^2} + \frac{y^2}{\pi} \int_{\mathbb{R}} g_1(x-\xi) \frac{d\xi}{\xi^2+y^2}$$

can be found in [22, (2.14), p. 262] (where a sign should be corrected and where the formula is attributed to L.F. Richardson in [23]). For a more recent, direct treatment of the Poisson kernel E_0 for the biharmonic operator in the half-plane we refer to [1, p. 781].

(c) If $n = 2$, the Poisson kernel E_{m-1} of the Dirichlet problem for $(\partial_x^2 + \partial_y^2)^m$ in $y > 0$ (i.e., with the boundary conditions $\partial_y^k E_{m-1}|_{y=0} = 0$, $k = 0, 1, \dots, m-2$, $\partial_y^{m-1} E_{m-1}|_{y=0} = \delta(x)$) is given by the formula

$$E_{m-1} = \frac{1}{\pi(m-1)!}\frac{y^m}{x^2+y^2},$$

see Example 5 in [30, p. 275].

Next, let us derive the Poisson kernel of the j-th Dirichlet problem, $j = 0, 1, \dots, m-1$, of the operator $(\Delta_n + \partial_y^2 - \xi^2)^m$ in $\mathbb{H}$ by Fourier transformation.

Proposition 9. *Let $m, n \in \mathbb{N}$. The* Poisson kernel *of the j-th Dirichlet problem, $j = 0, 1, \dots, m-1$, for the* iterated meta-harmonic operator $(\Delta_n + \partial_y^2 - \xi^2)^m$ *in $\mathbb{H}$, $\xi \neq 0$, is given by*

$$E_j = \frac{y^m |\xi|^{\frac{n+1}{2}}}{2^{\frac{n-1}{2}} \pi^{\frac{n+1}{2}} j!(m-1-j)!} (-\partial_y)^{m-j-1} \left(\frac{K_{\frac{n+1}{2}}(|\xi|\sqrt{|x|^2+y^2})}{(|x|^2+y^2)^{\frac{n+1}{4}}} \right) \tag{1}$$

or

$$E_j = \frac{-y^m |\xi|^{\frac{n-1}{2}}}{2^{\frac{n-1}{2}} \pi^{\frac{n+1}{2}} j!(m-1-j)!} (-\partial_y)^{m-j-1} \left(\frac{1}{y}\partial_y\right) \left(\frac{K_{\frac{n-1}{2}}(|\xi|\sqrt{|x|^2+y^2})}{(|x|^2+y^2)^{\frac{n-1}{4}}} \right) \tag{2}$$

Here, K_λ is the modified Bessel function of the second kind of order λ.

Proof. Setting

$$\begin{aligned} F_j &:= \frac{2}{\omega_{n+2}} \frac{y^m}{j!(m-1-j)!} (-\partial_y)^{m-j-1} \frac{1}{(|x|^2+y^2+s^2)^{\frac{n}{2}+1}} \\ &= -\frac{2}{n\omega_{n+2}} \frac{y^m}{j!(m-1-j)!} (-\partial_y)^{m-j-1} \left(\frac{1}{y}\partial_y\right) \frac{1}{(|x|^2+y^2+s^2)^{\frac{n}{2}}}, \end{aligned}$$

we obtain by means of Proposition 7:

$$\begin{aligned} &(\Delta_n + \partial_y^2 + \partial_s^2)^m F_j = 0 \\ &\qquad \partial_y^k F_j|_{y=0} = \delta(x,s)\delta_{jk}, \quad k = 0, 1, \dots, m-1. \end{aligned}$$

A partial Fourier transformation with respect to s yields for $E_j = \int_{\mathbb{R}} e^{-i\xi s} F_j \, ds$:

$$\begin{aligned} &(\Delta_n + \partial_y^2 - \xi^2)^m E_j = 0 \\ &\qquad \partial_y^k E_j|_{y=0} = \delta(x)\delta_{jk}, \quad k = 0, 1, \dots, m-1. \end{aligned}$$

By [12, 8.432 5, p. 917] we obtain

$$E_j = \frac{\Gamma(\frac{n}{2}+1)}{\pi^{\frac{n}{2}+1}} \frac{y^m}{j!(m-1-j)!} (-\partial_y)^{m-j-1} \left(\frac{2\sqrt{\pi}|\xi|^{\frac{n+1}{2}}}{2^{\frac{n+1}{2}}\Gamma(\frac{n}{2}+1)} \frac{K_{\frac{n+1}{2}}(|\xi|\sqrt{|x|^2+y^2})}{(|x|^2+y^2)^{\frac{n+1}{4}}} \right)$$

$$= \frac{y^m |\xi|^{\frac{n+1}{2}}}{2^{\frac{n-1}{2}} \pi^{\frac{n+1}{2}} j!(m-1-j)!} (-\partial_y)^{m-j-1} \left(\frac{K_{\frac{n+1}{2}}(|\xi|\sqrt{|x|^2+y^2})}{(|x|^2+y^2)^{\frac{n+1}{4}}} \right),$$

establishing (1). The second claim follows from the functional relation $(\frac{1}{z}\partial_z)(z^{-\lambda}K_\lambda(z)) = -z^{-\lambda-1}K_{\lambda+1}(z)$ (cf. [12, 8.486 15, p. 929]):

$$E_j = \frac{-\Gamma(\frac{n}{2}+1) y^m}{n\pi^{\frac{n}{2}+1} j!(m-1-j)!} (-\partial_y)^{m-j-1} \left(\frac{1}{y}\partial_y\right) \left(\frac{2\sqrt{\pi}|\xi|^{\frac{n-1}{2}}}{2^{\frac{n-1}{2}}\Gamma(\frac{n}{2})} \frac{K_{\frac{n-1}{2}}(|\xi|\sqrt{|x|^2+y^2})}{(|x|^2+y^2)^{\frac{n-1}{4}}} \right)$$
$$= \frac{-y^m |\xi|^{\frac{n-1}{2}}}{2^{\frac{n-1}{2}} \pi^{\frac{n+1}{2}} j!(m-1-j)!} (-\partial_y)^{m-j-1} \left(\frac{1}{y}\partial_y\right) \left(\frac{K_{\frac{n-1}{2}}(|\xi|\sqrt{|x|^2+y^2})}{(|x|^2+y^2)^{\frac{n-1}{4}}} \right). \quad \square$$

Remark 10. Let us mention a particular case of Proposition 9: Setting $m = 1$, $j = 0$, the Poisson kernel E_0 of the *metaharmonic operator* $\Delta_n + \partial_y^2 - \xi^2$ in $\mathbb{H}$ is given by

$$E_0 = \frac{y|\xi|^{\frac{n+1}{2}}}{2^{\frac{n-1}{2}}\pi^{\frac{n+1}{2}}} \frac{K_{\frac{n+1}{2}}\left(|\xi|\sqrt{|x|^2+y^2}\right)}{(|x|^2+y^2)^{\frac{n+1}{4}}},$$

see [5, Rem. 2, p. 321].

Proposition 9 remains valid if we substitute $p \in T^\Gamma = \mathbb{R}_+ + i\mathbb{R}$ ($\Gamma = \mathbb{R}_+ = (0,\infty)$, T^Γ the right half-plane) for $\xi \in \mathbb{R} \setminus \{0\}$. We obtain by analytic continuation:

Proposition 11. *The Poisson kernel E_j of the j-th Dirichlet problem, $j = 0, 1, \ldots, m-1$, of the* iterated meta-harmonic operator $(\Delta_n + \partial_y^2 - p^2)^m$, *$p \in T^\Gamma$, in the half-space $\mathbb{H}$ is given by*

$$E_j = \frac{-y^m p^{\frac{n-1}{2}}}{2^{\frac{n-1}{2}} \pi^{\frac{n+1}{2}} j!(m-1-j)!} (-\partial_y)^{m-j-1} \left(\frac{1}{y}\partial_y\right) \left(\frac{K_{\frac{n-1}{2}}(p\sqrt{|x|^2+y^2})}{(|x|^2+y^2)^{\frac{n-1}{4}}} \right)$$

(For even n the square root in $p^{\frac{n-1}{2}}$ is defined as usual.) Furthermore, $E_j \in \mathcal{H}(T_p^\Gamma)(\mathcal{D}'(\mathbb{H}_{xy}))$.

Proof. The integral representation

$$\frac{K_{\frac{n-1}{2}}(p\sqrt{|x|^2+y^2})}{(|x|^2+y^2)^{\frac{n-1}{4}}} = \frac{1}{2}\int_0^\infty t^{-\frac{n+1}{2}} e^{-\frac{p}{2}\left(t + \frac{|x|^2+y^2}{t}\right)}\, dt$$

in [12, 8.432 7, p. 917] can be interpreted as a vector-valued scalar product

$$\frac{1}{2}\left\langle 1(t), t^{-\frac{n+1}{2}} e^{-\frac{p}{2}t} \cdot e^{-\frac{p}{2t}(|x|^2+y^2)} \right\rangle$$

on $L^\infty(\mathbb{R}_{+,t}) \times L^1(\mathbb{R}_{+,t})(\mathcal{H}(T_p^\Gamma)(\mathcal{D}'(\mathbb{H}_{xy})))$. Here, $e^{-\frac{p}{2}t} \in \mathcal{H}(T_p^\Gamma)(L^1(\mathbb{R}_{+,t}))$, and

$$S(p,t,x,y) := e^{-\frac{p}{2t}(|x|^2+y^2)} t^{-\frac{n+1}{2}} \in \mathcal{H}(T_p^\Gamma)(\mathcal{C}_0(\mathbb{R}_{+,t})(\mathcal{D}'(\mathbb{H}_{xy})))$$
$$= \mathcal{H}(T_p^\Gamma)\hat{\otimes}_\varepsilon \mathcal{C}_0(\mathbb{R}_{+,t})\hat{\otimes}_\varepsilon \mathcal{D}'(\mathbb{H}_{xy}).$$

To prove this, we first show the following two auxiliary results:

Lemma 12. *Any complete, nuclear normal space of distributions has the ε-property.*

Proof. By the Kōmura Theorem [18, 21.7.1, p. 500], any such space F is isomorphic to a closed subspace of s^J for some index set J. Since the ε-property is preserved under taking products and subspaces, this implies that F has the ε-property. $\square$

Lemma 13. *$\mathcal{H}(T_p^\Gamma)\hat{\otimes}\mathcal{D}'(\mathbb{H}_{xy})$ has the ε-property.*

Proof. We have $\mathcal{H}(T^\Gamma) \subseteq \mathcal{E}(\Gamma \times \mathbb{R}^n)$, with the topology induced by $\mathcal{E}$. Both $\mathcal{E}$ and $\mathcal{D}$ are nuclear normal spaces of distributions, so $\mathcal{E}\hat{\otimes}\mathcal{D}'$ is nuclear, normal, and complete and has the ε-property by Lemma 12. As the ε-property is inherited by topological subspaces, the claim follows. $\square$

To establish $S(p,t,x,y) \in (\mathcal{H}(T_p^\Gamma)\hat{\otimes}\mathcal{D}'(\mathbb{H}_{xy}))(\mathcal{C}_0(\mathbb{R}_{+,t}))$ it therefore suffices to show that for $\mu \in \mathcal{M}^1(\mathbb{R}_{+,t})$ we have

$$\langle S(p,t,x,y), \mu(t)\rangle \in \mathcal{H}(T_p^\Gamma)\hat{\otimes}\mathcal{D}'(\mathbb{H}_{xy}).$$

Noting that $\mathcal{H}(T_p^\Gamma)$ has the ε-property, to see this it suffices in turn to show that

$$\langle \varphi(x,y), \langle S(p,t,x,y), \mu(t)\rangle\rangle \in \mathcal{H}(T_p^\Gamma), \tag{3}$$

for each $\varphi \in \mathcal{D}(\mathbb{H}_{xy})$. By Fubini's theorem ([26, p. 131, Corollaire]) this is equivalent to

$$\langle\langle S(p,t,x,y), \varphi(x,y)\rangle, \mu(t)\rangle \in \mathcal{H}(T_p^\Gamma).$$

In fact,

$$\begin{aligned}\langle t^{-\frac{n+1}{2}} e^{-\frac{p}{2t}(|x|^2+y^2)}, \varphi(x,y)\rangle &= \int\limits_{\mathbb{H}_{xy}\times\mathbb{R}_{+,t}} e^{-\frac{p}{2t}(|x|^2+y^2)} t^{-\frac{n+1}{2}} \varphi(x,y)\, dxdy \\ &= \int\limits_{\mathbb{H}_{xy}\times\mathbb{R}_{+,t}} e^{-\frac{p}{2}(|\xi|^2+\eta^2)} \varphi(\sqrt{t}\xi, \sqrt{t}\eta)\, d\xi d\eta,\end{aligned}$$

so that

$$\langle \mu(t), \langle S(p,t,x,y), \varphi(x,y)\rangle\rangle = \int\limits_{\mathbb{H}_{xy}\times\mathbb{R}_{+,t}} e^{-\frac{p}{2}(|\xi|^2+\eta^2)} \langle \mu(t), \varphi(\sqrt{t}\xi, \sqrt{t}\eta)\rangle\, d\xi d\eta.$$

Since the map $\mathbb{H} \to \mathbb{C}$, $(\xi,\eta) \mapsto \langle \mu(t), \varphi(\sqrt{t}\xi, \sqrt{t}\eta)\rangle$ is bounded by $\|\mu\|_1\|\varphi\|_\infty$, (3) follows by dominated convergence. Note that this argument in fact also shows that $S(p,t,x,y) \in \mathcal{H}(T_p^\Gamma)(\mathcal{BC}_b(\mathbb{R}_{+,t})(L^1(\mathbb{H}_{xy})))$. Here, the subscript b refers to the Buck topology, so $\mathcal{BC}_b(\mathbb{R}_+)' = \mathcal{M}(\mathbb{R}_+)$ (cf. [21, p. 6]).

Due to the continuity of the bilinear multiplication $\mathcal{H}(T^\Gamma) \times \mathcal{H}(T^\Gamma) \xrightarrow{\cdot} \mathcal{H}(T^\Gamma)$ and the continuity of the vector-valued multiplication $\mathcal{C}_0(\mathbb{R}_{+,t})(\mathcal{D}'(\mathbb{H}_{xy})) \times L^1(\mathbb{R}_{+,t}) \xrightarrow{\cdot} L^1(\mathbb{R}_{+,t})(\mathcal{D}'(\mathbb{H}_{xy}))$ we conclude by means of [28, Proposition 3, p. 37] that

$$e^{-\frac{pt}{2}-\frac{p}{2t}(|x|^2+y^2)} t^{-\frac{n+1}{2}} \in \mathcal{H}(T_p^\Gamma)(L^1(\mathbb{R}_{+,t})(\mathcal{D}'(\mathbb{H}_{xy}))).$$

Indeed, the assumptions '$\mathcal{H}(T^\Gamma)$ nuclear' and '$(\mathcal{H}(T^\Gamma))'_b$ nuclear' are fulfilled because of $\mathcal{H}(T^\Gamma) \cong \mathcal{O}'_C(\Gamma)$ (Definition and Theorem 3) and Lemma 5.

In virtue of

$$\mathcal{H}(T_p^\Gamma)(L^1(\mathbb{R}_{+,t})(\mathcal{D}'(\mathbb{H}_{xy}))) = L^1(\mathbb{R}_{+,t})(\mathcal{H}(T_p^\Gamma)(\mathcal{D}'(\mathbb{H}_{xy}))),$$

the final step consists in applying [27, Theorem 7.1, p. 31] to the vector-valued scalar product $\langle\, ,\, \rangle : L^\infty \times L^1(E) \to E$, with $E = \mathcal{H}(T_p^\Gamma)(\mathcal{D}'(\mathbb{H}_{xy}))$. $\square$

4. Transient response Dirichlet problems

In this section we study the transient response Dirichlet problem for the iterated wave operator $(\Delta_n + \partial_y^2 - \partial_t^2)^m$ and the iterated Klein-Gordon operator $(\Delta_n + \partial_y^2 - \partial_t^2 - \xi^2)^m$ in the half-space $y > 0$, more precisely in $\mathbb{H}_{yt} = \{(x, y, t) \in \mathbb{R}^{n+2} : y > 0,\ t > 0\}$.

We look for an explicit expression for the solution E_j to the j-th, $j = 0, 1, \dots, m-1$, (mixed) Cauchy-Dirichlet problem

$$\begin{aligned}
&(\Delta_n + \partial_y^2 - \partial_t^2 - \xi^2)^m E_j = 0 \qquad \text{in } \mathcal{D}'(\mathbb{H}_{yt})\\
&E_j|_{t=0} = \partial_t E_j|_{t=0} = \cdots = \partial_t^{2m-1} E_j|_{t=0} = 0 \ \text{ in } \mathcal{D}'(\mathbb{H}_1)\\
&\partial_y^k E_j|_{y=0} = \delta(x,t)\delta_{jk}, \quad k = 0, \dots, m-1, \ \text{ in } \mathcal{D}'(\mathbb{H}_2),
\end{aligned}$$

where $\mathbb{H}_1 = \{(x, y) \in \mathbb{R}^{n+1} : y > 0\}$, $\mathbb{H}_2 = \{(x, t) \in \mathbb{R}^{n+1} : t > 0\}$.

For the general theory of the mixed problem for constant coefficient, linear partial differential operators see [15, 12.9, p. 162–179] and [24, p. 57–118]. We call E_j *Poisson kernel* of the Cauchy-Dirichlet problem for the iterated wave operator and the iterated Klein-Gordon-operator if $\xi = 0$ or $\xi \neq 0$, respectively ([24, p. 94]).

Proposition 14. *The* Poisson kernel E_j, $j = 0, 1, \dots, m-1$, *of the* Cauchy-Dirichlet problem for the iterated wave operator $(\Delta_n + \partial_y^2 - \partial_t^2)^m$ *in the half space* $\mathbb{H}_{yt}$ *is given by*

$$E_j = \frac{-y^m}{2^{n-1}\pi^{\frac{n}{2}}\Gamma(\frac{n}{2}) j!(m-1-j)!}(-\partial_y)^{m-j-1}\left(\frac{1}{y}\partial_y\right)\left(\frac{\partial_t^{n-1}(t^2 - |x|^2 - y^2)_+^{\frac{n}{2}-1}}{(|x|^2 + y^2)^{\frac{n-1}{2}}} Y(t)\right),$$

where $x_+^\lambda := x^\lambda Y(x)$*. Furthermore,* $E_j \in \mathcal{S}'(\mathbb{R}_{+,t})(\mathcal{D}'(\mathbb{H}_{1,xy}))$*.*

Proof. Recall from Definition 2 that

$$\mathcal{S}'(\mathbb{R}_{+,t}) := \bigcap_{\tau \in (0,\infty)} e^{\tau t}\mathcal{S}'_t,$$

so by Definition and Theorem 3, the Laplace transform

$$\mathscr{L} : \mathcal{S}'(\mathbb{R}_{+,t})(\mathcal{D}'(\mathbb{H}_{xy})) \to \mathcal{H}(T_p^\Gamma)(\mathcal{D}'(\mathbb{H}_{xy}))$$

is an isomorphism.

Thus the inverse Laplace transform $E_j := \mathscr{L}^{-1}F_j$ of the Poisson kernel F_j of the j-th Dirichlet problem in the half-space of the iterated metaharmonic operator in Proposition 11 yields

$$\begin{aligned}
&(\Delta_n + \partial_y^2 - \partial_t^2)^m E_j = 0\\
&E_j|_{t=0} = \partial_t E_j|_{t=0} = \cdots = \partial_t^{2m-1} E_j|_{t=0} = 0\\
&\partial_y^k E_j|_{y=0} = \delta(x,t)\delta_{jk}, \quad k = 0, \dots, m-1,
\end{aligned}$$

and

$$E_j = \frac{-y^m}{2^{\frac{n-1}{2}}\pi^{\frac{n+1}{2}} j!(m-1-j)!}(-\partial_y)^{m-j-1}\left(\frac{1}{y}\partial_y\right)\left(\frac{\mathscr{L}^{-1}\left(p^{\frac{n-1}{2}}K_{\frac{n-1}{2}}(p\sqrt{|x|^2+y^2})\right)}{(|x|^2+y^2)^{\frac{n-1}{4}}}\right).$$

By [6, (5.19)] we have for any $S \in \mathcal{H}(T^\Gamma)$: $\mathscr{L}^{-1}(p^{n-1}S) = \partial_t^{n-1}\mathscr{L}^{-1}S$. Hence, we obtain by means of the transform pair

$$\begin{aligned}\mathscr{L}^{-1}\left(p^{\frac{n-1}{2}}K_{\frac{n-1}{2}}(p\sqrt{|x|^2+y^2})\right) &= \partial_t^{n-1}\mathscr{L}^{-1}\left(\frac{K_{\frac{n-1}{2}}(p\sqrt{|x|^2+y^2}))}{p^{\frac{n-1}{2}}}\right) \\ &= \partial_t^{n-1}\left(\frac{\sqrt{\pi}}{\Gamma(\frac{n}{2})}\frac{(t^2-|x|^2-y^2)_+^{\frac{n}{2}-1}}{(2\sqrt{|x|^2+y^2})^{\frac{n-1}{2}}}Y(t)\right).\end{aligned}$$

In fact, [6, (5.19)] is the inverse relation of

$$\left\langle 1(t), e^{-pt}\frac{(t^2-|x|^2-y^2)}{(|x|^2+y^2)^{\frac{n-1}{4}}}Y(t-\sqrt{|x|^2-y^2})\right\rangle \frac{\sqrt{\pi}}{2^{\frac{n-1}{2}}\Gamma(n/2)} = \frac{K_{\frac{n-1}{2}}(p\sqrt{|x|^2+y^2})}{p^{\frac{n-1}{2}}}$$

(which can be seen using [12, 8.432 3, p. 917] or [35, §6.15. (4), p. 172]). That this identity is in fact valid in $\mathcal{H}(T_p^\Gamma)\hat{\otimes}\mathcal{D}'(\mathbb{H}_{xy})$ can be concluded similarly to the proof of Proposition 11. This yields the formula stated in the Proposition. The fact that E_j belongs to $\mathcal{S}'(\mathbb{R}_{+,t})(\mathcal{D}'(\mathbb{H}_{1,xy}))$ now follows by inspection.

It remains to show that $\partial_t^k E_j\big|_{t=0} = 0$ for $0 \le k \le 2m-1$. For this we first note that by [15, Th. 12.9.12, p. 176] we have $E_j \in \mathcal{C}^\infty([0,\infty), \mathcal{D}'(\mathbb{H}_{xy}))$, so

$$\partial_t^k E_j(t) = \text{const}\cdot y^m(-\partial_y)^{m-j-1}\left(\frac{1}{y}\partial_y\right)\partial_t^{k+n-1}\frac{Y(t)(t^2-|x|^2-y^2)_+^{\frac{n}{2}-1}}{(|x|^2+y^2)^{\frac{n-1}{2}}} \in \mathcal{D}'(\mathbb{H}_{xy})$$

and for $\varphi \in \mathcal{D}(\mathbb{H}_{xy})$ we obtain

$$\begin{aligned}\langle\varphi, \partial_t^k E_j(t)\rangle &= \text{const}\cdot\partial_t^{k+n-1}Y(t)\left\langle \partial_y\frac{1}{y}\partial_y^{m-j-1}(y^m\varphi), \frac{(t^2-|x|^2-y^2)_+^{\frac{n}{2}-1}}{(|x|^2+y^2)^{\frac{n-1}{2}}}\right\rangle \\ &= \text{const}\cdot\partial_t^{k+n-1}\left(Y(t)\int\limits_{|x|^2+y^2\le t^2}\phi(x,y)\frac{(t^2-|x|^2-y^2)^{\frac{n}{2}-1}}{(|x|^2+y^2)^{\frac{n-1}{2}}}\,dxdy\right),\end{aligned}$$

where $\phi(x,y) := \partial_y\frac{1}{y}\partial_y^{m-j-1}(y^m\varphi)$. Applying the homothety $x = t\xi$, $y = t\eta$, $t > 0$ shows that the latter equals

$$\text{const}\cdot\partial_t^{k+n-1}\left(t_+^n\int\limits_{|\xi|^2+\eta^2\le 1}\phi(t\xi,t\eta)\frac{(1-|\xi|^2-\eta^2)^{\frac{n}{2}-1}}{(|\xi|^2+\eta^2)^{\frac{n-1}{2}}}\,dxdy\right),$$

so

$$\lim_{t\searrow 0}\langle\varphi, \partial_t^k E_j(t)\rangle = \text{const}\cdot\int\limits_{|\xi|^2+\eta^2\le 1}\left[\lim_{t\searrow 0}\partial_t^{k+n-1}t_+^n\phi(t\xi,t\eta)\right]\cdot\frac{(1-|\xi|^2-\eta^2)^{\frac{n}{2}-1}}{(|\xi|^2+\eta^2)^{\frac{n-1}{2}}}\,d\xi d\eta.$$

As ϕ vanishes at $t = 0$, together with all its derivatives, we indeed arrive at $\partial_t^k E_j\big|_{t=0} = 0$ for all k. $\square$

Remark 15. We single out two important special cases:

(a) $n = 1,\ m = 1,\ j = 0$:
The Poisson kernel of the mixed problem

$$\begin{aligned}
&(\partial_x^2 + \partial_y^2 - \partial_t^2)E_0 = 0, \quad x \in \mathbb{R},\ y > 0,\ t > 0,\\
&E_0|_{t=0} = \partial_t E_0|_{t=0} = 0\\
&E_0|_{y=0} = \delta(x,t)
\end{aligned}$$

in the half-space $y > 0$ is given by

$$E_0 = -\frac{1}{\pi}\partial_y \frac{Y(t)}{(t^2 - x^2 - y^2)_+^{\frac{1}{2}}} = -\frac{Y(t)}{\pi}\partial_y\left((t^2 - x^2 - y^2)_+^{-\frac{1}{2}}\right).$$

In [19, Ex. 405, p. 189] the solution U to the mixed problem with the temporally constant boundary value $U|_{y=0} = \delta(x)$ is presented. It emerges from E_0 by convolution with $\delta(x) \otimes Y(t)$, i.e.,

$$U = -\frac{Y(t)}{\pi}\partial_y\left((t^2 - x^2 - y^2)_+^{-\frac{1}{2}}\right) * (\delta(x) \otimes Y(t)) = \frac{yt_+}{\pi(x^2 + y^2)(t^2 - x^2 - y^2)_+^{1/2}}.$$

Note that our derivation differs essentially from that proposed in [19], where Fourier- and Laplace transformation are suggested to be applied with respect to different variables.

(b) $n = 2,\ m = 1,\ j = 0$:
The Poisson kernel of the mixed Cauchy-Dirichlet problem

$$\begin{aligned}
&(\Delta_2 + \partial_y^2 - \partial_t^2)E_0 = 0, \quad x \in \mathbb{R}^2,\ y > 0,\ t > 0,\\
&E_0|_{t=0} = \partial_t E_0|_{t=0} = 0\\
&E_0|_{y=0} = \delta(x,t)
\end{aligned}$$

in the half-space $y > 0$ is given by

$$E_0 = \frac{-Y(t)}{2\pi}\partial_y\left(\frac{1}{\sqrt{|x|^2 + y^2}}\partial_t(Y(t^2 - |x|^2 - y^2))\right) = \frac{-1}{2\pi t}\partial_y(\delta(t - \sqrt{|x|^2 + y^2})).$$

Note that E_0 is the negative derivative in the direction normal to the boundary of the Green-function of the mixed problem of $\Delta_2 + \partial_y^2 - \partial_t^2$ in the half-space $y > 0$ (compare [10, p. 92]). We obtain the solution U of the mixed problem with a temporally constant boundary value, $U|_{y=0} = \delta(x)$, by convolution of E_0 with $\delta(x) \otimes Y(t)$:

$$U = \frac{-Y(t)}{2\pi}\partial_y\left(\frac{Y(t^2 - |x|^2 - y^2)}{\sqrt{|x|^2 + y^2}}\right) = \frac{-1}{2\pi}\partial_y\left(\frac{Y\left(t - \sqrt{|x|^2 + y^2}\right)}{\sqrt{|x|^2 + y^2}}\right),$$

which coincides with [4, p. 7].

The main idea in deriving the Poisson kernel of the Cauchy-Dirichlet problem $(\Delta_n + \partial_y^2 - \partial_t^2)^m$ in the half-space $y > 0$ (cf. the proof of Proposition 14) is the application of the inverse Laplace transformation to the Poisson kernel of the Dirichlet problem of $(\Delta_n + \partial_y^2 - p^2)^m$ in $y > 0$. The Poisson kernel of the

Cauchy-Dirichlet problem of the iterated Klein-Gordon operator $(\Delta_{2n+1}+\partial_y^2-\partial_t^2-\xi^2)^m$ $(\xi>0)$ in the half-space $y>0$ can be derived by the same method, using the Poisson kernel of $(\Delta_{2n+1}+\partial_y^2-p^2-\xi^2)^m$ in $y>0$.

Proposition 16. *The j-th* Poisson kernel E_j*, $j=0,1,\dots,m-1$, of the Cauchy-Dirichlet problem of the* iterated Klein-Gordon operator $(\Delta_{2n+1}+\partial_y^2-\partial_t^2-\xi^2)^m$ *($m\in\mathbb{N}$, $n\in\mathbb{N}_0$, $\xi>0$) in the half-space $\mathbb{H}$ is given by*

$$E_j=\frac{-y^m}{(2\pi)^{n+\frac12}j!(m-1-j)!}(-\partial_y)^{m-j-1}\left(\frac1y\partial_y\right)\left(\frac{1}{(|x|^2+y^2)^{\frac n2}}\sum_{l=0}^n\binom{n}{l}\xi^{n-2l+1}\right.$$
$$\left.\partial_t^{2l}(Y(t)(t^2-|x|^2)-y^2)_+^{\frac n2-\frac14}J_{n-\frac12}(\xi\sqrt{t^2-|x|^2-y^2})\right),$$

where J_λ denotes the Bessel function of the first kind of order λ.

Proof. Similar to the proof of Proposition 14 we have by means of Proposition 11

$$\begin{aligned}E_j&=\frac{-y^m}{2^n\pi^{n+1}j!(m-1-j)!}(-\partial_y)^{m-j-1}\left(\frac1y\partial_y\right)\\&\qquad\left(\mathscr{L}^{-1}\left(\frac{(p^2+\xi^2)^{\frac n2}K_n(\sqrt{(p^2+\xi^2)(|x|^2+y^2)})}{(|x|^2+y^2)^{\frac n2}}\right)\right)\\&=\frac{-y^m}{2^n\pi^{n+1}j!(m-1-j)!}(-\partial_y)^{m-j-1}\left(\frac1y\partial_y\right)\\&\qquad\left(\sum_{l=0}^n\binom{n}{l}\xi^{2n-2l}\partial_t^{2l}\mathscr{L}^{-1}\left(\frac{K_n(\sqrt{(p^2+\xi^2)(|x|^2+y^2)})}{((p^2+\xi^2)(|x|^2+y^2))^{\frac n2}}\right)\right).\end{aligned}$$

By the formula

$$\mathscr{L}^{-1}\left(\frac{K_n\left(\beta\sqrt{(p^2+\xi^2)}\right)}{(p^2+\xi^2)^{\frac n2}}\right)=\sqrt{\frac{\pi}{2}}Y(t)\xi^{-n+\frac12}\beta^{-n}(t^2-\beta^2)_+^{\frac n2-\frac14}J_{n-\frac12}\left(\xi\sqrt{t^2-\beta^2}\right)$$

in [6, p. 125] we obtain

$$\begin{aligned}E_j&=\frac{-y^m}{(2\pi)^{n+\frac12}j!(m-1-j)!}\sum_{l=0}^n\binom{n}{l}\xi^{n-2l+\frac12}(-\partial_y)^{m-j-1}\left(\frac1y\partial_y\right)\partial_t^{2l}\\&\qquad\left(\frac{Y(t)}{(|x|^2+y^2)^{\frac n2}}(t^2-|x|^2-y^2)_+^{\frac n2-\frac14}J_{n-\frac12}\left(\xi\sqrt{t^2-|x|^2-y^2}\right)\right),\end{aligned}$$

establishing our claim. □

Remark 17.

(a) In the special case $n=0$, $m=1$, $j=0$ we obtain

$$E_0=\frac{Y(t)}{\sqrt{2\pi}}\xi^{1/2}\partial_y\left(\frac{J_{-1/2}\left(\xi\sqrt{t^2-x^2-y^2}\right)}{(t^2-x^2-y^2)_+^{1/4}}\right)=\frac{Y(t)}{\pi}\partial_y\left(\frac{\cos\left(\xi\sqrt{t^2-x^2-y^2}\right)}{(t^2-x^2-y^2)_+^{1/2}}\right)$$

as the Poisson kernel of the Cauchy-Dirichlet problem of $\partial_x^2+\partial_y^2-\partial_t^2-\xi^2$ ($\xi>0$) in $y>0$. The solution U to this problem in $y>0$ with the temporally constant boundary value $U|_{y=0}=\delta(x)$ emerges from E_0 by convolution with $\delta(x)\otimes Y(t)$, i.e., $U=E_0*(Y(t)\otimes\delta(x,y))$.

Note that for the Cauchy-Dirichlet problem of the related operator $\partial_x^2+\partial_y^2-\partial_t^2-b\partial_t$ in $y>0$ with a temporally constant boundary value, the solution is given explicitly in [19, Ex. 406, p. 189] in terms of elementary functions.

(b) The Poisson kernel of the Cauchy-Dirichlet problem of the iterated Klein-Gordon operator $(\Delta_{2n}+\partial_y^2-\partial_t^2-\xi^2)^m$ in odd space dimensions can be deduced from that in Proposition 16 by J. Hadamard's method of descent, i.e., by integration with respect to the variable x_{2n+1}.

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