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Existence of extremals for critical Trudinger-Moser inequalities via the method of energy estimate

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A B S T R A C T

We reprove the existence of extremals for the critical Trudinger-Moser inequality on a smooth bounded domain of $\mathbb{R}^2$ via the method of energy estimate, which was recently developed by Malchiodi-Martinazzi [11], Mancini-Martinazzi [12] and Mancini-Thizy [13]. For this purpose, unlike [12,13], it suffices to calculate local energy of maximizers for subcritical Trudinger-Moser functionals near the blow-up point.

1. Introduction

Let Ω be a smooth bounded domain of $\mathbb{R}^2$ and $W_0^{1,2}(\Omega)$ be a Sobolev space, the completion of all smooth functions with compact support in Ω under the norm

$$\|u\|_{W_0^{1,2}(\Omega)} = \|\nabla u\|_2 = \left(\int_\Omega |\nabla u|^2 dx\right)^{1/2},$$

where ∇ denotes the gradient operator. In literature, the Trudinger-Moser inequality [17,14,16,15,22] is written as

$$C_\alpha(\Omega) = \sup_{u\in W_0^{1,2}(\Omega),\, \|\nabla u\|_2\leq 1} \int_\Omega e^{\alpha u^2} dx < \infty, \quad \forall \alpha \leq 4\pi. \tag{1}$$

A direct method of variation gives extremals of $C_\alpha(\Omega)$ for any $\alpha < 4\pi$. When $\alpha = 4\pi$, the Trudinger-Moser functional $J_{4\pi}(u) = \int_\Omega e^{4\pi u^2} dx$ does not satisfy the Palais-Smale condition for $u \in W_0^{1,2}(\Omega)$ with $\|\nabla u\|_2 \leq 1$. As a consequence, whether extremals for $C_{4\pi}(\Omega)$ exist or not is a rather delicate problem. It was proved by

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Carleson-Chang [2] that $C_{4\pi}(\mathbb{B})$ has an extremal, where $\mathbb{B}$ is the unit disc in $\mathbb{R}^2$. This result was generalized by Flucher [7] to the case of general domain Ω. In particular, there holds the following:

Theorem A. *([2,7]). The supremum $C_{4\pi}(\Omega)$ is attained by some function $u_0 \in W_0^{1,2}(\Omega) \cap C^2(\overline{\Omega})$ such that $\|\nabla u_0\|_2 = 1$.*

Motivated by Ding-Jost-Li-Wang [3] and Adimurthi-Struwe [1], Li [9] was able to prove Theorem A by the method of blow-up analysis. Both methods of [2] and [9] are essentially the same. Roughly speaking, under the assumption that $C_{4\pi}(\Omega)$ has no extremal (or blow-up occur), an exact upper bound C^* of $C_{4\pi}(\Omega)$ can be derived; but a direct calculation shows $C_{4\pi}(\Omega) > C^*$, which is impossible unless no blow-up occur. Anyway $C_{4\pi}(\Omega)$ must be attained.

Based on works of Malchiodi-Martinazzi [11], Mancini-Martinazzi [12] were able to give a new proof of Theorem A in the case $\Omega = \mathbb{B}$ by the method of energy estimate. This method can be described as follows. For any $k \in \mathbb{N}$, there exists some radially symmetric function $u_k \in W_0^{1,2}(\mathbb{B}) \cap C^2(\overline{\mathbb{B}})$ such that

$$C_{4\pi-1/k}(\mathbb{B}) = \int_{\mathbb{B}} e^{(4\pi-1/k)u_k^2} dx$$

and that

$$\begin{cases} -\Delta u_k = \frac{1}{\lambda_k} u_k e^{(4\pi-1/k)u_k^2} \text{ in } \mathbb{B}, \\ \|\nabla u_k\|_2 = 1, \quad u_k > 0 \text{ in } \mathbb{B}, \\ \lambda_k = \int_{\mathbb{B}} u_k^2 e^{(4\pi-1/k)u_k^2} dx. \end{cases} \tag{2}$$

Set $v_k = \sqrt{4\pi - 1/k}\, u_k$. Assume $c_k = \max_\Omega v_k \to +\infty$ as $k \to \infty$. Using blow-up analysis and the topological fixed point theorem, they obtained the energy estimate

$$4\pi + \frac{4\pi}{c_k^4} + o\left(\frac{1}{c_k^4}\right) \leq \int_{\mathbb{B}} |\nabla v_k|^2 dx \leq 4\pi + \frac{6\pi}{c_k^4} + o\left(\frac{1}{c_k^4}\right). \tag{3}$$

This immediately leads to

$$4\pi + \frac{4\pi}{c_k^4} + o\left(\frac{1}{c_k^4}\right) \leq \int_{\mathbb{B}} |\nabla v_k|^2 dx = 4\pi - \frac{1}{k}, \tag{4}$$

which is impossible for large k. Therefore c_k must be bounded. Applying elliptic estimates to (2), one gets an extremal of $C_{4\pi}(\mathbb{B})$.

Our aim is to improve [12] to the case of general smooth bounded domain Ω. Similarly as above, by a direct method of variation, for any $k \in \mathbb{N}$, there exists some $u_k \in W_0^{1,2}(\Omega) \cap C^2(\overline{\Omega})$ such that

$$C_{4\pi-1/k}(\Omega) = \int_{\Omega} e^{(4\pi-1/k)u_k^2} dx \tag{5}$$

and that

$$\begin{cases} -\Delta u_k = \frac{1}{\lambda_k} u_k e^{(4\pi-1/k)u_k^2} \text{ in } \Omega, \\ \|\nabla u_k\|_2 = 1, \quad u_k > 0 \text{ in } \Omega, \\ \lambda_k = \int_{\Omega} u_k^2 e^{(4\pi-1/k)u_k^2} dx. \end{cases} \tag{6}$$

Then we shall prove the following:

Theorem 1. *Let* $u_k \in W_0^{1,2}(\Omega) \cap C^2(\overline{\Omega})$ *satisfy* (5) *and* (6)*, and let* $v_k = \sqrt{4\pi - 1/k}\, u_k$*. Suppose that* $c_k = \max_\Omega v_k \to +\infty$ *as* $k \to \infty$*. Then we have*

$$\int_\Omega |\nabla v_k|^2 dx \geq 4\pi + \frac{4\pi}{c_k^4} + o\left(\frac{1}{c_k^4}\right). \tag{7}$$

Clearly, by an obvious analog of (4), Theorem A follows from Theorem 1. The proof of Theorem 1 is based on techniques in [13]. We give its outline as follows. Let x_k be such that $c_k = \max_\Omega u_k = u_k(x_k)$. We assume up to a subsequence, $x_k \to x_0 \in \overline{\Omega}$ as $k \to \infty$. By a result of Gidas-Ni-Nirenberg [8], x_0 lies in the interior of Ω. Let B_k be a function radially symmetric with respect to x_k solving

$$\begin{cases} -\Delta B_k = \Lambda_k B_k e^{B_k^2} \ \text{ in } \mathbb{R}^2, \\ B_k(x_k) = c_k, \end{cases} \tag{8}$$

where $\Lambda_k = 1/\lambda_k$ and λ_k be defined as in (6). Firstly we expand B_k to a Taylor formula up to an error $O(c_k^{-7})$ near x_k, while in ([13], Step 3.2), only Taylor expansion up to an error $O(c_k^{-5})$ are given. Secondly, as in ([13], Step 3.3), we compare the behavior of v_k and B_k near x_k by employing ([5], Proposition 3.1). Finally we obtain (7) by combining the above two steps.

It should be mentioned that as we did in [20], the argument in this note can be adapted to reprove [10,19]. In view of [21], it can also be modified to reprove [18,6]. The remaining part of this note is to prove Theorem 1. Throughout this note, sequence and subsequence are not distinguished, various constants are often denoted by the same C, and $\mathbb{B}_x(r) \subset \mathbb{R}^2$ always denotes the disc centered at x with radius r.

2. Proof of Theorem 1

We now prove Theorem 1 by using the method of energy estimate [11–13]. To begin with, we let $u_k \in W_0^{1,2}(\Omega) \cap C^2(\overline{\Omega})$ satisfy (5) and (6). Then $v_k = \sqrt{4\pi - 1/k}\, u_k$ solves

$$\begin{cases} -\Delta v_k = \Lambda_k v_k e^{v_k^2} \ \text{ in } \Omega, \\ v_k > 0 \ \text{ in } \Omega, \quad v_k = 0 \ \text{ on } \partial\Omega, \\ \|\nabla v_k\|_2 = \sqrt{4\pi - 1/k}, \\ \Lambda_k = 1/\lambda_k. \end{cases} \tag{9}$$

By our assumption, $c_k = \max_\Omega v_k = v_k(x_k) \to +\infty$ as $k \to \infty$. According to Gidas-Ni-Nirenberg [8], one may assume $x_k \to x_0 \in \Omega$ as $k \to \infty$. Let

$$r_k^2 = 4\Lambda_k^{-1} c_k^{-2} \exp(-c_k^2). \tag{10}$$

It is known [1,4] that $r_k \to 0$ as $k \to \infty$, and that there exists a sequence of positive numbers (s_k) satisfying $s_k \to +\infty$ and

$$\|c_k(v_k(x_k + r_k \cdot) - c_k) - T_0\|_{C^2(\mathbb{B}_0(s_k))} \to 0 \tag{11}$$

as $k \to \infty$, where $T_0(x) = -\log(1 + |x|^2)$ is a solution of

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$$-\Delta T_0 = 4\exp(2T_0) \quad \text{in} \quad \mathbb{R}^2.$$

Let S_0 be a function radially symmetric with respect to the origin satisfying

$$\begin{cases} -\Delta S_0 = 4\exp(2T_0)(2S_0 + T_0^2 + T_0) \quad \text{in} \quad \mathbb{R}^2, \\ S_0(0) = 0. \end{cases} \tag{12}$$

Hereafter, we slightly abuse some notations. When a function u is radially symmetric with respect to $a \in \mathbb{R}^2$, we sometimes denote $u(|x-a|) = u(x-a)$ for $x \in \mathbb{R}^2$. Note that S_0 can be explicitly written out by Malchiodi-Martinazzi [11]. In particular

$$S_0(r) = O(\log r) \quad \text{as} \quad r \to \infty, \tag{13}$$

and

$$\int_{\mathbb{R}^2} \Delta S_0(y)dy = -4\pi. \tag{14}$$

Let $W_0(x) = W_0(|x|)$ be the radial solution of

$$\begin{cases} -\Delta W_0 = 4\exp(2T_0)(S_0 + 2S_0^2 + 4T_0S_0 + 2S_0T_0^2 + T_0^3 + \frac{1}{2}T_0^4 + 2W_0) \quad \text{in} \quad \mathbb{R}^2, \\ W_0(0) = 0. \end{cases} \tag{15}$$

According to Mancini-Martinazzi [12], there holds

$$W_0(r) = O(\log r) \quad \text{as} \quad r \to \infty, \tag{16}$$

and

$$\int_{\mathbb{R}^2} \Delta W_0(x)dx = -12\pi - \frac{2}{3}\pi^2. \tag{17}$$

We define three sequences of functions

$$T_k(x) = T_0\left(\frac{x-x_k}{r_k}\right), \ S_k(x) = S_0\left(\frac{x-x_k}{r_k}\right), \ W_k(x) = W_0\left(\frac{x-x_k}{r_k}\right). \tag{18}$$

Let $\tau \in (0,1)$ be a fixed real number. Take $r_{k,\tau} > 0$ such that $T_k(r_{k,\tau}) = -\tau c_k^2$. Obviously we have

$$r_{k,\tau}^2 = r_k^2 \exp(\tau c_k^2 + o_k(1)). \tag{19}$$

Let B_k be defined as in (8). Then B_k has the following expansion:

Proposition 2. *For any $y_k \in \mathbb{B}_{x_k}(r_{k,\tau})$, there holds*

$$B_k(y_k) = c_k + \frac{T_k(y_k)}{c_k} + \frac{S_k(y_k)}{c_k^3} + \frac{W_k(y_k)}{c_k^5} + O\left(\frac{1-T_k(y_k)}{c_k^7}\right).$$

Proof. We write

$$B_k = c_k + \frac{T_k}{c_k} + \frac{S_k}{c_k^3} + \frac{W_{1,k}}{c_k^5}. \tag{20}$$

Let $\rho_{1,k} > 0$ be defined by

$$\rho_{1,k} = \sup\left\{r \in (0, r_{k,\tau}] : |W_{1,k}(s) - W_k(s)| \leq 1 - T_k(s) \text{ for all } s \leq r\right\}. \tag{21}$$

We calculate on $\mathbb{B}_{x_k}(\rho_{1,k})$ that

$$B_k = c_k + \frac{T_k}{c_k} + \frac{S_k}{c_k^3} + O\left(\frac{1-T_k}{c_k^5}\right),$$

that

$$B_k^2 = c_k^2 + 2T_k + \frac{T_k^2 + 2S_k}{c_k^2} + \frac{2W_{1,k} + 2T_kS_k}{c_k^4} + O\left(\frac{1+T_k^2}{c_k^6}\right),$$

and that

$$\begin{aligned}\Lambda_k B_k \exp(B_k^2) = \Lambda_k c_k \exp(c_k^2)\exp(2T_k)\Bigg\{1 &+ \frac{T_k^2 + 2S_k + T_k}{c_k^2} \\ &+ \frac{S_k + 2W_{1,k} + 4T_kS_k + T_k^3 + 2T_k^2S_k + 2S_k^2 + \frac{1}{2}T_k^4}{c_k^4} \\ &+ O\left(\frac{(1+T_k^4)\exp(T_k^2/c_k^2)}{c_k^6}\right)\Bigg\}.\end{aligned}$$

This together with (10) and (20) gives

$$\begin{aligned}-\Delta W_{1,k} &= -c_k^5\left\{\Delta B_k + \frac{1}{c_k}\Delta T_k - \frac{1}{c_k^3}\Delta S_k\right\} \\ &= \frac{4\exp(2T_k)}{r_k^2}\Bigg\{2W_{1,k} + 4T_kS_k + \frac{1}{2}T_k^4 + 2T_k^2S_k + 2S_k^2 + S_k - T_k^3 \\ &\quad + O\left(\frac{(1+T_k^4)\exp(T_k^2/c_k^2)}{c_k^2}\right)\Bigg\}.\end{aligned}$$

In view of (15) and (18),

$$-\Delta W_k = \frac{4\exp(2T_k)}{r_k^2}\left(2W_k + 4T_kS_k + \frac{1}{2}T_k^4 + 2T_k^2S_k + 2S_k^2 + S_k - T_k^3\right).$$

Hence on $\mathbb{B}_{x_k}(\rho_{1,k})$, the difference between $W_{1,k}$ and W_k satisfies

$$-\Delta(W_{1,k} - W_k) = \frac{4\exp(2T_k)}{r_k^2}\left\{2(W_{1,k} - W_k) + O\left(\frac{(1+T_k^4)\exp(T_k^2/c_k^2)}{c_k^2}\right)\right\}. \tag{22}$$

Since on $\mathbb{B}_{x_k}(r_{k,\tau})$, $0 \geq T_k \geq -\tau c_k^2$, and thus $2 + T_k/c_k^2 \geq 2 - \tau > 1$. Hence we have for $r \leq \rho_{1,k}$,

$$\int\limits_{\mathbb{B}_{x_k}(r)} \frac{(1+T_k^4)\exp(2T_k+T_k^2/c_k^2)}{r_k^2}dx \leq \int\limits_{\mathbb{B}_{x_k}(r)} \frac{(1+T_k^4)\exp((2-\tau)T_k)}{r_k^2}dx$$
$$\leq C \int\limits_{\mathbb{B}_0(r/r_k)} \frac{1}{(1+|y|^2)^{\frac{3-\tau}{2}}}dy$$
$$\leq C\frac{(r/r_k)^2}{1+(r/r_k)^2}, \tag{23}$$

and by the differential mean value theorem,

$$\int\limits_{\mathbb{B}_{x_k}(r)} \frac{\exp(2T_k)}{r_k^2}|W_{1,k}-W_k|dx \leq \|(W_{1,k}-W_k)'\|_{C^0([0,\rho_{1,k}])} \int\limits_{\mathbb{B}_{x_k}(r)} \frac{\exp(2T_k)}{r_k^2}|x-x_k|dx$$
$$= \|(W_{1,k}-W_k)'\|_{C^0([0,\rho_{1,k}])} r_k \int\limits_{\mathbb{B}_0(r/r_k)} \frac{|y|}{(1+|y|^2)^2}dy$$
$$\leq Cr_k\|(W_{1,k}-W_k)'\|_{C^0([0,\rho_{1,k}])}\frac{(r/r_k)^3}{1+(r/r_k)^3}. \tag{24}$$

Using the divergence theorem, we have

$$\int\limits_{B_{x_k}(r)} \Delta(W_{1,k}-W_k)dx = 2\pi r(W_{1,k}-W_k)'(r). \tag{25}$$

Combining (23)-(25), we obtain for $r \leq \rho_{1,k}$,

$$r|(W_{1,k}-W_k)'(r)| \leq \frac{C(r/r_k)^2}{c_k^2(1+(r/r_k)^2)}$$
$$+Cr_k\|(W_{1,k}-W_k)'\|_{C^0([0,\rho_{1,k}])}\frac{(r/r_k)^3}{1+(r/r_k)^3}. \tag{26}$$

Now we *claim* that

$$r_k c_k^2\|(W_{1,k}-W_k)'\|_{C^0([0,\rho_{1,k}])} \leq C. \tag{27}$$

For otherwise we have

$$\Theta_k = r_k c_k^2\|(W_{1,k}-W_k)'\|_{C^0([0,\rho_{1,k}])} = r_k c_k^2|(W_{1,k}-W_k)'(a_k)| \to +\infty. \tag{28}$$

We may assume $\rho_{1,k}/r_k \to \theta_0 \in [0,+\infty]$ as $k\to\infty$. It follows from (26) that

$$\Theta_k \leq \frac{Ca_k/r_k}{1+(a_k/r_k)^2} + \Theta_k\frac{C(a_k/r_k)^2}{1+(a_k/r_k)^3}.$$

This together with (28) leads to

$$1/C \leq a_k/r_k \leq C \tag{29}$$

for some constant $C > 0$, and thus $\theta_0 > 0$. We set a sequence of functions

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$$w_k(t)=\frac{c_k^2}{\Theta_k}(W_{1,k}-W_k)(r_kt). \tag{30}$$

In view of (22), w_k satisfies

$$-\Delta w_k=4\exp(2\widetilde{T}_k)\left\{2w_k+O\left(\frac{(1+\widetilde{T}_k^4)\exp(\widetilde{T}_k^2/c_k^2)}{\Theta_k}\right)\right\}, \tag{31}$$

where $\widetilde{T}_k(t)=T_k(r_kt)$. Since $|w'(t)|\leq 1$ for all $t\in[0,\rho_{1,k}/r_k]$ and $w_k(0)=0$, we have that w_k is uniformly bounded in $(0,\rho_k/r_k)$. Then applying elliptic estimates to (31), we conclude

$$w_k\to w_0 \quad \text{in} \quad C^1_{\rm loc}(\mathbb{B}_0(\theta_0)), \tag{32}$$

and w_0 is a solution of

$$\begin{cases}-\Delta w_0=8\exp(2T_0)w_0 \quad \text{in} \quad \mathbb{B}_0(\theta_0)\\ w_0(0)=0.\end{cases}$$

Note also that w_0 is radially symmetric with respect to 0. The uniqueness of solutions to the ordinary differential equation (31) leads to $w_0\equiv 0$ on $[0,\theta_0)$. This together with (30), (32) and the Lebesgue dominated convergence theorem gives

$$\begin{aligned}\int\limits_{B_{x_k}(\rho_{1,k})}\frac{\exp(2T_k)}{r_k^2}|W_{1,k}-W_k|dx&=\frac{\Theta_k}{c_k^2}\int\limits_{|y|\leq\rho_{1,k}/r_k}\exp(2\widetilde{T}_k)|w_k|dy\\ &=o(\Theta_k/c_k^2).\end{aligned} \tag{33}$$

Replacing (24) with (33) in the proof of (26) and keeping in mind (28), we have

$$r|(W_{1,k}-W_k)'(r)|\leq o(\Theta_k/c_k^2) \tag{34}$$

for all $r\in[0,\rho_{1,k}]$. Taking $r=a_k$ in (34), we conclude $a_k/r_k=o(1)$, which contradicts (29). This confirms our claim (27).

Inserting (27) into (26), we have

$$r|(W_{1,k}-W_k)'(r)|\leq\frac{C}{c_k^2}\frac{(r/r_k)^2}{1+(r/r_k)^2} \tag{35}$$

for all $r\in[0,\rho_{1,k}]$. If $0\leq r\leq r_k$, then (35) gives $|(W_{1,k}-W_k)'(r)|\leq Cr/(r_k^2c_k^2)$, and thus

$$|(W_{1,k}-W_k)(r)|\leq\int_0^r\frac{Ct}{r_k^2c_k^2}dt\leq\frac{C}{c_k^2};$$

While if $r_k<r\leq\rho_{1,k}$, then it follows from (35) that $|(W_{1,k}-W_k)'(r)|\leq C/(c_k^2r)$, which leads to

$$|(W_{1,k}-W_k)(r)|\leq\int_{r_k}^r\frac{C}{c_k^2}\frac{1}{t}dt=\frac{C}{c_k^2}\log\frac{r}{r_k}$$

Combining the above two cases, we obtain

$$(W_{1,k} - W_k)(r) = O\left(\frac{1 - T_k(r)}{c_k^2}\right)$$

uniformly in $r \in [0, \rho_{1,k}]$. This together with (21) leads to $\rho_{1,k} = r_{k,\tau}$ and ends the proof of the proposition. □

Now we compare the values of v_k and B_k on $\mathbb{B}_{x_k}(r_{k,\tau})$. By (11), there exists a sequence of positive numbers $(\mathbf{s}_k)$ such that $\mathbf{s}_k \to +\infty$ and $v_k(x) = (1 + o_k(1))c_k$ uniformly in $x \in \mathbb{B}_{x_k}(r_k\mathbf{s}_k)$. For any positive number $\tau' < 1 - \tau$, we let $\varrho_{k,\tau'} > 0$ be defined as

$$\varrho_{k,\tau'} = \sup\{s : v_k(r) \geq \tau' c_k \text{ for all } r \leq s\}.$$

Clearly $\varrho_{k,\tau'} \geq r_k\mathbf{s}_k$. In particular $\varrho_{k,\tau'}/r_k \to +\infty$. By ([4], Proposition 2), there exists some constant C such that $|x - x_k||\nabla v_k(x)|v_k(x) \leq C$ for all $x \in \Omega$. Hence we have for all $x \in \mathbb{B}_{x_k}(\varrho_{k,\tau'})$,

$$|x - x_k||\nabla v_k(x)| \leq C c_k^{-1}.$$

It then follows from Proposition 2 and ([5], Proposition 3.1) that $\varrho_{k,\tau'} \geq r_{k,\tau}$ and

$$|v_k(x) - B_k(x)| \leq \frac{C|x - x_k|}{c_k r_{k,\tau}} \quad \text{for all} \quad x \in \mathbb{B}_{x_k}(r_{k,\tau}). \tag{36}$$

From now on, we restrict our considerations on the balls $\mathbb{B}_{x_k}(r_{k,\tau})$. Since (36) together with Proposition 2 leads to

$$v_k = c_k + \frac{T_k}{c_k} + \frac{S_k}{c_k^3} + \frac{W_k}{c_k^5} + O\left(\frac{1 - T_k}{c_k^7}\right) + O\left(\frac{|\cdot - x_k|}{c_k r_{k,\tau}}\right),$$

there holds

$$\begin{aligned} v_k^2 &= c_k^2 + 2T_k + \frac{T_k^2 + 2S_k}{c_k^2} + \frac{2T_kS_k + 2W_k}{c_k^4} + O\left(\frac{1 - T_k}{c_k^6}\right) + O\left(\frac{|\cdot - x_k|}{r_{k,\tau}}\right) \\ &= c_k^2\left(1 + \frac{2T_k}{c_k^2} + \frac{T_k^2 + 2S_k}{c_k^4} + O\left(\frac{1 + T_k^2}{c_k^6}\right) + O\left(\frac{|\cdot - x_k|}{c_k^2 r_{k,\tau}}\right)\right). \end{aligned} \tag{37}$$

By an inequality $|\exp(t) - 1 - t - t^2/2| \leq t^3\exp(t)$ for all $t \geq 0$, it follows from (37) that

$$\begin{aligned} \exp(v_k^2) = \exp(c_k^2 + 2T_k)\Bigg\{1 &+ \frac{T_k^2 + 2S_k}{c_k^2} + \frac{2T_kS_k + 2W_k + \frac{1}{2}(T_k^2 + 2S_k)^2}{c_k^4} \\ &+ O\left(\frac{(1 + T_k^6)\exp(T_k^2/c_k^2)}{c_k^6}\right) + O\left(\frac{|\cdot - x_k|}{r_{k,\tau}}\right)\Bigg\}. \end{aligned} \tag{38}$$

Combining (10), (37) and (38), we obtain

$$\begin{aligned} \Lambda_k v_k^2 \exp(v_k^2) = \frac{4\exp(2T_k)}{r_k^2}\Bigg\{1 &+ \frac{T_k^2 + 2S_k + 2T_k}{c_k^2} \\ &+ \frac{T_k^2 + 2S_k + 2W_k + \frac{1}{2}T_k^4 + 2T_k^2S_k + 2S_k^2 + 2T_k^3 + 6S_kT_k}{c_k^4} \end{aligned}$$

$$+O\left(\frac{(1+T_k^6)\exp(T_k^2/c_k^2)}{c_k^6}\right)+O\left(\frac{|\cdot-x_k|}{r_{k,\tau}}\right)\Bigg\}. \tag{39}$$

We now calculate integrals of all terms on the righthand side of (39). Firstly, by the definition of T_0 and T_k (see (18) above), there holds

$$\begin{aligned}\int_{\mathbb{B}_{x_k}(r_{k,\tau})}\frac{4\exp(2T_k)}{r_k^2}dx &= \int_{\mathbb{B}_0(r_{k,\tau}/r_k)} 4\exp(2T_0(y))dy\\ &= \int_{\mathbb{B}_0(r_{k,\tau}/r_k)}\frac{4}{(1+|y|^2)^2}dy = 4\pi+O\left(\frac{r_k}{r_{k,\tau}}\right).\end{aligned} \tag{40}$$

Secondly, in view of a direct calculation

$$\int_{\mathbb{R}^2} 4\exp(2T_0(y))T_0(y)dy = -\int_{\mathbb{R}^2}\frac{4\log(1+|y|^2)}{(1+|y|^2)^2}dy = -4\pi,$$

we have by (12), (13) and (14) that

$$\begin{aligned}&\int_{\mathbb{B}_{x_k}(r_{k,\tau})}\frac{4\exp(2T_k)}{r_k^2}\frac{T_k^2+2S_k+2T_k}{c_k^2}dx\\ &=\frac{4}{c_k^2}\int_{\mathbb{B}_0(r_{k,\tau}/r_k)}\exp(2T_0(y))(T_0^2(y)+2S_0(y)+2T_0(y))dy\\ &=\frac{4}{c_k^2}\int_{\mathbb{R}^2}\exp(2T_0(y))(T_0^2(y)+2S_0(y)+2T_0(y))dy+O\left(\frac{r_k}{c_k^2 r_{k,\tau}}\right)\\ &=\frac{1}{c_k^2}\left(\int_{\mathbb{R}^2}(-\Delta S_0(y))dy+\int_{\mathbb{R}^2}4\exp(2T_0(y))T_0(y)dy\right)+O\left(\frac{r_k}{c_k^2 r_{k,\tau}}\right)\\ &=O\left(\frac{r_k}{c_k^2 r_{k,\tau}}\right).\end{aligned} \tag{41}$$

Thirdly, since Mancini-Martinazzi [12] calculated

$$\int_{\mathbb{R}^2}4\exp(2T_0)(T_0^2+T_0^3+S_0+2S_0T_0)dy = -8\pi-\frac{2}{3}\pi^3,$$

we obtain by employing (15), (16) and (17),

$$\begin{aligned}&\int_{\mathbb{B}_{x_k}(r_{k,\tau})}\frac{4\exp(2T_k)}{r_k^2}\frac{T_k^2+2S_k+2W_k+\frac{1}{2}T_k^4+2T_k^2S_k+2S_k^2+2T_k^3+6S_kT_k}{c_k^4}dx\\ &=\frac{4}{c_k^4}\int_{\mathbb{B}_0(r_{k,\tau}/r_k)}\exp(2T_0)(T_0^2+2S_0+2W_0+\frac{1}{2}T_0^4+2T_0^2S_0+2S_0^2+2T_0^3+6S_0T_0)dy\\ &=\frac{1}{c_k^4}\left\{\int_{\mathbb{R}^2}(-\Delta W_0)dy+\int_{\mathbb{R}^2}4\exp(2T_0)(T_0^2+T_0^3+S_0+2S_0T_0)dy+O\left(\frac{r_k}{r_{k,\tau}}\right)\right\}\end{aligned}$$

$$= \frac{1}{c_k^4}\left\{(12\pi + \frac{2}{3}\pi^3) + (-8\pi - \frac{2}{3}\pi^3) + O\left(\frac{r_k}{r_{k,\tau}}\right)\right\}$$
$$= \frac{4\pi}{c_k^4} + O\left(\frac{r_k}{c_k^4 r_{k,\tau}}\right). \tag{42}$$

Fourthly, it is easy to estimate

$$\int_{\mathbb{B}_{x_k}(r_{k,\tau})} \frac{4\exp(2T_k)}{r_k^2}\frac{(1+T_k^6)\exp(T_k^2/c_k^2)}{c_k^6}dx = O\left(\frac{1}{c_k^6}\right). \tag{43}$$

Finally, in view of (19), we get

$$\begin{aligned}\int_{\mathbb{B}_{x_k}(r_{k,\tau})} \frac{4\exp(2T_k)}{r_k^2}\frac{|x-x_k|}{r_{k,\tau}}dx &= \frac{r_k}{r_{k,\tau}}\int_{\mathbb{B}_0(r_{k,\tau}/r_k)} 4\exp(2T_0(y))|y|dy \\ &= o\left(\frac{1}{c_k^4}\right).\end{aligned} \tag{44}$$

Combining (39)-(44), we conclude

$$\int_{\mathbb{B}_{x_k}(r_{k,\tau})} \Lambda_k v_k^2 \exp(v_k^2)dx = 4\pi + \frac{4\pi}{c_k^4} + o\left(\frac{1}{c_k^4}\right). \tag{45}$$

Since $\Lambda_k \geq 0$, we have by multiplying both sides of the equation (9) by v_k and integration by parts,

$$\int_\Omega |\nabla v_k|^2 dx = \int_\Omega \Lambda_k v_k^2 \exp(v_k^2)dx \geq \int_{\mathbb{B}_{x_k}(r_{k,\tau})} \Lambda_k v_k^2 \exp(v_k^2)dx. \tag{46}$$

Clearly (7) follows from (45) and (46), and the proof of Theorem 1 is completed. $\square$

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