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The Hájek-Rényi-Chow maximal inequality and a strong law of large numbers in Riesz spaces

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ABSTRACT

In this paper we generalize the Hájek-Rényi-Chow maximal inequality for submartingales to L^p type Riesz spaces with conditional expectation operators. As applications we obtain a submartingale convergence theorem and a strong law of large numbers in Riesz spaces. Along the way we develop a Riesz space variant of the Clarkson's inequality for $1 \le p \le 2$.

1. Introduction

In a Dedekind complete Riesz space E with weak order unit, say e, we say that T is a conditional expectation on E if T is a positive order continuous linear projection on E which maps weak order units to weak order units and has range, $R(T)$, a Dedekind complete Riesz subspace of E, see [13] for more details. It should be noted that the only conditional expectation operator which is also a band projection is the identity map. A conditional expectation operator T is said to be strictly positive if $T|f| = 0$ implies that $f = 0$. Every Archimedean Riesz space E can be extended uniquely (up to Riesz isomorphism) to a universally complete space E^u, see [23]. It was shown in [13] that the domain of a strictly positive conditional expectation operator T can be extended to its natural domain $L^1(T)$ in E^u. In particular $L^1(T) = dom(T) - dom(T)$ where $f \in dom(T)$ if $f \in E^u_+$, the positive cone of E^u, and there is an upwards directed net f_α in E_+ with the net Tf_α order bounded in E^u and in this case the value assigned to Tf is the order limit in E^u of the

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net Tf_α. Given that the above extensions can be made we will assume throughout that T is a conditional expectation operator acting on $L^1(T)$. The space E^u is an f-algebra with multiplication defined so that the chosen weak order unit, e, is the algebraic unit. Further, it was shown in [18] that $R(T) = \{Tf | f \in L^1(T)\}$ is an f-algebra and $L^1(T)$ is an $R(T)$-module with $R(T)$-valued norm $\|f\|_{1,T} := T|f|$. The space $L^2(T)$ was introduced in [19] and generalized to $L^p(T) = \{x \in L^1(T) : |x|^p \in L^1(T)\}, 1 < p < \infty$, in [2] where functional calculus was used to define $f(x) = x^p$ for $x \in E^u_+$. Much of the mathematical machinery needed to work in $L^p(T), 1 < p < \infty$, was developed in [10] even though such spaces were not considered there. Again these spaces are $R(T)$-modules with associated $R(T)$-valued norms $\|f\|_{p,T} := (T|f|^p)^{1/p}$. We note that in [18] the $R(T)$-module $L^\infty(T) = \{x \in L^1(T) : |x| \le y \text{ for some } y \in R(T)\}$ was considered, with $R(T)$-valued norm $\|f\|_{\infty,T} := \inf\{y \in R(T)_+ : |f| \le y\}$. Here we have that $L^\infty(T)$ is an f-algebra order dense in $L^1(T)$ and having $L^\infty(T) \subset L^p(T) \subset L^1(T)$ for all $1 < p < \infty$. Regarding L^p type spaces we also note the work of Boccuto, Candeloro and Sambucini in [3]. In Section 3, we generalize Clarkson's inequality to $L^p(T), 1 \le p \le 2$.

A filtration on a Dedekind complete Riesz space E with weak order unit is a family of conditional expectation operators $(T_i)_{i\in\mathbb{N}}$ defined on E having $T_iT_j = T_jT_i = T_i$ for all $i < j$. We say that a sequence of elements $(f_i)_{i\in\mathbb{N}}$ in a Riesz space is adapted to a filtration $(T_i)_{i\in\mathbb{N}}$ if $f_i \in R(T_i)$ for all $i \in \mathbb{N}$. A sequence $(f_i)_{i\in\mathbb{N}}$ is said to be predictable if $f_i \in R(T_{i-1})$ for each $i \in \mathbb{N}$. A Riesz space (sub, super) martingale is a double sequence $(f_i, T_i)_{i\in\mathbb{N}}$ with $(f_i)_{i\in\mathbb{N}}$ adapted to the filtration $(T_i)_{i\in\mathbb{N}}$ and $T_if_j(\ge, \le) = f_i$ for $i < j$. The fundamentals of such processes can be found in [12–14] as well as their continuous time versions in [8,9]. In Section 4, we give the Hájek-Rényi-Chow maximal inequality for Riesz space submartingales, see [4, Theorem 1] and [6, Proposition (6.1.4)] for measure theoretic versions. The Hájek-Rényi-Chow maximal inequality for submartingales has as a special case Doob's maximal inequality. We note that maximal inequalities have been obtained for Riesz space positive supermartingales in [11, Lemma 3.1] and for Riesz space quasi-martingales in [22, Theorem 6.2.10]. The Hájek-Rényi-Chow maximal inequality for submartingales is applied, in Theorem 5.1, to non-negative submartingales to obtain weighted convergence, via a proof which does not use of upcrossing. We note that this theorem can be deduced directly from [14, Theorem 3.5], which is, however, based on the Riesz space upcrossing theorem. For E a Dedekind complete Riesz space with weak order unit, e, and (B_n) an increasing sequence of bands in E, with associated band projections (P_n), it was proved in [21] that $x_n/b_n \to 0$, in order, as $n \to \infty$ if $x_n \in B_n$ with $P_nx_{n+1} = x_n$ and $|x_{n+1} - x_n| \le c_ne$, for all $n \in \mathbb{N}$. Here $c_n > 0$ and $0 < b_n \uparrow \infty$, for all $n \in \mathbb{N}$, with $\dfrac{1}{b_n}\left(\sum_{i=1}^{n} c_i^2\right)^{1/2} \to 0$ as $n \to \infty$. In Section 5, we conclude by giving Chow's strong law of large numbers in $L^p(T), 1 < p < \infty$, see [4,5] and [6, Theorems 6.1.8 and 6.1.9] for measure versions.

We note that, for Riesz space processes, a strong law of large numbers for ergodic processes was given in [15], a weak law of large number for mixingales in [16] and Bernoulli's law of large numbers in [17].

2. Weighted Cesàro means

In this section we give a version of Kronecker's Lemma for weighted Cesàro means in an Archimedean Riesz space.

Lemma 2.1. *Let E be an Archimedean Riesz space and (s_n) be a sequence in E_+ order convergent to 0. If b_n is a non-decreasing sequence of non-negative real numbers divergent to $+\infty$, then $\frac{1}{b_n}\sum_{i=1}^{n-1}(b_{i+1} - b_i)s_i$ converges to zero in order as $n \to \infty$.*

Proof. By the order convergence of (s_n) to 0, there is sequence (v_n) in E such that $s_n \le v_n \downarrow 0$, for $n \in \mathbb{N}$. As

$$0 \le \frac{1}{b_n} \sum_{i=1}^{n-1} (b_{i+1} - b_i) s_i \le \frac{1}{b_n} \sum_{i=1}^{n-1} (b_{i+1} - b_i) v_i =: z_n,$$

it suffices to show that $z_n \to 0$ in order. For $n \in \mathbb{N}$, let $N_n := \max\{j \in \mathbb{N} \,|\, j^2 b_j \le b_n\}$, then (N_n) is a non-decreasing sequence in $\mathbb{N}$ with $N_n \to \infty$ as $n \to \infty$ and $N_n < n$ for $n \ge 2$. Now, for $n \ge 2$,

$$\begin{aligned} z_n &= \frac{1}{b_n} \left(b_n v_{n-1} - b_1 v_1 + \sum_{i=2}^{N_n} b_i (v_{i-1} - v_i) + \sum_{i=N_n+1}^{n-1} b_i (v_{i-1} - v_i) \right) \\ &\le v_{n-1} + \frac{1}{b_n} \left(\sum_{i=2}^{N_n} b_{N_n} v_1 + \sum_{i=N_n+1}^{n-1} b_{n-1} (v_{i-1} - v_i) \right) \\ &\le v_{n-1} + \frac{N_n b_{N_n}}{b_n} v_1 + \frac{b_{n-1}}{b_n} v_{N_n} \le v_{n-1} + \frac{1}{N_n} v_1 + v_{N_n} \downarrow 0. \quad \square \end{aligned}$$

In [7, lemma 3.14], this result was proved for the case of $b_n = n, n \in \mathbb{N}$.

Lemma 2.2 *(Kronecker's Lemma). Let (x_n) be a summable sequence of elements in an Archimedean Riesz space E. Let $(b_n)_{n \in \mathbb{N}}$ be a non-decreasing sequence of non-negative real numbers divergent to $+\infty$. Then $\frac{1}{b_n} \sum_{i=1}^{n} b_i x_i \to 0$ in order as $n \to \infty$.*

Proof. Let $s_n := \sum_{i=n+1}^{\infty} x_i, n = 0, 1, 2, \ldots$, then $s_n \to 0$ in order and

$$\left| \frac{1}{b_n} \sum_{i=1}^{n} b_i x_i \right| = \frac{1}{b_n} \left| b_n s_n - b_1 s_0 - \sum_{i=1}^{n-1} (b_{i+1} - b_i) s_i \right| \le |s_n| + \frac{b_1}{b_n} |s_0| + \frac{1}{b_n} \sum_{i=1}^{n-1} (b_{i+1} - b_i) |s_i|$$

which converges to zero in order by Lemma 2.1. $\square$

3. Inequalities

The inequalities presented in this section form the foundation on which much of the rest of this paper is based.

Taking the product Riesz space $\mathcal{K} = [L^1(T)]^n$ with componentwise ordering and defining $\mathbb{F}(x_i)_{i=1}^n = (\frac{1}{n} \sum_{j=1}^n x_j)_{i=1}^n$ we have that $\mathbb{F}$ is a conditional expectation operator on $\mathcal{K}$. Hence from [10, Corollary 6.4] or [2, Theorem 3.7] we have the following theorem.

Theorem 3.1 *(Hölder's inequality for sums). Let T be a conditional expectation with natural domain $L^1(T)$ and $1 \le p, q \le \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $p = 1$ for $q = \infty$. Let $n \in \mathbb{N}$. If $x_i \in L^p(T)$ and $y_i \in L^q(T)$ for all $i \in \{1, 2, ..., n\}$, then $x_i y_i \in L^1(T)$ for each i, and*

$$\sum_{i=1}^{n} T|x_i y_i| \le \left(\sum_{i=1}^{n} T|x_i|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} T|y_i|^q \right)^{\frac{1}{q}}.$$

From [2, page 809] we have that

$$|x + y|^p + |x - y|^p \le 2^p (|x|^p + |y|^p)$$

for $1 < p < \infty$ with $x, y \in E^u$. This inequality, however, is inadequate for our purposes and we require a refined version, i.e. the Clarkson's inequalities for $1 < p < 2$. To this end we follow the approach of Ramaswamy [20].

Theorem 3.2 *(Clarkson's inequality). Let E be a Dedekind complete Riesz space with weak order unit, say e which we take as the multiplicative unit in the f-algebra E^u. For $x, y \in E^u$ and $1 \le p \le 2$ we have*

$$|x+y|^p + |x-y|^p \le 2(|x|^p + |y|^p). \tag{3.1}$$

Proof. For $p = 1$ the result follows from the triangle inequality while for $p = 2$ the result follows from $|f|^2 = f^2$, so we now consider only $1 < p < 2$. Taking $g(X) = X^{2/p}$ and $\mathbb{F}(X,Y) = (\frac{1}{2}(X+Y), \frac{1}{2}(X+Y))$ in Jensen's inequality of [10] on the Riesz space $F := E^u \times E^u$ with componentwise ordering, we have that $\mathbb{F}(g(|a|^p, |b|^p)) \ge g(\mathbb{F}(|a|^p, |b|^p))$ so $|a|^2+|b|^2 \ge 2^{(p-2)/p}(|a|^p+|b|^p)^{2/p}$ and $2^{(2-p)/2}(|a|^2+|b|^2)^{p/2} \ge |a|^p+|b|^p$. Setting $a = x+y$ and $b = x-y$ we have

$$|x+y|^p + |x-y|^p \le 2^{(2-p)/2}((x+y)^2 + (x-y)^2)^{p/2} = 2(x^2+y^2)^{p/2}. \tag{3.2}$$

We now apply the ∞ case of Hölder's inequality of [10] on the space F with conditional expectation $\mathbb{F}$ as above to get

$$\mathbb{F}((|x|^p, |y|^p)(|x|^{2-p}, |y|^{2-p})) \le \mathbb{F}((|x|^p, |y|^p))(|x|^{2-p} \vee |y|^{2-p}, |x|^{2-p} \vee |y|^{2-p}). \tag{3.3}$$

Here $|x|^{2-p} \vee |y|^{2-p} = (|x|^p \vee |y|^p)^{(2-p)/p} \le (|x|^p + |y|^p)^{(2-p)/p}$ by the commutation of multiplication and band projections in the f-algebra E^u. Hence from (3.3) we get

$$\frac{1}{2}(x^2+y^2) \le \frac{1}{2}(|x|^p+|y|^p)(|x|^p+|y|^p)^{(2-p)/p} = \frac{1}{2}(|x|^p+|y|^p)^{2/p},$$

which when combined with (3.2) gives (3.1). □

The strong law of large numbers for $p > 2$ will make use of Riesz space versions of Burkholder's inequality, [1, Theorem 16], which we give here for completeness.

Theorem 3.3 *(Burkholder's inequality). For $1 < p < \infty$, there are constants $c_p, C_p > 0$ such that*

$$C_p T|X_n|^p \le T|S_n^{\frac{1}{2}}|^p \le c_p T|X_n|^p,$$

for each $(X_n, T_n)_{n\in\mathbb{N}}$ a martingale in $L^p(T)$ compatible with T, i.e. $TT_n = T = T_n T$, for all $\in \mathbb{N}$. Here $S_n := \sum_{i=1}^{n}(X_i - X_{i-1})^2$ and $X_0 := 0$.

4. Hájek-Rényi-Chow maximal inequality

We now recall some well known results regarding band projections on a Dedekind complete Riesz space, E, with a weak order unit, say e. If $g \in E_+$ we denote the band projection onto the band generated by g by P_g. In this setting every band is a principal band and if B is a band in E with band projection Q onto E then a generator of the band is Qe. Moreover for $f \in E_+$ we have $P_g f = \sup_{n\in\mathbb{N}}(f \wedge (ng))$, see [24, Theorem 11.5]. Further if (f_n) is a sequence in E_+ then

$$\bigvee_{n=1}^{\infty} P_{f_n} = P_{\bigvee_{n=1}^{\infty} f_n} \tag{4.1}$$

since

$$P_{\bigvee_{n=1}^{\infty} f_n} e = \bigvee_{m=1}^{\infty} \left(e \wedge m \left(\bigvee_{n=1}^{\infty} f_n \right) \right) = \bigvee_{m,n=1}^{\infty} (e \wedge m f_n) = \bigvee_{n=1}^{\infty} P_{f_n} e.$$

We note however that for the case of infima only the following inequality can be assured

$$\bigwedge_{n=1}^{\infty} P_{f_n} \geq P_{\bigwedge_{n=1}^{\infty} f_n}. \tag{4.2}$$

For reference we note that if (g_n) is a sequence in E then $0 \leq P_{\bigwedge_{n=1}^{\infty} g_n^-} g_m^+ \leq P_{g_m^-} g_m^+ = 0$ giving

$$(I - P_{\bigwedge_{n=1}^{\infty} g_n^-}) \bigvee_{m=1}^{\infty} g_m^+ = \bigvee_{m=1}^{\infty} (I - P_{\bigwedge_{n=1}^{\infty} g_n^-}) g_m^+ = \bigvee_{m=1}^{\infty} g_m^+.$$

Hence

$$P_{\bigvee_{n=1}^{\infty} g_n^+} \leq I - P_{\bigwedge_{n=1}^{\infty} g_n^-}. \tag{4.3}$$

Using telescoping series we generalize [6, Lemma (6.1.1)] to vector lattices.

Lemma 4.1. *Let E be a Dedekind complete Riesz space weak order unit e. Let $(X_i) \subset E$ be a sequence in E and $g \in E$. Let $P_i := P_{(g-X_i)^+}, i \in \mathbb{N}$, be the band projection onto the band generated by $(g - X_i)^+$, then*

$$(I - Q_n) g \leq X_1 + \sum_{i=1}^{n-1} [Q_i (X_{i+1} - X_i)] - Q_n X_n, \tag{4.4}$$

where $Q_n := \prod_{j=1}^{n} P_j = P_{\left(g - \bigvee_{j=1}^{n} X_j\right)^+}, n \in \mathbb{N}$.

Proof. Let $Q_0 := I$. From the definition of P_i we have $P_j(g - X_j) = (g - X_j)^+$ and thus $(I - P_j)(g - X_j) = -(g - X_j)^- \leq 0$. However $Q_{j-1} - Q_j = Q_{j-1}(I - P_j)$, so applying Q_{j-1} to both sides of $(I - P_j)(g - X_j) \leq 0$, gives $(Q_{j-1} - Q_j)(g - X_j) \leq 0$. Hence $(Q_{j-1} - Q_j) g \leq (Q_{j-1} - Q_j) X_j$, which when summed over $j = 1, \ldots, n$ gives (4.4). □

If (f_i, T_i) is a submartingale in the Riesz space E then so is (f_i^+, T_i). To see this we observe that as T_j a positive operator and $f_j^+ \geq f_j$ so $T_i f_j^+ \geq T_i f_j \geq f_i$ and as $f_j^+ \geq 0$ so $T_i f_j^+ \geq 0$, for $i \leq j$. Hence $T_i f_j^+ \geq 0 \vee f_i = f_i^+$ for $i \leq j$.

Theorem 4.2 *(Hájek-Rényi-Chow maximal inequality). Let $(Y_i, T_i)_{i \in \mathbb{N}}$ be a submartingale in $L^1(T)$. For $(a_i)_{i \in \mathbb{N}}$ a non-decreasing sequence of positive real numbers and $g \in R(T_1)^+$ we have*

$$T_1 (I - U_n) g \leq \frac{Y_1^+}{a_1} + \sum_{i=1}^{n-1} T_1 \left[\frac{Y_{i+1}^+ - Y_i^+}{a_{i+1}} \right] \tag{4.5}$$

where $U_n := \prod_{i=1}^{n} P_{\left(g - \frac{Y_i}{a_i}\right)^+} = P_{\left(g - \bigvee_{i=1}^{n} \frac{Y_i}{a_i}\right)^+}$.

Proof. Let $Q = P_g$ be the band projection onto the band generated by g. Now as $g \in R(T_1)^+$ it follows that Q and T_1 commute, see [13, Theorem 3.2]. As (Y_i^+, T_i) is a submartingale, for $i \le j$, $T_i Y_j^+ \ge Y_i^+ = T_i Y_i^+$, hence

$$T_i(Y_{j+1}^+ - Y_j^+) \ge 0, \tag{4.6}$$

and thus

$$(I-Q)T_1(I-U_n)g = T_1(I-U_n)(I-Q)g = 0 \le (I-Q)\left(\frac{Y_1^+}{a_1} + \sum_{i=1}^{n-1} T_1\left[\frac{Y_{i+1}^+ - Y_i^+}{a_{i+1}}\right]\right). \tag{4.7}$$

Letting $X_i = Y_i^+/a_i, i \in \mathbb{N}$, in Lemma 4.1 we have, for $n \in \mathbb{N}$,

$$(I-Q_n)g \le \frac{Y_1^+}{a_1} + \sum_{i=1}^{n-1} Q_i\left(\frac{Y_{i+1}^+}{a_{i+1}} - \frac{Y_i^+}{a_i}\right) - Q_n\frac{Y_n^+}{a_n} \le \frac{Y_1^+}{a_1} + \sum_{i=1}^{n-1} Q_i\left(\frac{Y_{i+1}^+ - Y_i^+}{a_{i+1}}\right) \tag{4.8}$$

where $Q_i = P_{\left(g - \bigvee_{j=1}^{i} X_j\right)^+}$. Here $0 \le Q_i \le I$ and $T_i(Y_{i+1}^+ - Y_i^+) \ge 0$ so $Q_i T_i(Y_{i+1}^+ - Y_i^+) \le T_i(Y_{i+1}^+ - Y_i^+)$. Hence, as $T_1 = T_1 T_i$, from (4.6) and (4.8),

$$T_1(I-Q_n)g \le \frac{Y_1^+}{a_1} + \sum_{i=1}^{n-1} T_1\left(\frac{Y_{i+1}^+ - Y_i^+}{a_{i+1}}\right). \tag{4.9}$$

Since $g \ge 0$, we have

$$g \wedge \left(\left(g - \frac{Y_i}{a_i}\right) \vee 0\right) = \left(g \wedge \left(g - \frac{Y_i}{a_i}\right)\right) \vee (g \wedge 0) = \left(g - \left(0 \vee \frac{Y_i}{a_i}\right)\right) \vee 0,$$

giving $g \wedge \left(g - \frac{Y_i}{a_i}\right)^+ = \left(g - \frac{Y_i^+}{a_i}\right)^+$, thus $QU_n = QQ_n$. Now, applying Q to (4.9) and noting that $T_1 Q = QT_1$ we have

$$QT_1(I-U_n)g \le Q\left(\frac{Y_1^+}{a_1} + \sum_{i=1}^{n-1} T_1\left(\frac{Y_{i+1}^+ - Y_i^+}{a_{i+1}}\right)\right). \tag{4.10}$$

Combining (4.7) and (4.10) gives (4.5). □

5. Submartingale convergence

As an application of the Hájek-Rényi-Chow Maximal Inequality we give a weighted convergence theorem for submartingales, with a proof that is independent of upcrossing.

Theorem 5.1 *(Submartingale convergence). Let $p \ge 1$ and $(X_i, T_i)_{i \in \mathbb{N}}$ be a non-negative submartingale in $L^p(T)$. Let $(a_i)_{i \in \mathbb{N}}$ be a positive, non-decreasing, sequence of real numbers diverging to ∞. If*

$$\sum_{i=1}^{\infty} T_1\left(\frac{X_{i+1}^p - X_i^p}{a_{i+1}^p}\right) \tag{5.1}$$

converges in order, then $\frac{X_n}{a_n}$ tends to zero in order as n tends to ∞.

Proof. By [10, Corollary 4.5], (X_n^p, T_n) is a non-negative submartingale so $Z_i := (X_{i+1}^p - X_i^p)/a_{i+1}^p$ has $T_1 Z_i \geq 0$ and by assumption $\sum_{i=1}^\infty T_1 Z_i$ is order convergent, so by Lemma 2.2

$$\frac{T_1 X_{m+1}^p}{a_{m+1}^p} = \frac{1}{a_{m+1}^p} T_1 \left(X_1^p + \sum_{i=1}^m (X_{i+1}^p - X_i^p) \right) = \frac{X_1^p}{a_{m+1}^p} + \frac{1}{a_{m+1}^p} \sum_{i=1}^m a_{i+1}^p T_1 Z_i \to 0 \tag{5.2}$$

in order as $m \to \infty$.

By (4.3) and Theorem 4.2 applied to $(X_i^p)_{i=m}^n$, with $g = te$, $t \in \mathbb{R}$ where $t > 0$, for $n > m$, we have

$$T_1 P_{\left(\bigvee_{i=m}^n \left(\frac{X_i^p}{a_i^p} - te\right)^+\right)} te \leq T_1 \left(I - P_{\bigwedge_{i=m}^n \left(\frac{X_i^p}{a_i^p} - te\right)^-} \right) te \leq \frac{X_m^p}{a_m^p} + \sum_{i=m}^{n-1} T_1 Z_i. \tag{5.3}$$

Applying T_1 to (5.3) and taking the order limit as $n \to \infty$, by (4.1) we have

$$0 \leq t T_1 \left(\bigvee_{i=m}^\infty P_{\left(\frac{X_i^p}{a_i^p} - te\right)^+} \right) e \leq T_1 \left[\frac{X_m^p}{a_m^p} \right] + \sum_{i=m}^\infty T_1 Z_i. \tag{5.4}$$

Taking the order limit as $m \to \infty$ of (5.4), by (5.2), we have

$$0 \leq t T_1 \lim_{m\to\infty} \left(\bigvee_{i=m}^\infty P_{\left(\frac{X_i^p}{a_i^p} - te\right)^+} \right) e \leq \lim_{m\to\infty} T_1 \left[\frac{X_m^p}{a_m^p} \right] + \lim_{m\to\infty} \sum_{i=m}^\infty T_1 Z_i = 0. \tag{5.5}$$

Hence $T_1 \bigwedge_{m\in\mathbb{N}} \bigvee_{i=m}^\infty P_{\left(\frac{X_i^p}{a_i^p} - te\right)^+} e = 0$ and by the strict positivity of T_1, $\bigwedge_{m\in\mathbb{N}} \bigvee_{i=m}^\infty P_{\left(\frac{X_i^p}{a_i^p} - te\right)^+} e = 0$. Now, by (4.1) and (4.2),

$$0 \leq P_{\limsup_{i\to\infty}\left(\frac{X_i^p}{a_i^p} - te\right)^+} e \leq \bigwedge_{m\in\mathbb{N}} P_{\bigvee_{i=m}^\infty \left(\frac{X_i^p}{a_i^p} - te\right)^+} e = \bigwedge_{m\in\mathbb{N}} \bigvee_{i=m}^\infty P_{\left(\frac{X_i^p}{a_i^p} - te\right)^+} e = 0$$

and so $0 \leq \liminf_{i\to\infty} \frac{X_i^p}{a_i^p} \leq \limsup_{i\to\infty} \frac{X_i^p}{a_i^p} \leq te$ for all $t > 0$. Thus $\frac{X_i^p}{a_i^p} \to 0$ in order as $i \to \infty$. $\square$

Remark. Since (X_n^p, T_n) is a non-negative submartingale, by [10, Corollary 4.5], taking $Y_{j+1} = \sum_{i=1}^j Z_i$ in the above theorem, we have that (Y_j, T_j) is a T_1-bounded submartingale and Theorem 5.1 follows directly from [14, Theorem 3.5].

6. Chow's strong laws of large numbers

We recall that (Y_i, T_i) is a martingale difference sequence if (T_i) is a filtration, (Y_i) is adapted to (T_i) and $T_{i-1} Y_i = 0$ for $i \geq 2$. In Theorem 6.1, for $1 \leq p \leq 2$, and Corollary 6.2 and Theorem 6.3, for $p > 2$, Chow's strong law of large numbers is extended to martingale difference sequences in Riesz spaces.

Theorem 6.1. *Let $1 \leq p \leq 2$, and $(Y_n, T_n)_{n\in\mathbb{N}}$ be a martingale difference sequence in $L^p(T)$. Let $(a_i)_{i\in\mathbb{N}}$ be a positive, non-decreasing sequence of real numbers divergent to infinity with*

$$\sum_{i=1}^\infty T_1 \left(\frac{|Y_i|^p}{a_i^p} \right) \tag{6.1}$$

order convergent, then $\frac{1}{a_n}\sum_{i=1}^{n} Y_i \to 0$*, in order, as* n *tends to infinity.*

Proof. Let $X_n = \sum_{i=1}^{n} Y_i$ then $X_i + Y_{i+1} = X_{i+1}$ and $X_i - Y_{i+1} = 2X_i - X_{i+1}$ so Theorem 3.2 can be applied to give

$$|X_{i+1}|^p + |2X_i - X_{i+1}|^p \le 2(|X_i|^p + |Y_{i+1}|^p). \tag{6.2}$$

Now as (X_n, T_n) is a martingale, so $T_i(2X_i - X_{i+1}) = X_i$ and by functional calculus, see [10], $|X_i|^p \in R(T_i)$ giving $T_i|X_i|^p = |X_i|^p$, hence

$$T_i|X_i|^p = |T_i(2X_i - X_{i+1})|^p \le T_i|2X_i - X_{i+1}|^p \tag{6.3}$$

where the final inequality follows from Jensen's inequality, [10, Theorem 4.4]. Combining (6.2) and (6.3) we have

$$T_i|X_{i+1}|^p - T_i|X_i|^p \le 2T_i|Y_{i+1}|^p. \tag{6.4}$$

By [10, Corollary 4.5], $(|X_i|, T_i)$ and $(|X_i|^p, T_i)$ are submartingales so

$$0 \le \frac{T_i|X_{i+1}|^p - T_i|X_i|^p}{a_{i+1}^p} \le 2\frac{T_i|Y_{i+1}|^p}{a_{i+1}^p} \tag{6.5}$$

which, with (6.1), yields that $\sum_{i=1}^{\infty} \frac{T_i|X_{i+1}|^p - T_i|X_i|^p}{a_{i+1}^p}$ is order convergent. The theorem now follows from Theorem 5.1. □

We can now bootstrap on Theorem 6.1 to obtain a strong law for $p > 2$.

Corollary 6.2. *Let* $p > 2$ *and* $(Y_n, T_n)_{n\in\mathbb{N}}$ *be a martingale difference sequence in* $L^p(T_1)$*. Let* $(a_i)_{i\in\mathbb{N}}$ *be a positive, non-decreasing sequence of real numbers with* $\sum_{i=1}^{\infty} \frac{1}{a_i^k}$ *convergent in* $\mathbb{R}$*, and* $\sum_{i=1}^{\infty} T_1\left(\frac{|Y_i|^p}{a_i^\gamma}\right)$ *order convergent, where* $p \ge \gamma + (\frac{p}{2} - 1)k$*, then* $\frac{1}{a_n}\sum_{i=1}^{n} Y_i \to 0$*, in order, as* n *tends to infinity.*

Proof. From Hölder's inequality, Theorem 3.1, for $n > m$ we have

$$\sum_{i=m}^{n} T_1\frac{|Y_i|^2}{a_i^2} \le \left(\sum_{i=m}^{n} T_1\frac{|Y_i|^p}{a_i^\gamma}\right)^{\frac{2}{p}} \left(\sum_{i=m}^{n} \frac{e}{a_i^\delta}\right)^{1-\frac{2}{p}}$$

where $\delta = \frac{p-\gamma}{\frac{p}{2}-1} \ge k$ ensuring that $\sum_{i=1}^{\infty} \frac{1}{a_i^\delta}$ converges. Hence from Theorem 6.1 with $p = 2$, $\frac{1}{a_n}\sum_{i=1}^{n} Y_i \to 0$, in order, as n tends to infinity. □

From Corollary 6.2, if $p > 2$ and $(Y_n, T_n)_{n\in\mathbb{N}}$ is a martingale difference sequence in $L^p(T_1)$ with $\sum_{i=1}^{\infty} T_1\left(\frac{|Y_i|^p}{i^{1+\frac{p}{2}-\delta}}\right)$ order convergent for some $\delta > 0$ then $\frac{1}{n}\sum_{i=1}^{n} Y_i \to 0$, in order, as n tends to infinity. For this special case, of $a_i = i$, a more precise result can be given, as per [5,4].

Theorem 6.3. *Let $p > 2$ be a fixed number and let $(Y_n, T_n)_{n\in\mathbb{N}}$ be a martingale difference sequence in $L^p(T_1)$. If $\sum_{i=1}^{\infty} T_1\left(\frac{|Y_i|^p}{i^{1+\frac{p}{2}}}\right)$ converges in order then $\frac{1}{n}\sum_{i=1}^{n} Y_i \to 0$ in order as $n \to \infty$.*

Proof. Let $X_n =: \sum_{i=1}^{n} Y_i$ for $n \in \mathbb{N}$, then, from Theorem 5.1, it suffices to prove the convergence as $n \to \infty$ of

$$Z_n = \sum_{i=2}^{n} T_1\left(\frac{|X_i|^p - |X_{i-1}|^p}{i^p}\right) = \sum_{i=2}^{n-1}\left(\frac{1}{i^p} - \frac{1}{(i+1)^p}\right) T_1(|X_i|^p) + \frac{T_1(|X_n|^p)}{n^p} - \frac{T_1(|X_1|^p)}{2^p}.$$

Since each term in the above summations is non-negative we need only show the boundedness of $Z_n, n \in \mathbb{N}$. From Burkholder inequality, Theorem 3.3, there is $C_p > 0$ so that

$$C_p T_1|X_n|^p \le T_1\left(\sum_{i=1}^{n}|Y_i|^2\right)^{p/2}, \tag{6.6}$$

for all $n \in \mathbb{N}$. Applying Jensen's inequality of [10] we have

$$T_1\left(\sum_{i=1}^{n}|Y_i|^2\right)^{p/2} \le \left(\sum_{i=1}^{n} T_1|Y_i|^2\right)^{p/2}. \tag{6.7}$$

Now Hölder inequality, Theorem 3.1, gives

$$\left(\sum_{i=1}^{n} T_1|Y_i|^2\right)^{p/2} \le n^{\frac{p}{2}-1}\sum_{i=1}^{n} T_1|Y_i|^p. \tag{6.8}$$

Combining (6.6), (6.7) and (6.8) gives

$$\frac{T_1|X_n|^p}{n^p} \le \frac{1}{C_p n^{\frac{p}{2}+1}}\sum_{i=1}^{n} T_1|Y_i|^p. \tag{6.9}$$

From Kronecker's Lemma, Theorem 2.2, we have that $\frac{1}{n^{\frac{p}{2}+1}}\sum_{i=1}^{n} T_1|Y_i|^p \to 0$ in order as $n \to \infty$. Thus the left hand side of (6.9) is order bounded by say $h \in L^1(T_1)$ and

$$\sum_{i=2}^{n-1}\left(\frac{1}{i^p} - \frac{1}{(i+1)^p}\right) T_1(|X_i|^p) \le p\sum_{i=1}^{\infty}\frac{1}{i^{p/2}}h,$$

giving

$$Z_n \le p\sum_{i=1}^{\infty}\frac{1}{i^{p/2}}h + h - \frac{T_1(|X_1|^p)}{2^p}.$$

Here we have used that $n^{-p} - (n+1)^{-p} \le pn^{-p-1}, n \in \mathbb{N}$. $\square$

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