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Inverse problem for dynamical system associated with Jacobi matrices and classical moment problems ☆

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A B S T R A C T

We consider Hamburger, Stieltjes and Hausdorff moment problems, that are problems of the determination of a Borel measure supported on the real axis, on the semi-axis or on the interval $(0, 1)$, from a prescribed set of moments. We propose a unified approach to these three problems based on the use of the auxiliary dynamical system with the discrete time associated with a semi-infinite Jacobi matrix. It is shown that the set of moments determines the inverse dynamic data for such a system. Using the ideas of the Boundary Control method, for every $N \in \mathbb{N}$ we can recover the spectral measure of $N \times N$ block of Jacobi matrix, which is a solution to a truncated moment problem. This problem is reduced to the finite-dimensional generalized spectral problem, whose matrices are constructed from moments and are connected with the well-known Hankel matrices by simple formulas. Thus the results on existence of solutions to Hamburger, Stieltjes and Hausdorff moment problems are naturally provided in terms of these matrices. We also obtain results on uniqueness of the solution of the moment problems, where as a main tool we use Krein-type equations of inverse problem.

1. Introduction

The classical moment problem consists in the following: given the (real) numbers $s_0, s_1, s_2, \ldots$ which are called moments, find a Borel measure $d\rho$ such that

$$s_k = \int_{-\infty}^{\infty} \lambda^k \, d\rho(\lambda), \quad k = 0, 1, 2, \ldots. \tag{1.1}$$

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When $\operatorname{supp}\rho \subset \mathbb{R}$ the problem is called Hamburger moment problem, when $\operatorname{supp}\rho \subset [0, +\infty)$ the problem is called Stieltjes moment problem, and when $\operatorname{supp}\rho \subset [0, 1]$ the problem is called Hausforff moment problem. These problems have received a lot of attention in the last century, to mention [4,22,11,12,23,21] and references therein.

In the present paper we propose a unified approach to these classical moment problems based on considering an auxiliary dynamical system with discrete time governed by Jacobi matrix [14,17,19] and ideas of the Boundary Control (BC) method [5,7] of solving the inverse dynamic problems for hyperbolic dynamical systems. We restrict ourselves to the questions of existence and uniqueness of a solution. We also propose a procedure of recovering a measure which is a special solution to a truncated moment problem, i.e. when (1.1) holds for $k = 0, 1, \dots M$ for some fixed $M \in \mathbb{N}$. In the last case the special solution is given by a spectral measure of a finite Jacobi operator with Dirichlet boundary conditions, and thus has a form of finite sum of Dirac delta functions with some coefficients.

In the second section we consider initial-boundary value problems for dynamical systems with discrete time associated with semi-infinite and finite Jacobi matrices. Following [14,16,17] we derive a dynamic and spectral representations of their solutions, introduce the operators of the BC method and show that *the response operator*, i.e. the discrete analog of a dynamic Dirichlet-to-Neumann map for these systems (operators of this type are used as inverse data in dynamic inverse problems [5,7,15]) has a form of convolution. The kernel of the response operator, which is called *response vector*, admits a spectral representation in terms of a spectral measure of corresponding Jacobi matrix. This fact establishes the relationship between spectral (measure) and dynamic (response vector) data and gives a possibility to apply some ideas of the BC method [6,3] to solving the truncated moment problem.

In the third section we solve the truncated moment problem by extracting spectral data (i.e. the spectral measure of $N \times N$ block of Jacobi matrix) from the response vector. The main results are given in Theorems 3 and 4, which say that the solution to a truncated moment problem can be constructed by solving a finite dimensional generalized spectral problem, in which the matrices are connected with classical Hankel matrices (see [1,23]) constructed from moments by simple transformation. Then the results on the existence of solution to all three moment problems are given in terms of inequalities for these matrices, see also [13,10].

In the last section we obtain results on uniqueness of the solution to Hamburger, Stieltjes and Hausdorff moment problems, classical methods for these problems are described in [1,13,23]. The main tools in our considerations are classical Weyl-type results on the deficiency indices of Jacobi matrix [1,23] and Krein equations of inverse problem in dynamic form. For continuous systems such equations were derived firstly in [9] and in the framework of the BC method in [2,8]; for the discrete systems they were derived in [14,16,17]. We also compare the results on existence for Hausdorff moment problem obtained in the paper with classical results of Hausdorff [11,12,22].

2. Dynamical systems with discrete time associated with Jacobi matrix. Operators of the BC method

In this section we outline some results obtained in [14,16,17] on forward problems for dynamical systems with discrete time associated with finite and semi-infinite Jacobi matrices.

2.1. Finite Jacobi matrices

For a given sequence of positive numbers $\{a_0, a_1, \dots\}$ (in what follows we assume $a_0 = 1$) and real numbers $\{b_1, b_2, \dots\}$, we denote by A a semi-infinite Jacobi matrix

$$A = \begin{pmatrix} b_1 & a_1 & 0 & 0 & 0 & \dots \\ a_1 & b_2 & a_2 & 0 & 0 & \dots \\ 0 & a_2 & b_3 & a_3 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}. \tag{2.1}$$

For $N\in\mathbb{N}$, by A_N we denote the $N\times N$ Jacobi matrix which is a block of (2.1) consisting of the intersection of first N columns with first N rows of A.

Introduce the notation $\mathbb{N}_0=\mathbb{N}\cup\{0\}$, and consider a dynamical system with discrete time associated with a finite Jacobi matrix A_N:

$$\begin{cases} v_{n,t+1}+v_{n,t-1}-a_nv_{n+1,t}-a_{n-1}v_{n-1,t}-b_nv_{n,t}=0,\ t\in\mathbb{N}_0,\ n\in 1,\dots,N,\\ v_{n,\,-1}=v_{n,\,0}=0, \quad n=1,2,\dots,N+1,\\ v_{0,\,t}=f_t, \quad v_{N+1,\,t}=0, \quad t\in\mathbb{N}_0, \end{cases} \tag{2.2}$$

by an analogy with continuous problems [5,2,8], we treat the real sequence $f=(f_0,f_1,\dots)$ as a *boundary control*. Fixing a positive integer T we denote by $\mathcal{F}^T$ the *outer space* of the system (2.2): $\mathcal{F}^T:=\mathbb{R}^T$, $f\in\mathcal{F}^T$, $f=(f_0,\dots,f_{T-1})$ with the standard inner product $(f,g)_{\mathcal{F}^T}=(f,g)_{\mathbb{R}^T}$. The solution to (2.2) is denoted by v^f. Note that (2.2) is a discrete analog of an initial boundary value problem for a wave equation with a potential on an interval with the Dirichlet control at the left end and the Dirichlet condition at the right end. This observation makes it reasonable to refer to the solution v^f as to a wave. Since both variables in (2.2) are discrete, we have that the wave initiated at $t=0$ reaches the point $n=N$ at time $t=N$, which can be interpreted as the finiteness of the speed of a wave propagation. Note that in the similar model but with continuous time, so-called Krein-Stieltjes string [18], the speed of wave propagation is infinite.

Introduce the operator $A_{N,h}:\mathbb{R}^N\mapsto\mathbb{R}^N$, $h\in\mathbb{R}$ by the rule:

$$(A_{N,h}\psi)_n=\begin{cases} b_1\psi_1+a_1\psi_2, \quad n=1,\\ a_n\psi_{n+1}+a_{n-1}\psi_{n-1}+b_n\psi_n, \quad 2\leqslant n\leqslant N-1,\\ a_{N-1}\psi_{N-1}+(b_N-ha_N)\,\psi_N, \quad n=N. \end{cases}$$

Note that this operator corresponds to general boundary condition at the right end (see [4, Chapter 4]):

$$\psi_{N+1}+h\psi_N=0, \quad h\in\mathbb{R}, \tag{2.3}$$

the case $h=0$, which we deal with, is called Dirichlet, we set $A_N=A_{N,0}$:

$$(A_N\psi)_n=\begin{cases} b_1\psi_1+a_1\psi_2, \quad n=1,\\ a_n\psi_{n+1}+a_{n-1}\psi_{n-1}+b_n\psi_n, \quad 2\leqslant n\leqslant N-1,\\ a_{N-1}\psi_{N-1}+b_N\psi_N, \quad n=N. \end{cases}$$

Denote by $\phi=\{\phi_n\}$, $n=0,1,2,\dots$ the solution to the Cauchy problem for the following difference equation

$$\begin{cases} a_n\phi_{n+1}+a_{n-1}\phi_{n-1}+b_n\phi_n=\lambda\phi_n, \quad n\geqslant 1,\\ \phi_0=0,\ \phi_1=1, \end{cases} \tag{2.4}$$

where $\lambda\in\mathbb{C}$. Thus ϕ_n is a polynomial of degree $n-1$ in λ. Denote by $\{\lambda_k\}_{k=1}^N$ the roots of the equation $\phi_{N+1}(\lambda)=0$, it is known [1,23] that they are real and distinct. We introduce the vectors $\phi^k\in\mathbb{R}^N$ by the rule

$$\phi^k:=\begin{pmatrix}\phi_1(\lambda_k)\\ \phi_2(\lambda_k)\\ \cdot\\ \phi_N(\lambda_k)\end{pmatrix}, \quad k=1,\dots,N,$$

and define the numbers ω_k by

$$(\phi^k, \phi^l) = \delta_{kl}\omega_k, \quad k, l = 1, \dots, N, \tag{2.5}$$

where $(\cdot, \cdot)$ is a scalar product in $\mathbb{R}^N$.

Definition 1. The set of pairs

$$\{\lambda_k, \omega_k\}_{k=1}^N$$

is called Dirichlet spectral data of operator A_N.

Definition 2. For semi-infinite sequences $f = (f_0, f_1, \dots)$, $g = (g_0, g_1, \dots)$ we define the convolution $c = f * g = (c_0, c_1, \dots)$ by the formula

$$c_t = \sum_{s=0}^{t} f_s g_{t-s}, \quad t \in \mathbb{N} \cup \{0\}.$$

Denote by $\mathcal{T}_k(2\lambda)$ the Chebyshev polynomials of the second kind: they are obtained as a solution to the following Cauchy problem:

$$\begin{cases} \mathcal{T}_{t+1} + \mathcal{T}_{t-1} - \lambda \mathcal{T}_t = 0, \\ \mathcal{T}_0 = 0, \ \mathcal{T}_1 = 1. \end{cases} \tag{2.6}$$

In [14,17] the following formula for the solution $v^f = \left\{ v^f_{n,t} \right\}$, $n = 0, 1, \dots$; $t = -1, 0, 1, \dots$ was proved:

Proposition 1. *The solution to (2.2) admits the representation*

$$v^f_{n,t} = \begin{cases} \sum_{k=1}^N c^k_t \phi^k_n, & n = 1, \dots, N, \\ f_t, & n = 0, \end{cases} \qquad c^k = \frac{1}{\omega_k} \mathcal{T}(\lambda_k) * f. \tag{2.7}$$

The *inner space* of dynamical system (2.2) is denoted by $\mathcal{H}^N := \mathbb{R}^N$, $h \in \mathcal{H}^N$, $h = (h_1, \dots, h_N)^\tau$. By (2.7) we have that $v^f_{\cdot, T} \in \mathcal{H}^N$. For the system (2.2) the *control operator* $W_N^T : \mathcal{F}^T \mapsto \mathcal{H}^N$ is defined by the rule

$$W_N^T f := v^f_{n,\,T}, \quad n = 1, \dots, N.$$

The input $\longmapsto$ output correspondence in the system (2.2) is realized by a *response operator*: $R_N^T : \mathcal{F}^T \mapsto \mathbb{R}^T$, defined by the formula

$$\left(R_N^T f \right)_t = v^f_{1,\,t}, \quad t = 1, \dots, T. \tag{2.8}$$

This operator has a form of a convolution:

$$\left(R_N^T f \right)_t = \sum_{s=0}^{t} r_s f_{t-s-1} \quad \text{or} \quad R_N^T f = r^N * f_{\cdot -1},$$

where the convolution kernel is called a *response vector*: $r^N = (r_0^N, r_1^N, \dots, r_{T-1}^N)$. The response operator plays the role of dynamic inverse data [5,7], the corresponding inverse problems were studied in [14,17].

The *connecting operator* $C_N^T : \mathcal{F}^T \mapsto \mathcal{F}^T$ for the system (2.2) is defined via the quadratic form: for arbitrary $f, g \in \mathcal{F}^T$ one has that

$$\left(C_N^T f, g\right)_{\mathcal{F}^T} = \left(v^f_{\cdot,T}, v^g_{\cdot,T}\right)_{\mathcal{H}^N} = \left(W_N^T f, W_N^T g\right)_{\mathcal{H}^N}, \quad C_N^T = \left(W_N^T\right)^* W_N^T.$$

The speed of a wave propagation in the system (2.2) is finite, which implies the following dependence of inverse data on coefficients $\{a_n, b_n\}$: for $M \in \mathbb{N}$, $M \leqslant N$, the element $v^f_{1,2M-1}$ depends on $\{a_1, \dots, a_{M-1}\}$, $\{b_1, \dots, b_M\}$, on observing this we can formulate the following

Remark 1. The entries of the response vector $(r_0^N, r_1^N, \dots, r_{2N-2}^N)$ depend on $\{a_0, \dots, a_{N-1}\}$, $\{b_1, \dots, b_N\}$, and does not depend on the boundary condition at $n = N+1$. The entries beginning from r_{2N-1}^N do "feel" the boundary condition at $n = N+1$.

On introducing the special control $\delta = (1, 0, 0, \dots)$, one can see that the kernel of the response operator (2.8) is given by

$$r_{t-1}^N = \left(R_N^T \delta\right)_t = v^\delta_{1,\,t}, \quad t = 1, \dots. \tag{2.9}$$

The spectral function of operator A_N is introduced by the rule

$$\rho^N(\lambda) = \sum_{\{k \,|\, \lambda_k < \lambda\}} \frac{1}{\omega_k}, \tag{2.10}$$

then from (2.7), (2.9) we immediately deduce

Proposition 2. *The solution to (2.2), the response vector of (2.2) and entries of the matrix of the connecting operator C_N^T admit the following spectral representations:*

$$v^f_{n,t} = \int_{-\infty}^{\infty} \sum_{k=1}^{t} \mathcal{T}_k(\lambda) f_{t-k} \phi_n(\lambda)\, d\rho^N(\lambda), \quad n, t \in \mathbb{N}, \tag{2.11}$$

$$r_{t-1}^N = \int_{-\infty}^{\infty} \mathcal{T}_t(\lambda)\, d\rho^N(\lambda), \quad t \in \mathbb{N}, \tag{2.12}$$

$$\{C_N^T\}_{l+1,\,m+1} = \int_{-\infty}^{\infty} \mathcal{T}_{T-l}(\lambda) \mathcal{T}_{T-m}(\lambda)\, d\rho^N(\lambda), \; l, m = 0, \dots, T-1. \tag{2.13}$$

Details of the proof the reader can find in [16,17].

2.2. Semi-infinite Jacobi matrix

We consider an initial boundary value problem for a dynamical system with discrete time associated with a semi-infinite Jacobi matrix A:

$$\begin{cases} u_{n,\,t+1} + u_{n,\,t-1} - a_n u_{n+1,\,t} - a_{n-1} u_{n-1,\,t} - b_n u_{n,\,t} = 0, & n \in \mathbb{N},\, t \in \mathbb{N}_0, \\ u_{n,\,-1} = u_{n,\,0} = 0, & n \in \mathbb{N}, \\ u_{0,\,t} = f_t, & t \in \mathbb{N}_0, \end{cases} \tag{2.14}$$

which is a discrete analog of an initial boundary value problem for a wave equation with a potential on a half-line with the Dirichlet control at $n = 0$. The solution to (2.14) is denoted by $u^f_{n,\,t}$. We fix some positive integer T and denote by $\mathcal{F}^T$ the *outer space* of the system (2.14), the space of controls (inputs): $\mathcal{F}^T := \mathbb{R}^T$, $f \in \mathcal{F}^T$, $f = (f_0, \dots, f_{T-1})$. In [17] the following statement is proved.

Lemma 1. *A solution to (2.14) admits the representation*

$$u^f_{n,\,t} = \prod_{k=0}^{n-1} a_k f_{t-n} + \sum_{s=n}^{t-1} w_{n,\,s} f_{t-s-1}, \quad n, t \in \mathbb{N}, \tag{2.15}$$

where $w_{n,s}$ satisfies the Goursat problem

$$\begin{cases} w_{n,\,s+1} + w_{n,\,s-1} - a_n w_{n+1,\,s} - a_{n-1} w_{n-1,\,s} - b_n w_{n,\,s} = \\ = -\delta_{s,n}(1-a_n^2)\prod_{k=0}^{n-1} a_k,\ n, s \in \mathbb{N},\ s > n, \\ w_{n,\,n} - b_n \prod_{k=0}^{n-1} a_k - a_{n-1} w_{n-1,\,n-1} = 0, \quad n \in \mathbb{N}, \\ w_{0,\,t} = 0, \quad t \in \mathbb{N}_0. \end{cases}$$

The input $\longmapsto$ output correspondence in the system (2.14) is realized by a *response operator*: $R^T : \mathcal{F}^T \mapsto \mathbb{R}^T$ defined by the rule

$$\left(R^T f\right)_t = u^f_{1,\,t}, \quad t = 1, \ldots, T.$$

This operator plays the role of inverse data, the corresponding inverse problem was considered in [14,17]. Introduce the shift operator in the space of infinite sequences $l^\infty_{\mathbb{Z}}$, $(\ldots, a_{-1}, a_0, a_1, \ldots) = a \in l^\infty_{\mathbb{Z}}$:

$$K_m : l^\infty_{\mathbb{Z}} \mapsto l^\infty_{\mathbb{Z}}, \quad (K_m a)_l = a_{l-m}. \tag{2.16}$$

The convolution kernel of R^T is called a *response vector*, in accordance with (2.15) one has that $r = (r_0, r_1, \ldots, r_{T-1}) = (1, w_{1,1}, w_{1,2}, \ldots w_{1,T-1})$:

$$\left(R^T f\right)_t = u^f_{1,\,t} = f_{t-1} + \sum_{s=1}^{t-1} w_{1,\,s} f_{t-1-s} \quad t = 1, \ldots, T. \tag{2.17}$$

$$\left(R^T f\right) = r * K_1 f.$$

By choosing a special control $f = \delta = (1, 0, 0, \ldots)$, the kernel of the response operator can be determined as

$$\left(R^T \delta\right)_t = u^\delta_{1,\,t} = r_{t-1}, \quad t = 1, 2, \ldots.$$

For a fixed $T \in \mathbb{N}$ we introduce the *inner space* of the dynamical system (2.14) $\mathcal{H}^T := \mathbb{R}^T$, $h \in \mathcal{H}^T$, $h = (h_1, \ldots, h_T)$, the space of states. The wave $u^f_{\cdot,\,T}$ is considered as a state of the system (2.14) at the moment $t = T$. By (2.15) we have that $u^f_{\cdot,\,T} \in \mathcal{H}^T$. The input $\longmapsto$ state correspondence of the system (2.14) is realized by a *control operator* $W^T : \mathcal{F}^T \mapsto \mathcal{H}^T$, defined by the rule

$$\left(W^T f\right)_n := u^f_{n,\,T}, \quad n = 1, \ldots, T.$$

From (2.15) we deduce the representation for W^T:

$$\left(W^T f\right)_n = u^f_{n,\,T} = \prod_{k=0}^{n-1} a_k f_{T-n} + \sum_{s=n}^{T-1} w_{n,\,s} f_{T-s-1}, \quad n = 1, \ldots, T.$$

Or in matrix form:

$$W^T f = \begin{pmatrix} u_{1,T} \\ u_{2,T} \\ \cdot \\ u_{k,T} \\ \cdot \\ u_{T,T} \end{pmatrix} = \begin{pmatrix} w_{1,T-1} & w_{1,T-2} & w_{1,T-3} & \dots & \dots & 1 \\ w_{2,T-1} & w_{2,T-1} & \dots & \dots & a_1 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ w_{k,T-1} & \dots & \prod_{j=0}^{k-1} a_j & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \prod_{k=0}^{T-1} a_k & 0 & 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ \cdot \\ f_{T-k-1} \\ \cdot \\ f_{T-1} \end{pmatrix}. \tag{2.18}$$

The following statement proved in [17] is interpreted as a boundary controllability of the dynamical system (2.14):

Lemma 2. *The operator W^T is an isomorphism between $\mathcal{F}^T$ and $\mathcal{H}^T$.*

We introduce the *connecting operator* $C^T : \mathcal{F}^T \mapsto \mathcal{F}^T$ for the system (2.14), by the quadratic form: for arbitrary $f, g \in \mathcal{F}^T$ we define

$$\left(C^T f, g\right)_{\mathcal{F}^T} = \left(u^f_{\cdot,T}, u^g_{\cdot,T}\right)_{\mathcal{H}^T} = \left(W^T f, W^T g\right)_{\mathcal{H}^T}. \tag{2.19}$$

That is $C^T = \left(W^T\right)^* W^T$. The fact that the connecting operator can be represented in terms of inverse data is crucial in the BC method, the proof of the following theorem one can find in [14,17].

Theorem 1. *The connecting operator C^T is an isomorphism in $\mathcal{F}^T$, it admits the representation in terms of dynamic inverse data:*

$$C^T = C^T_{ij}, \quad C^T_{ij} = \sum_{k=0}^{T-\max i,j} r_{|i-j|+2k}, \quad r_0 = a_0 = 1, \tag{2.20}$$

$$C^T = \begin{pmatrix} r_0 + r_2 + \ldots + r_{2T-2} & r_1 + \ldots + r_{2T-3} & \dots & r_T + r_{T-2} & r_{T-1} \\ r_1 + r_3 + \ldots + r_{2T-3} & r_0 + \ldots + r_{2T-4} & \dots & \dots & r_{T-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ r_{T-3} + r_{T-1} + r_{T+1} & \dots & r_0 + r_2 + r_4 & r_1 + r_3 & r_2 \\ r_T + r_{T-2} & \dots & r_1 + r_3 & r_0 + r_2 & r_1 \\ r_{T-1} & r_{T-2} & \dots & r_1 & r_0 \end{pmatrix}.$$

One can observe [14] that C^T_{ij} satisfies the difference boundary problem:

Corollary 1. *The kernel of C^T satisfies*

$$\begin{cases} C^T_{i,j+1} + C^T_{i,j-1} - C^T_{i+1,j} - C^T_{i-1,j} = 0, \\ C^T_{i,T} = r_{T-i}, \ C^T_{T,j} = r_{T-j}, \ r_0 = 1. \end{cases}$$

With the matrix A we associate the symmetric operator A in the space l_2 (we keep the same notation), defined on finite sequences:

$$D(A) = \{\varkappa = (\varkappa_0, \varkappa_1, \ldots) \,|\, \varkappa_n = 0, \text{ for } n \geqslant N_0 \in \mathbb{N}\},$$

and given by the rule

$$\begin{aligned} (A\theta)_n &= a_n\theta_{n+1} + a_{n-1}\theta_{n-1} + b_n\theta_n, \quad n \geqslant 2, \\ (A\theta)_1 &= b_1\theta_1 + a_1\theta_2, \quad n = 1. \end{aligned}$$

By $[\cdot,\cdot]$ we denote the scalar product in l_2. For a given sequence $\varkappa = (\varkappa_1, \varkappa_2, \ldots)$ we define a new sequence

$$(G\varkappa)_n = \begin{cases} b_1\theta_1 + a_1\theta_2, & n = 1, \\ a_n\theta_{n+1} + a_{n-1}\theta_{n-1} + b_n\theta_n, & n \geqslant 2. \end{cases}$$

The adjoint operator $A^*\varkappa = G\varkappa$ is defined on the domain

$$D(A^*) = \{\varkappa = (\varkappa_0, \varkappa_1, \ldots) \in l_2 \,|\, (G\varkappa) \in l_2\}.$$

In the limit point at infinity case (i.e. A has deficiency indices $(0,0)$), then it is essentially self-adjoint. In the limit circle case (i.e. when A has deficiency indices $(1,1)$) we denote by $p(\lambda) = (p_1(\lambda), p_2(\lambda), \ldots)$, $q(\lambda) = (q_1(\lambda), q_2(\lambda), \ldots)$ two solutions of (2.4) satisfying Cauchy data $p_1(\lambda) = 1$, $p_2(\lambda) = \frac{\lambda - b_1}{a_1}$, $q_1(\lambda) = 0$, $q_2(\lambda) = \frac{1}{a_1}$. Then [21, Lemma 6.22]

$$D(A^*) = D(A) + \mathbb{R}p(0) + \mathbb{R}q(0).$$

All self-adjoint extensions of A are parameterized by $t \in \mathbb{R} \cup \{\infty\}$, are denoted by A_t and defined on the domain

$$D(A_t) = \begin{cases} D(A) + \mathbb{R}(q(0) + tp(0)), & t \in \mathbb{R} \\ D(A) + \mathbb{R}p(0), & t = \infty. \end{cases}$$

All the details the reader can find in [23,21]. We introduce the measure $d\rho_t(\lambda) = \left[dE_\lambda^{A_t}e_1, e_1\right]$, where $dE_\lambda^{A_t}$ is the projection-valued spectral measure of A_t such that $E_{\lambda-0}^{A_t} = E_\lambda^{A_t}$. The results of [4] and [23, Section 5] imply that $d\rho^N \to d\rho_{t^*}$ $*$-weakly as $N \to \infty$, where $t^* = -\lim_{n\to\infty} \frac{q_n(0)}{p_n(0)}$.

The Remark 1 in particular implies that

$$R^{2N-2} = R_N^{2N-2}, \tag{2.21}$$

$$u_{n,\,t}^f = v_{n,\,t}^f, \quad n \leqslant t \leqslant N, \quad \text{and} \quad W^N = W_N^N. \tag{2.22}$$

Thus due to (2.21), we have that $r_{t-1} = r_{t-1}^N$, $t = 1, \ldots, 2N$. On the other hand, taking into account (2.22), we can see that $C^T = C_N^T$ with $T \leqslant N$. Thus, in (2.12), (2.13) tending $N \to \infty$, we obtain

Proposition 3. *The entries of the response vector of (2.14) and of the matrix of the connecting operator C^T admit the spectral representation:*

$$r_{t-1} = \int_{-\infty}^{\infty} \mathcal{T}_t(\lambda)\, d\rho_{t^*}(\lambda), \quad t \in \mathbb{N}, \tag{2.23}$$

$$\{C^T\}_{l+1,\,m+1} = \int_{-\infty}^{\infty} \mathcal{T}_{T-l}(\lambda)\mathcal{T}_{T-m}(\lambda)\, d\rho_{t^*}(\lambda), \quad l, m = 0, \ldots, T-1. \tag{2.24}$$

Note that in (2.23) and (2.24) one can change $d\rho_{t^*}$ for $d\rho_t$, $t \in \mathbb{R} \cup \{\infty\}$ when A is in the limit circle case.

3. Truncated moment problem. Recovering spectral data from dynamic data

We make the following observation: in the classical moment problem [1,22,23] one answers a question of existence (and uniqueness) of a measure satisfying (1.1) for a given set of moments. In the dynamic inverse problem [14,17] one answers a question of existence (and possibility of recovering) of a Jacobi matrix for a given response vector (2.17). Results from the previous section implies the relationship between data of these two problems:

Remark 2. The spectral representation of response vector (2.23) implies that the knowledge of the set of moments $\{s_0, s_1, \ldots, s_{2N-2}\}$ is equivalent to the knowledge of $\{r_0, r_1, \ldots, r_{2N-2}\}$, where $N \in \mathbb{N} \cup \{\infty\}$.

Thus, the moment problem and the inverse dynamic problem for the system (2.14) being different, deal with essentially the same data.

Definition 3. By a solution of a truncated moment problem of order N we call a Borel measure $d\rho(\lambda)$ on $\mathbb{R}$ such that equalities (1.1) with this measure hold for $k = 0, 1, \ldots, 2N-2$.

In [17] the authors proved the following

Theorem 2. *The vector $(r_0, r_1, r_2, \ldots, r_{2N-2})$ is a response vector for the dynamical system (2.2) if and only if the matrix C^N defined by (2.24), (2.20) is positive definite.*

The necessary part of this statement is a simple consequence of the boundary controllability of system (2.14), see Lemma 2, and the definition of C^T (2.19). In the subsection 3.3 we outline the scheme of deriving of formulas for the entries of Jacobi matrix, which play an important role in the proof of the sufficient part of this theorem. All aforesaid, in particular, implies the following procedure of solving the truncated moment problem:

Procedure 1.

1) Calculate $(r_0, r_1, r_2, \ldots, r_{2N-2})$ from $\{s_0, s_1, \ldots, s_{2N-2}\}$ by using (2.23).
2) Recover $N \times N$ Jacobi matrix A_N using formulas for a_k, b_k from [17].
3) Recover spectral measure for finite Jacobi matrix A_N with prescribed arbitrary condition (2.3) at $n = N+1$.
3') Extend Jacobi matrix A_N in arbitrary way to *finite* Jacobi matrix A_M, $M > N$, prescribe arbitrary condition (2.3) at $n = M+1$ and recover spectral measure of A_M.
3") Extend Jacobi matrix A_N in arbitrary way to *semi-infinite* Jacobi matrix A, and recover spectral measure of A.

Every measure obtained in 3), $3'$), $3''$) provides a solution to the truncated moment problem.

Below we propose the different approach: using the ideas of the BC method we recover the spectral measure corresponding to Jacobi matrix A_N directly from moments (from the operator C^N), without recovering the Jacobi matrix itself.

Convention 1. We assume that controls $f \in \mathcal{F}^N$, $f = (f_0, \ldots, f_{N-1})$ are extended: $f = (f_{-1}, f_0, \ldots, f_{N-1}, f_N)$, where $f_{-1} = f_N = 0$.

We introduce the special space of controls $\mathcal{F}_0^N = \left\{f \in \mathcal{F}^N \mid f_0 = 0\right\}$ and the operators $D : \mathcal{F}^N \mapsto \mathcal{F}^N$, $\widehat{D} : \mathcal{H}^T \mapsto \mathcal{H}^T$ acting by

$$\left(Df\right)_t = f_{t+1} + f_{t-1}, \quad f \in \mathcal{F}^T,$$
$$\left(\widehat{D}h\right)_t = h_{t+1} + h_{t-1}, \quad h \in \mathcal{H}^T.$$

The following statement can be easily proved using arguments from [17] and representations (2.18) and (2.15):

Proposition 4.

1) *The operator W^N maps $\mathcal{F}_0^N$ isomorphically onto $\mathcal{H}^{N-1}$.*
2) *On the set $\mathcal{F}_0^N$ the following relation holds:*

$$W^N Df = DW^N f, \quad f \in \mathcal{F}_0^N. \tag{3.1}$$

Taking $f, g \in \mathcal{F}_0^N$ and evaluating the quadratic form, bearing in mind (3.1) and the equality $v_{N,N}^f = 0$ for $f \in \mathcal{F}_0^N$, which follows from (2.15), we obtain:

$$\begin{aligned}\left(C^N Df, f\right)_{\mathcal{F}^N} = \left(W^N Df, W^N g\right)_{\mathcal{H}^N} &= \left(DW^N f, W^N g\right)_{\mathcal{H}^N} \\ &= \left(A_{N-1} v^f, v^g\right)_{\mathcal{H}^N}.\end{aligned} \tag{3.2}$$

The last equality in (3.2) means that only A_{N-1} block of the whole matrix A_N is in use. Then it is possible to perform the spectral analysis of A_{N-1} using the classical variational approach, the controllability of the system (2.2) (see Proposition 4) and the representation (3.2), see also [6]. The spectral data of Jacobi matrix A_{N-1} with the Dirichlet boundary condition at $n = N$ can be recovered by the following

Procedure 2.

1) The first eigenvalue is given by

$$\lambda_1^{N-1} = \min_{f \in \mathcal{F}_0^N,\, (C^N f, f)_{\mathcal{F}^N} = 1} \left(C^N Df, f\right)_{\mathcal{F}^N}. \tag{3.3}$$

2) Let f^1, be the minimizer of (3.3), then

$$\omega_1 = \left(C^N f^1, f^1\right)_{\mathcal{F}^N}.$$

3) The second eigenvalue is given by

$$\lambda_2^{N-1} = \min_{\substack{f \in \mathcal{F}_0^N, (C^N f, f)_{\mathcal{F}^N} = 1 \\ (C^N f, f_1)_{\mathcal{F}^N} = 0}} \left(C^N Df, f\right)_{\mathcal{F}^N}. \tag{3.4}$$

4) Let f^2, be the minimizer of (3.4), then

$$\omega_2 = \left(C^N f^2, f^2\right)_{\mathcal{F}^N}.$$

Continuing this procedure, one recovers the set $\{\lambda_k^{N-1}, \omega_k\}_{k=1}^{N-1}$ and constructs the measure $d\rho^{N-1}(\lambda)$ by (2.10).

Remark 3. The measure, constructed by the above procedure solves the truncated moment problem for the set of moments $\{s_0, s_1, \ldots, s_{2N-4}\}$.

3.1. Euler-Lagrange equations

In this section we derive equations which can be thought of as Euler-Lagrange equations for the problem of the minimization of a functional $\left(C^N Df, f\right)_{\mathcal{F}^N}$ in $\mathcal{F}_0^N$ with the constrain $\left(C^N f, f\right)_{\mathcal{F}^N} = 1$, described in the previous section. Similar method of deriving equations which can be used for recovering of spectral data was used in [3].

By $f_k \in \mathcal{F}^N$, $k = 1, \dots, N$ we denote the control that drives system (2.2) to prescribed state ϕ^k (see (2.4)):

$$W^N f_k = \phi^k, \quad k = 1, \dots, N.$$

Due to Proposition 4, such a control exists and is unique for every k. We introduce the shift operator

$$V^N : \mathcal{F}^N \mapsto \mathcal{F}^N,$$
$$\left(V^N g\right)_n = g_{n-1}, \quad n = 1, \dots, N-1, \quad \left(V^N g\right)_0 = 0,$$

and denote by $E : \mathcal{F}^N \mapsto \mathcal{F}^{N+1}$ the embedding operator:

$$(Eg)_k = \begin{cases} g_k, & k = 0, 1, N-1, \\ 0 & k = N; \end{cases}$$

then the adjoint operator $E^* : \mathcal{F}^{N+1} \mapsto \mathcal{F}^N$ is a projection.

Theorem 3. *The spectrum of A_N and controls f_k, $k = 1, \dots, N$ are the spectrum and the eigenvectors of the following generalized spectral problem:*

$$\left(E^* \left(V^{N+1}\right)^* C^{N+1} E + C^N V^N\right) f_k = \lambda_k C^N f_k, \quad k = 1, \dots, N. \tag{3.5}$$

Proof. For $h \in \mathcal{F}^T$ we always assume that $h_{-1} = h_T = 0$ (see Agreement 1). For a fixed $k = 1, \dots, N$ we take $f_k \in \mathcal{F}^N$ such that $W^N f_k = v^{f_k}_{\cdot, N} = \phi^k$, then for arbitrary $g \in \mathcal{F}^N$ we can evaluate:

$$\begin{aligned} \left(\lambda_k C^N f_k, g\right)_{\mathcal{F}^N} &= \left(\lambda_k v^{f_k}_{\cdot, N}, v^g_{\cdot, N}\right)_{\mathcal{H}^N} = \left(\lambda_k \phi^k, v^g_{\cdot, N}\right)_{\mathcal{H}^N} = \left(A_N \phi^k, v^g_{\cdot, N}\right)_{\mathcal{H}^N} \\ &= \left(\left(A_N v^{f_k}\right)_{\cdot, N}, v^g_{\cdot, N}\right)_{\mathcal{H}^N} = \left(\left(D v^{f_k}\right)_{\cdot, N}, v^g_{\cdot, N}\right)_{\mathcal{H}^N} \\ &= \left(v^{f_k}_{\cdot, N+1}, v^g_{\cdot, N})\right)_{\mathcal{H}^N} + \left(v^{f_k}_{\cdot, N-1}, v^g_{\cdot, N}\right)_{\mathcal{H}^N}. \end{aligned} \tag{3.6}$$

We note that

$$\mathcal{H}^N \ni v^g_{\cdot, N} = \left(v^g_{1, N}, \dots, v^g_{N, N}\right),$$
$$\mathcal{H}^{N+1} \ni v^{V^{N+1} g}_{\cdot, N+1} = \left(v^g_{1, N}, \dots, v^g_{N, N}, 0\right).$$

That is why we can rewrite the first summand in the right hand side of (3.6) as

$$\left(v^{f_k}_{\cdot, N+1}, v^g_{\cdot, N}\right)_{\mathcal{H}^N} = \left(v^{f_k}_{\cdot, N+1}, v^{V^{N+1} g}_{\cdot, N+1}\right)_{\mathcal{H}^{N+1}} = \left(C^{N+1} f_k, V^{N+1} g\right)_{\mathcal{F}^{N+1}}. \tag{3.7}$$

Analogously:

$$\mathcal{H}^N \ni v^{f_k}_{\cdot,\,N-1} = \left(v^{f_k}_{1,\,N-1}, \dots, v^{f_k}_{N-1,N-1}, 0\right),$$
$$\mathcal{H}^N \ni v^{V^N f_k}_{\cdot,\,N} = \left(v^{f_k}_{1,\,N-1}, \dots, v^{f_k}_{N-1,N-1}, 0\right).$$

So we can rewrite the second summand in the right hand side of (3.6) as

$$\left(v^{f_k}_{\cdot,\,N-1}, v^{g}_{\cdot,\,N}\right)_{\mathcal{H}^N} = \left(v^{V^N f_k}_{\cdot,\,N}, v^{g}_{\cdot,\,N}\right)_{\mathcal{H}^N} = \left(C^N V^N f_k, g\right)_{\mathcal{F}^N}. \tag{3.8}$$

Finally from (3.6), (3.7) and (3.8) we deduce that

$$\left(\lambda_k C^N f_k, g\right)_{\mathcal{F}^N} = \left(C^{N+1} f_k, V^{N+1} g\right)_{\mathcal{F}^{N+1}} + \left(C^N V^N f_k, g\right)_{\mathcal{F}^N}. \tag{3.9}$$

Using operators E, E^* we can rewrite (3.9) in the form

$$\left(E^* \left(V^{N+1}\right)^* C^{N+1} E + C^N V^N\right) f_k = \lambda_k C^N f_k.$$

Thus the pair $\{f_k, \lambda_k\}$ provides the solution to (3.5). Now let the pair $\{f, \lambda\}$ be the solution to (3.5) with $f \in \mathcal{F}^N$ $f \neq f_k$, $\lambda \neq \lambda_k$ for any $k = 1 \dots, N$. Then $W^N f = v^f_{\cdot,\,N} = \sum_{k=1}^N a_k \phi^k$ for some $a_k \in \mathbb{R}$. We can evaluate for arbitrary $g \in \mathcal{F}^N$:

$$\begin{aligned}
0 &= \left(\left(E^* \left(V^{N+1}\right)^* C^{N+1} E + C^N V^N\right) f - \lambda C^N f, g\right)_{\mathcal{F}^N} \\
&= \left(C^{N+1} E f, V^{N+1} E g\right)_{\mathcal{F}^{N+1}} + \left(C^N V^N f, g\right)_{\mathcal{F}^N} - \lambda \left(v^f_{\cdot,\,N}, v^g_{\cdot,\,N}\right)_{\mathcal{H}^N} \\
&= \left(v^{Ef}_{\cdot,\,N+1}, v^{V^{N+1}Eg}_{\cdot,\,N+1}\right)_{\mathcal{H}^{N+1}} + \left(v^{V^N f}_{\cdot,\,N}, v^g_{\cdot,\,N}\right)_{\mathcal{H}^N} - \lambda \left(v^f_{\cdot,\,N}, v^g_{\cdot,\,N}\right)_{\mathcal{H}^N} \\
&= \left(v^{Ef}_{\cdot,\,N+1}, v^g_{\cdot,\,N+1}\right)_{\mathcal{H}^N} + \left(v^f_{\cdot,\,N-1}, v^g_{\cdot,\,N-1}\right)_{\mathcal{H}^N} - \lambda \left(v^f_{\cdot,\,N-1}, v^g_{\cdot,\,N-1}\right)_{\mathcal{H}^N} \\
&= \left(\left(A^N v^f\right)_{\cdot,\,N}, v^g_{\cdot,N}\right)_{\mathcal{H}^N} - \lambda \left(v^f_{\cdot,\,N}, v^g_{\cdot,\,N}\right)_{\mathcal{H}^N} \\
&= \left(A_N \sum_{k=1}^N a_k \phi^k - \lambda \sum_{k=1}^N a_k \phi^k, W^N g\right)_{\mathcal{H}^N} = \left(\sum_{k=1}^N a_k (\lambda_k - \lambda) \phi^k, W^N g\right)_{\mathcal{H}^N}.
\end{aligned}$$

From the above equality and Proposition 4 it follows that all a_k except one are equal to zero, and for such a_j, $\lambda = \lambda_j$, which completes the proof. □

Since (3.5) is linear, solving this equation one obtains spectrum $\{\lambda_k\}_{k=1}^N$ and controls $\left\{\widetilde{f}_k\right\}_{k=1}^N$, such that $W^T \widetilde{f}_k = \beta_k \phi^k$ for some $\beta_k \in \mathbb{R}\backslash\{0\}$, $k = 1, \dots, N$. Then the measure of operator A_N with Dirichlet boundary condition at $n = N + 1$ can be recovered by the following

Procedure 3.

1) Normalize controls $\widetilde{f}_k$ by the condition $\left(C^N \widetilde{f}_k, \widetilde{f}_k\right)_{\mathcal{F}^N} = 1$, $k = 1, \dots, N$.
2) Observe that $W^N \widetilde{f}^k = \alpha_k \phi^k$ for some $\alpha_k \in \mathbb{R}\backslash\{0\}$, where the constant is defined by $\alpha_k = \alpha_k \phi^k_1 = \left(W^N \widetilde{f}_k\right)_1 = \left(R \widetilde{f}_k\right)_N$, $k = 1, \dots, N$.
3) Coefficients (2.5) are given by $\omega_k = \alpha_k^2$, $k = 1, \dots, N$.
4) Recover the measure by (2.10).

Now we rewrite the generalized spectral problem (3.5) in more details and transfer the matrices in left and right-hand sides to Hankel matrices known from classical literature [1,23]. Note that the matrices in (3.5) has the following representations:

$$E^* = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ 0 & \dots & 1 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ 0 & \dots & 0 & 1 \\ 0 & \dots & 0 & 0 \end{pmatrix},$$

$$V^N = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ 0 & \dots & 1 & 0 \end{pmatrix}, \quad \left(V^N\right)^* = \begin{pmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ 0 & \dots & 0 & 1 \\ 0 & \dots & 0 & 0 \end{pmatrix},$$

$$C^N = \begin{pmatrix} c_{N,N} & \dots & \cdot & c_{N,1} \\ c_{N-1,N} & & \cdot & c_{N-1,1} \\ \cdot & \cdot & \cdot & \cdot \\ c_{1,N} & \dots & \cdot & c_{1,1} \end{pmatrix}, C^{N+1} = \begin{pmatrix} c_{N+1,N+1} & \dots & \cdot & c_{N+1,1} \\ c_{N,N+1} & & \cdot & c_{N,1} \\ \cdot & \cdot & \cdot & \cdot \\ c_{1,N+1} & \dots & \cdot & c_{1,1} \end{pmatrix}. \tag{3.10}$$

Here we used the notations for entries of C^N different from ones in (2.20) in order to show that C^N is a lower right block in C^{N+1}. The left hand side of (3.5) we denote by

$$B^N := E^* \left(V^{N+1}\right)^* C^{N+1} E + C^N V^N.$$

Proposition 5. *The matrix B^N is self-adjoint, it admits the following representation:*

$$B^N = \begin{pmatrix} c_{N,N+1} + c_{N,N-1} & c_{N,N} + c_{N,N-2} & \dots & c_{N,3} + c_{N,1} & c_{N,2} \\ c_{N-1,N+1} + c_{N-1,N-1} & \dots & \dots & c_{N-1,3} + c_{N-1,1} & c_{N-1,2} \\ \cdot & \cdot & \dots & \cdot & \cdot \\ c_{1,N+1} + c_{1,N-1} & c_{1,N} + c_{1,N-2} & \dots & \dots & c_{1,2} \end{pmatrix}. \tag{3.11}$$

Proof. We note that the matrices E, E^* and V^N, $\left(V^N\right)^*$ have one diagonal filled with ones and the other elements are zeros. Thus the multiplication by such a matrix leads to deleting a line or column from the original matrix (possibly with the addition of a zero line or column). Performing calculations we see that the first term in the right hand side of B^N is obtained by deleting last column and first row from C^{N+1} and the second term is obtained by deleting the first column and adding zero column to C^N. All aforesaid leads to the formula (3.11). We note that the representation (3.11) and Corollary 1 shows that B^T is self-adjoint matrix. □

Remark 4. The spectral problem (3.5) has a form

$$B^N f_k = \lambda_k C^N f_k, \quad k = 1, \dots, N, \qquad \text{where} \quad C^N > \mathbb{O}, \quad B^N = \left(B^N\right)^*. \tag{3.12}$$

Chebyshev polynomials of the second kind $\{\mathcal{T}_1(\lambda), \mathcal{T}_2(\lambda), \dots \mathcal{T}_n(\lambda)\}$ (see 2.6) are related to $\{1, \lambda, \lambda^{n-1}\}$ by the following relation

$$\begin{pmatrix} \mathcal{T}_1(\lambda) \\ \mathcal{T}_2(\lambda) \\ \dots \\ \mathcal{T}_n(\lambda) \end{pmatrix} = \Lambda_n \begin{pmatrix} 1 \\ \lambda \\ \dots \\ \lambda^{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ a_{21} & 1 & \dots & 0 \\ \dots & & & \\ a_{n1} & a_{n2} & \dots & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \lambda \\ \dots \\ \lambda^{n-1} \end{pmatrix}. \tag{3.13}$$

Proposition 6. *The entries of the matrix $\Lambda_n \in \mathbb{M}^{n-1}$ are given by*

$$\Lambda_n = a_{ij} = \begin{cases} 0, & \text{if } i > j, \\ 0, & \text{if } i+j \text{ is odd}, \\ C^j_{\frac{i+j}{2}}(-1)^{\frac{i+j}{2}+j}, \end{cases} \tag{3.14}$$

where C^k_n are binomial coefficients. The entries of the response vector are related to moments by the rule:

$$\begin{pmatrix} r_0 \\ r_1 \\ \dots \\ r_{n-1} \end{pmatrix} = \Lambda_n \begin{pmatrix} s_0 \\ s_1 \\ \dots \\ s_{n-1} \end{pmatrix}. \tag{3.15}$$

Proof. The formula (3.14) for entries of Λ_n is proved by direct calculations with the use of properties of Chebyshev polynomials. Then making use of (2.23) yields (3.15). $\square$

Introduce the following Hankel matrices

$$S^N_m := \begin{pmatrix} s_{2N-2+m} & s_{2N-3+m} & \dots & s_{N-1+m} \\ s_{2N-3+m} & \dots & \dots & \dots \\ \cdot & \cdot & \dots & s_{1+m} \\ s_{N-1+m} & \dots & s_{1+m} & s_m \end{pmatrix}, \quad m = 0, 1, \dots,$$

the matrix $J_N \in \mathbb{R}^{N\times N}$:

$$J_N = \begin{pmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 1 & 0 \\ \cdot & \cdot & \dots & \cdot \\ 0 & 1 & \dots & 0 \\ 1 & \dots & 0 & 0 \end{pmatrix}, \quad J_N J_N = I_N = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ 0 & \dots & 1 & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix},$$

and define

$$\widetilde{\Lambda}_N := J_N \Lambda_N J_N.$$

The remarkable fact is that the matrices B^N, C^N can be reduced to Hankel matrices by the same linear transformation:

Theorem 4. *The following relations hold:*

$$C^N = \widetilde{\Lambda}_N S^N_0 \left(\widetilde{\Lambda}_N\right)^*, \tag{3.16}$$

$$B^N = \widetilde{\Lambda}_N S^N_1 \left(\widetilde{\Lambda}_N\right)^*. \tag{3.17}$$

Then the generalized spectral problem (3.5) or (3.12) upon introducing the notation $g_k = \left(\widetilde{\Lambda}_N\right)^* f_k$ is equivalent to the following generalized spectral problem:

$$S^N_1 g_k = \lambda_k S^N_0 g_k. \tag{3.18}$$

Proof. Using (3.10) and the representation (2.24), we have that entries of C^N have a form:

$$c_{ij} = \int_{-\infty}^{\infty} \mathcal{T}_{N-i+1}(\lambda)\mathcal{T}_{N-j+1}(\lambda)\, d\rho(\lambda), \quad i,j = i,\dots,N.$$

Introducing the operation $\otimes : \mathbb{R}^N \times \mathbb{R}^N \mapsto \mathbb{M}^N$ by the following rule: for $a, b \in \mathbb{R}^N$ we set $a \otimes b = c \in \mathbb{M}^N$, where $c_{ij} = a_i b_j$, we see that in view of (2.24) the operator C^N has a form:

$$C^N = \int_{-\infty}^{\infty} \begin{pmatrix} \mathcal{T}_N(\lambda) \\ \mathcal{T}_{N-1}(\lambda) \\ \cdot \\ \mathcal{T}_1(\lambda) \end{pmatrix} \otimes \begin{pmatrix} \mathcal{T}_N(\lambda) \\ \mathcal{T}_{N-1}(\lambda) \\ \cdot \\ \mathcal{T}_1(\lambda) \end{pmatrix} d\rho(\lambda). \tag{3.19}$$

Using (3.13) we can rewrite (3.19) as

$$\begin{aligned} C^N &= \int_{-\infty}^{\infty} J_N \Lambda_N J_N J_N \begin{pmatrix} 1 \\ \lambda \\ \cdot \\ \lambda^{N-1} \end{pmatrix} \otimes J_N \Lambda_N J_N J_N \begin{pmatrix} 1 \\ \lambda \\ \cdot \\ \lambda^{N-1} \end{pmatrix} d\rho(\lambda) \\ &= \int_{-\infty}^{\infty} \widetilde{\Lambda}_N \begin{pmatrix} \lambda^{N-1} \\ \lambda^{N-2} \\ \cdot \\ 1 \end{pmatrix} \otimes \widetilde{\Lambda}_N \begin{pmatrix} \lambda^{N-1} \\ \lambda^{N-2} \\ \cdot \\ 1 \end{pmatrix} d\rho(\lambda) \\ &= \widetilde{\Lambda}_N \int_{-\infty}^{\infty} \begin{pmatrix} \lambda^{N-1} \\ \lambda^{N-2} \\ \cdot \\ 1 \end{pmatrix} \otimes \begin{pmatrix} \lambda^{N-1} \\ \lambda^{N-2} \\ \cdot \\ 1 \end{pmatrix} d\rho(\lambda) \left(\widetilde{\Lambda}_N\right)^* = \widetilde{\Lambda}_N S_0^N \left(\widetilde{\Lambda}_N\right)^*, \end{aligned}$$

which proves (3.16).

Using the representation of B^N (3.11) and (2.24) yields the following formula for entries b_{ij} of B^N:

$$b_{ij} = \int_{-\infty}^{\infty} \mathcal{T}_{N-i+1}(\lambda) \left(\mathcal{T}_{N-j+2}(\lambda) + \mathcal{T}_{N-j}(\lambda)\right) d\rho(\lambda), \quad i,j = i,\dots,N,$$

where we counted that $\mathcal{T}_0(\lambda) = 0$. Making use of (2.6) leads to:

$$b_{ij} = \int_{-\infty}^{\infty} \mathcal{T}_{N-i+1}(\lambda)\lambda\mathcal{T}_{N-j+1}(\lambda)\, d\rho(\lambda), \quad i,j = i,\dots,N. \tag{3.20}$$

Then using (3.13) and (3.20) we obtain:

$$\begin{aligned} B^N &= \int_{-\infty}^{\infty} J_N \Lambda_N J_N J_N \lambda \begin{pmatrix} 1 \\ \lambda \\ \cdot \\ \lambda^{N-1} \end{pmatrix} \otimes J_N \Lambda_N J_N J_N \begin{pmatrix} 1 \\ \lambda \\ \cdot \\ \lambda^{N-1} \end{pmatrix} d\rho(\lambda) \\ &= \int_{-\infty}^{\infty} \widetilde{\Lambda}_N \begin{pmatrix} \lambda^N \\ \lambda^N \\ \cdot \\ \lambda \end{pmatrix} \otimes \widetilde{\Lambda}_N \begin{pmatrix} \lambda^{N-1} \\ \lambda^{N-2} \\ \cdot \\ 1 \end{pmatrix} d\rho(\lambda) \end{aligned}$$

$$= \widetilde{\Lambda}_N \int_{-\infty}^{\infty} \begin{pmatrix} \lambda^N \\ \lambda^{N-1} \\ \cdot \\ \lambda \end{pmatrix} \otimes \begin{pmatrix} \lambda^{N-1} \\ \lambda^{N-2} \\ \cdot \\ 1 \end{pmatrix} d\rho(\lambda) \left(\widetilde{\Lambda}_N\right)^* = \widetilde{\Lambda}_N S_1^N \left(\widetilde{\Lambda}_N\right)^*,$$

which gives (3.17). Then (3.18) is a consequence of (3.16) and (3.17). □

3.2. Special cases: Hamburger, Stieltjes and Hausdorff moment problems

In the previous sections we constructed the special measure corresponding to operator A_N with Dirichlet boundary condition at $n = N + 1$ and which gives a solution of a truncated moment problem. Here we formulate several consequences of Theorems 3, 4.

Bearing in mind the relationship between elements of response vector and moments (2.23) and the formula (3.16), we can reformulate Theorem 2 as

Proposition 7. *The numbers $(s_0, s_1, s_2, \ldots, s_{2N-2})$ are the moments of the spectral measure corresponding to the Jacobi operator A_N with Dirichlet boundary condition at $n = N + 1$ if and only if*

$$\text{the matrix } S_0^N \text{ is positive definite.} \tag{3.21}$$

The Stieltjes moment problem is characterized by the positivity of a support of a measure. That means the positivity of a spectrum of A_N. The latter leads to the following

Proposition 8. *The numbers $(s_0, s_1, s_2, \ldots, s_{2N-1})$ are the moments of the spectral measure, supported on $(0, +\infty)$, corresponding to Jacobi operator A_N with Dirichlet boundary condition at $n = N + 1$ if and only if*

$$\text{matrices } S_0^N \text{ and } S_1^N \text{ are positive definite.} \tag{3.22}$$

In the Hausdorff moment problem the measure is supported on $(0, 1)$, which leads to the following

Proposition 9. *The numbers $(s_0, s_1, s_2, \ldots, s_{2N-1})$ are the moments of the spectral measure, supported on $(0, 1)$, corresponding to operator A_N with Dirichlet boundary condition at $n = N + 1$, if and only if the condition*

$$S_0^N \geqslant S_1^N > \mathbb{O} \tag{3.23}$$

holds.

Proof. From (3.18) it follows that

$$\lambda_k = \frac{\left(S_1^N g_k, g_k\right)}{\left(S_0^T g_k, g_k\right)}, \quad k = 1, \ldots, N.$$

Then the restriction $\lambda_k \in (0, 1)$ implies (3.23). □

Remark 5. Given an infinite sequence of moments, one can determine whether or not it is Hamburger or Stieltjes or Hausdorff moment sequence by verifying whether the condition (3.21) or (3.22) or (3.23) holds for all $N \in \mathbb{N}$.

The necessity of conditions (3.21), (3.22), (3.23) is clear. The condition (3.21) holding for all $N>0$ makes it possible to construct semi-infinite Jacobi matrix A. The finite measures constructed on each step $d\rho^N(\lambda)$ has proper supports in Stieltjes and Hausdorff cases, and as it was explained in the end of Section 2 these measures $d\rho^N$ converges $*$-weakly to $d\rho_{t^*}$ the spectral measure of some special self-adjoint extension of A, which has to have proper support as well.

3.3. Recovering Jacobi matrix, nonuniqueness of the solution of the truncated moment problem

As we mentioned, given a sequence of moments $\{s_0, s_1, \dots, s_{2N-2}\}$ or, equivalently, entries of the response vector, $\{r_0, r_1, \dots, r_{2N-2}\}$, it is possible to recover the Jacobi matrix A_N, see [14,17]. Below we outline the proof of this result (it is an important ingredient in the sufficient part of Theorem 2).

We rewrite W^N (2.18) as $W^N = \overline{W}^N J$, where

$$W^N f = \begin{pmatrix} 1 & w_{1,1} & w_{1,2} & \dots & w_{1,N-1} \\ 0 & a_1 & w_{2,2} & \dots & w_{2,N-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \dots & \prod_{j=1}^{k-1} a_j & \dots & w_{k,N-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & \prod_{j=1}^{N-1} a_j \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \dots & 1 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} f_0 \\ f_2 \\ \cdot \\ f_{N-k-1} \\ \cdot \\ f_{N-1} \end{pmatrix}.$$

Using the invertibility of W^T (cf. Lemma 2) we rewrite the definition of C^N in a form $\left((W^N)^{-1}\right)^* \times C^N (W^N)^{-1} = I$, which, in turn can be rewritten as

$$\left(\left(\overline{W}^N\right)^{-1}\right)^* \overline{C}^N \left(\overline{W}^N\right)^{-1} = I, \quad \overline{C}^N = JC^N J, \tag{3.24}$$

where entries $\overline{c}_{ij}$ of the matrix $\overline{C}^N$ are defined by the rile

$$\overline{c}_{ij} = \left\{C^N\right\}_{N+1-j, N+1-i}, \tag{3.25}$$

and $\left(\overline{W}^N\right)^{-1}$ has a form

$$\left(\overline{W}^N\right)^{-1} = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \dots & a_{1,N} \\ 0 & a_{2,2} & a_{2,3} & \dots & .. \\ \cdot & \cdot & \cdot & a_{N-1,N-1} & a_{N-1,N} \\ 0 & \dots & \dots & 0 & a_{N,N} \end{pmatrix}. \tag{3.26}$$

It is easy to see that diagonal elements of (3.26) satisfy the relation:

$$a_{k,k} = \left(\prod_{j=1}^{k-1} a_j\right)^{-1}. \tag{3.27}$$

Multiplying the k-th row of $\overline{W}^N$ by $k+1$-th column of $\left(\overline{W}^N\right)^{-1}$ leads to the relation

$$a_{k,k+1} a_1 \dots a_{k-1} + a_{k+1,k+1} w_{k,k} = 0,$$

from where we deduce that

$$a_{k,\,k+1}=-\left(\prod_{j=1}^{k}a_j\right)^{-2}a_k w_{k,\,k}. \tag{3.28}$$

Then we can rewrite (3.24) in the equivalent form:

$$\begin{pmatrix} a_{1,\,1} & 0 & . & 0\\ a_{1,\,2} & a_{2,\,2} & 0 & .\\ . & . & . & .\\ a_{1,\,N} & . & . & a_{N,\,N}\end{pmatrix}\begin{pmatrix} \overline{c}_{11} & .. & .. & \overline{c}_{1N}\\ .. & .. & .. & ..\\ . & . & . & .\\ \overline{c}_{N1} & .. & & \overline{c}_{NN}\end{pmatrix}\begin{pmatrix} a_{1,1} & a_{1,2} & .. & a_{1,N}\\ 0 & a_{2,2} & .. & a_{2,N}\\ . & . & . & .\\ 0 & \dots & \dots & a_{N,N}\end{pmatrix}=I \tag{3.29}$$

A direct consequence of the above equality is an equality for determinants:

$$\det\left(\left(\overline{W}^N\right)^{-1}\right)^*\det\overline{C}^N\det\left(\overline{W}^N\right)^{-1}=1,$$

which yields

$$a_{1,\,1}*\ldots*a_{N,\,N}=\left(\det\overline{C}^N\right)^{-\frac{1}{2}}.$$

From the above formula we derive that

$$a_{1,\,1}=\left(\det\overline{C}^1\right)^{-\frac{1}{2}},\quad a_{2,\,2}=\left(\frac{\det\overline{C}^2}{\det\overline{C}^1}\right)^{-\frac{1}{2}},\quad a_{k,\,k}=\left(\frac{\det\overline{C}^k}{\det\overline{C}^{k-1}}\right)^{-\frac{1}{2}}.$$

Combining latter relations with (3.27), we deduce that

$$\prod_{i=1}^{k-1}a_i=\left(\frac{\det\overline{C}^k}{\det\overline{C}^{k-1}}\right)^{\frac{1}{2}}.$$

Then we obtain that

$$a_k=\frac{\left(\det\overline{C}^{k+1}\right)^{\frac{1}{2}}\left(\det\overline{C}^{k-1}\right)^{\frac{1}{2}}}{\det\overline{C}^k},\quad k=1,\ldots,N-1, \tag{3.30}$$

where we set $\det C^0=1$, $\det C^{-1}=1$.

Now using (3.29) we write down the equation on the last column of $\left(\overline{W}^N\right)^{-1}$:

$$\begin{pmatrix} a_{1,\,1} & 0 & . & 0\\ a_{1,\,2} & a_{2,\,2} & 0 & .\\ . & . & . & .\\ a_{1,\,N-1} & . & . & a_{N-1,\,N-1}\end{pmatrix}\begin{pmatrix} \overline{c}_{1,\,1} & .. & .. & \overline{c}_{1,\,N}\\ .. & .. & .. & ..\\ . & . & . & .\\ \overline{c}_{N-1,\,1} & .. & & \overline{c}_{N-1,\,N-1}\end{pmatrix}\begin{pmatrix} a_{1,\,N}\\ a_{2,\,N}\\ .\\ a_{T,\,N}\end{pmatrix}=\begin{pmatrix}0\\0\\.\\0\end{pmatrix}.$$

Note that since we know $a_{N,\,N}$, we can rewrite the above equality in the form of equation on $(a_{1,\,N},\ldots,a_{N-1,\,N})^*$:

$$\begin{pmatrix} \overline{c}_{1,\,1} & .. & .. & \overline{c}_{1,\,N}\\ .. & .. & .. & ..\\ . & . & . & .\\ \overline{c}_{N-1,\,1} & .. & & \overline{c}_{N-1,\,N-1}\end{pmatrix}\begin{pmatrix} a_{1,\,N}\\ a_{2,\,N}\\ .\\ a_{N-1,\,N}\end{pmatrix}=-a_{N,\,N}\begin{pmatrix} a_{1,\,N}\\ a_{2,\,N}\\ .\\ a_{N-1,\,N}\end{pmatrix}. \tag{3.31}$$

Introduce the notation:

$$\overline{C}_k^{k-1} := \begin{pmatrix} \overline{c}_{1,\,1} & \cdots & \cdots & \overline{c}_{1,\,k-2} & \overline{c}_{1,\,k} \\ \vdots & \vdots & \vdots & \vdots & \\ \overline{c}_{k-1,\,1} & \cdots & & \overline{c}_{k-1,\,k-2} & \overline{c}_{k-1,\,k} \end{pmatrix},$$

that is $\overline{C}_k^{k-1}$ is constructed from the matrix $\overline{C}^{k-1}$ by substituting the last column by $(\overline{c}_{1,\,k},\dots,\overline{c}_{k-1,\,k})$. Then from (3.31) we deduce that:

$$a_{N-1,\,N} = -a_{N,\,N}\frac{\det \overline{C}_N^{N-1}}{\det \overline{C}^{N-1}}, \tag{3.32}$$

where we assumed that $\det C_0^{-1} = 0$. On the other hand (3.27), (3.28) yield

$$a_{N-1,\,N} = \left(\prod_{j=1}^{N-1} a_j\right)^{-1} \sum_{k=1}^{N-1} b_k. \tag{3.33}$$

Equating (3.32) and (3.33) gives equalities

$$\sum_{k=1}^{N-1} b_k = -\frac{\det \overline{C}_N^{N-1}}{\det \overline{C}^{T-1}}, \quad \sum_{k=1}^{N} b_k = -\frac{\det \overline{C}_{N+1}^{N}}{\det \overline{C}^{N}},$$

from where

$$b_k = -\frac{\det \overline{C}_{k+1}^{k}}{\det \overline{C}^{k}} + \frac{\det \overline{C}_k^{k-1}}{\det \overline{C}^{k-1}}, \quad k = 1,\dots,N. \tag{3.34}$$

Thus the matrix A_N can be recovered from $\{r_0, r_1, \dots, r_{2N-2}\}$ by use of (3.30), (3.34).

Remark 6. In order to apply the results of Theorems 3, 4 to the problem of reconstruction of spectral measure of A_N, one needs to know one extra moment, specifically s_{2N-1} (see the definition of S_1^N), than in the method based on direct calculation of A_N by formulas (3.30) and (3.34).

Denote by $M_+(\mathbb{R}\cup\{\infty\})$ the set of positive Borel measures on $\mathbb{R}\cup\{\infty\}$ and by $M_N \subset M_+(\mathbb{R}\cup\{\infty\})$ a subset such that $d\nu(\lambda) \in M_N$ is a solution of the truncated moment problem (1.1) of the order N. We used the BC method to construct the *special solution* of a truncated moment problem: for $N \in \mathbb{N}$ the set of moments $\{s_0, s_1, \dots, s_{2N-1}\}$ determines the measure $d\rho^N(\lambda) \in M_N$, where the constructed measure is a spectral measure of a finite Jacobi operator A_N with the Dirichlet condition at $n = N+1$. We point out that in our procedure we do not use the Jacobi matrix, but rather special Hankel matrices, constructed from moments.

Having constructed the Jacobi matrix A_N from the set $\{s_0, s_1, \dots, s_{2N-2}\}$ we can consider the operator $\widetilde{A}_N$ given by boundary condition (2.3) at $n = N+1$, or one can extend the matrix A_N in arbitrary way, keeping it to be Jacobi and taking a selfadjoint extension as it was described in the end of Section 2. Then the spectral measure of any of described operators also gives a solution to the Hamburger moment problem.

Remark 7. The spectral representation of (2.24) implies that M_N is a convex set in $M_+(\mathbb{R}\cup\{\infty\})$. It is also not hard to see that M_N is closed in the $*$-weak topology and obviously $M_{N_1} \subseteq M_{N_2}$ when $N_1 > N_2$. Taking N to infinity we deduce that the set of solutions M_∞ of the Hamburger moment problem (1.1) either convex and compact in weak topology, or consists of one element. The same arguments and spectral representation of B^T (3.20) shows that the set of solutions M_∞^s to Stieltjes moment problem either convex and compact in $*$-weak topology or consists of one element. More on this subject one can find in [23, Appendix B].

4. On the uniqueness of the solution of the Hamburger, Stieltjes and Hausdorff moment problems

We remind the reader that the moment problem is called *determinate* if it has only one solution, otherwise it is called *indeterminate*.

In this section we use complex-valued outer and inner spaces for the dynamical system (2.14): $\mathcal{F}^T = \mathcal{H}^T = \mathbb{C}^T$ with the scalar products $(f,g)_{\mathcal{F}^T} = \sum_{i=0}^{T-1} f_i\overline{g}_i$ and $(a,b)_{\mathcal{H}^T} = \sum_{i=1}^{T} a_i\overline{b_i}$. Although subsequently in applications of Krein equations to moment problems we use $\mathbb{R}$ rather than $\mathbb{C}$, the possibility to use complex controls could be important, see for example [16].

4.1. Krein equations

Let $\alpha, \beta \in \mathbb{R}$ and $y(\lambda)$ be a solution to a Cauchy problem for the following difference equation (we remind the agreement $a_0 = 1$):

$$\begin{cases} a_k y_{k+1} + a_{k-1}y_{k-1} + b_k y_k = \lambda y_k, \\ y_0 = \alpha, \ y_1 = \beta. \end{cases} \tag{4.1}$$

We set up the *special control problem*: to find a control $f^T \in \mathcal{F}^T$ that drives the system (2.14) to the prescribed state $\overline{y^T(\lambda)} := \left(\overline{y_1(\lambda)}, \dots, \overline{y_T(\lambda)}\right) \in \mathcal{H}^T$ at $t = T$:

$$W^T f^T = \overline{y^T(\lambda)}, \quad \left(W^T f^T\right)_k = \overline{y_k(\lambda)}, \quad k = 1, \dots, T. \tag{4.2}$$

Note that due to Lemma 2, this control problem has a unique solution $f^T = \left(W^T\right)^{-1} \overline{y^T(\lambda)}$. Let $\varkappa^T(\lambda)$ be a solution to

$$\begin{cases} \varkappa^T_{t+1} + \varkappa^T_{t-1} = \lambda \varkappa_t, \quad t = 0, \dots, T, \\ \varkappa^T_T = 0, \ \varkappa^T_{T-1} = 1. \end{cases} \tag{4.3}$$

One can easily see the relation with Chebyshev polynomials (2.6):

$$\varkappa^T_t(\lambda) = \mathcal{T}_{T-t}(\lambda), \quad t = 0, 1, \dots, T. \tag{4.4}$$

It is an important fact that the control f^T can be found as a solution to certain equation:

Theorem 5. *The control $f^T = \left(W^T\right)^{-1} \overline{y^T(\lambda)}$ solving the special control problem (4.2), is a unique solution to the following Krein-type equation in $\mathcal{F}^T$:*

$$C^T f^T = \beta \overline{\varkappa^T(\lambda)} - \alpha \left(R^T\right)^* \overline{\varkappa^T(\lambda)}. \tag{4.5}$$

Proof. Let f^T be a solution to (4.2). We observe that for any fixed $g \in \mathcal{F}^T$ we have that

$$u^g_{k,\,T} = \sum_{t=0}^{T-1} \left(u^g_{k,\,t+1} + u^g_{k,\,t-1} - \lambda u^g_{k,\,t}\right) \varkappa^T_t. \tag{4.6}$$

Indeed, changing the order of a summation in the right hand side of (4.6) and counting $u^g_{k,\,-1} = u^g_{k,\,0} = 0$ yields

$$\sum_{t=0}^{T-1} \left(u^g_{k,\,t+1} + u^g_{k,\,t-1} - \lambda u^g_{k,\,t}\right) \varkappa^T_t = \sum_{t=0}^{T-1} \left(\varkappa^T_{t+1} + \varkappa^T_{t-1} - \lambda \varkappa_t\right) u^g_{k,\,t} + u^g_{k,\,T} \varkappa^T_{T-1},$$

which gives (4.6) due to (4.3). Using this observation, we can evaluate

$$
\begin{aligned}
\left(C^T f^T, g\right)_{\mathcal{F}^T} &= \left(u^f(T), u^g(T)\right)_{\mathcal{F}^T} = \sum_{k=1}^{T} u^f_{k,\,T}\overline{u^g_{k\,T}} \\
&= \sum_{k=1}^{T} \overline{y_k(\lambda)u^g_{k,\,T}} = \sum_{k=1}^{T} \overline{y_k(\lambda)} \overline{\sum_{t=0}^{T-1}\left(u^g_{k,\,t+1} + u^g_{k,\,t-1} - \lambda u^g_{k,\,t}\right)\varkappa^T_t} \\
&= \sum_{t=0}^{T-1} \overline{\varkappa^T_t(\lambda)} \left(\sum_{k=1}^{T} \left(a_k \overline{u^g_{k+1,\,t}}\overline{y_k} + a_{k-1}\overline{u^g_{k-1,\,t}}\overline{y_k} + b_k \overline{u^g_{k,\,t}}\overline{y_k} - \overline{\lambda}\overline{u^g_{k,\,t}}\overline{y_k} \right)\right) \\
&= \sum_{t=0}^{T-1} \overline{\varkappa^T_t(\lambda)} \left(\sum_{k=1}^{T} \left(\overline{u^g_{k,\,t}}(a_k\overline{y_{k+1}} + a_{k-1}\overline{y_{k-1}} + b_k\overline{y_k} - \overline{\lambda y_k}\right) + \overline{u^g_{0,\,t}}\overline{y_1} \right. \\
&\qquad \left. + a_T u^g_{T+1,\,t}\overline{y_T} - \overline{u^g_{1,\,t}}\overline{y_0} - a_T\overline{u^g_{T,\,t}}\overline{y_{T+1}} \right) = \sum_{t=0}^{T-1} \overline{\varkappa^T_t(\lambda)}\left(\beta\overline{g_t} - \alpha\overline{(R^T g)}_t\right) \\
&= \left(\overline{\varkappa^T(\lambda)}, \left[\beta g - \alpha\left(R^T g\right)\right]\right)_{\mathcal{F}^T} = \left(\left[\beta\overline{\varkappa^T(\lambda)} - \alpha\left(\left(R^T\right)^* \overline{\varkappa^T(\lambda)}\right)\right], g\right)_{\mathcal{F}^T}.
\end{aligned}
$$

Which completes the proof due to the arbitrariness of g. $\square$

We consider two special solutions to (4.1): the first one $\varphi(\lambda)$ corresponds to the choice $\alpha = 0$, $\beta = 1$, the second one, $\xi(\lambda)$, corresponds to Cauchy data $\alpha = -1$, $\beta = 0$.

It is well-known fact [1,23] that the questions on the uniqueness of the solution to a moment problem are related to the deficiency indices of the operator A. Here we provide well-known results on discrete version of Weyl limit point-circle theory which answers the question on the index of A that will be subsequently used:

Proposition 10. *The limit circle case does hold (i.e. operator A is of indices $(1,1)$ if and only if one the following occurs:*

1) $\varphi(\lambda), \xi(\lambda) \in l^2$ *for some* $\lambda \in \mathbb{R}$*,*
2) $\varphi(\lambda), \varphi'(\lambda) \in l^2$ *for some* $\lambda \in \mathbb{R}$*,*
3) $\xi(\lambda), \xi'(\lambda) \in l^2$ *for some* $\lambda \in \mathbb{R}$*.*

4.2. Hamburger moment problem

Let in (4.1) $\alpha = 0$, $\beta = 1$, then the special control problem has a form:

$$W^T f^T_{01}(\lambda) = \overline{y}^T(\lambda) = \overline{(\varphi_1(\lambda), \dots, \varphi_T(\lambda))}. \tag{4.7}$$

The control f^T_{01} is a unique solution to (see (4.5) (4.4)) the equation

$$C^T f^T_{01}(\lambda) = \overline{\begin{pmatrix} \mathcal{T}_T(\lambda) \\ \mathcal{T}_{T-1}(\lambda) \\ \dots \\ \mathcal{T}_1(\lambda) \end{pmatrix}}. \tag{4.8}$$

Differentiating (4.7), (4.8) with respect to λ, we see that

$$W^T \left(f^T_{01}(\lambda)\right)' = \overline{\left(y^T\right)'(\lambda)} = \overline{(\varphi'_1(\lambda), \dots, \varphi'_T(\lambda))},$$

and the control $\left(f_{01}^T(\lambda)\right)'$ is a solution to

$$C^T\left(f^T(\lambda)\right)' = \overline{\begin{pmatrix} \mathcal{T}_T'(\lambda) \\ \mathcal{T}_{T-1}'(\lambda) \\ \dots \\ \mathcal{T}_1'(\lambda) \end{pmatrix}}.$$

Evaluating the quadratic form (2.19) we have that

$$\left(C^T f^T(\lambda), f^T(\lambda)\right)_{\mathcal{F}^T} = \left(\overline{(\varphi_1(\lambda),\dots,\varphi_T(\lambda))}, \overline{(\varphi_1(\lambda),\dots,\varphi_T(\lambda))}\right)_{\mathcal{H}^T} = \sum_1^T |\varphi_k(\lambda)|^2.$$

And similarly for the derivatives:

$$\left(C^T\left(f^T(\lambda)\right)', \left(f^T(\lambda)\right)'\right)_{\mathcal{F}^T} = \sum_1^T |\varphi_k'(\lambda)|^2.$$

It is known that

$$\mathcal{T}_{2n-1}(0) = (-1)^{n-1}, \quad \mathcal{T}_{2n}(0) = 0, \quad n \geqslant 1, \tag{4.9}$$
$$\mathcal{T}_{2n-1}'(0) = 0, \quad \mathcal{T}_{2n}'(0) = (-1)^{n-1}n, \quad n \geqslant 1. \tag{4.10}$$

We define the vectors

$$\Gamma_T := \begin{pmatrix} \mathcal{T}_T(0) \\ \mathcal{T}_{T-1}(0) \\ \dots \\ \mathcal{T}_1(0) \end{pmatrix}, \quad \Omega_T = \begin{pmatrix} \mathcal{T}_T'(0) \\ \mathcal{T}_{T-1}'(0) \\ \dots \\ \mathcal{T}_1'(0) \end{pmatrix}. \tag{4.11}$$

Using the above arguments we can state that

$$\sum_1^T |\varphi_k(0)|^2 = \left(\left(C^T\right)^{-1}\Gamma_T, \Gamma_T\right)_{\mathcal{F}^T}, \tag{4.12}$$
$$\sum_1^T |\varphi_k'(0)|^2 = \left(\left(C^T\right)^{-1}\Delta_T, \Delta_T\right)_{\mathcal{F}^T}.$$

Now we can use 2) from Proposition 10, and formulate the following

Proposition 11. *The Hamburger moment problem is indeterminate if and only if*

$$\lim_{T\to\infty}\left(\left(C^T\right)^{-1}\Gamma_T, \Gamma_T\right)_{\mathcal{F}^T} < +\infty, \quad \lim_{T\to\infty}\left(\left(C^T\right)^{-1}\Delta_T, \Delta_T\right)_{\mathcal{F}^T} < +\infty,$$

where Γ_T and Ω_T are defined by (4.11), (4.9), (4.10).

4.3. Stieltjes moment problem

It is known [1] that the Jacobi matrix in this case admits the special structure:

$$b_i = \frac{1}{m_i}\left(\frac{1}{l_{i-1}} + \frac{1}{l_i}\right), \quad i = 2,3,\dots, \quad b_1 = \frac{1}{m_1 l_1}, \tag{4.13}$$
$$a_i = \frac{1}{l_i\sqrt{m_i m_{i+1}}}, \quad i = 1,2,\dots$$

where l_i, m_i are positive and are interpreted as lengths of intervals and masses at the points x_j. The string is defined by the density $dM = \sum_{k=1}^{\infty} m_k \delta(x - x_k)$, where $0 = x_0 < x_1 < x_2 < \ldots < x_{N-1} < \ldots$, $l_i = x_i - x_{i-1}$, $i = 1, \ldots$. The inverse dynamic problem for the dynamical system corresponding to a finite Krein-Stieltjes string was studied in [18]. It is straightforward to check (see also [1]) that the following relations hold:

$$\varphi_n(0) = (-1)^n \sqrt{m_n}, \quad n = 1, 2, \ldots, \tag{4.14}$$

$$\xi_n(0) = (-1)^{n-1} \left(\sum_{j=1}^{n-1} l_j \right) \sqrt{m_n} \quad n = 1, 2, \ldots. \tag{4.15}$$

We define the mass and length of a segment of a string:

$$M_K = \sum_{k=1}^{K} m_k; \quad L_K = \sum_{k=1}^{K} l_k,$$

when $K = \infty$ above expressions correspond to the mass and the length of the whole string. Then formulas (4.14), (4.15) imply that

$$M_K = \sum_{n=1}^{K} \varphi_n^2(0), \quad L_K = -\frac{\xi_{K+1}(0)}{\varphi_{K+1}(0)}.$$

In [1,23] the following statement was proved:

Proposition 12. *The Stieltjes moment problem is indeterminate if and only if both length and mass of a string is finite:* M_∞, $L_\infty < +\infty$.

Note that the necessity of conditions M_∞, $L_\infty < +\infty$ is just a simple consequence of formulas for M_K and L_K.

For the mass of the segment of a string or of the whole string we have an expression (4.12). Now we obtain similar formula for the length. Denote by $h^T \in \mathcal{F}^T$ the (unique) control which drives the system (2.14) to the special state

$$W^T h^T = (0, \ldots, 0, 1) \in \mathcal{H}^T.$$

For arbitrary $q \in \mathcal{F}^T$ we have (see the representation (2.18)) that

$$\left(C^T h^T, q\right)_{\mathcal{F}^T} = \left(W^T h^T, W^T q\right)_{\mathcal{H}^T} = \prod_{i=0}^{T-1} a_i q_0.$$

The above relation implies that h^T can be found as a unique solution to Krein-type equation:

$$C^T h^T = \begin{pmatrix} \prod_{i=0}^{T-1} a_i \\ 0 \\ \ldots \\ 0 \end{pmatrix}. \tag{4.16}$$

Taking a control $f^T \in \mathcal{F}^T$ such that $W^T f^T = (\varphi_1(0), \ldots, \varphi_T(0))$ we have (see Theorem 5 and (4.16)) that

$$\varphi_T(0)=\left(\begin{pmatrix}0\\0\\\cdot\\1\end{pmatrix},\begin{pmatrix}\varphi_1(0)\\\varphi_2(0)\\\cdot\\\varphi_T(0)\end{pmatrix}\right)_{\mathcal{H}^T}=\left(W^Th^T,W^Tf^T\right)_{\mathcal{H}^T}$$
$$=\left(C^Th^T,f^T\right)_{\mathcal{F}^T}=\left(\begin{pmatrix}\prod_{i=0}^{T-1}a_i\\0\\\dots\\0\end{pmatrix},\left(C^T\right)^{-1}\begin{pmatrix}\mathcal{T}_T(0)\\\mathcal{T}_{T-1}(0)\\\dots\\\mathcal{T}_1(0)\end{pmatrix}\right)_{\mathcal{F}^T}.$$

Similarly, denote by g^T the control for which $W^Tg^T=(\xi_1(0),\dots,\xi_T(0))$. Then, using equations from Theorem 5 and (4.16) we have that

$$\xi_T(0)=\left(\begin{pmatrix}0\\0\\\cdot\\1\end{pmatrix},\begin{pmatrix}\xi_1(0)\\\xi_2(0)\\\cdot\\\xi_T(0)\end{pmatrix}\right)_{\mathcal{H}^T}=\left(W^Th^T,W^Tg^T\right)_{\mathcal{H}^T}$$
$$=\left(C^Th^T,g^T\right)_{\mathcal{F}^T}=-\left(\begin{pmatrix}\prod_{i=0}^{T-1}a_i\\0\\\dots\\0\end{pmatrix},\left(C^T\right)^{-1}\left(R^T\right)^*\begin{pmatrix}\mathcal{T}_T(0)\\\mathcal{T}_{T-1}(0)\\\dots\\\mathcal{T}_1(0)\end{pmatrix}\right)_{\mathcal{F}^T}.$$

The above arguments lead to the following expression for the length of the segment of the string:

$$L_K=-\frac{\xi_{K+1}(0)}{\varphi_{K+1}(0)}=\frac{\left(\left(C^{K+1}\right)^{-1}\left(R^{K+1}\right)^*\Gamma_{K+1},e_1\right)}{\left(\left(C^{K+1}\right)^{-1}\Gamma_{K+1},e_1\right)},$$

where we denoted $e_1=(1,0,\dots,0)\in\mathcal{F}^T$. The above arguments lead to the following statement:

Proposition 13. *The Stieltjes moment problem is indeterminate if and only if the following relations hold:*

$$M_\infty=\lim_{T\to\infty}\left(\left(C^T\right)^{-1}\Gamma_T,\Gamma_T\right)_{\mathcal{F}^T}<+\infty,$$
$$L_\infty=\lim_{K\to\infty}\frac{\left(\left(C^K\right)^{-1}\left(R^K\right)^*\Gamma_K,e_1\right)}{\left(\left(C^K\right)^{-1}\Gamma_K,e_1\right)}<+\infty.$$

4.4. Hausdorff moment problem

It is a special case of a Stieltjes moment problem. The necessary and sufficient conditions for a set of numbers to be moments of a measure supported on $(0,1)$ are obtained in [11,12], see also [22]. They are equivalent to inequality (3.23) holds for all $T\in\mathbb{N}$ since we get the limit measure as a limit of measures supported on $(0,1)$.

Proposition 14. *If the Halkel matrices S_0^T, S_1^T satisfy (3.23) for all $T\geqslant 2$, then there exists only one measure supported on $(0,1)$ which satisfies (1.1). In other words, the Hausdorff moment problem is determinate.*

Proof. Let us assume that the opposite is true and the Hausdorff moment problem is indeterminate, in this case by Proposition 13 the length and the mass of string determined by the matrix A constructed from moments, should be finite. Then for any fixed $T\in\mathbb{N}$ we have on the one hand that

$$\operatorname{tr} A_T = \sum_{n=1}^{T} \lambda_n^T. \tag{4.17}$$

On the other hand (see (4.13)),

$$\operatorname{tr} A_T = \frac{1}{m_1 l_1} + \sum_{n=2}^{T} \frac{1}{m_n}\left(\frac{1}{l_{n-1}} + \frac{1}{l_n}\right). \tag{4.18}$$

Since l_i, m_i are positive and by our assumption $\sum_{i=1}^{\infty} m_i < +\infty$, $\sum_{i=1}^{\infty} l_i < +\infty$, it immediately follows from (4.18) that $\operatorname{tr} A^T > T^2$ for sufficiently large T, and thus from (4.17) we can see that for such T the eigenvalues λ_k^T cannot be bounded by one. Which gives a contradiction. □

This statement (using a different approach) was proved in [13], see also [10].

Let $s \in \mathbb{R}^{\mathbb{N}}$ be a sequence, $s = (s_0, s_1, \ldots)$. One defines the difference operator $\Delta : \mathbb{R}^{\mathbb{N}} \mapsto \mathbb{R}^{\mathbb{N}}$ by the rule

$$(\Delta s)_n = s_{n+1} - s_n, \quad n = 0, 1, \ldots. \tag{4.19}$$

Hausdorff in [11,12] proved the following

Theorem 6. *A sequence $s = (s_0, s_1, \ldots) \in \mathbb{R}^{\mathbb{N}}$ is a moment sequence of a measure supported on $(0, 1)$ if and only if it is completely monotonic, i.e., its difference sequences satisfy the equalities*

$$(-1)^k (\Delta^k s)_n \geqslant 0, \quad \textit{for all } k, n \geqslant 0. \tag{4.20}$$

We will show that the Hausdorff condition (4.20) is a consequence of condition (3.23) holding far all $N \in \mathbb{N}$. On the other hand, the property (4.20) for finite k, n does not imply (3.23). The following two propositions confirm that.

Proposition 15. *The condition (3.23) implies the inequality (4.20) holds for k, n such that $k + n \leqslant 2N - 1$.*

Proof. According to definition (4.19)

$$\left(\Delta^2 s\right)_n = \left(\Delta\left(\Delta s\right)\right)_n = \left(\Delta s\right)_{n+1} - \left(\Delta s\right)_n = s_{n+2} - 2s_{n-1} + s_n,$$

continuing calculations yields

$$\left(\Delta^k s\right)_n = \sum_{i=0}^{k} (-1)^i C_k^i s_{n+k-i}. \tag{4.21}$$

For a given sequence $g = (g_0, g_1, \ldots) \in \mathbb{R}^{\mathbb{N}}$ by $(g_{i+j-2})_{1 \le i,j \le n+1}$ we denote the Hankel matrix, and the Hankel transform get map $g \in \mathbb{R}^{\mathbb{N}}$ to the sequence

$$h_n = \det(g_{i+j-2})_{1 \le i,j \le n+1}, \quad n = 0, 1, \ldots$$

The binomial transform get map a given sequence $g = (g_0, g_1, \ldots) \in \mathbb{R}^{\mathbb{N}}$ to the sequence

$$c_n = \sum_{k=0}^{n} C_n^k g_k, \quad n = 0, 1, \ldots \tag{4.22}$$

It is known [20] that the Hankel transform is invariant under the binomial transform, that is:

$$h_n = \det(g_{i+j-2})_{1\le i,j\le n+1} = \det(c_{i+j-2})_{1\le i,j\le n+1}. \tag{4.23}$$

Note that (4.23) remains valid if one replaces binomial transform (4.22) by the signed binomial transform introduced by the rule: for $g = (g_0, g_1, \dots) \in \mathbb{R}^{\mathbb{N}}$ one has

$$\widetilde{c}_n = \sum_{k=0}^{n} (-1)^k C_n^k g_k, \quad n = 0, 1, \dots$$

This fact and (4.21) imply that for Hankel matrices $J_N S_0^N J_N = (s_{i+j-2})_{1\le i,j\le N}$, $J_N S_1^N J_N = (s_{i+j-1})_{1\le i,j\le N}$ and their difference the following relations hold:

$$\det(s_{i+j+m-2})_{1\le i,j\le n+1} = \det\left(\left(\Delta^{i+j-2}s\right)_m\right)_{1\le i,j\le n+1}, \quad m = 0, 1, \tag{4.24}$$
$$\det(s_{i+j-2} - s_{i+j-1})_{1\le i,j\le n+1} = \det\left(\left(\Delta^{i+j-2}s\right)_0 - \left(\Delta^{i+j-2}s\right)_1\right)_{1\le i,j\le n+1},$$

where $n = 0, 1, \dots, N-1$. Then the Sylvester criterion of positivity of a matrix and the condition (3.23) imply that

$$\left(\left(\Delta^{i+j-2}s\right)_0\right)_{1\le i,j\le N} \geqslant \left(\left(\Delta^{i+j-2}s\right)_1\right)_{1\le i,j\le N} > 0.$$

The above inequalities imply that diagonal elements of matrices satisfy the same inequalities

$$\left(\Delta^{2i}s\right)_0 \geqslant \left(\Delta^{2i}s\right)_1 > 0, \quad i = 0, 1, \dots, N-1. \tag{4.25}$$

Using the definition of Δ (4.19) from (4.25) we derive that

$$\left(\Delta^{2i+1}s\right)_0 \leqslant 0, \quad i = 0, 1, \dots, N-1. \tag{4.26}$$

Note that inequalities in (4.25), (4.26) are only part of what we need to show in (4.20). To prove the other part we note that condition (3.23) implies that

$$S_{2m}^{N-m} \geqslant S_{1+2m}^{N-m} > 0, \quad m = 0, 1, \dots, N-1. \tag{4.27}$$

Like (4.25) was a consequence of (3.23), the inequality

$$\left(\Delta^{2i}s\right)_{2m} \geqslant \left(\Delta^{2i}s\right)_{2m+1} > 0, \quad i = 0, 1, \dots, N-m-1 \tag{4.28}$$

follows from (4.27). From (4.28) one obtains that

$$\left(\Delta^{2i+1}s\right)_{2m} \leqslant 0, \quad i = 0, 1, \dots, N-m-1. \tag{4.29}$$

The inequalities (4.28), (4.29) is exactly what we need to show in (4.20). □

Proposition 16. *The inequality (4.20) holding for all $k, n:\ k + n \leqslant 2N - 1$ does not imply (3.23).*

Proof. Consider the point mass measure concentrated at $\lambda = 1/2$: i.e. $d\rho(\lambda) = \delta(\lambda - 1/2)$. In this case all moments (1.1) are given by $s_k = (1/2)^k$. Using (4.19) we see that $\left(\Delta^k s\right)_n = (-1)^k (1/2)^{k+n}$, which agrees with (4.20).

Now we construct counterexample which works even for $N = 2$. To do so we slightly change the moment s_2: we take $s_0 = 1$, $s_1 = 1/2$, $s_2 = 1/4 + \varepsilon$, $s_3 = 1/8$. If ε is small enough then (4.20) remains valid, but (3.23) fails: indeed, $\det(s_{i+j-2})_{1\le i,j\le 2} = \varepsilon < 0$ if $\varepsilon < 0$. Therefore (3.23) doesn't follow from (4.20). □

Note that the above counterexample does not work in the case of infinite matrices: in this case the restriction on indices $k+n \leqslant 2N-1$ disappears and one should consider all k, n. Then according to (4.21)

$$\left(\Delta^k s\right)_n = \sum_{i=0}^{k}(-1)^i C_k^i s_{n+k-i} = (-1)^k (1/2)^{k+n} + (-1)^{n+k-2}\varepsilon C_{n+k-2}^k,$$

and we see that the second term in the right hand side of the above expression dominates if k, n are large enough. Therefore for such a moment sequence the condition (4.20) does not hold.

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