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# On the Kirchhoff type Choquard problem with Hardy-Littlewood-Sobolev critical exponent

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ABSTRACT

In this paper, we study the following Kirchhoff type Choquard problem:

$$-\left(a+b\int\limits_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x\right)\Delta u+u=\left(\int\limits_{\mathbb{R}^N}\frac{\beta F(u(y))+|u(y)|^{2^*_\mu}}{|x-y|^\mu}\mathrm{d}y\right)\times\left(\beta f(u)+2^*_\mu|u|^{2^*_\mu-2}u\right)\quad\text{in}\quad\mathbb{R}^N,\tag{0.1}$$

where $N\geq 3$, $a$, $b>0$ are constants, $\beta>0$ is a parameter. When $\mu\in(0,4]$, under suitable conditions on $b$, $\beta$ and $f$, we prove that (0.1) has a ground state solution. When $\mu>4$, we also obtain some related existence results.


## 1. Introduction

In this paper, we study the following Kirchhoff type Choquard problem:

$$-\left(a+b\int\limits_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x\right)\Delta u+u=\left(\int\limits_{\mathbb{R}^N}\frac{\beta F(u(y))+|u(y)|^{2^*_\mu}}{|x-y|^\mu}\mathrm{d}y\right)\times\left(\beta f(u)+2^*_\mu|u|^{2^*_\mu-2}u\right)\quad\text{in}\quad\mathbb{R}^N,\tag{1.1}$$

where $N\geq 3$, $\mu\in(0,N)$, $2^*_\mu=\frac{2N-\mu}{N-2}$ is the upper critical exponent of the Hardy-Littlewood-Sobolev inequality, $F$ is the primitive function of $f$.

The Choquard equation has several physical origins. For example, it can be used to describe the quantum mechanics of a polaron at rest. Also, it is known as the stationary Hartree equation, or the Schrödinger-

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Newton equation. In recent years, much interest has grown on Choquard equations. In [14], Lieb first proved the existence and uniqueness of ground state solutions to the following equation:

$$-\Delta u+u=\left(\int\limits_{\mathbb{R}^3}\frac{|u(y)|^p}{|x-y|^\mu}\mathrm{d}y\right)|u|^{p-2}u \quad \text{in} \quad \mathbb{R}^3.$$

In [15], Lions considered the existence of multiple solutions. In [16], the authors used variational methods to obtain the existence of ground state solutions to a more general Choquard equation. For other related results, we refer the readers to [1,6,8,18] for the subcritical case and [3,7,17] for the critical case. However, to our best knowledge, there are few results about the Kirchhoff type Choquard equation. The Kirchhoff type problem occurs in various branches of mathematical physics. It can be used to model suspension bridges. Also, it appears in other fields like biological systems, such as population density. Because of the presence of the nonlocal term, the Kirchhoff problem is not a pointwise identity, which causes additional mathematical difficulties. There are many papers focusing on the Kirchhoff type problem. See [5,10–12,21,23] for the subcritical case and [4,19,20,26–28] for the critical case. Recently, in [24], the authors studied a Dirichlet problem of the Kirchhoff type Choquard equation.

In this paper, we consider the Kirchhoff type Choquard problem in the whole space. By using variational methods, we obtain the existence of ground state solutions of (1.1) for the case $\mu\le 4$. Also, we consider related problems of (1.1) for the case $\mu>4$. To solve the problem, we assume the following conditions:

$(f_1)$ $f\in C(\mathbb{R},\mathbb{R})$ and $\lim_{u\to 0}\frac{f(u)}{u^{\frac{N-\mu}{N}}}=\lim_{u\to\infty}\frac{f(u)}{|u|^{2^*_\mu-2}u}=0$.

$(f_2)$ There exists $\xi>0$ such that $F(\xi)=\int_0^\xi f(s)\mathrm{d}s>0$.

$(f_2')$ $F(u)=\int_0^u f(s)\mathrm{d}s\ge 0$ for $u\in\mathbb{R}$. Moreover, there exists $\xi>0$ such that $F(\xi)=\int_0^\xi f(s)\mathrm{d}s>0$.

$(f_3)$ $\lim_{u\to+\infty}\frac{F(u)}{|u|^{\frac{2N-\mu}{N}}}=+\infty$ when $N\ge 5$, $\lim_{u\to+\infty}\frac{F(u)}{|u|^{\frac{8-\mu}{4}}|\ln u|^{\frac{8-\mu}{8}}}=+\infty$ when $N=4$, $\lim_{u\to+\infty}\frac{F(u)}{|u|^{\frac{2(6-\mu)}{3}}}=+\infty$ when $N=3$, where $F(u)=\int_0^u f(s)\mathrm{d}s$.

We first consider the case $\mu\in(0,4)$. In this case, we cannot derive the convergence of the PS sequence easily. To solve the problem, we have to estimate the PS sequence carefully. Our results are as follows.

**Theorem 1.1.** *Let $\mu\in(0,4)$ and $a$, $b>0$. Then*

(i) *there exists a large $\beta_0>0$ such that problem (1.1) has a ground state solution for $\beta>\beta_0$ if $(f_1)$ and $(f_2)$ hold;*

(ii) *for any $\beta>0$, problem (1.1) has a ground state solution if $(f_1)$ and $(f_3)$ hold.*

**Remark 1.1.** The condition $(f_2)$ or $(f_3)$ is used to estimate the energy level, which is crucial for the proof of the relative compactness of the PS sequence. In [3], the authors assumed the condition $F(u)=\int_0^u f(s)\mathrm{d}s\ge 0$ for $u\in\mathbb{R}$, which is essential when using the monotonicity trick originally due to [22]. In this paper, we use a direct method and remove the restrict condition.

We next consider the case $\mu=4$. In this case, it is hard to prove the geometric structure of the functional and the convergence of the PS sequence. Now we state our results. Define the best constant:

$$S_\mu:=\inf_{D^{1,2}(\mathbb{R}^N)\setminus\{0\}}\frac{\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x}{\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(y)|^{2^*_\mu}|u(x)|^{2^*_\mu}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2^*_\mu}}}.$$

**Theorem 1.2.** *Let $\mu = 4$, $a > 0$, $b \in \left(0, \frac{2}{S_\mu^2}\right)$ and $\beta > 0$. Assume that $(f_1)$ and $(f_3)$ hold. Then problem (1.1) has a ground state solution.*

**Theorem 1.3.** *Let $\mu = 4$, $a > 0$ and $b > \frac{2}{S_\mu^2}$. Assume that $(f_1)$ and $(f_2)$ hold. Then there exists a large $\beta_1 > 0$ such that problem (1.1) has a ground state solution for $\beta > \beta_1$.*

When $\mu > 4$, we cannot prove the boundedness of the PS sequence. Instead, we turn to consider the following problem:

$$-\left(a + b\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x\right)\Delta u + u = \lambda\left(\int_{\mathbb{R}^N} \frac{\beta F(u(y)) + |u(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}y\right) \times \left(\beta f(u) + 2_\mu^*|u|^{2_\mu^*-2}u\right) \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

where $\lambda \in [\frac{1}{2}, 1]$. Let

$$a_1 = \frac{(2_\mu^*)^{\frac{N-2}{\mu-4}}}{S_\mu^{\frac{2N-\mu}{\mu-4}}}\frac{\mu-4}{N-2}\left[\frac{N-\mu+2}{b(N-2)}\right]^{\frac{N-\mu+2}{\mu-4}},$$
$$b_1 = \frac{(2_\mu^*)^{\frac{N-2}{N-\mu+2}}}{S_\mu^{\frac{2N-\mu}{N-\mu+2}}}\frac{N-\mu+2}{N-2}\left[\frac{\mu-4}{a(N-2)}\right]^{\frac{\mu-4}{N-\mu+2}}. \tag{1.3}$$

**Theorem 1.4.** *Let $\mu > 4$. Assume that $(f_1)$ and $(f_2')$ hold. Then for $a > a_1$ (or $b > b_1$), there exists a large $\beta_2 > 0$ such that for $\beta > \beta_2$, problem (1.2) has at least a nontrivial solution $u_\lambda$ for almost every $\lambda \in [\frac{1}{2}, 1]$. Moreover, there exists a sequence $\{\lambda_n\} \subset [\frac{1}{2}, 1]$ such that $\lambda_n \uparrow 1$ as $n \uparrow \infty$ and $u_{\lambda_n}$ satisfies one of the following:*

(i) *$u_{\lambda_n} \to \infty$ in $H_r^1(\mathbb{R}^N)$ as $n \to \infty$;*
(ii) *$u_{\lambda_n}$ is bounded in $H_r^1(\mathbb{R}^N)$ and consequently, problem (1.1) has a nontrivial solution.*

When $\mu > 4$ and $\mathbb{R}^N$ is replaced by a bounded domain, problem (1.1) becomes the following problem:

$$-\left(a + b\int_\Omega |\nabla u|^2 \mathrm{d}x\right)\Delta u + u = \left(\int_\Omega \frac{\beta F(u(y)) + |u(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}y\right) \times \left(\beta f(u) + 2_\mu^*|u|^{2_\mu^*-2}u\right) \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega, \tag{1.4}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain with smooth boundary. In this case, we get the following results.

**Theorem 1.5.** *Let $\mu > 4$. Then*

(i) *for $a > a_1$ (or $b > b_1$), there exists a large $\beta_3 > 0$ such that for $\beta > \beta_3$, problem (1.4) has at least two nontrivial solutions if $(f_1)$ and $(f_2')$ hold;*
(ii) *for any $\beta > 0$, there exists a large $a_2 > 0$ (or a large $b_2 > 0$) such that for $a > a_2$ (or $b > b_2$), problem (1.4) has no nontrivial solutions if $(f_1)$ holds.*

The outline of this paper is as follows: in Section 2, we give some important lemmas; in Section 3, we prove Theorems 1.1; in Section 4, we prove Theorems 1.2-1.3; in Section 5, we prove Theorems 1.4-1.5.

**Notations:**

- $H^1(\mathbb{R}^N)$ denotes the Hilbert space with the norm $\|u\|^2 := \int_{\mathbb{R}^N}(a|\nabla u|^2 + |u|^2)\mathrm{d}x$. $D^{1,2}(\mathbb{R}^N) := \{u \in L^{2^*}(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\}$ denotes the Sobolev space with the norm $\|u\|^2_{D^{1,2}} := \int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x$.
- $\|u\|_s := \big(\int_{\mathbb{R}^N}|u|^s\mathrm{d}x\big)^{\frac{1}{s}}$, $2 \le s < \infty$.
- $C$ denotes a universal positive constant (possibly different).

## 2. Preliminary lemmas

We first introduce the following Hardy-Littlewood-Sobolev inequality.

**Lemma 2.1** *([13]). Let $s$, $t > 1$ and $\mu \in (0, N)$ with $\frac{1}{s} + \frac{1}{t} + \frac{\mu}{N} = 2$. Let $f \in L^s(\mathbb{R}^N)$ and $h \in L^t(\mathbb{R}^N)$. Then there exists a sharp constant $C(N, s, t, \mu)$ independent of $f$, $h$, such that*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{f(x)h(y)}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y \le C(N, s, t, \mu)\|f\|_s\|h\|_t. \tag{2.1}$$

*In particular, if $s = t = \frac{2N}{2N-\mu}$, then*

$$C(N, s, t, \mu) = C(N, \mu) = \pi^{\frac{\mu}{2}}\frac{\Gamma(\frac{N}{2}-\frac{\mu}{2})}{\Gamma(N-\frac{\mu}{2})}\left\{\frac{\Gamma(\frac{N}{2})}{\Gamma(N)}\right\}^{-1+\frac{\mu}{N}}.$$

Let $\frac{2N-\mu}{N} \le t \le 2^*_\mu$. By Lemma 2.1, for any $u \in H^1(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(y)|^t|u(x)|^t}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y < +\infty.$$

Also, for any $u \in D^{1,2}(\mathbb{R}^N)$,

$$\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(y)|^{2^*_\mu}|u(x)|^{2^*_\mu}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2^*_\mu}} \le [C(N, \mu)]^{\frac{1}{2^*_\mu}}\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x.$$

Define the best constant:

$$S_\mu := \inf_{D^{1,2}(\mathbb{R}^N)\setminus\{0\}}\frac{\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x}{\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(y)|^{2^*_\mu}|u(x)|^{2^*_\mu}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y\Big)^{\frac{1}{2^*_\mu}}}. \tag{2.2}$$

**Lemma 2.2** *([7]). Let $S = \inf_{u\in D^{1,2}(\mathbb{R}^N)\setminus\{0\}}\frac{\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x}{(\int_{\mathbb{R}^N}|u|^{2^*}\mathrm{d}x)^{\frac{2}{2^*}}}$. Then $S_\mu = \frac{S}{[C(N,\mu)]^{\frac{N-2}{2N-\mu}}}$ is achieved if and only if*

$$u(x) = C\left(\frac{b}{b^2 + |x-a|^2}\right)^{\frac{N-2}{2}}, \tag{2.3}$$

*where $a \in \mathbb{R}^N$, $b \in \mathbb{R}$, $C > 0$ is a constant.*

By [25], we know $S$ is attained by the function $U(x) = \frac{[N(N-2)]^{\frac{N-2}{4}}}{(1+|x|^2)^{\frac{N-2}{2}}}$. Then by Lemma 2.2, we get $U(x)$ is a minimizer for $S_\mu$. Let $\varepsilon > 0$ and $r_0 > 0$. Define $u_\varepsilon(x) = \psi(x)U_\varepsilon(x)$, where $U_\varepsilon(x) = \varepsilon^{-\frac{N-2}{2}}U(\frac{x}{\varepsilon})$, $\psi \in C_0^\infty(B_{2r_0}(0))$ such that $\psi(x) = 1$ on $B_{r_0}(0)$ and $0 \le \psi(x) \le 1$ on $\mathbb{R}^N$.

**Lemma 2.3** *([25]). For $\varepsilon > 0$ small,*

$$\int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 \mathrm{d}x = S^{\frac{N}{2}} + O(\varepsilon^{N-2}), \quad \int_{\mathbb{R}^N} |u_\varepsilon|^{2^*} \mathrm{d}x = S^{\frac{N}{2}} + O(\varepsilon^N),$$

$$\int_{\mathbb{R}^N} |u_\varepsilon|^2 \mathrm{d}x = \begin{cases} O(\varepsilon^2), & N \ge 5, \\ O(\varepsilon^2 |\ln \varepsilon|), & N = 4, \\ O(\varepsilon), & N = 3. \end{cases} \tag{2.4}$$

Similar to Lemma 2.4 in [24], we can use Lemmas 2.1-2.2 to obtain the following result. We omit the proof here.

**Lemma 2.4.**

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_\varepsilon(y)|^{2^*_\mu}|u_\varepsilon(x)|^{2^*_\mu}}{|x-y|^\mu} \mathrm{d}x\mathrm{d}y \ge [C(N,\mu)]^{\frac{N}{2}} S_\mu^{\frac{2N-\mu}{2}} - O(\varepsilon^{\frac{2N-\mu}{2}}). \tag{2.5}$$

**Lemma 2.5.** *Let $\beta > 0$. Assume that $(f_1)$ and $(f_3)$ hold. Then for any $L > 0$, there exists $\varepsilon_L > 0$ such that for $\varepsilon \in (0, \varepsilon_L)$,*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\beta^2 F(u_\varepsilon(y))F(u_\varepsilon(x)) + 2\beta F(u_\varepsilon(y))|u_\varepsilon(x)|^{2^*_\mu}}{|x-y|^\mu} \mathrm{d}x\mathrm{d}y$$

$$\ge \begin{cases} \frac{C_0 L}{2}\varepsilon^{\frac{2N-\mu}{N}}, & N \ge 5, \\ \frac{C_0 L}{2}\varepsilon^{\frac{8-\mu}{4}}|\ln\varepsilon|^{\frac{8-\mu}{8}}, & N = 4, \\ \frac{C_0 L}{2}\varepsilon^{\frac{6-\mu}{6}}, & N = 3, \end{cases} \tag{2.6}$$

*where $C_0 > 0$ is a constant independent of $L$, $\varepsilon$.*

**Proof.** By $(f_3)$, for any $L > 0$, there exists $R_L > 0$ such that for $u \ge R_L$, $F(u) \ge L|u|^{\frac{2N-\mu}{N}}$ when $N \ge 5$; $F(u) \ge L|u|^{\frac{8-\mu}{4}}|\ln u|^{\frac{8-\mu}{8}}$ when $N = 4$; $F(u) \ge L|u|^{\frac{2(6-\mu)}{3}}$ when $N = 3$. Choose $\varepsilon_0 \in (0,1)$ small such that $\left|\ln(\frac{\sqrt{2}}{\varepsilon})\right| \ge \frac{1}{2}|\ln\varepsilon|$ for $\varepsilon \in (0, \varepsilon_0)$. Let $\varepsilon_L = \min\left\{\varepsilon_0, 1, r_0, \frac{1}{2}\left(\frac{[N(N-2)]^{\frac{N-2}{4}}}{R_L}\right)^{\frac{2}{N-2}}\right\}$ and $\varepsilon \in (0, \varepsilon_L)$. Then $u_\varepsilon(x) \ge \frac{[N(N-2)]^{\frac{N-2}{4}}}{(2\varepsilon)^{\frac{N-2}{2}}} \ge R_L$ for $|x| \le \varepsilon$. Moreover, for $|x| \le \varepsilon$, $F(u_\varepsilon) \ge \frac{[N(N-2)]^{\frac{(N-2)(2N-\mu)}{4N}} L}{(2\varepsilon)^{\frac{(N-2)(2N-\mu)}{2N}}}$ when $N \ge 5$; $F(u_\varepsilon) \ge \frac{8^{\frac{8-\mu}{8}} L}{(2\varepsilon)^{\frac{8-\mu}{4}}}\left(\frac{1}{2}|\ln\varepsilon|\right)^{\frac{8-\mu}{8}}$ when $N = 4$; $F(u_\varepsilon) \ge \frac{3^{\frac{6-\mu}{6}} L}{(2\varepsilon)^{\frac{6-\mu}{3}}}$ when $N = 3$. By a direct calculation,

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\beta^2 F(u_\varepsilon(y))F(u_\varepsilon(x)) + 2\beta F(u_\varepsilon(y))|u_\varepsilon(x)|^{2^*_\mu}}{|x-y|^\mu} \mathrm{d}x\mathrm{d}y \\ &= \int_{B_\varepsilon(0)}\int_{B_\varepsilon(0)} \frac{\beta^2 F(u_\varepsilon(y))F(u_\varepsilon(x)) + 2\beta F(u_\varepsilon(y))|u_\varepsilon(x)|^{2^*_\mu}}{|x-y|^\mu} \mathrm{d}x\mathrm{d}y \end{aligned}$$

$$+ \int\limits_{B_{2r_0}(0)\setminus B_\varepsilon(0)} \int\limits_{B_{2r_0}(0)} \frac{\beta^2 F(u_\varepsilon(y))F(u_\varepsilon(x)) + 2\beta F(u_\varepsilon(y))|u_\varepsilon(x)|^{2^*_\mu}}{|x-y|^\mu} \mathrm{d}x\mathrm{d}y$$
$$+ \int\limits_{B_\varepsilon(0)} \int\limits_{B_{2r_0}(0)\setminus B_\varepsilon(0)} \frac{\beta^2 F(u_\varepsilon(y))F(u_\varepsilon(x)) + 2\beta F(u_\varepsilon(y))|u_\varepsilon(x)|^{2^*_\mu}}{|x-y|^\mu} \mathrm{d}x\mathrm{d}y. \tag{2.7}$$

By $(f_3)$, there exists $R_0 > 1$ such that $F(u) \geq 0$ for $u \geq R_0$. By $(f_1)$, there exists $M_0 > 0$ such that $|F(u)| \leq M_0|u|^{\frac{2N-\mu}{N}}$ for $0 \leq u \leq R_0$. Then $F(u) \geq -M_0|u|^{\frac{2N-\mu}{N}}$ for $u \geq 0$. Moreover, for $u$, $v \geq 0$,

$$F(u)F(v) \geq \min\{-M_0|v|^{\frac{2N-\mu}{N}}F(u), -M_0|u|^{\frac{2N-\mu}{N}}F(v)\}. \tag{2.8}$$

By Lemmas 2.1, 2.3,

$$\begin{aligned}
&\int\limits_{B_{2r_0}(0)\setminus B_\varepsilon(0)} \int\limits_{B_{2r_0}(0)} \frac{2\beta F(u_\varepsilon(y))|u_\varepsilon(x)|^{2^*_\mu}}{|x-y|^\mu} \mathrm{d}x\mathrm{d}y \\
&+ \int\limits_{B_\varepsilon(0)} \int\limits_{B_{2r_0}(0)\setminus B_\varepsilon(0)} \frac{2\beta F(u_\varepsilon(y))|u_\varepsilon(x)|^{2^*_\mu}}{|x-y|^\mu} \mathrm{d}x\mathrm{d}y \\
&\geq \int\limits_{B_{2r_0}(0)\setminus B_\varepsilon(0)} \int\limits_{B_{2r_0}(0)} \frac{-2\beta M_0|u_\varepsilon(y)|^{\frac{2N-\mu}{N}}|u_\varepsilon(x)|^{2^*_\mu}}{|x-y|^\mu} \mathrm{d}x\mathrm{d}y \\
&+ \int\limits_{B_\varepsilon(0)} \int\limits_{B_{2r_0}(0)\setminus B_\varepsilon(0)} \frac{-2\beta M_0|u_\varepsilon(y)|^{\frac{2N-\mu}{N}}|u_\varepsilon(x)|^{2^*_\mu}}{|x-y|^\mu} \mathrm{d}x\mathrm{d}y \\
&\geq -4\beta M_0 C(N,\mu)\||u_\varepsilon|^{\frac{2N-\mu}{N}}\|_{\frac{2N}{2N-\mu}} \||u_\varepsilon|^{2^*_\mu}\|_{\frac{2N}{2N-\mu}} \\
&\geq \begin{cases} -C'\varepsilon^{\frac{2N-\mu}{N}}, & N \geq 5, \\ -C'\varepsilon^{\frac{8-\mu}{4}}|\ln\varepsilon|^{\frac{8-\mu}{8}}, & N = 4, \\ -C'\varepsilon^{\frac{6-\mu}{6}}, & N = 3. \end{cases}
\end{aligned} \tag{2.9}$$

Also, by (2.8) and Lemmas 2.1, 2.3,

$$\begin{aligned}
&\int\limits_{B_{2r_0}(0)\setminus B_\varepsilon(0)} \int\limits_{B_{2r_0}(0)} \frac{\beta^2 F(u_\varepsilon(y))F(u_\varepsilon(x))}{|x-y|^\mu} \mathrm{d}x\mathrm{d}y \\
&+ \int\limits_{B_\varepsilon(0)} \int\limits_{B_{2r_0}(0)\setminus B_\varepsilon(0)} \frac{\beta^2 F(u_\varepsilon(y))F(u_\varepsilon(x))}{|x-y|^\mu} \mathrm{d}x\mathrm{d}y \\
&\geq -2\beta^2 M_0 C(N,\mu)\||u_\varepsilon|^{\frac{2N-\mu}{N}}\|_{\frac{2N}{2N-\mu}} \|F(u_\varepsilon)\|_{\frac{2N}{2N-\mu}} \\
&\geq \begin{cases} -C''\varepsilon^{\frac{2N-\mu}{N}}, & N \geq 5, \\ -C''\varepsilon^{\frac{8-\mu}{4}}|\ln\varepsilon|^{\frac{8-\mu}{8}}, & N = 4, \\ -C''\varepsilon^{\frac{6-\mu}{6}}, & N = 3. \end{cases}
\end{aligned} \tag{2.10}$$

Besides, there exist $C'$, $C''$ and $C_0 > 0$ such that

$$\begin{aligned}
&\int\limits_{B_\varepsilon(0)}\int\limits_{B_\varepsilon(0)} \frac{\beta^2 F(u_\varepsilon(y))F(u_\varepsilon(x))+2\beta F(u_\varepsilon(y))|u_\varepsilon(x)|^{2^*_\mu}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y\\
&\geq \int\limits_{B_\varepsilon(0)}\int\limits_{B_\varepsilon(0)} \frac{C' F(u_\varepsilon(y))|u_\varepsilon(x)|^{2^*_\mu}}{|x|^\mu+|y|^\mu}\mathrm{d}x\mathrm{d}y\\
&\geq \begin{cases} \frac{C'' L\varepsilon^{\frac{(2-N)(2N-\mu)}{2N}+\frac{\mu-2N}{2}}}{2\varepsilon^\mu}\int_{B_\varepsilon(0)}\int_{B_\varepsilon(0)}\mathrm{d}x\mathrm{d}y, & N\geq 5,\\ \frac{C'' L\varepsilon^{\frac{\mu-8}{4}+\frac{\mu-8}{2}}|\ln\varepsilon|^{\frac{8-\mu}{8}}}{2\varepsilon^\mu}\int_{B_\varepsilon(0)}\int_{B_\varepsilon(0)}\mathrm{d}x\mathrm{d}y, & N=4,\\ \frac{C'' L\varepsilon^{\frac{\mu-6}{3}+\frac{\mu-6}{2}}}{2\varepsilon^\mu}\int_{B_\varepsilon(0)}\int_{B_\varepsilon(0)}\mathrm{d}x\mathrm{d}y, & N=3,\end{cases}\\
&= \begin{cases} C_0 L\varepsilon^{\frac{2N-\mu}{N}}, & N\geq 5,\\ C_0 L\varepsilon^{\frac{8-\mu}{4}}|\ln\varepsilon|^{\frac{8-\mu}{8}}, & N=4,\\ C_0 L\varepsilon^{\frac{6-\mu}{6}}, & N=3.\end{cases}
\end{aligned} \tag{2.11}$$

Combining (2.7) and (2.9)-(2.11), we get the result. $\square$

Let $H(u)=\beta F(u)+|u|^{2^*_\mu}$ and $h(u)=\frac{\partial H(u)}{\partial u}$. Define the functional on $H^1_r(\mathbb{R}^N)$ by

$$I(u)=\frac{1}{2}\|u\|^2+\frac{b}{4}\left(\int\limits_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x\right)^2-\frac{1}{2}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{H(u(y))H(u(x))}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y. \tag{2.12}$$

Then $I: H^1_r(\mathbb{R}^N)\mapsto\mathbb{R}$ is of class $C^1$ and critical points of $I$ are solutions of (1.1).

**Lemma 2.6.** *Assume that $(f_1)$ hold. If $u$ is a critical point of $I$, then*

$$\begin{aligned}
&a(N-2)\int\limits_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x+N\int\limits_{\mathbb{R}^N}|u|^2\mathrm{d}x+b(N-2)\left(\int\limits_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x\right)^2\\
&=(2N-\mu)\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{H(u(y))H(u(x))}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y.
\end{aligned} \tag{2.13}$$

*Moreover,*

$$\begin{aligned}
I(u)=&\frac{a(N+2-\mu)}{2(2N-\mu)}\int\limits_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x+\frac{N-\mu}{2(2N-\mu)}\int\limits_{\mathbb{R}^N}|u|^2\mathrm{d}x\\
&+\frac{b(4-\mu)}{4(2N-\mu)}\left(\int\limits_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x\right)^2.
\end{aligned} \tag{2.14}$$

**Proof.** By a standard argument, we can derive the Pohožaev type identity (2.13). We omit the proof here. By (2.13), we get (2.14). $\square$

Similar to Lemma 2.6 in [24], we can get the following results.

**Lemma 2.7.** *Assume that* $(f_1)$ *holds. If* $u_n \rightharpoonup u$ *weakly in* $H_r^1(\mathbb{R}^N)$*, let* $v_n = u_n - u$*, then*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{H(u_n(y))H(u_n(x))}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{H(u(y))H(u(x))}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y$$
$$= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_n(y)|^{2_\mu^*}|v_n(x)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y + o_n(1),$$

*and*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{H(u_n(y))h(u_n(x))u_n(x)}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{H(u(y))h(u(x))u(x)}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y$$
$$= 2_\mu^* \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_n(y)|^{2_\mu^*}|v_n(x)|^{2_\mu^*}}{|x-y|^\mu} + o_n(1).$$

Let $P = \{u \in H_r^1(\mathbb{R}^N) \setminus \{0\} : J(u) = 0\}$, where

$$J(u) = \frac{a(N-2)}{2}\int_{\mathbb{R}^N} |\nabla u|^2\mathrm{d}x + \frac{N}{2}\int_{\mathbb{R}^N} |u|^2\mathrm{d}x + \frac{b(N-2)}{2}\left(\int_{\mathbb{R}^N} |\nabla u|^2\mathrm{d}x\right)^2$$
$$- \frac{2N-\mu}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{H(u(y))H(u(x))}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y. \tag{2.15}$$

When $P \neq \emptyset$, let $p = \inf_{u\in P} I(u)$.

**Lemma 2.8.** *Let* $\mu \in (0, 4]$ *and* $\beta > 0$*. Assume that* $(f_1)$ *holds. If* $P \neq \emptyset$*, then* $p > 0$*.*

**Proof.** By Lemmas 2.1, 2.6 and $(f_1)$, there exists $C > 0$ such that

$$a(N-2)\int_{\mathbb{R}^N} |\nabla u|^2\mathrm{d}x + N\int_{\mathbb{R}^N} |u|^2\mathrm{d}x$$
$$\leq (2N-\mu)C(N,\mu)\|H(u)\|^2_{\frac{2N}{2N-\mu}}$$
$$\leq C\left(\int_{\mathbb{R}^N} |u|^2\mathrm{d}x\right)^{\frac{2N-\mu}{N}} + C\left(\int_{\mathbb{R}^N} |u|^{2^*}\mathrm{d}x\right)^{\frac{2N-\mu}{N}}. \tag{2.16}$$

By (2.16) and the definition of $S$, we get there exists $M > 0$ such that

$$\|u\|^2 \leq M\left(\|u\|^{\frac{2(2N-\mu)}{N}} + \|u\|^{\frac{2(2N-\mu)}{N-2}}\right). \tag{2.17}$$

Then there exists $m > 0$ such that $\|u\| \geq m$. Since $u \in P$ is arbitrary, by (2.14), we get $p > 0$. $\square$

**Lemma 2.9.** *Let* $\mu \in (0, 4]$ *and* $\beta > 0$*. Assume that* $(f_1)$ *holds. If* $p$ *is attained by* $u \in P$ *with* $P \neq \emptyset$*, then* $I'(u) = 0$*.*

**Proof.** We first prove $J'(u) \neq 0$. Otherwise,

$$-(N-2)\left(a+2b\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x\right)\Delta u + Nu = (2N-\mu)\left(\int_{\mathbb{R}^N}\frac{H(u(y))}{|x-y|^\mu}\mathrm{d}y\right)h(u).$$

Then we have the Pohozǎev type identity:

$$\begin{aligned}&\frac{a(N-2)^2}{2}\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x + \frac{N^2}{2}\int_{\mathbb{R}^N}|u|^2\mathrm{d}x + b(N-2)^2\left(\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x\right)^2\\ &= \frac{(2N-\mu)^2}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{H(u(y))H(u(x))}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y.\end{aligned} \tag{2.18}$$

Since $u \in P$, by (2.18),

$$\begin{aligned}&\frac{a(N-2)^2}{2}\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x + \frac{N^2}{2}\int_{\mathbb{R}^N}|u|^2\mathrm{d}x + b(N-2)^2\left(\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x\right)^2\\ &= \frac{a(N-2)(2N-\mu)}{2}\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x + \frac{N(2N-\mu)}{2}\int_{\mathbb{R}^N}|u|^2\mathrm{d}x\\ &\quad + \frac{b(N-2)(2N-\mu)}{2}\left(\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x\right)^2,\end{aligned}$$

a contradiction with $u \neq 0$.

Now we prove $I'(u) = 0$. By the Lagrange multiplier rule, there exists $\lambda \in \mathbb{R}$ such that $I'(u) - \lambda J'(u) = 0$. Then

$$\begin{aligned}&-\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x\right)\Delta u + u - \left(\int_{\mathbb{R}^N}\frac{H(u(y))}{|x-y|^\mu}\mathrm{d}y\right)h(u)\\ &= -\lambda(N-2)\left(a+2b\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x\right)\Delta u + \lambda N u\\ &\quad - \lambda(2N-\mu)\left(\int_{\mathbb{R}^N}\frac{H(u(y))}{|x-y|^\mu}\mathrm{d}y\right)h(u).\end{aligned}$$

Moreover, we have the Pohozǎev type identity:

$$\begin{aligned}&\frac{a(N-2)}{2}\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x + \frac{N}{2}\int_{\mathbb{R}^N}|u|^2\mathrm{d}x + \frac{b(N-2)}{2}\left(\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x\right)^2\\ &- \frac{2N-\mu}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{H(u(y))H(u(x))}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y\end{aligned}$$

$$= \frac{a\lambda(N-2)^2}{2}\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x + \frac{\lambda N^2}{2}\int_{\mathbb{R}^N}|u|^2\mathrm{d}x + b\lambda(N-2)^2\left(\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x\right)^2$$
$$- \frac{\lambda(2N-\mu)^2}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{H(u(y))H(u(x))}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y. \tag{2.19}$$

Since $u \in P$, by (2.19),

$$\frac{a\lambda(N-2)(N-\mu+2)}{2}\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x + \frac{\lambda N(N-\mu)}{2}\int_{\mathbb{R}^N}|u|^2\mathrm{d}x$$
$$+ \frac{b(N-2)(4-\mu)}{2}\left(\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x\right)^2 = 0.$$

By $\mu \in (0,4]$ and $u \neq 0$, we get $\lambda = 0$. So $I'(u) = 0$. □

## 3. The case $\mu \in (0,4)$

When $\mu < 4$, we have $22^*_\mu > 4$. We first prove Theorem 1.1 $(i)$. Let

$$c_0 = \frac{a(N-\mu+2)}{2(2N-\mu)}\left(\frac{a}{2^*_\mu C(N,\mu)}\right)^{\frac{N-2}{N-\mu+2}} S^{\frac{2N-\mu}{N-\mu+2}}$$
$$+ \frac{b(4-\mu)}{4(2N-\mu)}\left(\frac{a}{2^*_\mu C(N,\mu)}\right)^{\frac{2(N-2)}{N-\mu+2}} S^{\frac{2(2N-\mu)}{N-\mu+2}}. \tag{3.1}$$

**Lemma 3.1.** *Assume that $(f_1)$-$(f_2)$ hold. Then there exists a large $\beta'' > 0$ such that $P \neq \emptyset$ for $\beta > \beta''$.*

**Proof.** For $R > 0$, define $w_R(x) = \xi$ for $|x| \le R$, $w_R(x) = 0$ for $|x| \ge R+1$, $w_R(x) = \xi(R+1-|x|)$ for $R \le |x| \le R+1$. Then $w_R \in H^1_r(\mathbb{R}^N)$. We note that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(w_R(y))F(w_R(x))}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y - \int_{B_R(0)}\int_{B_R(0)}\frac{F(w_R(y))F(w_R(x))}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y$$
$$= \int_{B_{R+1}(0)}\int_{B_{R+1}(0)\setminus B_R(0)}\frac{F(w_R(y))F(w_R(x))}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y$$
$$+ \int_{B_{R+1}(0)\setminus B_R(0)}\int_{B_R(0)}\frac{F(w_R(y))F(w_R(x))}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y. \tag{3.2}$$

By $(f_2)$, we have $F(w_R) > 0$ for $|x| \le R$. Then

$$\int_{B_R(0)}\int_{B_R(0)}\frac{F(w_R(y))F(w_R(x))}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y$$
$$\ge \int_{B_R(0)}\int_{B_R(0)}\frac{CF(w_R(y))F(w_R(x))}{(|x|+|y|)^\mu}\mathrm{d}x\mathrm{d}y$$

$$\geq \frac{C}{(2R)^{\mu}}\left(\int\limits_{B_R(0)} F(\xi)\mathrm{d}x\right)^2 = \frac{CF^2(\xi)|B_R(0)|^2}{(2R)^{\mu}}. \tag{3.3}$$

By Lemma 2.1,

$$\begin{aligned}
&\int\limits_{B_{R+1}(0)}\int\limits_{B_{R+1}(0)\setminus B_R(0)} \frac{F(w_R(y))F(w_R(x))}{|x-y|^{\mu}}\mathrm{d}x\mathrm{d}y \\
&\leq C(N,\mu)\left(\int\limits_{B_{R+1}(0)} |F(w_R)|^{\frac{2N}{2N-\mu}}\mathrm{d}y\right)^{\frac{2N-\mu}{2N}} \\
&\quad\times\left(\int\limits_{B_{R+1}(0)\setminus B_R(0)} |F(w_R)|^{\frac{2N}{2N-\mu}}\mathrm{d}y\right)^{\frac{2N-\mu}{2N}} \\
&\leq C(N,\mu)\max_{s\in[0,\xi]}|F(s)|^2|B_{R+1}(0)|^{\frac{2N-\mu}{2N}}|B_{R+1}(0)-B_R(0)|^{\frac{2N-\mu}{2N}}.
\end{aligned} \tag{3.4}$$

Similarly,

$$\begin{aligned}
&\int\limits_{B_{R+1}(0)\setminus B_R(0)}\int\limits_{B_R(0)} \frac{F(w_R(y))F(w_R(x))}{|x-y|^{\mu}}\mathrm{d}x\mathrm{d}y \\
&\leq C(N,\mu)\max_{s\in[0,\xi]}|F(s)|^2|B_R(0)|^{\frac{2N-\mu}{2N}}|B_{R+1}(0)-B_R(0)|^{\frac{2N-\mu}{2N}}.
\end{aligned} \tag{3.5}$$

From (3.2)-(3.5), we can choose $R_0 > 0$ large such that for $R > R_0$,

$$\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{F(w_R(y))F(w_R(x))}{|x-y|^{\mu}}\mathrm{d}x\mathrm{d}y > 0. \tag{3.6}$$

Let $R > R_0$. Then

$$\begin{aligned}
&\int\limits_{\mathbb{R}^N} \frac{H(w_R(y))H(w_R(x))}{|x-y|^{\mu}}\mathrm{d}x\mathrm{d}y \\
&= \beta^2\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{F(w_R(y))F(w_R(x))}{|x-y|^{\mu}}\mathrm{d}x\mathrm{d}y + 2\beta\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{F(w_R(y))|w_R(x)|^{2^*_\mu}}{|x-y|^{\mu}}\mathrm{d}x\mathrm{d}y \\
&\quad+\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{|w_R(y)|^{2^*_\mu}|w_R(x)|^{2^*_\mu}}{|x-y|^{\mu}}\mathrm{d}x\mathrm{d}y.
\end{aligned} \tag{3.7}$$

By (3.6)-(3.7), there exist $\beta' > 0$ and $\kappa > 0$ such that for $\beta > \beta'$,

$$\int\limits_{\mathbb{R}^N} \frac{H(w_R(y))H(w_R(x))}{|x-y|^{\mu}}\mathrm{d}x\mathrm{d}y \geq \kappa\beta^2. \tag{3.8}$$

By a direct calculation,

$$J\left(w_R\left(\frac{\cdot}{t}\right)\right) = \frac{a(N-2)t^{N-2}}{2}\int_{\mathbb{R}^N}|\nabla w_R|^2\mathrm{d}x + \frac{Nt^N}{2}\int_{\mathbb{R}^N}|w_R|^2\mathrm{d}x + \frac{b(N-2)t^{2N-4}}{2}\left(\int_{\mathbb{R}^N}|\nabla w_R|^2\mathrm{d}x\right)^2 - \frac{2N-\mu}{2}t^{2N-\mu}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{H(w_R(y))H(w_R(x))}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y. \tag{3.9}$$

By (3.8)-(3.9), we know there exists a large $\beta'' > 0$ such that $J(w_R) < 0$ for $\beta > \beta''$. Let $\beta > \beta''$. By (3.7), we get $J\left(w_R\left(\frac{\cdot}{t}\right)\right) > 0$ for $t > 0$ small. Then there exists $t_0 \in (0,1)$ such that $J\left(w_R\left(\frac{\cdot}{t_0}\right)\right) = 0$. So $P \neq \emptyset$. □

**Lemma 3.2.** *Assume that* $(f_1)$-$(f_2)$ *hold. Then there exists* $\beta_0 > \beta''$ *such that* $p < c_0$ *for* $\beta > \beta_0$.

**Proof.** By Lemma 3.1, we get $w_R\left(\frac{\cdot}{t_0}\right) \in P$ for $\beta > \beta''$. By the definition of $p$, we have $p \le I\left(w_R\left(\frac{\cdot}{t_0}\right)\right) \le \sup_{t\ge 0} I\left(w_R\left(\frac{\cdot}{t}\right)\right)$. We note that

$$I\left(w_R\left(\frac{\cdot}{t}\right)\right) = \frac{at^{N-2}}{2}\int_{\mathbb{R}^N}|\nabla w_R|^2\mathrm{d}x + \frac{t^N}{2}\int_{\mathbb{R}^N}|w_R|^2\mathrm{d}x + \frac{bt^{2N-4}}{4}\left(\int_{\mathbb{R}^N}|\nabla w_R|^2\mathrm{d}x\right)^2 - \frac{t^{2N-\mu}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{H(w_R(y))H(w_R(x))}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y. \tag{3.10}$$

Let

$$M_0 = \max\left\{\frac{a}{2}\int_{\mathbb{R}^N}|\nabla w_R|^2\mathrm{d}x, \frac{1}{2}\int_{\mathbb{R}^N}|w_R|^2\mathrm{d}x, \frac{b}{4}\left(\int_{\mathbb{R}^N}|\nabla w_R|^2\mathrm{d}x\right)^2\right\}. \tag{3.11}$$

Then by (3.8) and (3.10)-(3.11), we get there exists $L_0 > 0$ such that

$$p \le \sup_{t\ge 0} I\left(w_R\left(\frac{x}{t}\right)\right) \le M_0(t^{N-2} + t^N + t^{2N-4}) - \frac{\kappa\beta^2 t^{2N-\mu}}{2} \le \frac{L_0}{\beta^{\frac{2(N-2)}{N-\mu+2}}} + \frac{L_0}{\beta^{\frac{2N}{N-\mu}}} + \frac{L_0}{\beta^{\frac{4(N-2)}{4-\mu}}}.$$

So there exists $\beta_0 > \beta''$ such that $p < c_0$ for $\beta > \beta_0$. □

**Lemma 3.3.** *Let* $\beta > \beta_0$. *Assume that* $(f_1)$ *holds. If* $\{u_n\} \subset P$ *is a bounded sequence such that* $I(u_n) \to p \in (0, c_0)$, *then* $\{u_n\}$ *converges strongly in* $H_r^1(\mathbb{R}^N)$ *up to a subsequence.*

**Proof.** We assume $u_n \rightharpoonup u$ weakly in $H_r^1(\mathbb{R}^N)$. We prove $J(u) \ge 0$. Otherwise, $J(u) < 0$. We note that

$$J\left(u\left(\frac{\cdot}{t}\right)\right) = \frac{a(N-2)t^{N-2}}{2}\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x + \frac{Nt^N}{2}\int_{\mathbb{R}^N}|u|^2\mathrm{d}x$$

$$+\frac{b(N-2)t^{2N-4}}{2}\left(\int\limits_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x\right)^2$$
$$-\frac{2N-\mu}{2}t^{2N-\mu}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{H(u(y))H(u(x))}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y.$$

Then $J\left(u\left(\frac{\cdot}{t}\right)\right)>0$ for $t>0$ small. Moreover, there exists $t_0\in(0,1)$ such that $J\left(u\left(\frac{\cdot}{t_0}\right)\right)=0$. So by $I(u_n)\to p$ and $J(u_n)=0$,

$$\begin{aligned}
p=&\lim_{n\to\infty}\left(I(u_n)-\frac{1}{2N-\mu}J(u_n)\right)\\
=&\frac{a(N-\mu+2)}{2(2N-\mu)}\lim_{n\to\infty}\int\limits_{\mathbb{R}^N}|\nabla u_n|^2\mathrm{d}x+\frac{N-\mu}{2(2N-\mu)}\lim_{n\to\infty}\int\limits_{\mathbb{R}^N}|u_n|^2\mathrm{d}x\\
&+\frac{b(4-\mu)}{4(2N-\mu)}\lim_{n\to\infty}\left(\int\limits_{\mathbb{R}^N}|\nabla u_n|^2\mathrm{d}x\right)^2\\
\geq&\frac{a(N-\mu+2)}{2(2N-\mu)}\int\limits_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x+\frac{N-\mu}{2(2N-\mu)}\int\limits_{\mathbb{R}^N}|u|^2\mathrm{d}x\\
&+\frac{b(4-\mu)}{4(2N-\mu)}\lim_{n\to\infty}\left(\int\limits_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x\right)^2\\
>&I\left(u\left(\frac{\cdot}{t_0}\right)\right)-\frac{1}{2N-\mu}J\left(u\left(\frac{\cdot}{t_0}\right)\right)=I\left(u\left(\frac{\cdot}{t_0}\right)\right)\geq p,
\end{aligned}$$

a contradiction. Let $v_n=u_n-u$. By $J(u)\geq 0$, $J(u_n)=0$ and Lemma 2.7,

$$\begin{aligned}
0\geq&J(u_n)-J(u)\\
\geq&\frac{a(N-2)}{2}\int\limits_{\mathbb{R}^N}|\nabla v_n|^2\mathrm{d}x+\frac{N}{2}\int\limits_{\mathbb{R}^N}|v_n|^2\mathrm{d}x+\frac{b(N-2)}{2}\left(\int\limits_{\mathbb{R}^N}|\nabla v_n|^2\mathrm{d}x\right)^2\\
&-\frac{2N-\mu}{2}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{|v_n(y)|^{2_\mu^*}|v_n(x)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y+o_n(1).
\end{aligned}\tag{3.12}$$

Then by Lemma 2.1,

$$\begin{aligned}
&a\lim_{n\to\infty}\int\limits_{\mathbb{R}^N}|\nabla v_n|^2\mathrm{d}x\\
&\leq a\lim_{n\to\infty}\int\limits_{\mathbb{R}^N}|\nabla v_n|^2\mathrm{d}x+b\lim_{n\to\infty}\left(\int\limits_{\mathbb{R}^N}|\nabla v_n|^2\mathrm{d}x\right)^2\\
&\leq 2_\mu^*C(N,\mu)\left(\lim_{n\to\infty}\int\limits_{\mathbb{R}^N}|v_n|^{2^*}\mathrm{d}x\right)^{\frac{2N-\mu}{N}}\leq\frac{2_\mu^*C(N,\mu)}{S^{\frac{2N-\mu}{N-2}}}\left(\lim_{n\to\infty}\int\limits_{\mathbb{R}^N}|\nabla v_n|^2\mathrm{d}x\right)^{\frac{2N-\mu}{N-2}}.
\end{aligned}\tag{3.13}$$

Assume that $\lim_{n\to\infty}\int_{\mathbb{R}^N}|\nabla v_n|^2\mathrm{d}x = l$ holds. If $l>0$, by (3.13), we obtain that $l \geq \left(\frac{a}{2_\mu^* C(N,\mu)}\right)^{\frac{N-2}{N-\mu+2}} \times S^{\frac{2N-\mu}{N-\mu+2}}$. Then by $I(u_n)\to p$ and $J(u_n)=0$,

$$p = \lim_{n\to\infty}\left(I(u_n) - \frac{1}{2N-\mu}J(u_n)\right)$$
$$\geq \lim_{n\to\infty}\left[\frac{a(N-\mu+2)}{2(2N-\mu)}\int\limits_{\mathbb{R}^N}|\nabla u_n|^2\mathrm{d}x + \frac{b(4-\mu)}{4(2N-\mu)}\left(\int\limits_{\mathbb{R}^N}|\nabla u_n|^2\mathrm{d}x\right)^2\right] \geq c_0,$$

a contradiction. So $l=0$. Moreover, $\lim_{n\to\infty}\int_{\mathbb{R}^N}|v_n|^{2^*}\mathrm{d}x = 0$. By Lemma 2.1,

$$\lim_{n\to\infty}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{|v_n(y)|^{2_\mu^*}|v_n(x)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y = 0. \tag{3.14}$$

From (3.12) and (3.14), we get $u_n \to u$ in $H_r^1(\mathbb{R}^N)$. $\square$

**Proof of Theorem 1.1** (*i*)**.** Let $\beta > \beta_0$. By Lemmas 2.8 and 3.1-3.2, we get $p\in(0,c_0)$. By the definition of $p$, there exists $\{u_n\}\subset P$ such that $I(u_n)\to p$. Since $\{u_n\}\subset P$, by Lemma 2.6, we get $\|u_n\|$ is bounded. Then by Lemma 3.3, there exists $u\in H_r^1(\mathbb{R}^N)$ such that $u_n\to u$ in $H_r^1(\mathbb{R}^N)$. So $p = I(u)$ with $u\in P$. By Lemma 2.9, we get $I'(u)=0$. $\square$

Now we prove Theorem 1.1 (*ii*). Let

$$h(t) = \frac{aS^{\frac{N}{2}}}{2}t^{N-2} + \frac{bS^N}{4}t^{2N-4} - \frac{[C(N,\mu)]^{\frac{N}{2}}S_\mu^{\frac{2N-\mu}{2}}}{2}t^{2N-\mu}. \tag{3.15}$$

Obviously, we have $\sup_{t\geq 0}h(t) > 0$.

**Lemma 3.4.** *Let $\beta>0$. Assume that $(f_1)$ and $(f_3)$ hold. Then $P\neq\emptyset$ and $p<\sup_{t\geq 0}h(t)$.*

**Proof.** We first prove $P\neq\emptyset$. By a direct calculation,

$$\begin{aligned} J\left(u_\varepsilon\left(\frac{\cdot}{t}\right)\right) =& \frac{a(N-2)t^{N-2}}{2}\int\limits_{\mathbb{R}^N}|\nabla u_\varepsilon|^2\mathrm{d}x + \frac{Nt^N}{2}\int\limits_{\mathbb{R}^N}|u_\varepsilon|^2\mathrm{d}x \\ &+ \frac{b(N-2)t^{2N-4}}{2}\left(\int\limits_{\mathbb{R}^N}|\nabla u_\varepsilon|^2\mathrm{d}x\right)^2 \\ &- \frac{2N-\mu}{2}t^{2N-\mu}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{H(u_\varepsilon(y))H(u_\varepsilon(x))}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y. \end{aligned} \tag{3.16}$$

By Lemmas 2.3-2.4, there exists $\varepsilon_1\in(0,1)$ such that $\|u_\varepsilon\|^2 \leq \frac{3aS^{\frac{N}{2}}}{2}$ and $\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_\varepsilon(y)|^{2_\mu^*}|u_\varepsilon(x)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y \geq \frac{[C(N,\mu)]^{\frac{N}{2}}S_\mu^{\frac{2N-\mu}{2}}}{2}$ for $\varepsilon\in(0,\varepsilon_1)$. By $(f_1)$, for any $\delta>0$, there exists $C_\delta>0$ such that $|F(u)|\leq\delta|u|^{2_\mu^*} + C_\delta|u|^{\frac{2N-\mu}{N}}$ for $u\in\mathbb{R}$. Then by Lemmas 2.1 and 2.3,

$$\begin{aligned}
&\left| \int\limits_{\mathbb{R}^N} \int\limits_{\mathbb{R}^N} \frac{\beta^2 F(u_\varepsilon(y))F(u_\varepsilon(x)) + 2\beta F(u_\varepsilon(y))|u_\varepsilon(x)|^{2^*_\mu}}{|x-y|^\mu} \mathrm{d}x\mathrm{d}y \right| \\
&\leq \beta^2 C(N,\mu)\|F(u_\varepsilon)\|^2_{\frac{2N}{2N-\mu}} + 2\beta C(N,\mu)\|F(u_\varepsilon)\|_{\frac{2N}{2N-\mu}} \||u_\varepsilon|^{2^*_\mu}\|_{\frac{2N}{2N-\mu}} \\
&\leq CC_\delta^2 \left( \int\limits_{\mathbb{R}^N} |u_\varepsilon|^2 \mathrm{d}x \right)^{\frac{2N-\mu}{N}} + C\delta^2 \left( \int\limits_{\mathbb{R}^N} |u_\varepsilon|^{2^*} \mathrm{d}x \right)^{\frac{2N-\mu}{N}} \\
&\quad + C \left[ C_\delta \left( \int\limits_{\mathbb{R}^N} |u_\varepsilon|^2 \mathrm{d}x \right)^{\frac{2N-\mu}{2N}} + \delta \left( \int\limits_{\mathbb{R}^N} |u_\varepsilon|^{2^*} \mathrm{d}x \right)^{\frac{2N-\mu}{2N}} \right] \left( \int\limits_{\mathbb{R}^N} |u_\varepsilon|^{2^*} \mathrm{d}x \right)^{\frac{2N-\mu}{2N}} \\
&\leq CC_\delta^2 \varepsilon^{\frac{2N-\mu}{N}} + CC_\delta \varepsilon^{\frac{2N-\mu}{2N}} + C\delta^2 + C\delta. \qquad (3.17)
\end{aligned}$$

Thus, there exist $\delta_0 > 0$ and $\varepsilon_2 \in (0, \varepsilon_1)$ such that for $\delta \in (0, \delta_0)$ and $\varepsilon \in (0, \varepsilon_2)$,

$$\int\limits_{\mathbb{R}^N} \int\limits_{\mathbb{R}^N} \frac{H(u_\varepsilon(y))H(u_\varepsilon(x))}{|x-y|^\mu} \mathrm{d}x\mathrm{d}y \geq \frac{[C(N,\mu)]^{\frac{N}{2}} S_\mu^{\frac{2N-\mu}{2}}}{4}. \qquad (3.18)$$

Since $\|u_\varepsilon\|^2 \leq \frac{3aS^{\frac{N}{2}}}{2}$, by (3.16) and (3.18), there exists $t' > 1$ large such that $J\left(u_\varepsilon\left(\frac{\cdot}{t'}\right)\right) < 0$. Let $\varepsilon \in (0, \varepsilon_2)$. By (3.16), there exists $t'' \in (0,1)$ small such that $J\left(u_\varepsilon\left(\frac{\cdot}{t''}\right)\right) > 0$. So there exists $t''' \in (t', t'')$ such that $J\left(u_\varepsilon\left(\frac{\cdot}{t'''}\right)\right) = 0$. Then $P \neq \emptyset$.

By the definition of $p$, we get $p \leq I\left(u_\varepsilon\left(\frac{\cdot}{t'''}\right)\right) \leq \sup_{t\geq 0} I\left(u_\varepsilon\left(\frac{\cdot}{t}\right)\right)$. Also,

$$\begin{aligned}
I\left(u_\varepsilon\left(\frac{\cdot}{t}\right)\right) =& \frac{at^{N-2}}{2} \int\limits_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 \mathrm{d}x + \frac{t^N}{2} \int\limits_{\mathbb{R}^N} |u_\varepsilon|^2 \mathrm{d}x + \frac{bt^{2N-4}}{4} \left( \int\limits_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 \mathrm{d}x \right)^2 \\
&- \frac{t^{2N-\mu}}{2} \int\limits_{\mathbb{R}^N} \int\limits_{\mathbb{R}^N} \frac{H(u_\varepsilon(y))H(u_\varepsilon(x))}{|x-y|^\mu} \mathrm{d}x\mathrm{d}y. \qquad (3.19)
\end{aligned}$$

Since $\|u_\varepsilon\|^2 \leq \frac{3aS^{\frac{N}{2}}}{2}$, by (3.18)-(3.19), there exist a small $t_1 \in (0,1)$ and a large $t_2 > 1$ independent of $\varepsilon$ such that

$$\sup_{t \in [0,t_1] \cup [t_2, +\infty)} I\left(u_\varepsilon\left(\frac{\cdot}{t}\right)\right) \leq \frac{1}{2} \sup_{t\geq 0} h(t). \qquad (3.20)$$

Let

$$\begin{aligned}
y(t) =& \frac{at^{N-2}}{2} \int\limits_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 \mathrm{d}x + \frac{t^N}{2} \int\limits_{\mathbb{R}^N} |u_\varepsilon|^2 \mathrm{d}x + \frac{bt^{2N-4}}{4} \left( \int\limits_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 \mathrm{d}x \right)^2 \\
&- \frac{t^{2N-\mu}}{2} \int\limits_{\mathbb{R}^N} \int\limits_{\mathbb{R}^N} \frac{|u_\varepsilon(y)|^{2^*_\mu} |u_\varepsilon(x)|^{2^*_\mu}}{|x-y|^\mu} \mathrm{d}x\mathrm{d}y.
\end{aligned}$$

By Lemmas 2.3-2.4, there exists $\varepsilon_3 \in (0, \varepsilon_2)$ such that for $\varepsilon \in (0, \varepsilon_3)$,

$$\max_{t\in[t_1,t_2]} y(t) \le \sup_{t\ge 0} h(t) + C\varepsilon^{N-2} + C\varepsilon^{\frac{2N-\mu}{2}} + \begin{cases} C\varepsilon^2, & N\ge 5,\\ C\varepsilon^2|\ln\varepsilon|, & N=4,\\ C\varepsilon, & N=3. \end{cases} \tag{3.21}$$

By (3.21) and Lemma 2.5, we derive that $\sup_{t\in[t_1,t_2]} I\left(u_\varepsilon\left(\frac{\cdot}{t}\right)\right) < \sup_{t\ge 0} h(t)$ for $\varepsilon>0$ small. Together with (3.20), we get $p < \sup_{t\ge 0} h(t)$. $\square$

**Lemma 3.5.** *Let $\beta>0$. Assume that $(f_1)$ holds. If $\{u_n\}\subset P$ is a bounded sequence such that $I(u_n)\to p\in \left(0, \sup_{t\ge 0} l(t)\right)$, then $\{u_n\}$ converges strongly in $H_r^1(\mathbb{R}^N)$ up to a subsequence.*

**Proof.** We assume $u_n \rightharpoonup u$ weakly in $H_r^1(\mathbb{R}^N)$. Similar to the argument of Lemma 3.3, we can prove $J(u)\ge 0$. Recall that $J(u_n)=0$. Let $v_n = u_n - u$. Then by Lemma 2.7,

$$\begin{aligned} 0 \ge & J(u_n) - J(u)\\ \ge & \frac{a(N-2)}{2}\int_{\mathbb{R}^N} |\nabla v_n|^2 \mathrm{d}x + \frac{N}{2}\int_{\mathbb{R}^N} |v_n|^2\mathrm{d}x + \frac{b(N-2)}{2}\left(\int_{\mathbb{R}^N}|\nabla v_n|^2\mathrm{d}x\right)^2\\ & - \frac{2N-\mu}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_n(y)|^{2_\mu^*}|v_n(x)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y + o_n(1). \end{aligned} \tag{3.22}$$

By (3.22) and Lemma 2.1,

$$\begin{aligned} & a(N-2)\lim_{n\to\infty}\int_{\mathbb{R}^N}|\nabla v_n|^2\mathrm{d}x + b(N-2)\lim_{n\to\infty}\left(\int_{\mathbb{R}^N}|\nabla v_n|^2\mathrm{d}x\right)^2\\ \le & \frac{(2N-\mu)C(N,\mu)}{S^{\frac{2N-\mu}{N-2}}}\left(\lim_{n\to\infty}\int_{\mathbb{R}^N}|\nabla v_n|^2\mathrm{d}x\right)^{\frac{2N-\mu}{N-2}}. \end{aligned} \tag{3.23}$$

Assume that $\lim_{n\to\infty}\int_{\mathbb{R}^N}|\nabla v_n|^2\mathrm{d}x = l$ holds. If $l>0$, then by (3.23),

$$a(N-2)l + b(N-2)l^2 \le \frac{(2N-\mu)C(N,\mu)}{S^{\frac{2N-\mu}{N-2}}} l^{\frac{2N-\mu}{N-2}}. \tag{3.24}$$

By (3.24) and Lemma 2.2, we have $h'\left(\frac{l^{\frac{1}{N-2}}}{S^{\frac{N}{2(N-2)}}}\right)\le 0$. By the structure of $h$, we know $h(t)$ attains its maximum at a unique $T\in(0,+\infty)$ and $h'(T)=0$. Moreover, $h'(t)>0$ for $t\in(0,T)$ and $h'(t)<0$ for $t\in(T,+\infty)$. Then $\frac{l^{\frac{1}{N-2}}}{S^{\frac{N}{2(N-2)}}}\ge T$. By $I(u_n)\to p$ and $J(u_n)=0$,

$$\begin{aligned} p = & \lim_{n\to\infty}\left(I(u_n) - \frac{1}{2N-\mu}J(u_n)\right)\\ = & \lim_{n\to\infty}\left[\frac{a(N-\mu+2)}{2(2N-\mu)}\int_{\mathbb{R}^N}|\nabla u_n|^2\mathrm{d}x + \frac{b(4-\mu)}{4(2N-\mu)}\left(\int_{\mathbb{R}^N}|\nabla u_n|^2\mathrm{d}x\right)^2\right]\\ \ge & \frac{a(N-\mu+2)}{2(2N-\mu)}T^{N-2}S^{\frac{N}{2}} + \frac{b(4-\mu)}{4(2N-\mu)}T^{2N-4}S^N \end{aligned}$$

$$=h(T)-\frac{1}{2N-\mu}(h'(T),T)=h(T)=\sup_{t\geq 0}h(t),$$

a contradiction. So $l=0$ and $\lim_{n\to\infty}\int_{\mathbb{R}^N}|v_n|^{2^*}\mathrm{d}x=0$. Together with (3.22) and Lemma 2.1, we get $\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_n(y)|^{2^*_\mu}|v_n(x)|^{2^*_\mu}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y=0$ and $u_n\to u$ in $H^1_r(\mathbb{R}^N)$. $\square$

**Proof of Theorem 1.1** *(ii)***.** By Lemmas 2.8 and 3.4, we get $p\in\left(0,\sup_{t\geq 0}h(t)\right)$. By the definition of $p$, there exists $\{u_n\}\subset P$ such that $I(u_n)\to p$. Since $\{u_n\}\subset P$, by Lemma 2.6, we know $\|u_n\|$ is bounded. By Lemma 3.5, we have $u_n\to u$ in $H^1_r(\mathbb{R}^N)$. Then $p=I(u)$ with $u\in P$. By Lemma 2.9, we get $I'(u)=0$. $\square$

## 4. The case $\mu=4$

When $\mu=4$, we have $2^*_\mu=2$. We first consider the case $b<\frac{2}{S^2_\mu}$.

**Lemma 4.1.** *Let $\beta>0$. Assume that $(f_1)$ and $(f_3)$ hold. Then $P\neq\emptyset$ and $p<\frac{a^2S^2_\mu}{4(2-bS^2_\mu)}$.*

**Proof.** We first prove $P\neq\emptyset$. By $b<\frac{2}{S^2_\mu}$ and Lemmas 2.2-2.4, there exists $\varepsilon_1\in(0,1)$ such that $\|u_\varepsilon\|^2\leq\frac{3aS^{\frac{N}{2}}}{2}$ for $\varepsilon\in(0,\varepsilon_1)$. Moreover, for $\varepsilon\in(0,\varepsilon_1)$,

$$2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_\varepsilon(y)|^2|u_\varepsilon(x)|^2}{|x-y|^4}\mathrm{d}x\mathrm{d}y-b\left(\int_{\mathbb{R}^N}|\nabla u_\varepsilon|^2\mathrm{d}x\right)^2\geq\frac{(2-bS^2_\mu)S^N}{2S^2_\mu}. \tag{4.1}$$

By (3.17), for any $\delta>0$, there exists $C_\delta>0$ such that

$$\begin{aligned}&\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\beta^2F(u_\varepsilon(y))F(u_\varepsilon(x))+2\beta F(u_\varepsilon(y))|u_\varepsilon(x)|^2}{|x-y|^4}\mathrm{d}x\mathrm{d}y\right|\\&\leq CC^2_\delta\varepsilon^{\frac{2(N-2)}{N}}+CC_\delta\varepsilon^{\frac{N-2}{N}}+C\delta^2+C\delta.\end{aligned} \tag{4.2}$$

By (4.1)-(4.2), we derive that there exist $\delta_0>0$ and $\varepsilon_2\in(0,\varepsilon_1)$ such that for $\delta\in(0,\delta_0)$ and $\varepsilon\in(0,\varepsilon_2)$,

$$2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{H(u_\varepsilon(y))H(u_\varepsilon(x))}{|x-y|^4}\mathrm{d}x\mathrm{d}y-b\left(\int_{\mathbb{R}^N}|\nabla u_\varepsilon|^2\mathrm{d}x\right)^2\geq\frac{(2-bS^2_\mu)S^N}{4S^2_\mu}. \tag{4.3}$$

Recall that $\|u_\varepsilon\|^2\leq\frac{3aS^{\frac{N}{2}}}{2}$. Then by (3.16) and (4.3), there exists $t'>1$ large such that $J\left(u_\varepsilon\left(\frac{\cdot}{t'}\right)\right)<0$. Let $\varepsilon\in(0,\varepsilon_2)$. By (3.16), there exists $t''\in(0,1)$ small such that $J\left(u_\varepsilon\left(\frac{\cdot}{t''}\right)\right)>0$. So there exists $t'''\in(t',t'')$ such that $J\left(u_\varepsilon\left(\frac{\cdot}{t'''}\right)\right)=0$. Then $P\neq\emptyset$.

Now we prove $p<\frac{a^2S^2_\mu}{4(2-bS^2_\mu)}$. Since $u_\varepsilon\left(\frac{\cdot}{t'''}\right)\in P$, by the definition of $p$, we have $p\leq I\left(u_\varepsilon\left(\frac{\cdot}{t'''}\right)\right)\leq\sup_{t\geq 0}I\left(u_\varepsilon\left(\frac{\cdot}{t}\right)\right)$. Since $\|u_\varepsilon\|^2\leq\frac{3aS^{\frac{N}{2}}}{2}$, by (3.19) and (4.3), there exist a small $t_1\in(0,1)$ and a large $t_2>1$ such that

$$\sup_{t\in[0,t_1]\cup[t_2,+\infty)}I\left(u_\varepsilon\left(\frac{\cdot}{t}\right)\right)<\frac{a^2S^2_\mu}{4(2-bS^2_\mu)}. \tag{4.4}$$

Let

$$
\begin{aligned}
y(t) =&\frac{at^{N-2}}{2}\int_{\mathbb{R}^N}|\nabla u_\varepsilon|^2\mathrm{d}x+\frac{t^N}{2}\int_{\mathbb{R}^N}|u_\varepsilon|^2\mathrm{d}x\\
&-t^{2N-4}\left[\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_\varepsilon(y)|^2|u_\varepsilon(x)|^2}{|x-y|^4}\mathrm{d}x\mathrm{d}y-\frac{b}{4}\left(\int_{\mathbb{R}^N}|\nabla u_\varepsilon|^2\mathrm{d}x\right)^2\right].
\end{aligned}
$$

By Lemmas 2.2-2.4, there exists $\varepsilon_3 \in (0, \varepsilon_2)$ such that for $\varepsilon \in (0, \varepsilon_3)$,

$$
\begin{aligned}
\max_{t\in[t_1,t_2]} y(t) \le& \frac{a^2\left(\int_{\mathbb{R}^N}|\nabla u_\varepsilon|^2\mathrm{d}x\right)^2}{4\left[2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_\varepsilon(y)|^2|u_\varepsilon(x)|^2}{|x-y|^4}\mathrm{d}x\mathrm{d}y-b\left(\int_{\mathbb{R}^N}|\nabla u_\varepsilon|^2\mathrm{d}x\right)^2\right]}\\
&+\frac{t_2^N}{2}\int_{\mathbb{R}^N}|u_\varepsilon|^2\mathrm{d}x \le \frac{a^2S_\mu^2}{4(2-bS_\mu^2)}+C\varepsilon^{N-2}+C\varepsilon^2.
\end{aligned}
\tag{4.5}
$$

Since $N \ge 5$, by (4.5) and Lemma 2.5, we derive that for $\varepsilon > 0$ small,

$$
\sup_{t\in[t_1,t_2]} I\left(u_\varepsilon\left(\frac{\cdot}{t}\right)\right) \le \frac{a^2S_\mu^2}{4(2-bS_\mu^2)}+2C\varepsilon^2-\frac{C_0Lt_1^{2N-4}}{4}\varepsilon^{\frac{2N-4}{N}} < \frac{a^2S_\mu^2}{4(2-bS_\mu^2)}.
$$

Together with (4.4), we get $p < \frac{a^2S_\mu^2}{4(2-bS_\mu^2)}$. $\square$

**Lemma 4.2.** *Let $\beta > 0$. Assume that $(f_1)$ holds. If $\{u_n\} \subset P$ is a bounded sequence such that $I(u_n) \to p \in \left(0, \frac{a^2S_\mu^2}{4(2-bS_\mu^2)}\right)$, then $\{u_n\}$ converges strongly in $H_r^1(\mathbb{R}^N)$ up to a subsequence.*

**Proof.** We assume $u_n \rightharpoonup u$ weakly in $H_r^1(\mathbb{R}^N)$. Similar to the argument of Lemma 3.3, we have $J(u) \ge 0$. Recall that $J(u_n) = 0$. Let $v_n = u_n - u$. By Lemma 2.7, we have

$$
\begin{aligned}
0 \ge& J(u_n) - J(u)\\
\ge& \frac{a(N-2)}{2}\int_{\mathbb{R}^N}|\nabla v_n|^2\mathrm{d}x+\frac{N}{2}\int_{\mathbb{R}^N}|v_n|^2\mathrm{d}x+\frac{b(N-2)}{2}\left(\int_{\mathbb{R}^N}|\nabla v_n|^2\mathrm{d}x\right)^2\\
&-(N-2)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_n(y)|^2|v_n(x)|^2}{|x-y|^4}\mathrm{d}x\mathrm{d}y+o_n(1).
\end{aligned}
\tag{4.6}
$$

By (4.6) and Lemma 2.1,

$$
\begin{aligned}
&aS\left(\lim_{n\to\infty}\int_{\mathbb{R}^N}|v_n|^{2^*}\mathrm{d}x\right)^{\frac{2}{2^*}}+bS^2\lim_{n\to\infty}\left(\lim_{n\to\infty}\int_{\mathbb{R}^N}|v_n|^{2^*}\mathrm{d}x\right)^{\frac{4}{2^*}}\\
&\le 2C(N,\mu)\left(\lim_{n\to\infty}\int_{\mathbb{R}^N}|v_n|^{2^*}\mathrm{d}x\right)^{\frac{4}{2^*}}.
\end{aligned}
\tag{4.7}
$$

Assume that $\lim_{n\to\infty}\int_{\mathbb{R}^N}|v_n|^{2^*}\mathrm{d}x = l$ holds. If $l > 0$, then $\lim_{n\to\infty}\int_{\mathbb{R}^N}|\nabla v_n|^2\mathrm{d}x \ge Sl^{\frac{2}{2^*}} \ge \frac{aS_\mu^2}{2-bS_\mu^2}$. By $I(u_n) \to p$ and $J(u_n) = 0$,

$$
\begin{aligned}
p= & \lim_{n\to\infty}\left(I(u_n)-\frac{1}{2(N-2)}J(u_n)\right) \\
= & \lim_{n\to\infty}\left(\frac{a}{4}\int\limits_{\mathbb{R}^N}|\nabla u_n|^2\mathrm{d}x+\frac{N-4}{4(N-2)}\int\limits_{\mathbb{R}^N}|u_n|^2\mathrm{d}x\right) \\
\geq & \frac{a}{4}\lim_{n\to\infty}\int\limits_{\mathbb{R}^N}|\nabla v_n|^2\mathrm{d}x\geq\frac{a^2S_\mu^2}{4(2-bS_\mu^2)},
\end{aligned}
$$

a contradiction. So $l=0$. By (4.6), we get $u_n\to u$ in $H_r^1(\mathbb{R}^N)$. □

**Proof of Theorem 1.2.** By Lemmas 2.8 and 4.1, we know $p\in\left(0,\frac{a^2S_\mu^2}{4(2-bS_\mu^2)}\right)$. By the definition of $p$, there exists $\{u_n\}\subset P$ such that $I(u_n)\to p$. Since $\{u_n\}\subset P$, by Lemma 2.6, we know $\|u_n\|$ is bounded. By Lemma 4.2, we have $u_n\to u$ in $H_r^1(\mathbb{R}^N)$. Then $p=I(u)$ with $u\in P$. By Lemma 2.9, we get $I'(u)=0$. □

Now we consider the case $b>\frac{2}{S_\mu^2}$. By the definition of $S_\mu$,

$$
b\left(\int\limits_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x\right)^2-2\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{|u(y)|^2|u(x)|^2}{|x-y|^4}\mathrm{d}x\mathrm{d}y\geq\left(b-\frac{2}{S_\mu^2}\right)\left(\int\limits_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x\right)^2. \tag{4.8}
$$

**Lemma 4.3.** *Assume that $(f_1)$ and $(f_2)$ hold. Then there exists a large $\beta_1>0$ such that $P\neq\emptyset$ for $\beta>\beta_1$.*

**Proof.** For $R>0$, define $w_R(x)=\xi$ for $|x|\leq R$, $w_R(x)=0$ for $|x|\geq R+1$, $w_R(x)=\xi(R+1-|x|)$ for $R\leq|x|\leq R+1$. Then $w_R\in H_r^1(\mathbb{R}^N)$. By (3.8)-(3.9), there exists a large $\beta_1>0$ such that $J(w_R)<0$ for $\beta>\beta_1$. Also, $J\left(u\left(\frac{\cdot}{t}\right)\right)>0$ for $t>0$ small. Then there exists $t_0\in(0,1)$ such that $J\left(w_R\left(\frac{\cdot}{t_0}\right)\right)=0$. So $P\neq\emptyset$. □

**Lemma 4.4.** *Let $\beta>\beta_1$. If $\{u_n\}\subset P$ is a bounded sequence such that $I(u_n)\to p$, then $\{u_n\}$ converges strongly in $H_r^1(\mathbb{R}^N)$ up to a subsequence.*

**Proof.** We assume $u_n\rightharpoonup u$ weakly in $H_r^1(\mathbb{R}^N)$. Similar to the argument of Lemma 3.3, we get $J(u)\geq 0$. Let $v_n=u_n-u$. Then

$$
\begin{aligned}
0\geq & J(u_n)-J(u) \\
\geq & \frac{a(N-2)}{2}\int\limits_{\mathbb{R}^N}|\nabla v_n|^2\mathrm{d}x+\frac{N}{2}\int\limits_{\mathbb{R}^N}|v_n|^2\mathrm{d}x+\frac{b(N-2)}{2}\left(\int\limits_{\mathbb{R}^N}|\nabla v_n|^2\mathrm{d}x\right)^2 \\
& -(N-2)\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{|v_n(y)|^2|v_n(x)|^2}{|x-y|^4}\mathrm{d}x\mathrm{d}y+o_n(1).
\end{aligned} \tag{4.9}
$$

Since $b>\frac{2}{S_\mu^2}$, by (4.8)-(4.9), we get $u_n\to u$ in $H_r^1(\mathbb{R}^N)$. □

**Proof of Theorem 1.3.** Let $\beta>\beta_1$. By Lemmas 2.8 and 4.3, we know $p>0$. By the definition of $p$, there exists $\{u_n\}\subset P$ such that $I(u_n)\to p$. Since $\{u_n\}\subset P$, by Lemma 2.6, we know $\|u_n\|$ is bounded. By Lemma 4.4, we have $u_n\to u$ in $H_r^1(\mathbb{R}^N)$. Then $p=I(u)$ with $u\in P$. By Lemma 2.9, we get $I'(u)=0$. □

## 5. The case $\mu > 4$

When $\mu > 4$, we have $22^*_\mu < 4$. For $\lambda \in [\frac{1}{2}, 1]$, define $I_\lambda(u) = A(u) - \lambda B(u)$, $u \in H^1_r(\mathbb{R}^N)$. Here

$$A(u) = \frac{1}{2}\|u\|^2 + \frac{b}{4}\left(\int\limits_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x\right)^2,$$
$$B(u) = \frac{1}{2}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{H(u(y))H(u(x))}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y. \tag{5.1}$$

Then $I_\lambda : H^1_r(\mathbb{R}^N) \mapsto \mathbb{R}$ is of class $C^1$ and critical points of $I_\lambda$ are solutions of (1.2). In order to prove Theorem 1.4, we need the following result.

**Theorem 5.1** *([9]). Let $(X, \|.\|_X)$ be a Banach space and let $J \subset \mathbb{R}^+$ be an interval. Consider a family $(J_\lambda)_{\lambda \in J}$ of $C^1$-functionals on $X$ of the form*

$$J_\lambda(u) = A(u) - \lambda B(u), \quad \forall\ \lambda \in J,$$

*where $B(u) \geq 0$ for any $u \in X$ and either $A(u) \to +\infty$ or $B(u) \to +\infty$ as $\|u\|_X \to \infty$. Assume there exist two points $v_1$, $v_2$ in $X$ such that*

$$c_\lambda = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J_\lambda(\gamma(t)) > \max\{J_\lambda(v_1), J_\lambda(v_2)\}, \quad \forall\ \lambda \in J,$$

*where $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = v_1, \gamma(1) = v_2\}$. Then for almost every $\lambda \in J$, there is a sequence $\{v_n\} \subset X$ such that $\{v_n\}$ is bounded, $J_\lambda(v_n) \to c_\lambda$ and $J'_\lambda(v_n) \to 0$ in $X^{-1}$. Moreover, the map $\lambda \to c_\lambda$ is continuous from the left.*

**Lemma 5.1.** *Let $\lambda \in [\frac{1}{2}, 1]$. Assume that $(f_1)$ holds. If $\{u_n\} \subset H^1_r(\mathbb{R}^N)$ is a sequence such that $\|u_n\|$ is bounded, $I_\lambda(u_n) \to c$ and $I'_\lambda(u_n) \to 0$, then there exists $a_1 > 0$ (or $b_1 > 0$) such that $\{u_n\}$ converges strongly in $H^1_r(\mathbb{R}^N)$ up to a subsequence for $a > a_1$ (or $b > b_1$).*

**Proof.** Let $\lambda \in [\frac{1}{2}, 1]$. Since $\|u_n\|$ is bounded, we assume $u_n \rightharpoonup u_\lambda$ weakly in $H^1_r(\mathbb{R}^N)$. Let $\tilde{A}(u) = \frac{1}{2}\|u\|^2 + \frac{bA}{2}\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x$, where $A = \lim_{n\to\infty}\int_{\mathbb{R}^N} |\nabla u_n|^2 \mathrm{d}x$. For $\lambda \in [\frac{1}{2}, 1]$, define $\tilde{I}_\lambda(u) = \tilde{A}(u) - \lambda B(u)$, where $u \in H^1_r(\mathbb{R}^N)$. Then $\tilde{I}'_\lambda(u_n) \to 0$. By Lemma 2.7,

$$\begin{aligned} o_n(1) &= \big(\tilde{I}'_\lambda(u_n), u_n\big) - \big(\tilde{I}'_\lambda(u_\lambda), u_\lambda\big) \\ &= \|v_n\|^2 + bA\int\limits_{\mathbb{R}^N} |\nabla v_n|^2 \mathrm{d}x - 2^*_\mu \lambda \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{|v_n(y)|^{2^*_\mu}|v_n(x)|^{2^*_\mu}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y + o_n(1). \end{aligned} \tag{5.2}$$

By (5.2) and Lemma 2.1,

$$\begin{aligned} &\lim_{n\to\infty}\|v_n\|^2 + b\left(\lim_{n\to\infty}\int\limits_{\mathbb{R}^N} |\nabla v_n|^2 \mathrm{d}x\right)^2 \\ &\leq 2^*_\mu C(N,\mu)\left(\lim_{n\to\infty}\int\limits_{\mathbb{R}^N} |v_n|^{2^*}\mathrm{d}x\right)^{\frac{2N-\mu}{N}} \leq \frac{2^*_\mu C(N,\mu)}{S^{\frac{2N-\mu}{N-2}}}\left(\lim_{n\to\infty}\int\limits_{\mathbb{R}^N} |\nabla v_n|^2 \mathrm{d}x\right)^{\frac{2N-\mu}{N-2}}. \end{aligned} \tag{5.3}$$

By (5.3) and the Young's inequality,

$$
\begin{aligned}
&\lim_{n\to\infty}\|v_n\|^2 + b\left(\lim_{n\to\infty}\int\limits_{\mathbb{R}^N}|\nabla v_n|^2\mathrm{d}x\right)^2\\
&\le a_1\lim_{n\to\infty}\int\limits_{\mathbb{R}^N}|\nabla v_n|^2\mathrm{d}x + b\left(\lim_{n\to\infty}\int\limits_{\mathbb{R}^N}|\nabla v_n|^2\mathrm{d}x\right)^2,
\end{aligned}\tag{5.4}
$$

or

$$
\begin{aligned}
&\lim_{n\to\infty}\|v_n\|^2 + b\left(\lim_{n\to\infty}\int\limits_{\mathbb{R}^N}|\nabla v_n|^2\mathrm{d}x\right)^2\\
&\le a\lim_{n\to\infty}\int\limits_{\mathbb{R}^N}|\nabla v_n|^2\mathrm{d}x + b_1\left(\lim_{n\to\infty}\int\limits_{\mathbb{R}^N}|\nabla v_n|^2\mathrm{d}x\right)^2,
\end{aligned}\tag{5.5}
$$

where $a_1$, $b_1$ are given in (1.3). Then $u_n \to u$ in $H_r^1(\mathbb{R}^N)$ for $a > a_1$ (or $b > b_1$). $\square$

**Lemma 5.2.** *Let $a > a_1$ (or $b > b_1$). Assume that $(f_1)$ and $(f_2')$ hold. Then there exist a large $\beta_2 > 0$ and $\varrho_0 > 0$ such that for $\beta > \beta_2$ and almost every $\lambda \in [\frac{1}{2}, 1]$, there exists a sequence $\{u_n\} \subset H_r^1(\mathbb{R}^N)$ such that $\|u_n\|$ is bounded, $I_\lambda(u_n) \to c_\lambda \ge \varrho_0$ and $I_\lambda'(u_n) \to 0$. Moreover, the map $\lambda \to c_\lambda$ is continuous from the left.*

**Proof.** By (5.1) and $(f_2')$, we have $B(u) \ge 0$ and $A(u) \to +\infty$ as $\|u\| \to \infty$. For $R > 0$, define $w_R(x) = \xi$ for $|x| \le R$, $w_R(x) = 0$ for $|x| \ge R+1$, $w_R(x) = \xi(R+1-|x|)$ for $R \le |x| \le R+1$. Then $w_R \in H_r^1(\mathbb{R}^N)$. By $(f_2')$, we get

$$
\begin{aligned}
&\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{H(w_R(y))H(w_R(x))}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y\\
&\ge \beta^2\int\limits_{B_R(0)}\int\limits_{B_R(0)}\frac{F(w_R(y))F(w_R(x))}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y > 0.
\end{aligned}\tag{5.6}
$$

Then there exists a large $\beta_2 > 0$ such that $I_\lambda(w_R) \le I_{\frac{1}{2}}(w_R) < 0$ for $\beta > \beta_2$. Let $\beta > \beta_2$. By Lemma 2.1, we get $|B(u)| \le \frac{1}{2}C(N,\mu)\|H(u)\|^2_{\frac{2N}{2N-\mu}}$. Then by $(f_1)$, there exists $C' > 0$ such that

$$
\begin{aligned}
|B(u)| \le& C\left(\int\limits_{\mathbb{R}^N}|u|^2\mathrm{d}x\right)^{\frac{2N-\mu}{N}} + C\left(\int\limits_{\mathbb{R}^N}|u|^{2^*}\mathrm{d}x\right)^{\frac{2N-\mu}{N}}\\
\le& C'\left(\|u\|^{\frac{2(2N-\mu)}{N}} + \|u\|^{\frac{2(2N-\mu)}{N-2}}\right).
\end{aligned}\tag{5.7}
$$

By (5.7), we have $I_\lambda(u) \ge \frac{1}{2}\|u\|^2 - C'\left(\|u\|^{\frac{2(2N-\mu)}{N}} + \|u\|^{\frac{2(2N-\mu)}{N-2}}\right)$. Then there exist $\rho_0 \in (0, \|w_R\|)$ and $\varrho_0 > 0$ independent of $\lambda$ such that $I_\lambda(u) \ge \varrho_0$ for $\|u\| = \rho_0$. Also, $I_\lambda(0) = 0$. By Theorem 5.1, for almost every $\lambda \in [\frac{1}{2}, 1]$, there exists a sequence $\{u_n\} \subset H_r^1(\mathbb{R}^N)$ such that $\|u_n\|$ is bounded, $I_\lambda(u_n) \to c_\lambda \ge \varrho_0$ and $I_\lambda'(u_n) \to 0$. Moreover, the map $\lambda \to c_\lambda$ is continuous from the left. $\square$

**Proof of Theorem 1.4.** Let $a > a_1$ (or $b > b_1$) and $\beta > \beta_2$. By Lemmas 5.1-5.2, for almost every $\lambda \in \left[\frac{1}{2}, 1\right]$, there exists a sequence $\{u_n\} \subset H_r^1(\mathbb{R}^N)$ such that $u_n \to u_\lambda$ in $H_r^1(\mathbb{R}^N)$, $I_\lambda(u_n) \to c_\lambda \geq \varrho_0 > 0$ and $I'_\lambda(u_n) \to 0$. Then $I_\lambda(u_\lambda) = c_\lambda \geq \varrho_0$ and $I'_\lambda(u_\lambda) = 0$ for almost every $\lambda \in \left[\frac{1}{2}, 1\right]$, that is, problem (1.2) has at least a nontrivial solution $u_\lambda$ for almost every $\lambda \in [\frac{1}{2}, 1]$. By Lemma 5.2, there exist $\lambda_n \uparrow 1$ and $\{u_{\lambda_n}\} \subset H_r^1(\mathbb{R}^N) \setminus \{0\}$ such that $I_{\lambda_n}(u_{\lambda_n}) = c_{\lambda_n} \to c_1 \geq \varrho_0$ and $I'_{\lambda_n}(u_{\lambda_n}) = 0$. When $\|u_{\lambda_n}\|$ is bounded, we have $I(u_{\lambda_n}) \to c_1 \geq \varrho_0$ and $I'(u_{\lambda_n}) \to 0$. By Lemma 5.1, there exists $u \in H_r^1(\mathbb{R}^N)$ such that $u_{\lambda_n} \to u$ in $H_r^1(\mathbb{R}^N)$. Then $I(u) = c_1 > 0$ and $I'(u) = 0$, that is, problem (1.1) has at least a nontrivial solution. $\square$

Now we prove Theorem 1.5. Define the functional on $H_0^1(\Omega)$ by

$$J(u) = \frac{1}{2}\|u\|^2_{H_0^1} + \frac{b}{4}\left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)^2 - \frac{1}{2}\int_\Omega\int_\Omega \frac{H(u(y))H(u(x))}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y, \tag{5.8}$$

where $\|u\|_{H_0^1} := \left(\int_\Omega a|\nabla u|^2 + |u|^2 \mathrm{d}x\right)^{\frac{1}{2}}$ is the norm on $H_0^1(\Omega)$. Then $J : H_0^1(\Omega) \mapsto \mathbb{R}$ is of class $C^1$ and critical points of $J$ are solutions of (1.4). Let $\lambda_1 = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 \mathrm{d}x}{\int_\Omega |u|^2 \mathrm{d}x}$. Then

$$a\int_\Omega |\nabla u|^2 \mathrm{d}x \leq \|u\|^2_{H_0^1} \leq \left(a + \frac{1}{\lambda_1}\right)\int_\Omega |\nabla u|^2 \mathrm{d}x. \tag{5.9}$$

**Proof of Theorem 1.5** ($i$)**.** Let $a > a_1$ (or $b > b_1$). Without loss of generality, we may assume $0 \in \Omega$. Choose $r > 0$ such that $B_{2r}(0) \subset \Omega$. Define a function $w_r \in H_0^1(\Omega)$ such that $w_r(x) = \xi$ for $|x| \leq r$ and $w_r(x) = 0$ for $|x| \geq 2r$. By $(f_2')$, we get

$$\int_\Omega\int_\Omega \frac{H(w_r(y))H(w_r(x))}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y \geq \beta^2 \int_{B_r(0)}\int_{B_r(0)} \frac{F(w_r(y))F(w_r(x))}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y > 0.$$

Then there exists a large $\beta_3 > 0$ such that $J(w_r) < 0$ for $\beta > \beta_3$.

Let $\beta > \beta_3$. We prove $J(u) \to +\infty$ as $\|u\|_{H_0^1} \to \infty$. In fact, by $(f_1)$ and Lemma 2.1, there exists $C_1 > 0$ such that

$$\begin{aligned}
&\int_\Omega\int_\Omega \frac{H(u(y))H(u(x))}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y \\
&\leq C_1\left(\int_\Omega |u|^2 \mathrm{d}x\right)^{\frac{2N-\mu}{N}} + C_1\left(\int_\Omega |u|^{2^*} \mathrm{d}x\right)^{\frac{2N-\mu}{N}} \\
&\leq \frac{C_1}{\lambda_1^{\frac{2N-\mu}{N}}}\left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)^{\frac{2N-\mu}{N}} + \frac{C_1}{S^{\frac{2N-\mu}{N-2}}}\left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)^{\frac{2N-\mu}{N-2}}.
\end{aligned} \tag{5.10}$$

By (5.9)-(5.10),

$$\begin{aligned}
J(u) \geq &\frac{1}{2}\|u\|^2_{H_0^1} + \frac{b}{4\left(a + \frac{1}{\lambda_1}\right)^2}\|u\|^4_{H_0^1} - \frac{C_1}{2a^{\frac{2N-\mu}{N}}\lambda_1^{\frac{2N-\mu}{N}}}\|u\|_{H_0^1}^{\frac{2(2N-\mu)}{N}} \\
&- \frac{C_1}{2a^{\frac{2N-\mu}{N-2}}S^{\frac{2N-\mu}{N-2}}}\|u\|_{H_0^1}^{\frac{2(2N-\mu)}{N-2}}.
\end{aligned} \tag{5.11}$$

Then $J(u) \to +\infty$ as $\|u\|_{H_0^1} \to \infty$.

Since $J(w_r) < 0$ and $J(u) \to +\infty$ as $\|u\|_{H_0^1} \to \infty$, by the Ekeland variational principle, there exist a constant $d_0 < 0$ and a bounded sequence $\{u_n\} \subset H_0^1(\Omega)$ such that $J(u_n) \to d$ and $J'(u_n) \to 0$. Similar to the argument of Lemma 5.1, we obtain that there exists $u_1 \in H_0^1(\Omega)$ such that $u_n \to u_1$ in $H_0^1(\Omega)$. Then $J(u_1) < 0$ and $J'(u_1) = 0$. By (5.9)-(5.10), there exists $C_2 > 0$ such that

$$J(u) \geq \frac{1}{2}\|u\|_{H_0^1}^2 - C_2\left(\|u\|_{H_0^1}^{\frac{2(2N-\mu)}{N}} + \|u\|_{H_0^1}^{\frac{2(2N-\mu)}{N-2}}\right).$$

Then there exist $\sigma_0 \in \left(0, \|w_r\|_{H_0^1}\right)$ and $\eta_0 > 0$ such that $J(u) \geq \eta_0$ for $\|u\|_{H_0^1} = \sigma_0$. Also, $J(0) = 0$. By the mountain pass lemma in [2], there exists $\{u_n\} \subset H_0^1(\Omega)$ such that $J(u_n) \to c \geq \eta_0$ and $J'(u_n) \to 0$. Since $J(u) \to +\infty$ as $\|u\|_{H_0^1} \to \infty$, we know $\|u_n\|_{H_0^1}$ is bounded. Similar to the argument of Lemma 5.1, we derive that there exists $u_2 \in H_0^1(\Omega)$ such that $u_n \to u_2$ in $H_0^1(\Omega)$. Then $J(u_2) > 0$ and $J'(u_2) = 0$. $\square$

**Proof of Theorem 1.5** $(ii)$**.** Assume that $(J'(u), u) = 0$ holds. By Lemma 2.1,

$$\begin{aligned}
&\|u\|_{H_0^1}^2 + b\left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)^2 \\
&\leq \beta^2 C(N,\mu)\|F(u)\|_{L^{\frac{2N}{2N-\mu}}(\Omega)}\|f(u)u\|_{L^{\frac{2N}{2N-\mu}}(\Omega)} + 2_\mu^* C(N,\mu)\||u|^{2_\mu^*}\|_{L^{\frac{2N}{2N-\mu}}(\Omega)}^2 \\
&\quad + \beta 2_\mu^* C(N,\mu)\|F(u)\|_{L^{\frac{2N}{2N-\mu}}(\Omega)}\||u|^{2_\mu^*}\|_{L^{\frac{2N}{2N-\mu}}(\Omega)} \\
&\quad + \beta C(N,\mu)\|f(u)u\|_{L^{\frac{2N}{2N-\mu}}(\Omega)}\||u|^{2_\mu^*}\|_{L^{\frac{2N}{2N-\mu}}(\Omega)}.
\end{aligned} \tag{5.12}$$

By $(f_1)$, we get $\max\{|F(u)|, |f(u)u|\} \leq C|u|^{\frac{2N-\mu}{N}} + C|u|^{2_\mu^*}$ for $u \in \mathbb{R}$. Then by (5.12), there exists $C_3 > 0$ such that

$$\begin{aligned}
&\|u\|_{H_0^1}^2 + b\left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)^2 \\
&\leq C_3\left[\left(\int_\Omega |u|^2 \mathrm{d}x\right)^{\frac{2N-\mu}{N}} + \left(\int_\Omega |u|^{2^*} \mathrm{d}x\right)^{\frac{2N-\mu}{N}}\right] \\
&\leq \frac{C_3}{\lambda_1^{\frac{2N-\mu}{N}}}\left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)^{\frac{2N-\mu}{N}} + \frac{C_3}{S^{\frac{2N-\mu}{N-2}}}\left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)^{\frac{2N-\mu}{N-2}}.
\end{aligned} \tag{5.13}$$

Since $1 < \max\left\{\frac{2N-\mu}{N}, \frac{2N-\mu}{N-2}\right\} < 2$, by (5.13) and the Young's inequality, we obtain that there exists a constant $a_2 > 0$ such that

$$\|u\|_{H_0^1}^2 + b\left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)^2 \leq b\left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)^2 + a_2\int_\Omega |\nabla u|^2 \mathrm{d}x, \tag{5.14}$$

or there exists a constant $b_2 > 0$ such that

$$\|u\|_{H_0^1}^2 + b\left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)^2 \le \frac{a}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x + b_2\left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)^2. \tag{5.15}$$

Let $a > a_2$ (or $b > b_2$). By (5.14)-(5.15), we get $\|u\|_{H_0^1} = 0$. $\square$

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