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# Spectrum preservers revisited ☆

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ABSTRACT

We classify spectrum preserving additive bijections which map the set of linear (possibly unbounded) maps on a complex Banach space $\mathcal{X}$ to the set of linear (possibly unbounded) maps on a complex Banach space $\mathcal{Y}$.

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## 1. Introduction

The spectrum of unbounded linear operators on Hilbert or Banach spaces plays an important role in different areas, for example in the theory of partial differential equations (consider, e.g., Sturm-Liouville theory), in the mathematical formulation of quantum mechanics (see, e.g. [7,12]), or in financial mathematics (analysis of valuation of financial derivatives, see, e.g. [1,4]). Due to its importance in analysis, quite a few papers were investigating maps (usually linear) which preserve the spectrum. This was done on operator algebras (see, e.g., [6,9,13]) and on more general Banach algebras (see, e.g., [2]). However, there are not many results on preservers connected with sets of unbounded operators (for some results see [14,15]). It is the aim of this paper to extend the study of spectrum preservers to unbounded linear operators.

Let $\mathcal{X}$ be a complex Banach space. A (perhaps unbounded) operator $T$ on $\mathcal{X}$ is a linear map defined on a subspace $\mathrm{Dom}(T) \subseteq \mathcal{X}$ with the range, $\mathrm{Im}(T)$, contained in $\mathcal{X}$. The set of all such operators will be denoted

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by $\mathrm{Unb}(\mathcal{X})$ and, except for 0-dimensional $\mathcal{X}$, properly contains the set $\mathbb{B}(\mathcal{X})$ of all bounded (everywhere defined) operators. Two operators are equal if they have the same domain and agree on all the vectors from their common domain. Hence the identity on $\mathcal{X}$ and its restriction to a proper subspace are not the same operators. Ordinary algebraic operations with operators are the usual ones with the following domain considerations:

$$\mathrm{Dom}(T+S) = \mathrm{Dom}(T) \cap \mathrm{Dom}(S).$$
$$\mathrm{Dom}(ST) = \{x \in \mathrm{Dom}(T);\ \ Tx \in \mathrm{Dom}(S)\}.$$

In particular the distributive law $A(B+C) = AB + AC$ may fail in the set $\mathrm{Unb}(\mathcal{X})$ because of domain restrictions. For example, if the range of $B$ is nonzero and $\mathrm{Im}(B) \cap \mathrm{Dom}(A) = \{0\}$, then $AB - AB = 0|_{\mathrm{Ker}\,B}$ (i.e., a zero map on the null space of $B$) but $A(B-B) = 0|_{\mathrm{Dom}(B)} \neq 0|_{\mathrm{Ker}\,B}$.

Note that an everywhere defined operator which maps each vector to 0 is an additive identity in the set $\mathrm{Unb}(\mathcal{X})$; we denote it as usual by 0. However, the set $\mathrm{Unb}(\mathcal{X})$ *is not a vector space.* Scalar multiplication and addition are well-defined and satisfy the usual properties (commutativity, associativity, distributivity) but additive inverses may not exist: if the domain of $A$ is a proper subspace then no operator $T$ satisfies $A + T = 0$. We do have $A + (-A) = 0|_{\mathrm{Dom}(A)}$ but this is a restriction of 0 operator, hence different from it. Instead $\mathrm{Unb}(\mathcal{X})$ behaves like a stack of vector spaces indexed by the domains of the operators, with addition providing a way to move 'down' the stack. We record the following useful property:

$$A + 0|_{\mathcal{D}} = A|_{\mathrm{Dom}(A)\cap\mathcal{D}}.$$

The resolvent set of $T$ is the set of all $\lambda \in \mathbb{C}$ such that $T - \lambda\mathrm{I}$ is a bijective map from $\mathrm{Dom}(T)$ onto $\mathcal{X}$ and whose inverse is bounded (see a book by Rudin [11, 13.26]). More precisely there exists a bounded linear operator $B$ such that

$$B(T - \lambda\mathrm{I}\,) = \mathrm{I}\,|_{\mathrm{Dom}(T)} \quad \text{and} \quad (T - \lambda\mathrm{I}\,)B = \mathrm{I}\,.$$

The complement of the resolvent set is called the spectrum of $T$ and denoted by $\mathrm{Sp}\,(T)$. Similarly to bounded operators the spectrum of unbounded operators is always closed, however, unlike with bounded operators, the spectrum of unbounded operators in infinite-dimensional Hilbert spaces may be any closed subset of $\mathbb{C}$, including the empty set and $\mathbb{C}$ (see [11, Exercises 17–20, p. 365]).

A rank-one operator $T$ maps its domain into a one-dimensional subspace $\mathbb{C}x$. Hence, for every $u \in \mathrm{Dom}(T)$ we can find a scalar $f(u) \in \mathbb{C}$ with $Tu = f(u)x$. The map $f$ is clearly a linear functional defined on $\mathrm{Dom}(T)$. Hence rank-one operators take the form

$$T = x \otimes f \colon u \mapsto f(u)x.$$

## 2. Statement of the main result

Let $\mathcal{X}$ and $\mathcal{Y}$ be complex Banach spaces. We say that a map $\Phi\colon \mathrm{Unb}(\mathcal{X}) \to \mathrm{Unb}(\mathcal{Y})$ is additive if $C = A+B$ implies $\Phi(C) = \Phi(A) + \Phi(B)$ for $A, B, C \in \mathrm{Unb}(\mathcal{X})$. Moreover, we say that a map $\Phi$ preserves spectrum if $\mathrm{Sp}\,(X) = \mathrm{Sp}\,(\Phi(X))$ for every $X \in \mathrm{Unb}(\mathcal{X})$. Before stating our main result, let us give an example of a spectrum preserving bijective linear map.

**Example 1.** Consider a linear bounded bijection $A\colon \mathcal{X} \to \mathcal{Y}$. For $T \in \mathrm{Unb}(\mathcal{X})$ with $\mathrm{Dom}(T) = \mathcal{D} \subseteq \mathcal{X}$, let $B := ATA^{-1}$ be an operator in $\mathcal{Y}$ with domain $\mathrm{Dom}(B) := A(\mathcal{D}) \subseteq \mathcal{Y}$, which maps $y = Ax \in \mathrm{Dom}(B)$ into $ATA^{-1}y = ATx \in \mathcal{Y}$.

Clearly, $T\colon \mathcal{D} \to \mathcal{X}$ is bijective if and only if $ATA^{-1}\colon \mathrm{Dom}(B) \to \mathcal{Y}$ is. Also, the algebraic inverse, $T^{-1}$ is bounded if and only if $(ATA^{-1})^{-1} = AT^{-1}A^{-1}$ is bounded. Moreover, $A\mathrm{I}_{\mathcal{X}}A^{-1} = \mathrm{I}_{\mathcal{Y}}$. Hence, $X \mapsto AXA^{-1}$ is a spectrum preserving linear bijection from $\mathrm{Unb}\,(\mathcal{X})$ to $\mathrm{Unb}\,(\mathcal{Y})$.

Our main result states that Example 1 exhibits all additive spectrum preserving bijections.

**Main Theorem.** *Let $\mathcal{X}, \mathcal{Y}$ be complex Banach spaces with $\dim \mathcal{X} \geq 2$ and let $\Phi\colon \mathrm{Unb}\,(\mathcal{X}) \to \mathrm{Unb}\,(\mathcal{Y})$ be a spectrum preserving additive bijection. Then there exists a bounded linear bijection $Z\colon \mathcal{X} \to \mathcal{Y}$ such that*

$$\Phi(X) = ZXZ^{-1}.$$

Let us mention that in the case of *bounded operators*, additive spectrum preserving bijections can be also of the form $X \mapsto ZX^*Z^{-1}$ where $X^*$ is the adjoint operator, see [9]. The lack of this extra map in our main Theorem shows yet again that the adjoint cannot be (uniquely) defined for every unbounded operator.

From our main Theorem we can immediately conclude the following:

(a) Every spectrum preserving additive bijection on $\mathrm{Unb}\,(\mathcal{X})$ is linear and maps bounded operators onto bounded operators and unbounded ones onto unbounded ones.

(b) Every spectrum preserving additive bijection on $\mathrm{Unb}\,(\mathcal{X})$ preserves the parts in which the spectrum of an operator is traditionally decomposed (point, continuous, residual, ...).

We postpone the proof till some auxiliary results are proven. Also, let us mention that the majority of the arguments, with the exception of the concluding ones, are valid if one assumes surjectivity only. We will utilize this fact in the concluding section where we list some important examples of Banach spaces (which include Hilbert spaces) where our theorem holds without imposing injectivity in advance.

## 3. General properties of additive maps on $\mathrm{Unb}\,(\mathcal{X})$

Throughout this section, $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces and $\Phi\colon \mathrm{Unb}\,(\mathcal{X}) \to \mathrm{Unb}\,(\mathcal{Y})$ is an additive map. To avoid trivialities we also assume without further notice that $\dim \mathcal{X} \geq 2$. We investigate the properties of $\Phi$, some of which are self-evident for additive maps on vector spaces. However, as the set $\mathrm{Unb}\,(\mathcal{X})$ is not a vector space but rather a disjoint union of vector spaces we need to prove them from scratch.

To each additive $\Phi\colon \mathrm{Unb}\,(\mathcal{X}) \to \mathrm{Unb}\,(\mathcal{Y})$ we associate an operation, denoted by $\mathcal{D} \mapsto \widehat{\mathcal{D}}$ which maps subspaces of $\mathcal{X}$ into subspaces of $\mathcal{Y}$ by the following rule: Given a subspace $\mathcal{D} \subseteq \mathcal{X}$ let

$$\widehat{\mathcal{D}} := \mathrm{Dom}(\Phi(0|_{\mathcal{D}})) \subseteq \mathcal{Y}.$$

**Lemma 2.** *The following hold for every additive $\Phi\colon \mathrm{Unb}\,(\mathcal{X}) \to \mathrm{Unb}\,(\mathcal{Y})$.*

(i) $\Phi(0|_{\mathcal{D}}) = 0|_{\widehat{\mathcal{D}}}$ *for each subspace* $\mathcal{D} \subseteq \mathcal{X}$.
(ii) $\mathrm{Dom}(A) = \mathrm{Dom}(B)$ *implies* $\mathrm{Dom}(\Phi(A)) = \mathrm{Dom}(\Phi(B))$ *for* $A, B \in \mathrm{Unb}\,(\mathcal{X})$.
(iii) $\Phi(-A) = -\Phi(A)$.
(iv) $\widehat{\mathcal{D}_1 \cap \mathcal{D}_2} = \widehat{\mathcal{D}_1} \cap \widehat{\mathcal{D}_2}$ *for subspaces* $\mathcal{D}_1, \mathcal{D}_2 \subseteq \mathcal{X}$. *In particular, if* $\mathcal{D}_1 \subseteq \mathcal{D}_2$ *then also* $\widehat{\mathcal{D}_1} \subseteq \widehat{\mathcal{D}_2}$.

**Proof.** (i). Apply $\Phi$ on $0|_{\mathcal{D}} = 0|_{\mathcal{D}} + 0|_{\mathcal{D}}$ and add $-\Phi(0|_{\mathcal{D}})$ whose domain $\widehat{\mathcal{D}}$ coincides with the domain of $\Phi(0|_{\mathcal{D}})$.

(ii). Let $\mathcal{D}$ be the common domain for $A$ and $B$. Then $\Phi(A) = \Phi(A + 0|_{\mathcal{D}}) = \Phi(A) + \Phi(0|_{\mathcal{D}}) = \Phi(A) + 0|_{\widehat{\mathcal{D}}}$ by (i) above. It follows that $\mathrm{Dom}(\Phi(A)) \subseteq \widehat{\mathcal{D}} = \mathrm{Dom}(0|_{\widehat{\mathcal{D}}})$.

Conversely, $0|_{\widehat{\mathcal{D}}} = \Phi(0|_{\mathcal{D}}) = \Phi(A + (-A)) = \Phi(A) + \Phi(-A)$ implies that the domain of $0|_{\widehat{\mathcal{D}}}$ equals the intersection of the domains of $\Phi(A)$ and $\Phi(-A)$. Thus, $\mathrm{Dom}(\Phi(A)) \supseteq \mathrm{Dom}(0|_{\widehat{\mathcal{D}}}) = \widehat{\mathcal{D}}$. Combining with the above gives $\mathrm{Dom}(\Phi(A)) = \widehat{\mathcal{D}}$. The same derivation is applicable with any $B$ in place of $A$ as long as $B$ and $A$ have the same domain.

(iii). $A$ and $-A$ share the same domain $\mathcal{D}$, so by (ii) above the same holds for $\Phi(A)$ and $\Phi(-A)$, and also for $\Phi(A + (-A)) = \Phi(0|_{\mathcal{D}}) = 0|_{\widehat{\mathcal{D}}}$. The rest is trivial.

(iv). Apply additive $\Phi$ on $0|_{\mathcal{D}_1 \cap \mathcal{D}_2} = 0|_{\mathcal{D}_1} + 0|_{\mathcal{D}_2}$. □

**Lemma 3.** *Let $\mathcal{X}$ be a Banach space and let $\Phi\colon \mathrm{Unb}\,(\mathcal{X}) \to \mathrm{Unb}\,(\mathcal{X})$ be an additive map which fixes all operators with one-dimensional domain. Then $\Phi$ is the identity.*

**Proof.** Choose any operator $T \in \mathrm{Unb}\,(\mathcal{X})$ and any nonzero vector $x \in \mathrm{Dom}(T) \subseteq \mathcal{X}$; denote by $\langle x \rangle$ the one-dimensional space spanned by $x$. Then, $T|_{\langle x \rangle} = T + 0|_{\langle x \rangle}$. By the assumptions,

$$T|_{\langle x \rangle} = \Phi(T|_{\langle x \rangle}) = \Phi(T + 0|_{\langle x \rangle}) = \Phi(T) + \Phi(0|_{\langle x \rangle}) = \Phi(T) + 0|_{\langle x \rangle} = \Phi(T)|_{\mathrm{Dom}(\Phi(T)) \cap \langle x \rangle}.$$

Clearly then, $x \in \mathrm{Dom}(\Phi(T))$ and $Tx = \Phi(T)x$. It follows that domain of $\Phi(T)$ contains $\mathrm{Dom}(T)$ and both operators $T$ and $\Phi(T)$ agree on $\mathrm{Dom}(T)$.

If $x \notin \mathrm{Dom}(T)$ then $T + 0|_{\langle x \rangle} = 0|_0$ (a zero operator on 0-dim space). Note that $0|_0 = 0|_{\langle x \rangle} + 0|_{\langle y \rangle}$ where $x, y$ are arbitrarily chosen linearly independent vectors, so $\Phi(0|_0) = \Phi(0|_{\langle x \rangle}) + \Phi(0|_{\langle y \rangle}) = 0|_{\langle x \rangle} + 0|_{\langle y \rangle} = 0|_0$ is also fixed by $\Phi$. Hence, we may repeat the above arguments to deduce that $x \notin \mathrm{Dom}(\Phi(T))$. Therefore, $T = \Phi(T)$. □

**Lemma 4.** *Let $\Phi\colon \mathrm{Unb}\,(\mathcal{X}) \to \mathrm{Unb}\,(\mathcal{X})$ be an additive map which fixes each everywhere defined bounded operator. If an operator $A \in \mathrm{Unb}\,(\mathcal{X})$ has a closed domain $\mathcal{D}$ and admits a bounded extension to $\mathcal{X}$, then $\Phi(A) = A|_{\mathcal{D}'}$ is a restriction to a subdomain $\mathcal{D}' \subseteq \mathcal{D}$.*

**Proof.** Let $\mathcal{D}$ be a closed domain of $A \in \mathrm{Unb}\,(\mathcal{X})$. By the Hahn-Banach theorem, $\mathcal{D}$ is the intersection of the kernels of a collection of bounded linear functionals $\left(f_\alpha\right)_\alpha$. Note that

$$\mathcal{D} \subseteq \mathrm{Ker}\,(x \otimes f_\alpha) \text{ is equivalent to } 0|_{\mathcal{D}} = 0|_{\mathcal{D}} + x \otimes f_\alpha. \tag{1}$$

Applying $\Phi$ on $0|_{\mathcal{D}} = 0|_{\mathcal{D}} + x \otimes f_\alpha$ and taking into account that $x \otimes f_\alpha$ is a bounded operator on $\mathcal{X}$, so kept fixed by $\Phi$, we obtain $0|_{\mathcal{D}'} = 0|_{\mathcal{D}'} + \Phi(x \otimes f_\alpha) = 0|_{\mathcal{D}'} + x \otimes f_\alpha$ where $0|_{\mathcal{D}'} = \Phi(0|_{\mathcal{D}})$. Consequently by (1),

$$\mathcal{D}' \subseteq \bigcap \mathrm{Ker}\,(x \otimes f_\alpha) = \mathcal{D}. \tag{2}$$

Since $\mathcal{D}'$ is the domain of $\Phi(0|_{\mathcal{D}})$, Lemma 2 (ii) implies that $\mathcal{D}'$ is also the domain of $\Phi(A)$.

Now, let $A_0$ be a bounded, everywhere defined extension of $A$. Then, applying $\Phi$ on $A = A_0 + 0|_{\mathcal{D}}$ and using the fact that a bounded operator $A_0$ is fixed by $\Phi$, we get $\Phi(A) = A_0 + 0|_{\mathcal{D}'} = A_0|_{\mathcal{D}'} = A|_{\mathcal{D}'}$ where the last equality is a consequence of (2). □

Unlike the preceding results, we now impose surjectivity or injectivity on $\Phi$.

**Lemma 5.** *If $\Phi\colon \mathrm{Unb}\,(\mathcal{X}) \to \mathrm{Unb}\,(\mathcal{Y})$ is an additive surjection, then $\Phi(0) = 0$ and $\Phi(0|_0) = 0|_0$. Hence, if $\mathrm{Dom}(A) = \mathcal{X}$, then also $\mathrm{Dom}(\Phi(A)) = \mathcal{Y}$.*

**Proof.** Choose $A \in \mathrm{Unb}\,(\mathcal{X})$ such that $\Phi(A) = 0$ and apply $\Phi$ on $A + 0 = A$. For the second claim, choose $A$ with $\Phi(A) = 0|_0$ and apply $\Phi$ on $A + 0|_0 = 0|_0$. The last claim follows from Lemma 2 (ii) by the fact that $\mathrm{Dom}(0) = \mathcal{X}$ and $\mathrm{Dom}(\Phi(0)) = \mathrm{Dom}(0|_{\mathcal{Y}}) = \mathcal{Y}$. □

We remark that Lemma 4 applies to each bounded operator defined on a closed and complemented subspace $\mathcal{D} \subseteq \mathcal{X}$. This fact will be used in the next corollary.

**Corollary 6.** *If an injective additive* $\Phi\colon \mathrm{Unb}\,(\mathcal{X}) \to \mathrm{Unb}\,(\mathcal{X})$ *fixes each everywhere defined bounded operator, then* $\Phi$ *is the identity.*

**Proof.** Choose any operator $A$ with one-dimensional domain $\mathcal{D}$. By Lemma 4, $\Phi(A) = A|_{\mathcal{D}'}$ for some subspace $\mathcal{D}' \subseteq \mathcal{D}$. The only possibilities are $\mathcal{D}' = 0$ or $\mathcal{D}' = \mathcal{D}$. The former case implies $\Phi(A) = 0|_0$; applying $\Phi$ on $A + 0|_0 = 0|_0$ gives $\Phi(A) = \Phi(0|_0) = 0|_0$, contradicting injectivity of $\Phi$.

The result then follows by Lemma 3. $\square$

Recall that the adjoint $A^*$ can also be defined for operators $A \in \mathrm{Unb}\,(\mathcal{X})$ with dense domain: $\mathrm{Dom}(A^*)$ consists of all functionals $f \in \mathcal{X}^*$ such that the functional $x \mapsto f(Ax)$ is bounded; then it can be uniquely extended to a bounded functional denoted by $(A^* f) \in \mathcal{X}^*$. Here it is essential that the domain is dense for otherwise the extension of $x \mapsto f(Ax)$ is not unique and consequently $A^* f$ is not well-defined. The next lemma builds along these assertions.

**Lemma 7.** *Assume* $\mathcal{X}^*$ *is the dual of a Banach space* $\mathcal{X}$. *If* $\Phi\colon \mathrm{Unb}\,(\mathcal{X}) \to \mathrm{Unb}\,(\mathcal{X}^*)$ *is an additive injection, then* $\Phi(T) = T^*$ *cannot hold for each bounded everywhere defined operator* $T$ *on* $\mathcal{X}$.

**Proof.** Suppose otherwise that $\Phi(T) = T^*$ does hold for each everywhere defined bounded operator $T$ on $\mathcal{X}$. Choose $A \in \mathrm{Unb}\,(\mathcal{X})$ with one-dimensional domain, $\mathrm{Dom}(A) = \langle x_0 \rangle$, and let

$$B = \Phi(A) \in \mathrm{Unb}\,(\mathcal{X}^*).$$

In particular, $\mathrm{Dom}(A)$ is complemented and in fact there exist at least two closed hyperplanes $\mathcal{W}_1, \mathcal{W}_2$ with

$$\mathcal{X} = \langle x_0 \rangle \oplus \mathcal{W}_i; \qquad i = 1, 2.$$

With respect to these two decompositions consider the two collections of bounded everywhere defined operators

$$T_\lambda^{(i)} := A \oplus \lambda \mathrm{I}_{\mathcal{W}_i}; \qquad \lambda \in \mathbb{C},\ i = 1, 2.$$

Observe that $A = T_\lambda^{(i)} + 0|_{\langle x_0 \rangle}$. Applying additive $\Phi$ gives, with the help of Lemma 2,

$$B = \left(T_\lambda^{(i)}\right)^* + 0|_{\mathcal{D}'} = \left(T_\lambda^{(i)}\right)^*|_{\mathcal{D}'}$$

for some subspace $\mathcal{D}' \subseteq \mathcal{X}^*$. In particular, if $f \in \mathrm{Dom}(B)$, and $x_i \in \mathcal{W}_i$, then $(Bf)\colon x_i \mapsto \lambda f(x_i)$, so

$$(Bf)|_{\mathcal{W}_i} = \lambda f|_{\mathcal{W}_i}$$

and since the parameter $\lambda \in \mathbb{C}$ is arbitrary we see that $f(\mathcal{W}_i) = 0$. Due to $\mathcal{X} = \mathcal{W}_1 + \mathcal{W}_2$ this implies that $f(\mathcal{X}) = f(\mathcal{W}_1 + \mathcal{W}_2) = 0$ so $\mathrm{Dom}(B) = 0$ is zero-dimensional. Therefore, $B = \Phi(A) = 0|_0$ contradicting injectivity of $\Phi$. $\square$

## 4. Spectral properties of unbounded operators

**Lemma 8.** *Suppose* $T$ *is an operator whose domain is a subspace of strictly smaller Hamel dimension than* $\mathcal{X}$. *Then* $\mathrm{Sp}\,(T) = \mathbb{C}$.

**Proof.** If $(\lambda \mathrm{I} - T)\colon \mathrm{Dom}(T) \to \mathcal{X}$ would be invertible for some $\lambda$ then its inverse $B$ would bijectively map space $\mathcal{X}$ onto the space $\mathrm{Dom}(T)$ of strictly smaller dimension, a contradiction. Hence, $\lambda\mathrm{I} - T$ is always noninvertible. $\square$

The next three results are motivated by Lemma 1, Lemma 4 and Theorem 1 in the paper by Jafarian and Sourour [6].

**Lemma 9.** $\mathrm{Sp}\,(A+T) \subseteq \mathrm{Sp}\,(T)$ *for all* $T \in \mathrm{Unb}\,(\mathcal{X})$ *if and only if* $A = 0_{\mathcal{X}}$.

**Proof.** For the nontrivial implication, we can follow the proof of Lemma 1 in [6] to show that the range of $A$ must be 0, so $A = 0|_{\mathcal{D}}$ for some domain $\mathcal{D}$. Suppose $\mathcal{D} \neq \mathcal{X}$. Then $A + I = I|_{\mathcal{D}}$ is not surjective, and hence not invertible. Thus $0 \in \mathrm{Sp}\,(A+I)$, so $\mathrm{Sp}\,(A+I)$ is not a subset of $\mathrm{Sp}\,(I)$, a contradiction. Hence $\mathcal{D} = \mathcal{X}$. $\square$

**Lemma 10.** *Let* $f\colon \mathrm{Dom}(f) \to \mathbb{C}$ *be a linear functional and let* $T$ *be an operator on* $\mathcal{X}$ *with* $\mathrm{Dom}(T) \subseteq \mathrm{Dom}(f)$. *If there exists* $\lambda_0 \in \mathbb{C} \setminus \mathrm{Sp}\,(T)$ *for which* $f \circ (\lambda_0\mathrm{I} - T)^{-1}$ *is a bounded functional, then* $f \circ (\lambda\mathrm{I} - T)^{-1}$ *is a bounded functional for every* $\lambda \in \mathbb{C} \setminus \mathrm{Sp}\,(T)$.

**Proof.** By the assumption, there exists a constant $\kappa$ so that

$$\sup_{x\neq 0} \frac{|f\big((\lambda_0\mathrm{I} - T)^{-1}x\big)|}{\|x\|} = \kappa.$$

Write $y = (\lambda_0\mathrm{I} - T)^{-1}x \in \mathrm{Dom}(T) \subseteq \mathrm{Dom}(f)$ to deduce that

$$\kappa = \sup_{y\in\mathrm{Dom}(T)\setminus\{0\}} \frac{|f\big(y\big)|}{\|(\lambda_0\mathrm{I} - T)y\|}$$

or equivalently,

$$|f\big(y\big)| \le \kappa\|(\lambda_0\mathrm{I} - T)y\|; \qquad y \in \mathrm{Dom}(T).$$

Choose now any $\lambda \in \mathbb{C} \setminus \mathrm{Sp}\,(T)$. Then, $(\lambda\mathrm{I} - T)$ is invertible with bounded inverse, hence there exists a constant $\mu = \|(\lambda\mathrm{I} - T)^{-1}\|$ such that

$$\|y\| \le \mu\|(\lambda\mathrm{I} - T)y\|; \qquad y \in \mathrm{Dom}(T). \tag{3}$$

It follows that

$$\begin{aligned} |f(y)| &\le \kappa\|(\lambda_0\mathrm{I} - T)y\| \le \kappa|\lambda_0 - \lambda| \cdot \|y\| + \kappa\|(\lambda\mathrm{I} - T)y\| \\ &\le \kappa\big(\mu\,|\lambda_0 - \lambda| + 1\big) \cdot \|(\lambda\mathrm{I} - T)y\| \end{aligned}$$

where at the end we used (3). This shows that $f \circ (\lambda\mathrm{I} - T)^{-1}$ is also bounded. $\square$

**Lemma 11.** *Let* $T, x \otimes f \in \mathrm{Unb}\,(\mathcal{X})$ *with* $x \neq 0$.

(i) *If* $\mathrm{Dom}(T)$ *is not contained in* $\mathrm{Dom}(f)$ *then*

$$\mathrm{Sp}\,(T + x \otimes f) \supseteq \mathbb{C} \setminus \mathrm{Sp}\,(T).$$

(ii) *If* $\mathrm{Dom}(T) \subseteq \mathrm{Dom}(f)$ *and there exists* $\lambda_0 \in \mathbb{C} \setminus \mathrm{Sp}\,(T)$ *such that the linear functional* $f \circ (\lambda_0 \mathrm{I} - T)^{-1} \colon \mathcal{X} \to \mathbb{C}$ *is unbounded, then again*

$$\mathrm{Sp}\,(T + x \otimes f) \supseteq \mathbb{C} \setminus \mathrm{Sp}\,(T).$$

(iii) *If* $\mathrm{Dom}(T) \subseteq \mathrm{Dom}(f)$ *and there exists* $\lambda_0 \in \mathbb{C} \setminus \mathrm{Sp}\,(T)$ *such that the linear functional* $f \circ (\lambda_0 \mathrm{I} - T)^{-1} \colon \mathcal{X} \to \mathbb{C}$ *is bounded, then*

$$\mathrm{Sp}\,(T + x \otimes f) \setminus \mathrm{Sp}\,(T) = \{\lambda \in \mathbb{C} \setminus \mathrm{Sp}\,(T);\ \ f\big((\lambda \mathrm{I} - T)^{-1} x\big) = 1\}.$$

**Proof.** The lemma is vacuously true if $\mathrm{Sp}\,(T) = \mathbb{C}$, so we suppose $\mathrm{Sp}\,(T) \neq \mathbb{C}$. Let $\mathcal{D} = \mathrm{Dom}(T) \cap \mathrm{Dom}(f)$ and let $\lambda \in \mathbb{C} \setminus \mathrm{Sp}\, T$. Since $(\lambda \mathrm{I} - T) \colon \mathrm{Dom}(T) \to \mathcal{X}$ is bijective with bounded inverse we have

$$\lambda \mathrm{I} - (T + x \otimes f) = (\lambda \mathrm{I} - T)(\mathrm{I}\,|_{\mathcal{D}} - y \otimes f|_{\mathcal{D}}); \qquad y = (\lambda \mathrm{I} - T)^{-1} x \in \mathrm{Dom}(T). \tag{4}$$

Temporarily assume that the left-hand side of (4) is invertible. Then, by bijectivity of $(\lambda \mathrm{I} - T) \colon \mathrm{Dom}(T) \to \mathcal{X}$ we must have that the second factor on the right-hand side of (4), i.e.,

$$A = I|_{\mathcal{D}} - y \otimes f|_{\mathcal{D}}$$

maps $\mathcal{D}$ bijectively onto $\mathrm{Dom}(T)$. In particular, as $y = (\lambda \mathrm{I} - T)^{-1} x \in \mathrm{Dom}(T)$, there exists nonzero $z \in \mathcal{D}$ with

$$y = Az = z - f(z) y.$$

It follows that $z \parallel y$. Then, $y \in \mathcal{D}$ so $\mathcal{D}$ is an invariant subspace for $A$. However, under assumption (i), it cannot be mapped by $A$ onto $\mathrm{Dom}(T)$ because $\mathcal{D}$ is a proper subspace of $\mathrm{Dom}(T)$, a contradiction. Therefore, under assumption (i), $\lambda \mathrm{I} - (T + x \otimes f)$ is never invertible if $\lambda \notin \mathrm{Sp}\,(T)$.

Henceforth, we assume $\mathrm{Dom}(T) \subseteq \mathrm{Dom}(f)$ and the left-hand side of (4) may no longer be invertible. Recall that $y \in \mathrm{Dom}(T) \subseteq \mathrm{Dom}(f)$. If $1 - f(y) = 0$ then $Ay = y - f(y)y = 0$, so $A$ is not injective and hence also $\lambda \mathrm{I} - T - x \otimes f = (\lambda \mathrm{I} - T)A$ is not invertible. If $1 - f(y) \neq 0$ define the operator $B \colon \mathrm{Dom}(T) \to \mathrm{Dom}(T)$ by

$$B = I|_{\mathrm{Dom}(T)} + \frac{1}{1 - f(y)}\, y \otimes f|_{\mathrm{Dom}(T)}.$$

One readily verifies that $BAz = z$ and $ABz = z$ for all $z \in \mathrm{Dom}(T)$, so

$$(\lambda \mathrm{I} - T)A \colon \mathrm{Dom}(T) \to \mathcal{X}$$

is a bijective map whose algebraic inverse equals

$$B(\lambda \mathrm{I} - T)^{-1} = (\lambda \mathrm{I} - T)^{-1} + \frac{y}{1 - f(y)} \big(f \circ (\lambda \mathrm{I} - T)^{-1}\big).$$

This algebraic inverse is bounded if and only if $g_\lambda := \big(f \circ (\lambda \mathrm{I} - T)^{-1}\big) \colon \mathcal{X} \to \mathbb{C}$ is a bounded linear functional. Now apply Lemma 10. We deduce that either $g_\lambda$ is always unbounded for $\lambda \in \mathbb{C} \setminus \mathrm{Sp}\,(T)$ in which case $\mathrm{Sp}\,(T + x \otimes f) \supseteq \mathbb{C} \setminus \mathrm{Sp}\,(T)$ as claimed in (ii) or else $g_\lambda$ is always bounded in which case $\mathrm{Sp}\,(T + x \otimes f) \setminus \mathrm{Sp}\,(T) = \{\lambda \in \mathbb{C} \setminus \mathrm{Sp}\,(T);\ \ f\big((\lambda \mathrm{I} - T)^{-1} x\big) = 1\}$ as claimed in (iii). □

**Remark 12.** By setting $T = 0$ in Lemma 11 one sees that if $A$ is rank-one operator that is not defined on all of $\mathcal{X}$ then $\mathrm{Sp}\,(A) = \mathbb{C}$. By setting $T = 0$ and $\lambda_0 = 1$ in item (ii) of this lemma one also observes that if $A$ is an everywhere defined unbounded rank-one operator then again $\mathrm{Sp}\,(A) = \mathbb{C}$.

**Lemma 13.** *Suppose $A = x \otimes f$ is a rank-one operator. If $T$ is an operator in $\mathcal{X}$ and if $c \in \mathbb{C} \setminus \{0, 1\}$, then one of the following holds:*

(i) $\mathrm{Sp}\,(T + A) \cap \mathrm{Sp}\,(T + cA) \subseteq \mathrm{Sp}\,(T)$.
(ii) $\mathbb{C} \setminus \mathrm{Sp}\,(T) \subseteq \mathrm{Sp}\,(T + A) \cap \mathrm{Sp}\,(T + cA)$.

**Proof.** If $\mathrm{Dom}(T)$ is not contained in $\mathrm{Dom}(f)$ or if the functional $\big(f \circ (\lambda\mathrm{I} - T)^{-1}\big)$ is not bounded for some $\lambda \in \mathbb{C} \setminus \mathrm{Sp}\,(T)$, then, by (i)–(ii) of Lemma 11 we get the second option.

However, if $\mathrm{Dom}(T) \subseteq \mathrm{Dom}(f)$ and $\big(f \circ (\lambda\mathrm{I} - T)^{-1}\big)$ is bounded for some $\lambda \notin \mathrm{Sp}\,(T)$, then (iii) of Lemma 11 holds. Clearly, with $\lambda \notin \mathrm{Sp}\,(T)$ we cannot have simultaneously $f\big((\lambda\mathrm{I} - T)^{-1}x\big) = f\big((\lambda\mathrm{I} - T)^{-1}(cx)\big) = 1$. So, $\mathrm{Sp}\,(T + A) \cap \mathrm{Sp}\,(T + cA)$ does not contain points outside $\mathrm{Sp}\,(T)$. □

The next lemma is a folklore result in the case of bounded everywhere defined operators (see, e.g., [6] for linear maps $\Phi$ and Omladič and Šemrl [9] for additive $\Phi$). We present the proof based on ideas from [6,9].

**Lemma 14.** *The following are equivalent for an operator $A$ in $\mathcal{X}$:*

(i) *$A$ is a bounded, everywhere defined operator of rank-one or zero.*
(ii) $\mathrm{Sp}\,(A) \neq \mathbb{C}$ *and for every operator $T$ and every $c \neq 1$ we have*

$$\mathrm{Sp}\,(T + A) \cap \mathrm{Sp}\,(T + cA) \subseteq \mathrm{Sp}\,(T).$$

(iii) $\mathrm{Sp}\,(A) \neq \mathbb{C}$ *and for every operator $T$ we have*

$$\mathrm{Sp}\,(T + A) \cap \mathrm{Sp}\,(T + 2A) \subseteq \mathrm{Sp}\,(T).$$

**Proof.** (i) $\Longrightarrow$ (ii). The case $A = 0$ is trivial, so we may write $A = x \otimes f$ for some nonzero vector $x$ and some nonzero bounded functional $f$. Clearly, $\mathrm{Sp}\,(A) \subseteq \{0, f(x)\} \neq \mathbb{C}$. Let $T \in \mathrm{Unb}\,(\mathcal{X})$ and let $\lambda \notin \mathrm{Sp}\,(T)$. Since $f$ is bounded, (iii) of Lemma 11 applies, by which $\lambda \in \mathrm{Sp}\,(T + cA)$ if and only if $cf((\lambda\mathrm{I} - T)^{-1}x) = 1$. Thus $\lambda \in \mathbb{C} \setminus \mathrm{Sp}\,(T)$ cannot belong to $\mathrm{Sp}\,(T + cA)$ for two distinct values of $c$.

(ii) $\Longrightarrow$ (iii). Clear.

(iii) $\Longrightarrow$ (i). We prove the contrapositive. If $A = 0|_{\mathcal{D}}$ with $\mathcal{D} \neq \mathcal{X}$, then $\mathrm{Sp}\,(A) = \mathbb{C}$, contradicting the assumptions in (iii). Likewise we get a contradiction if $A$ is a rank-one operator that is either unbounded or not defined on all of $\mathcal{X}$ (see Remark 12). Now suppose $\mathrm{rank}\, A \geq 2$. Then there exist linearly independent $x_1, x_2 \in \mathcal{X}$ such that $Ax_1 = y_1$ and $Ax_2 = y_2$ are linearly independent. Let $\mathcal{M} = \mathrm{span}\,\{x_1, x_2, y_1, y_2\}$, and let $\mathcal{M}^\perp \subseteq \mathcal{X}$ be a closed complement for $\mathcal{M}$. Define a bounded linear operator $T$ by $T|_{\mathcal{M}^\perp} = \mathrm{I}\,|_{\mathcal{M}^\perp}$, $Tx_1 = -y_1$, and $Tx_2 = -2y_2$. If $\dim \mathcal{M} = 2$ this completely defines $T$; otherwise we have the following two cases.

If $\dim \mathcal{M} = 4$ define $Ty_1 = x_1$, $Ty_2 = x_2$.

If $\dim \mathcal{M} = 3$ we may assume without loss of generality that $y_1 = a_1x_1 + a_2x_2 + by_2$ for some $a_1, a_2, b \in \mathbb{C}$; note that not both $a_1, a_2$ are zero because $y_1, y_2$ are linearly independent. Let $i$ be the index for which $|a_i|$ is the smallest (or let $i = 1$ if $|a_1| = |a_2|$). Define $Ty_2 = x_i$. Note that the matrix of $T|_{\mathcal{M}}$ with respect to the basis $x_1, x_2, y_2$ is

$$\begin{bmatrix} -a_1 & 0 & * \\ -a_2 & 0 & * \\ -b & -2 & 0 \end{bmatrix},$$

where one $*$ is 0 and the other is 1.

Note that in every case $T$ is invertible, so $0 \notin \mathrm{Sp}\,(T)$. However $(T+A)(x_1)=0$ and $(T+2A)(x_2)=0$, so $0 \in \mathrm{Sp}\,(T+A) \cap \mathrm{Sp}\,(T+2A)$. Thus $\mathrm{Sp}\,(T+A) \cap \mathrm{Sp}\,(T+2A)$ is not a subset of $\mathrm{Sp}\,(T)$. □

**Lemma 15.** *Let $A, B \in \mathrm{Unb}\,(\mathfrak{X})$ with $\mathrm{Sp}\,(A) = \mathrm{Sp}\,(B) \neq \mathbb{C}$. The following are equivalent:*

(i) $\mathrm{Sp}\,(A + x \otimes f) = \mathrm{Sp}\,(B + x \otimes f)$ *for every bounded everywhere defined rank-one operator* $x \otimes f$.
(ii) $A = B$.

**Proof.** We only need to prove (i) $\Longrightarrow$ (ii). Choose $\lambda \notin \mathrm{Sp}\,(A)$ and assume, to reach a contradiction, that the vectors $(\lambda \mathrm{I} - A)^{-1}x$ and $(\lambda \mathrm{I} - B)^{-1}x$ are linearly independent for some vector $x$. By Hahn-Banach theorem we can find a bounded functional $f$ with

$$f((\lambda \mathrm{I} - A)^{-1}x) = 0 \qquad \text{and} \qquad f((\lambda \mathrm{I} - B)^{-1}x) = 1.$$

Since $x \otimes f$ is a bounded rank-one operator item (iii) of Lemma 11 applies by which $\lambda \notin \mathrm{Sp}\,(A + x \otimes f) \setminus \mathrm{Sp}\,(A)$ while $\lambda \in \mathrm{Sp}\,(B + x \otimes f) \setminus \mathrm{Sp}\,(B)$ contradicting item (i) of the present lemma. Hence, $(\lambda \mathrm{I} - A)^{-1}x$ and $(\lambda \mathrm{I} - B)^{-1}x$ are parallel for every vector $x$. In fact, they must be equal for otherwise we could find a bounded linear functional $f$ with $f((\lambda \mathrm{I} - A)^{-1}x) = 1 \neq f((\lambda \mathrm{I} - B)^{-1}x)$, again violating (i) of the present lemma.

This shows that

$$(\lambda \mathrm{I} - A)^{-1} = (\lambda \mathrm{I} - B)^{-1}.$$

Taking the inverse we deduce that $(\lambda \mathrm{I} - A) = (\lambda \mathrm{I} - B)$ with $\mathrm{Dom}(A) = \mathrm{Dom}(B)$, so indeed $A = B$. □

We are now ready to prove the Main Theorem. We will prove a little more in that we will require injectivity of $\Phi$ only at the very end of the proof. In the next section we will review some important spaces where the injectivity assumption is redundant.

**Proof of Main Theorem.** Let $\Phi\colon \mathrm{Unb}\,(\mathfrak{X}) \to \mathrm{Unb}\,(\mathfrak{Y})$ be an additive surjection which preserves the spectrum. By the equivalence (i) $\Longleftrightarrow$ (iii) in Lemma 14, $\Phi$ maps the subset $\mathfrak{F}_1(\mathfrak{X})$ of everywhere defined, bounded operators on $\mathfrak{X}$ with rank at most one surjectively onto $\mathfrak{F}_1(\mathfrak{Y})$. It follows by additivity that $\Phi$ maps the subset $\mathfrak{F}(\mathfrak{X})$ of bounded, everywhere defined operators with finite rank surjectively onto $\mathfrak{F}(\mathfrak{Y})$. By [9, Lemma 2.3], the restriction $\Psi = \Phi|_{\mathfrak{F}(\mathfrak{X})}$ of $\Phi$ to $\mathfrak{F}(\mathfrak{X})$ is a linear map. By Lemma 9 (applied on $\Phi$), this restriction cannot annihilate a nonzero operator, so it is injective and hence bijective. Also, it maps rank-one idempotents from $\mathfrak{F}_1(\mathfrak{X})$ onto rank-one idempotents because these are the only operators in $\mathfrak{F}_1(\mathfrak{X})$ whose spectrum contains 1. It then follows from [5, Theorem 1.4] (this is formulated for weakly continuous linear maps on $\mathbb{B}(\mathfrak{X})$ but the proof is valid also for linear bijections from $\mathfrak{F}(\mathfrak{X})$ to $\mathfrak{F}(\mathfrak{Y})$; see also [8, Main Theorem, p. 250]) that there either exists a bounded linear bijection $Z\colon \mathfrak{X} \to \mathfrak{Y}$ such that

$$\Phi(X) = ZXZ^{-1}; \qquad X \in \mathfrak{F}(\mathfrak{X}) \tag{5}$$

or else there exists a bounded linear bijection $Z\colon \mathfrak{X}^* \to \mathfrak{Y}$ (here, $\mathfrak{X}^*$ is a dual space) such that

$$\Phi(X) = ZX^*Z^{-1}; \qquad X \in \mathfrak{F}(\mathfrak{X}). \tag{6}$$

Assume firstly case (5). Then, $Z^{-1}\Phi(\cdot)Z\colon \mathrm{Unb}\,(\mathfrak{X}) \to \mathrm{Unb}\,(\mathfrak{X})$ fixes each everywhere defined bounded rank-one operator and, by Example 1, is an additive spectrum preserving surjection from $\mathrm{Unb}\,(\mathfrak{X})$ onto $\mathrm{Unb}\,(\mathfrak{X})$. Apply Lemma 15, with $B = Z^{-1}\Phi(A)Z \in \mathrm{Unb}\,(\mathfrak{X})$ to deduce that it fixes each operator $A \in \mathrm{Unb}\,(\mathfrak{X})$ whose spectrum is different from $\mathbb{C}$. Since we assumed $\Phi$ is also injective, Corollary 6 shows $Z^{-1}\Phi(X)Z = X$ for every $X \in \mathrm{Unb}\,(\mathcal{X})$.

Assume lastly case (6). Again, by Example 1, $X \mapsto Z^{-1}\Phi(X)Z$ is a spectrum preserving additive surjection from $\mathrm{Unb}\,(\mathcal{X})$ onto $\mathrm{Unb}\,(\mathcal{X}^*)$, which maps bounded everywhere defined rank-one $X = x \otimes f$ into $X^* = f \otimes \kappa_x$, where $\kappa_x := \kappa(x)$ and $\kappa\colon \mathcal{X} \to \mathcal{X}^{**}$ is a natural embedding.

Take $A \in \mathrm{Unb}\,(\mathcal{X})$ and let $B = Z^{-1}\Phi(A)Z \in \mathrm{Unb}\,(\mathcal{X}^*)$. Assume $\lambda \notin \mathrm{Sp}\,(A) = \mathrm{Sp}\,(B)$. Then, $(\lambda \mathrm{I} - A)^{-1}$ is bounded and everywhere defined. In particular, there exists its adjoint operator $((\lambda\mathrm{I} - A)^{-1})^*$. Assume $((\lambda\mathrm{I} - A)^{-1})^* f$ and $(\lambda\mathrm{I} - B)^{-1} f$ are linearly independent functionals for some $f \in \mathcal{X}^*$. Then (see, e.g., a book by Pedersen [10, 2.4.3]) there exists $x \in \mathcal{X}$ which lies in the kernel of the first functional but not in the kernel of the second one. We may assume $x$ is such that $\kappa_x((\lambda\mathrm{I} - B)^{-1} f) = 1$. Then, $\lambda \in \mathrm{Sp}\,(B + f \otimes \kappa_x) = \mathrm{Sp}\,(Z^{-1}\Phi(A + x \otimes f)Z)$ but $f((A - \lambda\mathrm{I})^{-1}x) = \kappa_x\big(((A - \lambda\mathrm{I})^{-1})^* f\big) = 0$, so $\lambda \notin \mathrm{Sp}\,(A + x \otimes f)$, a contradiction. As before in the proof of Lemma 15 we deduce that

$$(\lambda\mathrm{I} - B)^{-1} = ((\lambda\mathrm{I} - A)^{-1})^*; \qquad B = Z^{-1}\Phi(A)Z. \tag{7}$$

In particular, if $A$ is bounded and everywhere defined, then $((\lambda\mathrm{I} - A)^{-1})^* = \big((\lambda\mathrm{I} - A)^*\big)^{-1}$ and so, after taking inverses,

$$Z^{-1}\Phi(A)Z = B = A^*.$$

By Lemma 7 this is impossible to hold with injective $\Phi$. Case (6) is hence impossible for bijective spectrum preserving $\Phi$. $\square$

## 5. Concluding examples and remarks

Let us conclude by proving that for some important Banach spaces, which include Hilbert spaces, the injectivity assumption in our main theorem can be removed. To do this let us first define a class of operators with a special property.

**Definition 16.** An operator $T$ is called *spectrally indecomposable* if there does not exist $n \geq 2$ and operators $T_1, \ldots, T_n$ with $\mathrm{Sp}\,(T_i) \neq \mathbb{C}$, $i = 1, \ldots, n$, such that $T = T_1 + \cdots + T_n$.
The antonym of spectrally indecomposable is *spectrally decomposable*.

**Remark 17.** In the definition, $\mathrm{Dom}(T_i)$ can be bigger than $\mathrm{Dom}(T)$ as long as $\mathrm{Dom}(T) = \mathrm{Dom}(T_1) \cap \cdots \cap \mathrm{Dom}(T_n)$ holds. Moreover, we also allow decompositions of the form $T = 0 + T$, where $0$ is a zero operator on $\mathcal{X}$. In particular, every spectrally indecomposable operator satisfies $\mathrm{Sp}\,(T) = \mathbb{C}$.

We supply a few examples, and a proposition, that spectrally (in)decomposable operators may be tricky to describe.

**Example 18.** The spectrally indecomposable operators on a finite dimensional Banach space $\mathcal{X}$ are exactly the operators whose domain is a proper subspace of $\mathcal{X}$. This follows from the fact that each everywhere defined operator on $\mathcal{X}$ has a bounded spectrum and that the sum of two such operators is again defined everywhere.

**Example 19.** An example of a spectrally indecomposable operator on an infinite-dimensional Banach space $\mathcal{X}$ is an operator $T = x \otimes f$ where $x \in \mathcal{X} \setminus \{0\}$ and where $f$ is an unbounded linear functional defined everywhere on $\mathcal{X}$ (such $f$ always exists: choose a normalized Hamel basis and define $f$ to be unbounded on it).

Otherwise, we could decompose $T = T_1 + \cdots + T_n$. Since $\mathrm{Dom}(T_i) \supseteq \mathrm{Dom}(T) = \mathcal{X}$, and since there exists $\lambda_i \in \mathbb{C}$ with $(\lambda_i \mathrm{I} - T_i)\colon \mathrm{Dom}(T_i) = \mathcal{X} \to \mathcal{X}$ invertible we see that $(\lambda_i \mathrm{I} - T_i)^{-1}$ is an everywhere defined, bounded surjective operator, hence also its inverse is bounded and as such $T_i$ is bounded. Thus, $x \otimes f$ is a finite sum of bounded operators, a contradiction because $f$ is not bounded.

**Example 20.** It was shown recently [3] that there exists an infinite-dimensional Banach space $\mathcal{X}$ such that every bounded injective operator on it is automatically surjective.

It follows that no operator whose domain is a proper subspace can have a bounded inverse and so every $T \in \mathrm{Unb}\,(\mathcal{X})$ with $\mathrm{Dom}(T) \subsetneqq \mathcal{X}$ is spectrally indecomposable.

We next give a simple but useful sufficient criterion for spectral decomposability.

**Proposition 21.** *Let $\mathcal{X}$ be a Banach space. Suppose there exist proper subspaces $\mathcal{D}_1, \mathcal{D}_2 \subseteq \mathcal{X}$ with $\dim \mathcal{D}_1 \cap \mathcal{D}_2 = 1$ and bijective operators $A_j\colon \mathcal{D}_j \to \mathcal{X}$ $(j = 1, 2)$ with bounded inverses. Then every $T \in \mathrm{Unb}\,(\mathcal{X})$ with one-dimensional domain is spectrally decomposable.*

**Proof.** First note that, given any nonzero $x, y \in \mathcal{X}$, there exists a bounded invertible operator $A \in \mathbb{B}(\mathcal{X})$ satisfying $Ax = y$; we shall make repeated use of this fact.

Let $e$ be a unit vector in $\mathcal{D}_1 \cap \mathcal{D}_2$. Suppose the domain of $T \in \mathrm{Unb}\,(\mathcal{X})$ is $\mathbb{C}x$ (we may assume $x$ is a unit vector) and $Tx = y$. Choose a nonzero vector $v \in \mathcal{X}$ so that $v + y \neq 0$. Choose invertible operators $S, B_1, B_2 \in \mathbb{B}(\mathcal{X})$ such that

$$Se = x, \quad B_1 A_1 e = v + y, \quad B_2 A_2 e = -v. \tag{8}$$

Then $\mathrm{Dom}(B_1 A_1 S^{-1}) = S\mathcal{D}_1$ and $\mathrm{Dom}(B_2 A_2 S^{-1}) = S\mathcal{D}_2$, so

$$\mathrm{Dom}(B_1 A_1 S^{-1} + B_2 A_2 S^{-1}) = S\mathcal{D}_1 \cap S\mathcal{D}_2 = S(\mathcal{D}_1 \cap \mathcal{D}_2) = \mathbb{C}x.$$

Moreover, $(B_1 A_1 S^{-1} + B_2 A_2 S^{-1})x = y$, so $T = B_1 A_1 S^{-1} + B_2 A_2 S^{-1}$. Since $B_j A_j S^{-1}\colon S\mathcal{D}_j \to \mathcal{X}$ is bijective with bounded inverse $SA_j^{-1}B_j^{-1}$ for $j = 1, 2$, it follows that $T$ is spectrally decomposable. $\square$

We apply this proposition to a standard sequence space $\ell^p$ to show that the injectivity assumption can be removed from Main Theorem.

**Proposition 22.** *Let $1 \leq p \leq \infty$. Every operator $T \in \mathrm{Unb}\,(\ell^p)$ with one-dimensional domain is spectrally decomposable. Consequently, if $\mathcal{Y}$ is any Banach space and an additive surjection $\Phi\colon \mathrm{Unb}\,(\ell^p) \to \mathrm{Unb}\,(\mathcal{Y})$ preserves the spectrum, then the conclusion of Main Theorem is valid.*

**Proof.** Let $e_1, e_2, \ldots$ be the standard basis for $\mathcal{X} = \ell^p$. Let $\mathcal{D}_1 = \mathrm{Span}\,\{e_{2j-1};\ \ j \in \mathbb{N}\}$ and $\mathcal{D}_2 = \mathrm{Span}\,\{e_1, e_{2j};\ \ j \in \mathbb{N}\}$, so $\mathcal{D}_1 \cap \mathcal{D}_2 = \mathbb{C}e_1$ is one-dimensional. Define $A_1\colon \mathcal{D}_1 \to \mathcal{X}$ by

$$A_1 \left( \sum_{k=1}^{\infty} \gamma_k e_{2k-1} \right) = \sum_{k=1}^{\infty} \gamma_k e_k; \tag{9}$$

define $A_2\colon \mathcal{D}_2 \to \mathcal{X}$ by

$$A_2 \left( \gamma e_1 + \sum_{k=1}^{\infty} \gamma_k e_{2k} \right) = \gamma e_1 + \sum_{k=1}^{\infty} \gamma_k e_{k+1}.$$

Then $A_1, A_2$ are surjective isometries, so their inverses are bounded. Now apply Proposition 21.

To prove the final statement recall that the proof of Main Theorem establishes the following: given a surjective additive spectrum preserver, either (5) or (7) holds for all $X, A \in \mathrm{Unb}\,(\ell^p)$ with spectrum not equal to $\mathbb{C}$ (injectivity assumption was only used after this point). We now consider the two cases separately.

Case (7) is impossible. Namely, it implies that $((\lambda \mathrm{I} - A)^{-1})^*$ is injective for every $A \in \mathrm{Unb}\,(\mathcal{X})$ with $\mathrm{Sp}\,(A) \neq \mathbb{C}$ and every $\lambda \notin \mathrm{Sp}\,(A)$. However, there does exist invertible $A = A_1$, defined by (9), whose domain is not dense. It follows that there exists a nonzero linear bounded functional $f$ which annihilates $\mathrm{Dom}(A_1)$ and consequently, taking $\lambda = 0 \notin \mathrm{Sp}\,(A_1)$, we would have $((\lambda \mathrm{I} - A_1)^{-1})^* f = 0$, a contradiction.

Case (5) implies, by additivity, that $Z^{-1}\Phi(\cdot)Z$ must also fix every spectrally decomposable operator. In particular, $Z^{-1}\Phi(\cdot)Z$ fixes each operator with one-dimensional domain. The rest follows from Lemma 3. □

We have a similar result for finite-dimensional Banach spaces. However, the proof is different. Note that in finite-dimensional space $\mathcal{X}$, the set $\mathrm{Unb}\,(\mathcal{X})$ consists of bounded linear operators with restricted domains.

**Proposition 23.** *Let $\Phi\colon \mathrm{Unb}\,(\mathcal{X}) \to \mathrm{Unb}\,(\mathcal{Y})$ be an additive spectrum preserving surjection and assume $\mathcal{X}$ is finite-dimensional. Then the conclusion of Main Theorem is valid.*

**Proof.** As in the proof of Proposition 22, either (5) or (7) holds for all $X, A \in \mathrm{Unb}\,(\mathcal{X})$ with spectrum not equal to $\mathbb{C}$. In particular, $Z\colon \mathcal{X} \to \mathcal{Y}$ establishes an isomorphism of finite-dimensional Banach spaces.

Case (7) is impossible. Namely it would imply that $Z^{-1}\Phi(A)Z = A^*$ for each everywhere defined $A \in \mathrm{Unb}\,(\mathcal{X})$. Consider a proper nontrivial subspace $0 \neq \mathcal{D}' \subsetneq \mathcal{X}^*$. By surjectivity, there must exist $T \in \mathrm{Unb}\,(\mathcal{X})$ with $Z^{-1}\Phi(T)Z = 0|_{\mathcal{D}'}$. By Lemma 2, $\Phi(0|_{\mathcal{D}}) = 0|_{\mathcal{D}'}$ where $\mathcal{D} = \mathrm{Dom}(T) \neq \mathcal{X}$. Now consider any everywhere defined $A$ with $A|_{\mathcal{D}} = 0$, i.e., with $0|_{\mathcal{D}} + A = 0|_{\mathcal{D}}$. Applying $\Phi$ gives $0|_{\mathcal{D}'} + A^* = 0|_{\mathcal{D}'}$ so that $f(\mathrm{Im}\,A) = 0$ for every $f \in \mathcal{D}' \subsetneq \mathcal{X}^*$. Since the images of everywhere defined operators $A$ with $A|_{\mathcal{D}} = 0$ cover the whole space $\mathcal{X}$ we see that $f = 0$, i.e., $\mathcal{D}'$ is zero-dimensional, a contradiction.

Case (5) implies $Z^{-1}\Phi(\cdot)Z$ fixes operators defined everywhere on $\mathcal{X}$. We claim that $Z^{-1}\Phi(\cdot)Z$ must fix every operator. To see this, let $\mathcal{D} \subsetneq \mathcal{X}$ be a proper subspace of maximal possible dimension such that not every operator with domain equal $\mathcal{D}$ is fixed and let $A\colon \mathcal{D} \to \mathcal{X}$ be mapped into $B = \Phi(A) \neq A$. We know from Lemma 4 that $B = A|_{\widehat{\mathcal{D}}}$ where $\widehat{\mathcal{D}} \subsetneq \mathcal{D}$ is a proper subspace. By surjectivity, there must exist $T \neq A$ with $\Phi(T) = A$. However, due to the assumptions on $\mathcal{D}$, we have $\dim \mathrm{Dom}(T) \leq \dim \mathcal{D}$. And since $\Phi(T)$ is again a restriction of $T$ to a properly contained subdomain, we see that $\dim \mathrm{Dom}(\Phi(T)) < \dim \mathrm{Dom}(T) \leq \dim \mathcal{D} = \dim \mathrm{Dom}(A)$, contradicting the fact that $\Phi(T) = A$. □

Finally, with the following examples we show that the surjectivity assumption *is essential.*

**Example 24.** Let $\mathcal{X}$ be finite-dimensional. Then every operator in $\mathcal{X}$ whose domain is a proper subspace is spectrally indecomposable, see Example 18. Hence a map which is identity on operators defined everywhere on $\mathcal{X}$ and maps operators $T$ with $\mathrm{Dom}(T) \subsetneq \mathcal{X}$ into $0|_0$ (the zero operator with domain equal $\{0\}$) is a linear map and preserves the spectrum because $\mathrm{Sp}\,(T) = \mathbb{C}$ if and only if $\mathrm{Dom}(T)$ is a proper subspace of $\mathcal{X}$.

**Example 25.** Note that, in general, spectrum preserving maps do not have such a nice classification if we consider them only on everywhere defined (possibly unbounded) operators.

Namely, $V = \{A \in \mathrm{Unb}\,(\mathcal{X});\ \ \mathrm{Dom}(A) = \mathcal{X}\}$ does form a legitimate vector space, with additive identity $0_{\mathcal{X}}$. The subset $M \subseteq V$ of spectrally decomposable operators (with domain $\mathcal{X}$) is a subspace. Let $N \subseteq V$ be its complemented subspace (which necessarily consists of spectrally indecomposable operators) so that $M \cap N = 0$ and $M + N = V$.

Take an arbitrary linear bijection $\psi\colon N \to N$ and define $\Phi$ on $V$ by $\Phi(A + B) = A + \psi(B)$ for all $A \in M$, $B \in N$. Then $\Phi$ is a bijective linear map which preserves the spectrum on $V$.

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