J. Math. Anal. Appl. 491 (2020) 124399

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Necessary and sufficient conditions for unique solution to functional equations of Poincaré type

Chin-Yuan Hu [a], Gwo Dong Lin [b,c,*]

[a] Department of Business Education, National Changhua University of Education, Changhua 50058, Taiwan
[b] Social and Data Science Research Center, Hwa-Kang Xing-Ye Foundation, Taipei 10659, Taiwan
[c] Institute of Statistical Science, Academia Sinica, Taipei 11529, Taiwan

A R T I C L E I N F O

Article history:
Received 21 February 2020
Available online 17 July 2020
Submitted by V. Pozdnyakov

Keywords:
Distributional equation
Poincaré's functional equation
Laplace–Stieltjes transform
Probability generating function
Characterization of distributions

A B S T R A C T

Distributional equation is an important tool in the characterization theory because many characteristic properties of distributions can be transferred to such equations. Using a novel and natural approach, we retreat a remarkable distributional equation whose corresponding functional equation in terms of Laplace–Stieltjes transform is of the Poincaré type. The necessary and sufficient conditions for the equation to have a *unique* distributional solution with finite *variance* are provided. This complements the previous results which involve at most the *mean* of the distributional solution. Besides, more general distributional (or functional) equations are investigated as well.

1. Introduction

One useful method to characterize probability distributions is through suitable distributional equations (see, e.g., [18,19,11,7,9], and the references therein). In this paper, we will retreat a remarkable distributional equation described below.

Let X and T be two nonnegative random variables having distributions F and F_T, respectively, denoted $X \sim F_X = F$, $T \sim F_T$. Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of independent and identically distributed (i.i.d.) random variables having distribution F on $\mathbb{R}_+ \equiv [0, \infty)$, and let $\{T_i\}_{i=1}^{\infty}$ be another sequence of i.i.d. random variables having distribution F_T. Moreover, let N be a random variable taking values in $\mathbb{N}_0 \equiv \{0, 1, 2, \dots\}$, and assume that all the random variables X, X_i, T, T_i, N are independent. For given T and N, we will investigate the distributional equation

* Corresponding author.
E-mail addresses: buhuua@gmail.com (C.-Y. Hu), gdlin@stat.sinica.edu.tw (G.D. Lin).

https://doi.org/10.1016/j.jmaa.2020.124399

$$X \stackrel{\mathrm{d}}{=} \sum_{i=1}^{N} T_i X_i, \tag{1}$$

where '$\stackrel{\mathrm{d}}{=}$' means equality in distribution and the summation is zero if N takes value 0. For applications of Eq (1) in various fields, see, e.g., the survey paper by Liu [14].

Let P_N denote the probability generating function (pgf) of N and let $\hat{F}$ be the Laplace–Stieltjes transform of $X \sim F$; namely, $P_N(t) = \mathbf{E}[t^N] = \sum_{k=0}^{\infty} \Pr(N = k)t^k,\ t \in [0, 1]$, where $0^0 \equiv 1$, and $\hat{F}(s) = \mathbf{E}[\exp(-sX)]$, $s \geq 0$. Then the distributional equation (1) can be further transferred to the following functional equation in terms of $\hat{F}, F_T$ and P_N:

$$\hat{F}(s) = P_N\Big(\int_0^{\infty} \hat{F}(ts)dF_T(t)\Big) = P_N\big(\mathbf{E}[\exp(-sTX)]\big), \quad s \geq 0. \tag{2}$$

When F_T is a degenerate distribution at $p \in (0, 1)$, namely, $\Pr(T = p) = 1$, Eq (1) reduces to $X \stackrel{\mathrm{d}}{=} \sum_{i=1}^{N} pX_i$, and Eq (2) is exactly the Poincaré functional equation

$$\hat{F}(s) = P_N\big(\hat{F}(ps)\big), \quad s \geq 0, \tag{3}$$

which arises in the Galton–Watson processes [16,17]. So we call the general Eq (2) a functional equation of the Poincaré type.

It is seen that once Eq (1) or Eq (2) has a solution $X \sim F$, each constant multiplication of X also plays a solution to Eq (1). However, the solution might be unique, provided we fix the mean of the distributions. In this sense, Eq (1) or Eq (2) becomes a characteristic property of the distributional solution. A typical example is the classical characterization of the exponential distribution through Eq (3), where we can take N obeying the geometric distribution: $\Pr(N = n) = p(1 - p)^{n-1},\ n \geq 1$; see, e.g., Azlarov and Volodin [1], p. 79.

The properties of the solutions $X \sim F$ heavily depend on those of the given T and N. Some results about Eqs (1) and (2) are available in the literature. For example, denote the counting number $\tilde{N} = \sum_{i=1}^{N} \mathbb{I}_{\{T_i>0\}}$, where $\mathbb{I}_A$ is the indicator function of the set A. Then for given N and $T \sim F_T$ with the conditions $\Pr(\tilde{N} = 0 \text{ or } 1) < 1$ and $\Pr(T = 0 \text{ or } 1) < 1$ (which are used to exclude some trivial cases), the following results hold (see [15], Theorem 1.1, and the references therein):

(i) Eq (1) (or Eq (2)) has a solution $0 \leq X \sim F$ iff the random variables N and T together satisfy the conditions

$$\Pr(T > 0)\mathbf{E}[N] > 1, \quad \mathbf{E}[N]\mathbf{E}[T^{\alpha}] = 1 \quad \text{and} \quad \mathbf{E}[T^{\alpha} \log T] \leq 0 \quad \text{for some } \alpha \in (0, 1]; \tag{4}$$

(ii) Eq (1) (or Eq (2)) has a solution $0 \leq X \sim F$ with finite *mean* iff the random variables N and T together satisfy the conditions

$$\Pr(T > 0)\mathbf{E}[N] > 1, \quad \mathbf{E}[N]\mathbf{E}[T] = 1, \quad \mathbf{E}[N \log^{+} N] < \infty \quad \text{and} \quad \mathbf{E}[T \log T] < 0, \tag{5}$$

where $\log^{+} x = \log x$ if $x \geq 1$ and $\log^{+} x = 0$ otherwise, and $0 \log 0 \equiv 0$.

One of the main purposes in this paper is to find the necessary and sufficient conditions for which Eq (2) (or Eq (1)) has a *unique* solution F (on $\mathbb{R}_+$) with a fixed mean and finite *variance*, and hence it can be used to characterize distributions. This complements the above results (i) and (ii) which involve at most the *mean* of the distributional solution. Our approach is different from the previous ones and is somehow more natural. Moreover, some general cases are also investigated. The main results are stated in the next section, while their proofs are given in Section 4. The needed lemmas are provided in Section 3. Finally, we have some discussions in Section 5.

2. Main results

We start with the simplest case Eq (1) (or Eq (2)). More complicated cases will follow.

Theorem 1. *Let $0 \leq X \sim F$ with Laplace–Stieltjes transform $\hat{F}$ and $\mu = \mathbf{E}[X] \in (0, \infty)$. Let $T \geq 0$ and $N \geq 0$ be two given random variables, where N takes values in $\mathbb{N}_0$ and has pgf P_N. Then for given μ, the random variables N and T together satisfy the conditions*

$$\mathbf{E}[N]\mathbf{E}[T] = 1, \quad 0 < \mathbf{E}[T^2] < \mathbf{E}[T] < 1, \quad \textit{and} \quad \mathbf{E}[N^2] < \infty \tag{6}$$

iff the functional equation (2) has exactly one solution F with mean μ and a finite variance. Moreover, the variance is of the form

$$Var(X) = \frac{(\mathbf{E}[T])^2\, Var(N) + \mathbf{E}[N]\, Var(T)}{1 - \mathbf{E}[N]\mathbf{E}[T^2]} \cdot \mu^2 \tag{7}$$

with $\mathbf{E}[N] = 1/\mathbf{E}[T]$.

Unlike the previous results (i) and (ii) (which assume some initial conditions on $\tilde{N}$ and T to exclude the trivial cases), we don't assume explicitly any initial condition in Theorem 1. But each of the sufficiency and necessity parts does imply implicitly the following: $\Pr(N = 1) < 1$ or $\Pr(T = 1) < 1$. To see this, if on the contrary $\Pr(N = 1) = \Pr(T = 1) = 1$, then the second condition in (6) fails to hold. Moreover, in this case, Eqs (1) and (2) reduce to the identities $X \overset{\mathrm{d}}{=} X_1$ and $\hat{F}(s) = \hat{F}(s),\ s \geq 0$, respectively, so the solution to Eq (2) is not unique, a contradiction to the assumption in the sufficiency part.

The following result is about a characterization of degenerate distributions.

Corollary 1. *Under the setting of Theorem 1, the functional equation (2) has exactly one solution F degenerate at mean $\mu = \mathbf{E}[X]$ iff the random variables N and T are degenerate at $\mathbf{E}[N]$ and $\mathbf{E}[T]$, respectively, and $\mathbf{E}[T] = (\mathbf{E}[N])^{-1} \in (0, 1)$; precisely, $\Pr(N = n_0) = 1$ for some integer $n_0 \geq 2$ and $\Pr(T = 1/n_0) = 1$.*

When $\Pr(T = p) = 1$ for some $p \in (0, 1)$ in Theorem 1, we are able to rewrite Hu and Cheng's [6] Theorem 1 with $\alpha = 1$ and Corollary 1 as follows.

Corollary 2. *Let $p \in (0, 1)$ and $\mu \in (0, \infty)$ be two constants. Let $N \geq 0$ be a random variable taking values in $\mathbb{N}_0$ and let $0 \leq X \sim F$ with mean μ and Laplace–Stieltjes transform $\hat{F}$. Then for given μ, the random variable N satisfies the conditions*

$$\mathbf{E}[N] = 1/p \quad \textit{and} \quad \mathbf{E}[N^2] < \infty$$

iff the Poincaré functional equation (3) has exactly one solution F with mean μ and a finite variance. Moreover, the variance is equal to

$$Var(X) = \frac{p^2\, Var(N)}{1 - p} \cdot \mu^2.$$

It is seen that the set of conditions (5) is stronger than (4), while (6) is stronger than both (4) and (5), as seen below. This in turn implies that the solution F in Theorem 1 belongs to the classes of the previous solutions to Eq (2) under conditions (4) or (5).

Proposition 1. *Suppose that N and T are two nonnegative random variables satisfying $\mathbf{E}[N]\mathbf{E}[T] = 1$ and $0 < \mathbf{E}[T^2] < \mathbf{E}[T] < 1$. Then $\Pr(T > 0)\mathbf{E}[N] > 1$ and $\mathbf{E}[T \log T] < 0$.*

If, in addition to (1), assume that $N \geq m$, where $m \geq 1$ is an integer, then we can split the RHS of (1) into two parts:

$$X \stackrel{\mathrm{d}}{=} \sum_{i=1}^{m} T_i X_i + \sum_{j=1}^{N^*} T_{j+m} X_{j+m}, \tag{8}$$

where $N^* = N - m \geq 0$. This is equivalent to study the functional equation

$$\begin{aligned} \hat{F}(s) &= \Big(\int_0^\infty \hat{F}(ts) dF_T(t) \Big)^m P_N \Big(\int_0^\infty \hat{F}(ts) dF_T(t) \Big) \\ &= \big(\mathbf{E}[\exp(-sTX)] \big)^m P_N \big(\mathbf{E}[\exp(-sTX)] \big), \quad s \geq 0, \end{aligned} \tag{9}$$

where $N \geq 0$ as in Eq (2).

In the next two theorems, we consider Eq (9) for the cases $m = 1$ and $m \geq 2$, separately (see the explanations right after Theorem 3).

Theorem 2. *Let $0 \leq X \sim F$ with Laplace–Stieltjes transform $\hat{F}$ and $\mu = \mathbf{E}[X] \in (0, \infty)$. Let $T \geq 0$ and $N \geq 0$ be two given random variables, where N takes values in $\mathbb{N}_0$ and has pgf P_N. Then for given μ, the random variables N and T together satisfy the conditions*

$$\mathbf{E}[N] = \frac{1 - \mathbf{E}[T]}{\mathbf{E}[T]}, \quad 0 < \mathbf{E}[T^2] < \mathbf{E}[T] < 1, \quad \textit{and} \quad \mathbf{E}[N^2] < \infty \tag{10}$$

iff the functional equation (9) with $m = 1$ has exactly one solution F with mean μ and a finite variance. Moreover, the variance is of the form

$$Var(X) = \frac{(\mathbf{E}[T])^2 \, Var(N) + \mathbf{E}[N+1] \, Var(T)}{1 - \mathbf{E}[N+1]\mathbf{E}[T^2]} \cdot \mu^2 \tag{11}$$

with $\mathbf{E}[N] = (1 - \mathbf{E}[T])/\mathbf{E}[T]$.

Corollary 3. *Under the setting of Theorem 2, the functional equation (9) with $m = 1$ has exactly one solution F degenerate at mean $\mu = \mathbf{E}[X]$ iff the random variables N and T are degenerate at $\mathbf{E}[N]$ and $\mathbf{E}[T]$, respectively, and $\mathbf{E}[T] = 1/(\mathbf{E}[N] + 1) \in (0, 1)$; precisely, $\Pr(N = n_0) = 1$ for some integer $n_0 \geq 1$ and $\Pr(T = 1/(n_0 + 1)) = 1$.*

When $\Pr(T = p) = 1$ for some $p \in (0, 1)$, Theorem 2 reduces to the following.

Corollary 4. *Under the setting of Corollary 2, the random variable N satisfies the conditions*

$$\mathbf{E}[N] = (1 - p)/p \quad \textit{and} \quad \mathbf{E}[N^2] < \infty$$

iff the functional equation

$$\hat{F}(s) = \hat{F}(ps) P_N \big(\hat{F}(ps) \big), \quad s \geq 0,$$

has exactly one solution F with mean μ and a finite variance. Moreover, the variance equals

$$Var(X) = \frac{p^2 \, Var(N)}{1 - p} \cdot \mu^2.$$

Theorem 3. *Let $0 \le X \sim F$ with Laplace–Stieltjes transform $\hat{F}$ and $\mu = \mathbf{E}[X] \in (0, \infty)$. Let $T \ge 0$ and $N \ge 0$ be two given random variables, where N takes values in $\mathbb{N}_0$ and has pgf P_N. Assume further that $m \ge 2$ is an integer. Then for given μ, the random variables N and T together satisfy the conditions*

$$\mathbf{E}[N] = \frac{1 - m\mathbf{E}[T]}{\mathbf{E}[T]}, \quad 0 < \mathbf{E}[T^2] < \mathbf{E}[T] \le \frac{1}{m} < 1, \quad \textit{and} \quad \mathbf{E}[N^2] < \infty \tag{12}$$

iff the functional equation (9) has exactly one solution F with mean μ and a finite variance. Moreover, the variance is of the form

$$Var(X) = \frac{(\mathbf{E}[T])^2\, Var(N) + \mathbf{E}[N+m]\, Var(T)}{1 - \mathbf{E}[N+m]\mathbf{E}[T^2]} \cdot \mu^2 \tag{13}$$

with $\mathbf{E}[N] = (1 - m\mathbf{E}[T])/\mathbf{E}[T]$.

Note that we don't exclude the case $N = 0$ in Theorem 3, because when $N = 0$, Eq (9) with $m \ge 2$ is not a trivial case. Besides, when $N = 0$, $\mathbf{E}[T]$ in (12) equals $1/m\,(\le 1/2 < 1)$, while the first two conditions in (10) fail to hold together. Therefore, Theorem 2 is not a special case of Theorem 3; namely, we cannot derive Theorem 2 from Theorem 3 by just letting $m = 1$. On the other hand, it is seen that Eq (9) with $m \ge 2$ and $N = 0$ is equivalent to Eq (2) with $N = m \ge 2$.

Corollary 5. *Under the setting of Theorem 3, the functional equation (9) has exactly one solution F degenerate at mean $\mu = \mathbf{E}[X]$ iff the random variables N and T are degenerate at $\mathbf{E}[N]$ and $\mathbf{E}[T]$, respectively, and $\mathbf{E}[T] = 1/(\mathbf{E}[N] + m) \in (0, 1/m]$; precisely, $\Pr(N = n_0) = 1$ for some integer $n_0 \ge 0$ and $\Pr(T = 1/(n_0 + m)) = 1$.*

When $\Pr(T = p) = 1$ for some $p \in (0, 1/m]$, Theorem 3 reduces to the following.

Corollary 6. *Under the setting of Corollary 2, assume, in addition, $p \in (0, 1/m]$, where $m \ge 2$ is an integer. Then for given μ, the random variable N satisfies the conditions*

$$\mathbf{E}[N] = (1 - mp)/p \quad \textit{and} \quad \mathbf{E}[N^2] < \infty$$

iff the functional equation

$$\hat{F}(s) = (\hat{F}(ps))^m P_N\big(\hat{F}(ps)\big), \quad s \ge 0,$$

has exactly one solution F with mean μ and a finite variance. Moreover, the variance equals

$$Var(X) = \frac{p^2\, Var(N)}{1 - p} \cdot \mu^2.$$

The distributional equations (1) and (8) are homogeneous cases, because $X, X_1, X_2, \ldots$ are i.i.d random variables. We now consider a nonhomogeneous case defined below. In addition to the setting for Eq (1), suppose $0 \le B \sim F_B$ is another random variable independent of all X, X_i, T, T_i, N. We will find necessary and sufficient conditions on B, T and N for which the distributional equation

$$X \stackrel{\mathrm{d}}{=} B + \sum_{i=1}^{N} T_i X_i, \tag{14}$$

has a solution $X \sim F$ with finite *variance*. Like Eq (2), Eq (14) has the functional form

$$\hat{F}(s) = \hat{F_B}(s) \cdot P_N\Big(\int_0^\infty \hat{F}(ts)dF_T(t)\Big) \tag{15}$$
$$= \hat{F_B}(s) \cdot P_N\big(\mathbf{E}[\exp(-sTX)]\big), \quad s \geq 0.$$

Theorem 4. *Let $0 \leq X \sim F$ with Laplace–Stieltjes transform $\hat{F}$ and $\mu = \mathbf{E}[X] \in (0,\infty)$. Let $T \geq 0$ and $N \geq 0$ be two given random variables with finite variances, where N takes values in $\mathbb{N}_0$ and has pgf P_N. Suppose that $0 \leq B \sim F_B$ is another random variable with mean $\mathbf{E}[B] > 0$ and a finite variance. Assume further that (i) $\Pr(N = 0) < 1$, $\Pr(T = 0) < 1$ and (ii) $Var(B) + Var(T) + Var(N) > 0$. Then for given μ, the following statements are true.*
(a) The random variables B, N and T together satisfy the conditions

$$\mu = \frac{\mathbf{E}[B]}{1 - \mathbf{E}[N]\mathbf{E}[T]}, \quad 0 < \mathbf{E}[N]\mathbf{E}[T] < 1, \quad \text{and} \quad 0 < \mathbf{E}[N]\mathbf{E}[T^2] < 1 \tag{16}$$

iff the functional equation (15) has one solution F with mean μ and a finite variance. Moreover, the variance is of the form

$$Var(X) = \frac{Var(B) + \mu^2(\mathbf{E}[T])^2\, Var(N) + \mu^2\mathbf{E}[N]\, Var(T)}{1 - \mathbf{E}[N]\mathbf{E}[T^2]} \tag{17}$$

with $\mu = \mathbf{E}[B]/(1 - \mathbf{E}[N]\mathbf{E}[T])$.
(b) If, in addition to (16), $\mathbf{E}[T^2] < \mathbf{E}[T]$, then the solution F to Eq (15) is unique.

The purpose of the assumptions (i) and (ii) in Theorem 4 is to exclude the trivial cases:
(a) if $N = 0$ or $T = 0$, Eq (14) reduces to the equality $X \stackrel{\mathrm{d}}{=} B$;
(b) if $\mathrm{Var}(B) + \mathrm{Var}(T) + \mathrm{Var}(N) = 0$, all B, T, N have degenerate distributions, and so does the solution X.

The following interesting theorem points out the one-to-one correspondence between solutions to Eq (1) and Eq (18) defined below, where $\alpha \in (0, 1)$.

Theorem 5. *Let $\alpha \in (0, 1)$ and let $0 \leq T_\alpha \sim H_\alpha$ have the stable distribution with Laplace–Stieltjes transform $\hat{H}_\alpha(s) = \exp(-s^\alpha)$, $s \geq 0$. Then, under the setting of Eq (1) with given N and T, $X_* \sim F_*$ is a solution to Eq (1) with a mean $\mu \in (0,\infty)$ iff $X_\alpha \sim F_\alpha$, where $X_\alpha = T_\alpha X_*^{1/\alpha}$ and T_α is independent of X_*, is a solution to the distributional equation*

$$X \stackrel{\mathrm{d}}{=} \sum_{i=1}^{N} T_i^{1/\alpha} X_i \tag{18}$$

with $\lim_{s\to 0^+}(1 - \hat{F}_\alpha(s))/s^\alpha = \mu \in (0,\infty)$.

3. Lemmas

To prove the main results, we need some lemmas in the sequel. Recall that the pgf P_N of a random variable N taking values in $\mathbb{N}_0$ is an absolutely monotone function on $[0, 1]$ with $P_N(1) = 1$, because $P_N(t) = \mathbf{E}[t^N] = \sum_{n=0}^{\infty} r_n t^n$, $t \in [0, 1]$, with each $r_n = \Pr(N = n) \geq 0$. For the first two lemmas, see, e.g., Steutel and van Harn [20], pp. 483–484; Lemma 1 is the so-called Bernstein Theorem.

Lemma 1. *The Laplace–Stieltjes transform $\hat{F}$ of a nonnegative random variable $X \sim F$ is a completely monotone function on $[0,\infty)$ with $\hat{F}(0) = 1$, and vice versa.*

Lemma 2. *Let Q be a pgf on $[0,1]$ and let $\rho_1,\ \rho_2$ be two completely monotone functions on $[0,\infty)$ with $\rho_1(0)=\rho_2(0)=1$. Then each of the composition function $Q\circ\rho_1$ and the product function $\rho_1\rho_2$ is completely monotone on $[0,\infty)$, and is the Laplace–Stieltjes transform of a nonnegative random variable.*

Lemma 3. *If $a,b\in[0,1]$ and $t\ge 1$ are three real numbers, then $|a^t-b^t|\le t|a-b|$.*

Proof. If $t=1$, the result is trivial. Suppose now that $t>1$. There are two possible cases for a and b: (i) $0\le a\le b\le 1$ and (ii) $0\le b\le a\le 1$. It suffices to prove Case (i), because Case (ii) follows from Case (i) immediately by the symmetry property. Consider the function: $g(x)=x^t-tx,\ x\in[0,1]$. Since $g'(x)=t(x^{t-1}-1)\le 0,\ x\in[0,1]$, the function g is decreasing on $[0,1]$. Therefore, $g(a)\ge g(b)$ for Case (i). That is, $a^t-ta\ge b^t-tb$ for $0\le a\le b\le 1$. Equivalently, $b^t-a^t\le t(b-a)$ or

$$|a^t-b^t|=b^t-a^t\le t(b-a)=t|a-b|\quad\text{for}\ \ 0\le a\le b\le 1.$$

The proof is complete.

For a proof of the next crucial lemma, see Eckberg [2], Guljas et al. [4] or Hu and Lin [8].

Lemma 4. *Let $0\le X\sim F$ have a finite positive second moment. Then its Laplace–Stieltjes transform satisfies*

$$\hat{F}(s)\le 1-\frac{\mu_1^2}{\mu_2}+\frac{\mu_1^2}{\mu_2}e^{-(\mu_2/\mu_1)s},\quad s\ge 0, \tag{19}$$

where $\mu_j,\ j=1,2$, is the jth moment of X.

Note that in Lemma 4, if the variance of X is zero, then $\mu_2=\mu_1^2$ and X is degenerate at the mean $\mu_1>0$. In this case, (19) becomes an equality: $\hat{F}(s)=e^{-\mu_1 s},\ s\ge 0$.

The next two lemmas are taken from Lin [13,10]. The sufficiency parts of Corollaries 1, 3, and 5 can be proved directly by using Lemma 5.

Lemma 5. *Let g be a nonnegative function defined on $[0,\infty)$ and let g satisfy (i) $g(0)=1$, (ii) $g'(0)=b\in\mathbb{R}\equiv(-\infty,\infty)$ and (iii) for some positive real $r\ne 1$, $g(rx)=(g(x))^r$, $x\ge 0$. Then g is the exponential function $g(x)=e^{bx}$, $x\ge 0$.*

Lemma 6. *Let $0\le X\sim F$ with Laplace–Stieltjes transform $\hat{F}$. Then for each integer $n\ge 1$, the nth moment $\mathbf{E}[X^n]=\lim_{s\to 0^+}(-1)^n\hat{F}^{(n)}(s)=(-1)^n\hat{F}^{(n)}(0^+)$ (finite or infinite).*

For $0\le X\sim F$ with finite positive mean μ_1, we define the first-order equilibrium distribution by $F_{(1)}(x)=\mu_1^{-1}\int_0^x\overline{F}(y)\,dy,\ x\ge 0$, where $\overline{F}(x)=1-F(x)$. The high-order equilibrium distributions are defined iteratively. Namely, the nth-order equilibrium distribution is $F_{(n)}(x)=\mu_{(n-1)}^{-1}\int_0^x\overline{F}_{(n-1)}(y)\,dy$, $x\ge 0$, provided the mean $\mu_{(n-1)}$ of $F_{(n-1)}$ is finite (equivalently, the nth moment $\mu_n=\mathbf{E}[X^n]$ of F is finite). For the next relationship between the means of $\{F_{(n)}\}$ and moments of F, see, e.g., Lin [12], p. 265, or Harkness and Shantaram [5].

Lemma 7. *Let $0\le X\sim F$ have the nth moment $\mu_n\in(0,\infty)$ for some $n\ge 2$. Then the mean of the $(n-1)$th-order equilibrium distribution $F_{(n-1)}$ is equal to $\mu_{(n-1)}=\mu_n/(n\,\mu_{n-1})$.*

Lemma 8. *Let $0\le X\sim F$ with finite mean $\mu\in(0,\infty)$ and let $X_{(1)}\sim F_{(1)}$ have the first-order equilibrium distribution. Then for $s>0$, the following statements are true:*

(i) $(1-\hat{F}(s))/s = \int_0^\infty e^{-sx}(1-F(x))dx$;
(ii) $\hat{F}_{(1)}(s) = (1-\hat{F}(s))/(\mu s) \le 1$;
(iii) $(\hat{F}(s)-1+\mu s)/s^2 = \mu\int_0^\infty e^{-sx}(1-F_{(1)}(x))dx$;
(iv) $\lim_{s\to 0^+}(1-\hat{F}(s))/s = \mu$ *and* $\lim_{s\to 0^+}(\hat{F}(s)-1+\mu s)/s^2 = \mathbf{E}[X^2]/2$ *(finite or infinite).*

Proof. For part (i), see Lin [12], p. 260, or Feller [3], p. 435. Parts (ii)–(iv) follow from the definition of equilibrium distribution and Lemmas 6 and 7 immediately.

The next two lemmas are key tools to prove the main results.

Lemma 9. *Let* $0 \le Y_n \sim G_n$, $n = 0, 1, 2, \ldots$, *be a sequence of random variables having the same first two finite moments, say* μ_1 *and* μ_2. *Suppose that their Laplace–Stieltjes transforms* $\{\hat{G}_n\}_{n=0}^\infty$ *form a decreasing sequence of functions. Then the limiting function* $\lim_{n\to\infty}\hat{G}_n(s) = \hat{G}_\infty(s)$, $s \ge 0$, *exists and is the Laplace–Stieltjes transform of a nonnegative random variable, say* Y_∞, *which has the mean* $\mathbf{E}[Y_\infty] = \mu_1$ *and second moment* $\mathbf{E}[Y_\infty^2] \in [\mu_1^2, \mu_2]$.

Proof. For each fixed $s \ge 0$, $\hat{G}_n(s) \in [0, 1]$, $n \ge 0$, so the decreasing sequence $\{\hat{G}_n(s)\}_{n=0}^\infty$ has a limit, denoted $\hat{G}_\infty(s) = \lim_{n\to\infty}\hat{G}_n(s)$. On the other hand, we have, by Jensen's inequality and the assumption,

$$e^{-\mu_1 s} \le \hat{G}_n(s) \le \hat{G}_0(s), \quad s \ge 0,\ n \ge 1. \tag{20}$$

Therefore, the limiting function $\hat{G}_\infty$ satisfies $e^{-\mu_1 s} \le \hat{G}_\infty(s) \le \hat{G}_0(s)$, $s \ge 0$, and hence $\lim_{s\to 0^+}\hat{G}_\infty(s) = 1 = \hat{G}_\infty(0)$. By the continuity theorem for Laplace–Stieltjes transforms (see, e.g., [20], p. 479), we conclude that $\hat{G}_\infty$ is the Laplace–Stieltjes transform of a nonnegative random variable, denoted Y_∞. It remains to verify $\mathbf{E}[Y_\infty] = \mu_1$ and $\mathbf{E}[Y_\infty^2] \in [\mu_1^2, \mu_2]$. From (20) it follows that the limiting function $\hat{G}_\infty$ satisfies

$$\frac{1-e^{-\mu_1 s}}{s} \ge \frac{1-\hat{G}_\infty(s)}{s} \ge \frac{1-\hat{G}_0(s)}{s}, \quad s > 0, \tag{21}$$

and

$$\frac{e^{-\mu_1 s}-1+\mu_1 s}{s^2} \le \frac{\hat{G}_\infty(s)-1+\mu_1 s}{s^2} \le \frac{\hat{G}_0(s)-1+\mu_1 s}{s^2}, \quad s > 0. \tag{22}$$

Finally, applying Lemma 8(iv) first to (21) gets $\mathbf{E}[Y_\infty] = \mu_1$ and then to (22) yields $\mu_1^2/2 \le \mathbf{E}[Y_\infty^2]/2 \le \mu_2/2$. This completes the proof.

Lemma 10. *Let* $W_1 \sim F_{W_1}$ *and* $W_2 \sim F_{W_2}$ *be two nonnegative random variables with the same mean* $\mu_W \in (0, \infty)$, *and let* $0 \le Z^* \sim F_{Z^*}$ *have a mean* $\mu_{Z^*} \in (0, 1)$. *Assume further that the Laplace–Stieltjes transforms of* W_1 *and* W_2 *satisfy*

$$|\hat{F}_{W_1}(s) - \hat{F}_{W_2}(s)| \le \int_0^\infty |\hat{F}_{W_1}(ts) - \hat{F}_{W_2}(ts)|dF_{Z^*}(t), \quad s \ge 0, \tag{23}$$

or, equivalently,

$$\big|\mathbf{E}[\exp(-sW_1)] - \mathbf{E}[\exp(-sW_2)]\big| \le \big|\mathbf{E}[\exp(-sZ^*W_1)] - \mathbf{E}[\exp(-sZ^*W_2)]\big|, \quad s \ge 0.$$

Then $\hat{F}_{W_1} = \hat{F}_{W_2}$ *and hence* $F_{W_1} = F_{W_2}$.

Proof. For each $s > 0$, applying the inequality (23) $(n-1)$ more times yields

$$|\hat{F}_{W_1}(s) - \hat{F}_{W_2}(s)| \le \int_0^\infty \cdots \int_0^\infty |\hat{F}_{W_1}(t_1\cdots t_n s) - \hat{F}_{W_2}(t_1\cdots t_n s)| dF_{Z^*}(t_1)\cdots dF_{Z^*}(t_n)$$

$$= \int_{0+}^\infty \cdots \int_{0+}^\infty \left|\frac{\hat{F}_{W_1}(t_1\cdots t_n s) - \hat{F}_{W_2}(t_1\cdots t_n s)}{\mu_W t_1\cdots t_n s}\right| \mu_W t_1\cdots t_n s dF_{Z^*}(t_1)\cdots dF_{Z^*}(t_n). \tag{24}$$

We now estimate the first part of the integrand:

$$\left|\frac{\hat{F}_{W_1}(t_1\cdots t_n s) - \hat{F}_{W_2}(t_1\cdots t_n s)}{\mu_W t_1\cdots t_n s}\right| = \left|\frac{1-\hat{F}_{W_1}(t_1\cdots t_n s)}{\mu_W t_1\cdots t_n s} - \frac{1-\hat{F}_{W_2}(t_1\cdots t_n s)}{\mu_W t_1\cdots t_n s}\right|$$

$$\le \left|\frac{1-\hat{F}_{W_1}(t_1\cdots t_n s)}{\mu_W t_1\cdots t_n s}\right| + \left|\frac{1-\hat{F}_{W_2}(t_1\cdots t_n s)}{\mu_W t_1\cdots t_n s}\right| \le 2. \tag{25}$$

The last inequality is due to Lemma 8(ii). Combining (24) and (25) together leads to

$$|\hat{F}_{W_1}(s) - \hat{F}_{W_2}(s)| \le 2\mu_W s \int_{0+}^\infty \cdots \int_{0+}^\infty t_1\cdots t_n dF_{Z^*}(t_1)\cdots dF_{Z^*}(t_n)$$
$$= 2\mu_W s(\mathbf{E}[Z^*])^n \longrightarrow 0 \quad \text{as } n\to\infty,$$

in which the last conclusion follows from the assumption $\mathbf{E}[Z^*] = \mu_{Z^*} \in (0,1)$. Therefore, $\hat{F}_{W_1} = \hat{F}_{W_2}$. This completes the proof.

4. Proofs of main results

Proof of Theorem 1. (Sufficiency) Suppose that Eq (2) has exactly one solution $0 \le X \sim F$ with mean $\mu \in (0,\infty)$ and a finite variance (and hence $\mathbf{E}[X^2] \in (0,\infty)$). Then we want to prove that the conditions (6) hold true.

Rewrite Eq (2) as

$$\hat{F}(s) = P_N\Big(\int_0^\infty \hat{F}(ts)dF_T(t)\Big) = \sum_{n=0}^\infty \Pr(N=n)\Big(\int_0^\infty \hat{F}(ts)dF_T(t)\Big)^n, \quad s\ge 0.$$

Differentiating twice the above equation with respect to s, we have, for $s > 0$,

$$\hat{F}'(s) = \sum_{n=1}^\infty \Pr(N=n)n\Big(\int_0^\infty \hat{F}(ts)dF_T(t)\Big)^{n-1}\int_0^\infty \hat{F}'(ts)tdF_T(t), \tag{26}$$

$$\hat{F}''(s) = \sum_{n=2}^\infty \Pr(N=n)n(n-1)\Big(\int_0^\infty \hat{F}(ts)dF_T(t)\Big)^{n-2}\Big(\int_0^\infty \hat{F}'(ts)tdF_T(t)\Big)^2$$
$$+\sum_{n=1}^\infty \Pr(N=n)n\Big(\int_0^\infty \hat{F}(ts)dF_T(t)\Big)^{n-1}\int_0^\infty \hat{F}''(ts)t^2dF_T(t). \tag{27}$$

Letting $s\to 0^+$ in (26) and (27) yields, respectively,

$$\hat{F}'(0^+) = \hat{F}'(0^+)\mathbf{E}[N]\mathbf{E}[T],$$
$$\hat{F}''(0^+) = \mathbf{E}[N(N-1)](\hat{F}'(0^+)\mathbf{E}[T])^2 + \hat{F}''(0^+)\mathbf{E}[N]\mathbf{E}[T^2].$$

Equivalently, we have, by Lemma 6,

$$\mu = \mu\, \mathbf{E}[N]\mathbf{E}[T], \tag{28}$$
$$\mathbf{E}[X^2] = \mathbf{E}[N(N-1)](\mu\, \mathbf{E}[T])^2 + \mathbf{E}[X^2]\mathbf{E}[N]\mathbf{E}[T^2]. \tag{29}$$

From (28) and (29) it follows that $\mathbf{E}[N]\mathbf{E}[T] = 1$ (which implies that $\mathbf{E}[N], \mathbf{E}[T] > 0$) and $\mathbf{E}[N^2] < \infty$ because $\mu, \mathbf{E}[X^2] \in (0, \infty)$. It remains to prove that $0 < \mathbf{E}[T^2] < \mathbf{E}[T] < 1$.

Since $\mathbf{E}[N(N-1)] \geq 0$, we have by (29) that $\mathbf{E}[N]\mathbf{E}[T^2] \leq 1$, and hence $\mathbf{E}[T^2] \leq \mathbf{E}[T]$ due to the fact $\mathbf{E}[N]\mathbf{E}[T] = 1$. Namely,

$$0 < (\mathbf{E}[T])^2 \leq \mathbf{E}[T^2] \leq \mathbf{E}[T], \tag{30}$$

from which we further have $0 < \mathbf{E}[T] \leq 1$. We now prove that $\mathbf{E}[T] < 1$. Suppose on the contrary $\mathbf{E}[T] = 1$. Then $\mathbf{E}[N] = 1$ (by the fact $\mathbf{E}[N]\mathbf{E}[T] = 1$) and from (30) it follows that $\mathbf{E}[T^2] = 1$, $\mathrm{Var}(T) = 0$ and $\Pr(T = 1) = 1$. Plugging these in (29) yields

$$1 = (\mathbf{E}[N])^2 \leq \mathbf{E}[N^2] = \mathbf{E}[N] = 1, \tag{31}$$

which implies $\Pr(N = 1) = 1$ as in the case of T. These together imply that Eq (2) is an identity for any $0 \leq X \sim F$ as described before, which contradicts the unique solution to Eq (2). So we conclude that $\mathbf{E}[T] \in (0, 1)$.

Finally, we prove $\mathbf{E}[T^2] < \mathbf{E}[T]$. Suppose on the contrary $\mathbf{E}[T^2] = \mathbf{E}[T]$. Then (31) follows from (29) again (using $\mathbf{E}[N]\mathbf{E}[T] = 1$) and hence $\Pr(N = 1) = 1$. This is impossible because $\mathbf{E}[T] \in (0, 1)$ and $\mathbf{E}[N]\mathbf{E}[T] = 1$. The proof of the sufficiency part is complete.

(Necessity) Suppose that the conditions (6) hold true. Then we will prove the existence of a solution F to Eq (2) with mean μ and a finite variance.

Set first

$$\mu_1 = \mu \quad \text{and} \quad \mu_2 = \frac{\mathbf{E}[N(N-1)](\mathbf{E}[T])^2}{1 - \mathbf{E}[N]\mathbf{E}[T^2]} \cdot \mu_1^2. \tag{32}$$

Note that the denominator $1 - \mathbf{E}[N]\mathbf{E}[T^2] = 1 - \mathbf{E}[T^2]/\mathbf{E}[T] > 0$ by (6) and that $\mu_2 \geq \mu_1^2$ by the facts: $\mathbf{E}[N^2] \geq (\mathbf{E}[N])^2$ and $\mathbf{E}[T^2] \geq (\mathbf{E}[T])^2$. Therefore, the RHS of (19) with μ_1, μ_2 defined in (32) is a bona fide Laplace–Stieltjes transform, say $\hat{F}_0$, of a nonnegative random variable $Y_0 \sim F_0$ (by Lemma 1). Namely,

$$\hat{F}_0(s) = 1 - \frac{\mu_1^2}{\mu_2} + \frac{\mu_1^2}{\mu_2} e^{-(\mu_2/\mu_1)s}, \quad s \geq 0.$$

Next, using the initial $Y_0 \sim F_0$ we define iteratively the sequence of random variables $Y_n \sim F_n$, $n = 1, 2, \ldots$, through Laplace–Stieltjes transforms:

$$\hat{F}_n(s) = P_N\left(\int_0^\infty \hat{F}_{n-1}(ts)\,dF_T(t)\right) = P_N\big(\mathbf{E}[\exp(-sTY_{n-1})]\big),\ n \geq 1, \tag{33}$$

which is well-defined due to Lemma 2. Differentiating twice the above equation with respect to s and letting $s \to 0^+$, we have, for $n \geq 1$,

$$\hat{F}'_n(0^+) = \hat{F}'_{n-1}(0^+)\mathbf{E}[N]\mathbf{E}[T] = \hat{F}'_{n-1}(0^+), \tag{34}$$

$$\hat{F}''_n(0^+) = \mathbf{E}[N(N-1)](\hat{F}'_{n-1}(0^+)\mathbf{E}[T])^2 + \hat{F}''_{n-1}(0^+)\mathbf{E}[N]\mathbf{E}[T^2]. \tag{35}$$

With the help of Lemma 6 and by induction on n, we can show through (34) and (35) that $\mathbf{E}[Y_n] = \mathbf{E}[Y_0] = \mu_1$, $\mathbf{E}[Y_n^2] = \mathbf{E}[Y_0^2] = \mu_2$ (defined in (32)) for all $n \geq 1$ and hence

$$\mathrm{Var}(Y_n) = \mu_2 - \mu_1^2 = \frac{(\mathbf{E}[T])^2\mathrm{Var}(N) + \mathbf{E}[N]\mathrm{Var}(T)}{1 - \mathbf{E}[N]\mathbf{E}[T^2]} \cdot \mu_1^2, \quad n \geq 0. \tag{36}$$

Moreover, by Lemma 4, we first have $\hat{F}_1 \leq \hat{F}_0$, and then by the iteration (33), $\hat{F}_n \leq \hat{F}_{n-1}$ for all $n \geq 2$ (due to the absolute monotonicity of P_N). Namely, $\{Y_n\}_{n=0}^{\infty}$ is a sequence of nonnegative random variables having the same first two moments μ_1, μ_2, and their Laplace–Stieltjes transforms $\{\hat{F}_n\}$ are decreasing. Therefore, Lemma 9 applies. Denote the limit of $\{\hat{F}_n\}$ by $\hat{F}_\infty$, which is the Laplace–Stieltjes transform of a nonnegative random variable $Y_\infty \sim F_\infty$ with $\mathbf{E}[Y_\infty] = \mu_1$ and $\mathbf{E}[Y_\infty^2] \in [\mu_1^2, \mu_2]$. Consequently, it follows from (33) that the limit F_∞ is a solution to Eq (2) with mean μ and a finite variance. Applying Lemma 6 to Eq (2) again (with $F = F_\infty$), we conclude that $\mathbf{E}[Y_\infty^2] = \mu_2$ as given in (32), and hence the solution $Y_\infty \sim F_\infty$ has the required variance as shown in (7) or (36).

Finally, we prove the uniqueness of the solution to Eq (2). Suppose there are two solutions to Eq (2), denoted $0 \leq X \sim F_X$ and $0 \leq Y \sim F_Y$. We want to show that $F_X = F_Y$. As before, with the help of Lemma 6 and the conditions (6), we have from Eq (2) that

$$\mathbf{E}[X] = \mathbf{E}[Y] = \mu_1 = \mu, \quad \mathbf{E}[X^2] = \mathbf{E}[Y^2] = \mu_2 = \frac{\mathbf{E}[N(N-1)](\mathbf{E}[T])^2}{1 - \mathbf{E}[N]\mathbf{E}[T^2]} \cdot \mu_1^2.$$

Let $W_1 \sim F_{W_1}, W_2 \sim F_{W_2}$ have the first-order equilibrium distributions of X, Y, respectively. By Lemma 8(ii), their Laplace-Stieltjes transforms are of the form:

$$\hat{F}_{W_1}(s) = \frac{1 - \hat{F}_X(s)}{\mu s}, \quad \hat{F}_{W_2}(s) = \frac{1 - \hat{F}_Y(s)}{\mu s}, \ s > 0. \tag{37}$$

Therefore, it remains to prove that $\hat{F}_{W_1}(s) = \hat{F}_{W_2}(s)$, $s > 0$.

From Lemma 7 it follows that $\mathbf{E}[W_1] = \mathbf{E}[W_2] = \mu_2/(2\mu_1) \equiv \mu_W \in (0, \infty)$. Using Eq (2), we first estimate the difference between $\hat{F}_X$ and $\hat{F}_Y$ as follows: for $s > 0$,

$$|\hat{F}_X(s) - \hat{F}_Y(s)| = \left| P_N\Big(\int_0^\infty \hat{F_X}(ts)dF_T(t)\Big) - P_N\Big(\int_0^\infty \hat{F_Y}(ts)dF_T(t)\Big)\right|$$

$$= \left|\sum_{n=0}^\infty \Pr(N=n)\Big[\Big(\int_0^\infty \hat{F_X}(ts)dF_T(t)\Big)^n - \Big(\int_0^\infty \hat{F_Y}(ts)dF_T(t)\Big)^n\Big]\right|$$

$$\leq \sum_{n=0}^\infty \Pr(N=n)\cdot n \int_0^\infty |\hat{F}_X(ts) - \hat{F}_Y(ts)|dF_T(t)$$

$$= \mathbf{E}[N]\int_0^\infty |\hat{F}_X(ts) - \hat{F}_Y(ts)|dF_T(t)$$

$$= \frac{1}{\mathbf{E}[T]}\int_0^\infty |\hat{F}_X(ts) - \hat{F}_Y(ts)|dF_T(t),$$

in which the inequality follows from Lemma 3, while the last equality is due to the condition $\mathbf{E}[N]\mathbf{E}[T] = 1$ in (6). Therefore, we have, for $s > 0$,

$$\left|\frac{\hat{F}_X(s) - \hat{F}_Y(s)}{\mu s}\right| \le \int_0^\infty \left|\frac{\hat{F}_X(ts) - \hat{F}_Y(ts)}{\mu st}\right| \frac{t}{\mathbf{E}[T]} dF_T(t) \equiv \int_0^\infty \left|\frac{\hat{F}_X(ts) - \hat{F}_Y(ts)}{\mu st}\right| dF_{Z^*}(t),$$

where $Z^* \sim F_{Z^*}$ has the length-biased distribution of $T \sim F_T$ and $\mathbf{E}[Z^*] = \mathbf{E}[T^2]/\mathbf{E}[T] < 1$. Equivalently, we have, by (37), that

$$|\hat{F}_{W_1}(s) - \hat{F}_{W_2}(s)| \le \int_0^\infty |\hat{F}_{W_1}(ts) - \hat{F}_{W_2}(ts)| dF_{Z^*}(t), \ \ s > 0.$$

Lemma 10 applies and hence $\hat{F}_{W_1} = \hat{F}_{W_2}$. This proves the uniqueness of the solution to Eq (2). The proof of the necessity part is complete.

Proof of Proposition 1. (i) Since $\mathbf{E}[N]\mathbf{E}[T] = 1$, we have $\mathbf{E}[N],\ \mathbf{E}[T] > 0$ and hence $\Pr(T > 0) > 0$. Rewrite

$$1 = \mathbf{E}[N]\mathbf{E}[T] = \mathbf{E}[N]\mathbf{E}[T|\, T > 0] \Pr(T > 0).$$

Therefore, $0 < \Pr(T > 0)\mathbf{E}[N] = (\mathbf{E}[T|\, T > 0])^{-1}$. We will show that $\mathbf{E}[T|\, T > 0] < 1$. Write

$$\mathbf{E}[T|\, T > 0] = \frac{1}{\Pr(T > 0)} \int_{0^+}^\infty t dF_T(t) = \frac{\mathbf{E}[T]}{\Pr(T > 0)}.$$

Similarly, $\mathbf{E}[T^2|\, T > 0] = \mathbf{E}[T^2]/\Pr(T > 0)$. From the condition $\mathbf{E}[T^2] < \mathbf{E}[T]$ it then follows that $(\mathbf{E}[T|\, T > 0])^2 \le \mathbf{E}[T^2|\, T > 0] < \mathbf{E}[T|\, T > 0]$. Consequently, $\mathbf{E}[T|\, T > 0] < 1$. This proves the first conclusion of the proposition.

(ii) We next prove the second conclusion $\mathbf{E}[T \log T] < 0$. Note first that the function $g(t) = t^2 - t - t \log t \ge 0$ for $t > 0$. So we have $\mathbf{E}[T^2 - T - T \log T \,|\, T > 0] \ge 0$. Equivalently,

$$\mathbf{E}[T^2 - T \,|\, T > 0] \ge \mathbf{E}[T \log T \,|\, T > 0].$$

Finally, by the condition $0 < \mathbf{E}[T^2] < \mathbf{E}[T] < 1$, we have

$$\begin{aligned} 0 > \mathbf{E}[T^2 - T] &= \mathbf{E}[T^2 - T \,|\, T > 0] \Pr(T > 0) \\ &\ge \mathbf{E}[T \log T \,|\, T > 0] \Pr(T > 0) = \mathbf{E}[T \log T]. \end{aligned}$$

This completes the proof.

Proof of Theorem 2. Note that Eq (9) with $m = 1$ is equivalent to

$$\hat{F}(s) = P_{N+1}\Big(\int_0^\infty \hat{F}(ts) dF_T(t)\Big) = P_{N+1}\big(\mathbf{E}[\exp(-sTX)]\big), \quad s \ge 0,$$

and that $\mathrm{Var}(N + 1) = \mathrm{Var}(N)$ in (11). Therefore, Theorem 2 follows from Theorem 1 by replacing N by $N + 1$ taking values in $\mathbb{N} \equiv \{1, 2, 3, \ldots\}$. The proof is complete.

Proof of Theorem 3. The proof is similar to that of Theorem 1. We give the details here for completeness.

(Sufficiency) Suppose that Eq (9) with $m \geq 2$ has exactly one solution $0 \leq X \sim F$ with mean $\mu \in (0, \infty)$ and a finite variance (and hence $\mathbf{E}[X^2] \in (0, \infty)$). Then we want to prove that the conditions (12) hold true.

Rewrite Eq (9) with $m \geq 2$ as

$$\begin{aligned}\hat{F}(s) &= \left(\int_0^\infty \hat{F}(ts)dF_T(t)\right)^m P_N\left(\int_0^\infty \hat{F}(ts)dF_T(t)\right)\\ &= \sum_{n=0}^\infty \Pr(N=n)\left(\int_0^\infty \hat{F}(ts)dF_T(t)\right)^{n+m}, \quad s \geq 0.\end{aligned}$$

Differentiating twice the above equation with respect to s, we have, for $s > 0$,

$$\hat{F}'(s) = \sum_{n=0}^\infty \Pr(N=n)(n+m)\left(\int_0^\infty \hat{F}(ts)dF_T(t)\right)^{n+m-1}\int_0^\infty \hat{F}'(ts)tdF_T(t), \tag{38}$$

$$\begin{aligned}\hat{F}''(s) &= \sum_{n=0}^\infty \Pr(N=n)(n+m)(n+m-1)\left(\int_0^\infty \hat{F}(ts)dF_T(t)\right)^{n+m-2}\left(\int_0^\infty \hat{F}'(ts)tdF_T(t)\right)^2\\ &\quad+\sum_{n=0}^\infty \Pr(N=n)(n+m)\left(\int_0^\infty \hat{F}(ts)dF_T(t)\right)^{n+m-1}\int_0^\infty \hat{F}''(ts)t^2dF_T(t).\end{aligned} \tag{39}$$

Letting $s \to 0^+$ in (38) and (39) yields, respectively,

$$\begin{aligned}\hat{F}'(0^+) &= \hat{F}'(0^+)\mathbf{E}[N+m]\mathbf{E}[T],\\ \hat{F}''(0^+) &= \mathbf{E}[(N+m)(N+m-1)](\hat{F}'(0^+)\mathbf{E}[T])^2 + \hat{F}''(0^+)\mathbf{E}[N+m]\mathbf{E}[T^2].\end{aligned}$$

Equivalently, we have, by Lemma 6,

$$\mu = \mu\ \mathbf{E}[N+m]\mathbf{E}[T], \tag{40}$$

$$\mathbf{E}[X^2] = \mathbf{E}[(N+m)(N+m-1)](\mu\,\mathbf{E}[T])^2 + \mathbf{E}[X^2]\mathbf{E}[N+m]\mathbf{E}[T^2]. \tag{41}$$

Therefore, from (40) and (41) it follows that $\mathbf{E}[N+m]\mathbf{E}[T] = 1$ and $\mathbf{E}[N^2] < \infty$ because $\mu, \mathbf{E}[X^2] \in (0, \infty)$. It remains to prove that $0 < \mathbf{E}[T^2] < \mathbf{E}[T] \leq 1/m < 1$.

Since $\mathbf{E}[(N+m)N] \geq 0$, it further follows from (41) that $\mathbf{E}[N+m]\mathbf{E}[T^2] \leq 1$, and hence $\mathbf{E}[T^2] \leq \mathbf{E}[T]$ because $\mathbf{E}[N+m]\mathbf{E}[T] = 1$. The latter also implies that $\mathbf{E}[T] > 0$, and hence

$$0 < (\mathbf{E}[T])^2 \leq \mathbf{E}[T^2] \leq \mathbf{E}[T] = \frac{1}{\mathbf{E}[N+m]} \leq \frac{1}{m} < 1.$$

Finally, we prove $\mathbf{E}[T^2] < \mathbf{E}[T]$. Suppose on the contrary $\mathbf{E}[T^2] = \mathbf{E}[T]$. Then from (41) it follows (by using $\mathbf{E}[N+m]\mathbf{E}[T] = 1$) that

$$\mathbf{E}[(N+m)(N+m-1)] = 0.$$

This is impossible because $N \geq 0$ and $m \geq 2$. The proof of the sufficiency part is complete.

(Necessity) Suppose that the conditions (12) hold true. Then we will prove the existence of a solution F to Eq (9) with $m \geq 2$, which has mean μ and a finite variance.

Set first

$$\mu_1 = \mu \quad \text{and} \quad \mu_2 = \frac{\mathbf{E}[(N+m)(N+m-1)](\mathbf{E}[T])^2}{1-\mathbf{E}[N+m]\mathbf{E}[T^2]} \cdot \mu_1^2. \tag{42}$$

Note that the denominator $1-\mathbf{E}[N+m]\mathbf{E}[T^2] = 1-\mathbf{E}[T^2]/\mathbf{E}[T] > 0$ by (12) and that $\mu_2 \geq \mu_1^2$ by the facts: $\mathbf{E}[(N+m)^2] \geq (\mathbf{E}[N+m])^2$ and $\mathbf{E}[T^2] \geq (\mathbf{E}[T])^2$. Therefore, the RHS of (19) with μ_1, μ_2 defined in (42) is a bona fide Laplace–Stieltjes transform, say $\hat{F}_0$, of a nonnegative random variable $Y_0 \sim F_0$. Namely,

$$\hat{F}_0(s) = 1 - \frac{\mu_1^2}{\mu_2} + \frac{\mu_1^2}{\mu_2} e^{-(\mu_2/\mu_1)s}, \quad s \geq 0.$$

Next, using the initial $Y_0 \sim F_0$ we define iteratively the sequence of random variables $Y_n \sim F_n$, $n = 1, 2, \ldots$, through Laplace–Stieltjes transforms:

$$\hat{F}_n(s) = \Big(\int_0^\infty \hat{F}_{n-1}(ts)dF_T(t)\Big)^m P_N\Big(\int_0^\infty \hat{F}_{n-1}(ts)dF_T(t)\Big), \quad n \geq 1, \tag{43}$$

which is well-defined due to Lemma 2. Differentiating twice the above equation with respect to s and letting $s \to 0^+$, we have, for $n \geq 1$,

$$\hat{F}_n'(0^+) = \hat{F}_{n-1}'(0^+)\mathbf{E}[N+m]\mathbf{E}[T] = \hat{F}_{n-1}'(0^+), \tag{44}$$

$$\hat{F}_n''(0^+) = \mathbf{E}[(N+m)(N+m-1)](\hat{F}_{n-1}'(0^+)\mathbf{E}[T])^2 + \hat{F}_{n-1}''(0^+)\mathbf{E}[N+m]\mathbf{E}[T^2]. \tag{45}$$

With the help of Lemma 6 and by induction on n, we can show through (44) and (45) that $\mathbf{E}[Y_n] = \mathbf{E}[Y_0] = \mu_1$, $\mathbf{E}[Y_n^2] = \mathbf{E}[Y_0^2] = \mu_2$ (defined in (42)) for all $n \geq 1$ and hence

$$\mathrm{Var}(Y_n) = \mu_2 - \mu_1^2 = \frac{(\mathbf{E}[T])^2\mathrm{Var}(N) + \mathbf{E}[N+m]\mathrm{Var}(T)}{1-\mathbf{E}[N+m]\mathbf{E}[T^2]} \cdot \mu_1^2, \quad n \geq 0. \tag{46}$$

Moreover, by Lemma 4, we first have $\hat{F}_1 \leq \hat{F}_0$, and then by the iteration (43), $\hat{F}_n \leq \hat{F}_{n-1}$ for all $n \geq 2$. Namely, $\{Y_n\}_{n=0}^\infty$ is a sequence of nonnegative random variables having the same first two moments μ_1, μ_2, and their Laplace–Stieltjes transforms $\{\hat{F}_n\}$ are decreasing. Therefore, Lemma 9 applies. Denote the limit of $\{\hat{F}_n\}$ by $\hat{F}_\infty$, which is the Laplace–Stieltjes transform of a nonnegative random variable $Y_\infty \sim F_\infty$ with $\mathbf{E}[Y_\infty] = \mu_1$ and $\mathbf{E}[Y_\infty^2] \in [\mu_1^2, \mu_2]$. Consequently, it follows from (43) that the limit F_∞ is a solution to Eq (9) with $m \geq 2$, which has mean μ and a finite variance. Applying Lemma 6 to Eq (9) with $m \geq 2$ again, we conclude that $\mathbf{E}[Y_\infty^2] = \mu_2$ as given in (42), and hence the solution $Y_\infty \sim F_\infty$ has the required variance as shown in (13) or (46).

Finally, we prove the uniqueness of the solution to Eq (9) with $m \geq 2$. Suppose that there are two solutions, denoted $0 \leq X \sim F_X$ and $0 \leq Y \sim F_Y$. We want to show $F_X = F_Y$. As before, with the help of Lemma 6 and the conditions (12), we have from Eq (9) with $m \geq 2$ that

$$\mathbf{E}[X] = \mathbf{E}[Y] = \mu_1 = \mu, \quad \mathbf{E}[X^2] = \mathbf{E}[Y^2] = \mu_2 = \frac{\mathbf{E}[(N+m)(N+m-1)](\mathbf{E}[T])^2}{1-\mathbf{E}[N+m]\mathbf{E}[T^2]} \cdot \mu_1^2.$$

Let $W_1 \sim F_{W_1}, W_2 \sim F_{W_2}$ have the first-order equilibrium distributions of X, Y, respectively. By Lemma 8(ii), their Laplace–Stieltjes transforms are of the form:

$$\hat{F}_{W_1}(s) = \frac{1-\hat{F}_X(s)}{\mu s}, \quad \hat{F}_{W_2}(s) = \frac{1-\hat{F}_Y(s)}{\mu s}, \quad s > 0. \tag{47}$$

Therefore, it remains to prove that $\hat{F}_{W_1}(s) = \hat{F}_{W_2}(s)$, $s > 0$.

From Lemma 7 it follows that $\mathbf{E}[W_1] = \mathbf{E}[W_2] = \mu_2/(2\mu_1) \equiv \mu_W \in (0, \infty)$. Using Eq (9) with $m \geq 2$, we first estimate the difference between $\hat{F}_X$ and $\hat{F}_Y$ as follows: for $s > 0$,

$$
\begin{aligned}
|\hat{F}_X(s) - \hat{F}_Y(s)| &= \Big|\sum_{n=0}^{\infty} \Pr(N = n)\Big[\Big(\int_0^{\infty} \hat{F}_X(ts)dF_T(t)\Big)^{n+m} - \Big(\int_0^{\infty} \hat{F}_Y(ts)dF_T(t)\Big)^{n+m}\Big]\Big| \\
&\leq \sum_{n=0}^{\infty} \Pr(N = n) \cdot (n + m) \int_0^{\infty} |\hat{F}_X(ts) - \hat{F}_Y(ts)|dF_T(t) \\
&= \mathbf{E}[N + m] \int_0^{\infty} |\hat{F}_X(ts) - \hat{F}_Y(ts)|dF_T(t) \\
&= \frac{1}{\mathbf{E}[T]} \int_0^{\infty} |\hat{F}_X(ts) - \hat{F}_Y(ts)|dF_T(t),
\end{aligned}
$$

in which the inequality follows from Lemma 3, while the last equality is due to the condition $\mathbf{E}[N+m]\mathbf{E}[T] = 1$ in (12). Therefore, we have, for $s > 0$,

$$
\left|\frac{\hat{F}_X(s) - \hat{F}_Y(s)}{\mu s}\right| \leq \int_0^{\infty} \left|\frac{\hat{F}_X(ts) - \hat{F}_Y(ts)}{\mu st}\right| \frac{t}{\mathbf{E}[T]} dF_T(t) \equiv \int_0^{\infty} \left|\frac{\hat{F}_X(ts) - \hat{F}_Y(ts)}{\mu st}\right| dF_{Z^*}(t),
$$

where $Z^* \sim F_{Z^*}$ has the length-biased distribution of $T \sim F_T$ and $\mathbf{E}[Z^*] = \mathbf{E}[T^2]/\mathbf{E}[T] < 1$. Equivalently, we have, by (47), that

$$
|\hat{F}_{W_1}(s) - \hat{F}_{W_2}(s)| \leq \int_0^{\infty} |\hat{F}_{W_1}(ts) - \hat{F}_{W_2}(ts)|dF_{Z^*}(t), \quad s > 0.
$$

Lemma 10 applies and hence $\hat{F}_{W_1} = \hat{F}_{W_2}$. This proves the uniqueness of the solution to Eq (9) with $m \geq 2$. The proof of the necessity part is complete.

Proof of Theorem 4. We first sketch the proof of part (a).

(Sufficiency) Suppose that Eq (15) has one solution $0 \leq X \sim F$ with mean $\mu \in (0, \infty)$ and a finite variance (and hence $\mathbf{E}[X^2] \in (0, \infty)$). Then we want to prove that the conditions (16) hold true.

Rewrite Eq (15) as

$$
\hat{F}(s) = \hat{F}_B(s) \sum_{n=0}^{\infty} \Pr(N = n)\Big(\int_0^{\infty} \hat{F}(ts)dF_T(t)\Big)^n, \quad s \geq 0.
$$

Differentiating twice the above equation with respect to s and letting $s \to 0^+$ yield

$$
\begin{aligned}
\hat{F}'(0^+) &= \hat{F}_B'(0^+) + \hat{F}'(0^+)\mathbf{E}[N]\mathbf{E}[T], \\
\hat{F}''(0^+) &= \hat{F}_B''(0^+) + 2\hat{F}_B'(0^+)\hat{F}'(0^+)\mathbf{E}[N]\mathbf{E}[T] \\
&\quad + \mathbf{E}[N(N-1)](\hat{F}'(0^+)\mathbf{E}[T])^2 + \hat{F}''(0^+)\mathbf{E}[N]\mathbf{E}[T^2].
\end{aligned}
$$

Equivalently, we have, by Lemma 6,

$$\mu = \mathbf{E}[B] + \mu\, \mathbf{E}[N]\mathbf{E}[T] > \mu\, \mathbf{E}[N]\mathbf{E}[T], \tag{48}$$

$$\begin{aligned}\mathbf{E}[X^2] &= \mathbf{E}[B^2] + 2\mu\, \mathbf{E}[B]\mathbf{E}[N]\mathbf{E}[T] + \mathbf{E}[N(N-1)](\mu\, \mathbf{E}[T])^2 + \mathbf{E}[X^2]\mathbf{E}[N]\mathbf{E}[T^2] \\ &= \mathrm{Var}(B) + \mu^2(\mathbf{E}[T])^2\mathrm{Var}(N) + \mu^2(1-\mathbf{E}[N](\mathbf{E}[T])^2) + \mathbf{E}[X^2]\mathbf{E}[N]\mathbf{E}[T^2]. \end{aligned} \tag{49}$$

Therefore, $0 < \mathbf{E}[N]\mathbf{E}[T] < 1$ by (48) and $\mu = \mathbf{E}[B]/(1 - \mathbf{E}[N]\mathbf{E}[T])$. Moreover, it follows from (48) and (49) that

$$\mathrm{Var}(X)(1 - \mathbf{E}[N]\mathbf{E}[T^2]) = \mathrm{Var}(B) + \mu^2(\mathbf{E}[T])^2\mathrm{Var}(N) + \mu^2\mathbf{E}[N]\mathrm{Var}(T) \in (0,\infty).$$

The last conclusion is due to the assumptions. Thus, $0 < \mathbf{E}[N]\mathbf{E}[T^2] < 1$, and we have the variance of X as required in (17). The proof of the sufficiency part of (a) is complete.

(Necessity) Suppose that the conditions (16) hold true. Then we will prove the existence of a solution F to Eq (15), which has mean μ and a finite variance.

Set first

$$\mu_1 = \mu = \frac{\mathbf{E}[B]}{1-\mathbf{E}[N]\mathbf{E}[T]}, \tag{50}$$

$$\mu_2 = \frac{\mathrm{Var}(B) + \mu^2(\mathbf{E}[T])^2\mathrm{Var}(N) + \mu^2(1-\mathbf{E}[N](\mathbf{E}[T])^2)}{1-\mathbf{E}[N]\mathbf{E}[T^2]}. \tag{51}$$

Note that the denominators $1 - \mathbf{E}[N]\mathbf{E}[T],\ 1 - \mathbf{E}[N]\mathbf{E}[T^2] > 0$ by (16) and that $\mu_2 \ge \mu_1^2$ by the fact $\mathrm{Var}(T) \ge 0$. Therefore, the RHS of (19) with (50) and (51) is a bona fide Laplace–Stieltjes transform, say $\hat{F}_0$, of a nonnegative random variable $Y_0 \sim F_0$. Namely,

$$\hat{F}_0(s) = 1 - \frac{\mu_1^2}{\mu_2} + \frac{\mu_1^2}{\mu_2}e^{-(\mu_2/\mu_1)s}, \quad s \ge 0.$$

Next, using the initial $Y_0 \sim F_0$ we define iteratively the sequence of random variables $Y_n \sim F_n$, $n = 1, 2, \ldots$, through Laplace–Stieltjes transforms:

$$\hat{F}_n(s) = \hat{F}_B(s)P_N\Big(\int_0^\infty \hat{F}_{n-1}(ts)dF_T(t)\Big), \quad n \ge 1, \tag{52}$$

which is well-defined due to Lemma 2. Differentiating twice the above equation with respect to s and letting $s \to 0^+$, we have, for $n \ge 1$,

$$\hat{F}_n'(0^+) = \hat{F}_B'(0^+) + \hat{F}_{n-1}'(0^+)\mathbf{E}[N]\mathbf{E}[T], \tag{53}$$

$$\begin{aligned}\hat{F}_n''(0^+) &= \hat{F}_B''(0^+) + 2\hat{F}_B'(0^+)\hat{F}_{n-1}'(0^+)\mathbf{E}[N]\mathbf{E}[T] \\ &\quad + \mathbf{E}[N(N-1)](\hat{F}_{n-1}'(0^+)\mathbf{E}[T])^2 + \hat{F}_{n-1}''(0^+)\mathbf{E}[N]\mathbf{E}[T^2]. \end{aligned} \tag{54}$$

With the help of Lemma 6 and by induction on n, we can show through (53) and (54) that $\mathbf{E}[Y_n] = \mathbf{E}[Y_0] = \mu_1$, $\mathbf{E}[Y_n^2] = \mathbf{E}[Y_0^2] = \mu_2$ (defined in (50) and (51)) for all $n \ge 1$ and hence

$$\mathrm{Var}(Y_n) = \mu_2 - \mu_1^2 = \frac{\mathrm{Var}(B) + \mu^2(\mathbf{E}[T])^2\mathrm{Var}(N) + \mu^2\mathbf{E}[N]\mathrm{Var}(T)}{1-\mathbf{E}[N]\mathbf{E}[T^2]}, \quad n \ge 0. \tag{55}$$

Moreover, by Lemma 4, we first have $\hat{F}_1 \le \hat{F}_0$, and then by the iteration (52), $\hat{F}_n \le \hat{F}_{n-1}$ for all $n \ge 2$. Namely, $\{Y_n\}_{n=0}^\infty$ is a sequence of nonnegative random variables having the same first two moments μ_1, μ_2, and their Laplace–Stieltjes transforms $\{\hat{F}_n\}$ are decreasing. Therefore, Lemma 9 applies. Denote the limit of $\{\hat{F}_n\}$ by $\hat{F}_\infty$, which is the Laplace–Stieltjes transform of a nonnegative random variable $Y_\infty \sim F_\infty$ with $\mathbf{E}[Y_\infty] = \mu_1$ and $\mathbf{E}[Y_\infty^2] \in [\mu_1^2, \mu_2]$. Consequently, it follows from (52) that the limit F_∞ is a solution to Eq (15), which has mean μ and a finite variance. Applying Lemma 6 to Eq (15) again, we conclude that $\mathbf{E}[Y_\infty^2] = \mu_2$ as given in (51), and hence the solution $Y_\infty \sim F_\infty$ has the required variance as shown in (17) or (55). This proves the necessity part of (a).

For part (b), the proof of the uniqueness of the solution to Eq (15) is similar to that of Theorem 1, and is omitted. The proof of the theorem is complete.

Proof of Theorem 5. In view of Eqs (1) and (18), we want to prove the equivalence of the two functional equations:

$$\hat{F}_*(s) = P_N\Big(\int_0^\infty \hat{F}_*(ts)dF_T(t)\Big), \quad s \ge 0, \tag{56}$$

with $\lim_{s\to 0^+}(1-\hat{F}_*(s))/s = \mu \in (0,\infty)$, and

$$\hat{F}_\alpha(s) = P_N\Big(\int_0^\infty \hat{F}_\alpha(t^{1/\alpha}s)dF_T(t)\Big), \quad s \ge 0, \tag{57}$$

with $\lim_{s\to 0^+}(1-\hat{F}_\alpha(s))/s^\alpha = \mu \in (0,\infty)$. Here

$$\hat{F}_\alpha(s) = \mathbf{E}\big[\exp(-sX_\alpha)\big] = \mathbf{E}\big[\exp(-sT_\alpha X_*^{1/\alpha})\big] = \int_0^\infty e^{-s^\alpha x}dF_*(x) = \hat{F}_*(s^\alpha), \quad s \ge 0. \tag{58}$$

Suppose $\hat{F}_*$ satisfies (56) with $\lim_{s\to 0^+}(1-\hat{F}_*(s))/s = \mu$. Then it follows from (58) that

$$\hat{F}_\alpha(s) = \hat{F}_*(s^\alpha) = P_N\Big(\int_0^\infty \hat{F}_*(ts^\alpha)dF_T(t)\Big) = P_N\Big(\int_0^\infty \hat{F}_\alpha(t^{1/\alpha}s)dF_T(t)\Big), \quad s \ge 0,$$

and $\lim_{s\to 0^+}(1-\hat{F}_\alpha(s))/s^\alpha = \lim_{s\to 0^+}(1-\hat{F}_*(s^\alpha))/s^\alpha = \lim_{s\to 0^+}(1-\hat{F}_*(s))/s = \mu$. This means that (56) implies (57). The converse implication can be proved similarly.

5. Discussions

We have to mention that under the setting of Eq (1), the distributional equation is different from the following one (in which all i.i.d. T_i are replaced by the same T):

$$X \overset{\mathrm{d}}{=} T\sum_{i=1}^N X_i. \tag{59}$$

It is seen that Eq (59) has instead the corresponding functional form

$$\hat{F}(s) = \int_0^\infty P_N\big(\hat{F}(ts)\big)dF_T(t) = \int_0^\infty P_N\big(\mathbf{E}[\exp(-stX)]\big)dF_T(t), \quad s \ge 0, \tag{60}$$

which is not equal to Eq (2) in general. But when T is degenerate at $p \in (0, 1)$, Eq (60) also reduces to the Poincaré functional equation (3) as Eq (2) does. Therefore, Eq (60) is another generalization of the Poincaré functional equation (3).

The solutions to Eqs (2) and (60) are distinct in general; in fact, the second moments of the distributional solutions are different from each other. More precisely, the second moment of the solution $X \sim F$ (with mean μ) to Eq (60) is of the form

$$\mathbf{E}[X^2] = \frac{\mathbf{E}[N(N-1)]\mathbf{E}[T^2]}{1 - \mathbf{E}[N]\mathbf{E}[T^2]} \cdot \mu^2,$$

which is greater than or equal to that in (32) because $\mathbf{E}[T^2] \geq (\mathbf{E}[T])^2$.

It is interesting, however, that the necessary and sufficient conditions for Eq (60) to have exactly one solution with finite variance are the same as those for Eq (2). Namely, we have the following result. The proof is similar to that of Theorem 1 and is omitted.

Theorem 6. *Under the setting of Theorem 1 with given μ, the random variables N and T together satisfy the conditions (6) iff the functional equation (60) has exactly one solution F with mean μ and a finite variance. Moreover, the variance is of the form*

$$Var(X) = \frac{\mathbf{E}[N^2]\mathbf{E}[T^2] - 1}{1 - \mathbf{E}[N]\mathbf{E}[T^2]} \cdot \mu^2$$

with $\mathbf{E}[N] = 1/\mathbf{E}[T]$.

Theorem 5 about Eq (1) has a parallel result for Eq (59), which extends both Theorems 1 and 2 of Hu and Cheng [6]. In this regard, see also Hu and Lin [7], Section 4, for characterizations of the so-called semi-Mittag-Leffler distributions.

Analogously, Eq (14) is different from the following:

$$X \stackrel{\mathrm{d}}{=} B + T \sum_{i=1}^{N} X_i,$$

which has the corresponding functional form

$$\begin{aligned} \hat{F}(s) &= \hat{F_B}(s) \cdot \int_0^\infty P_N\big(\hat{F}(ts)\big) dF_T(t) \qquad (61) \\ &= \hat{F_B}(s) \cdot \int_0^\infty P_N\big(\mathbf{E}[\exp(-stX)]\big) dF_T(t), \quad s \geq 0. \end{aligned}$$

For this case, we have the next result analogous to Theorem 4. The proof is also omitted.

Theorem 7. *Under the setting of Theorem 4, the following statements are true.*
(a) The random variables B, N and T together satisfy the conditions (16) iff the functional equation (61) has one solution F with mean μ and a finite variance. Moreover, the variance is of the form

$$Var(X) = \frac{Var(B) + \mathbf{E}[B](2\mu - \mathbf{E}[B]) + (\mathbf{E}[N^2]\mathbf{E}[T^2] - 1)\mu^2}{1 - \mathbf{E}[N]\mathbf{E}[T^2]}$$

with $\mu = \mathbf{E}[B]/(1 - \mathbf{E}[N]\mathbf{E}[T])$*.*
(b) If, in addition to (16), $\mathbf{E}[T^2] < \mathbf{E}[T]$*, then the solution* F *to Eq (61) with given mean is unique.*

Finally, we remark that two different distributional equations may be transferred to the same functional form which of course leads to the same distributional solution. This means that a probability distribution may have several kinds of characteristic properties.

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