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Optimal Existence Results for nth Order Periodic Boundary Value Difference Equations[1]

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1. INTRODUCTION

The study of difference equations has had important growth in the last years, not only as a fundamental tool in the discretization of a differential equation, but also as a useful model for several economical and population problems. Models in traffic in channels or the study of the logistic equation give us an idea of the importance of this theory. These and other examples coupled with the basic theory of this type of equations can be found in the classical monograph by S. Goldberg [9] and in the more recent books by V. Lakshmikantham and D. Trigiante [11] and S. N. Elaydi [7].

The second order centered Dirichlet difference problem

$$u_{k+1} - 2u_k + u_{k-1} = f(k, u_k);$$
$$k \in \{1, \dots, N\}, u(0) = u(N+1) = 0,$$

is studied in [14]. There, the authors obtain the existence of solutions lying in the sector formed by a lower solution α and an upper solution β such that $\alpha \le \beta$.

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For first order periodic problems, the existence of a pair of ordered lower and upper solutions without any other assumption over the continuous function f does not imply in general the existence of a solution of the periodic problem lying between α and β. This result is proved in [6] and makes evident the differences with classical results for differential equations [13].

The second order periodic problem is treated in [12], in which maximum principles are obtained studying the sign of the Green function. Higher order equations have been studied, for instance in [1, 3] where, among others, the focal boundary conditions are treated, and in [4] considering in this last case lower and upper solutions. Multiplicity results for higher order equations can be found in [2].

This paper is devoted to the study of the nth order difference equation

$$u(k+n) = f(k, u(k), u(k+1), \dots, u(k+n)), \qquad k \in \{0, \dots, N-1\},$$

with periodic boundary conditions.

Supposing the existence of a pair of ordered lower and upper solutions, we obtain optimal existence results for those problems. To this end, we give optimal discernment to assure the validity of comparison results for the linear operator

$$T_n[K_0, \dots, K_n]u(k) \equiv u(k+n) + \sum_{i=0}^{n} K_i u(k+i),$$

in the space of periodic functions.

We prove that such existence results are equivalent to finding the values of K_i for which T_n is an inverse positive operator (if the lower solution is under the upper solution) in the space of periodic solutions. This assertion is proved in Section 2, where existence of extremal solutions (each other solution in $[\alpha, \beta]$ lies between those solutions) is warranted via monotone iterative techniques derived from the comparison results of operator T_n. In Section 3 conditions on f are weakened in two senses: the first one is not imposing on function f a lateral Lipschitz condition in the space variables, for this we cannot assure the existence of extremal solutions; the second point of view, in the way of Heikkilä and Lakshmikantham [10], consists of relaxing their continuity properties. In this case the existence of extremal solutions is warranted; however, it is not given as the limit of a sequence of approximate solutions. In both sections the optimality of the obtained results is exposed.

In Section 4 we present a topological property verified by the values of $K_0, \dots, K_n$ for which the operator T_n satisfies a comparison result and, in Section 5, we obtain the expression of the Green function for the linear problem

$$T_n[K_0, \dots, K_n]u(k) = \sigma(k),$$
$$k \in \{0, \dots, N-1\};\ u(i) - u(N+i) = \gamma_i,\ i = 0, \dots, n-1.$$

This expression is deduced from the solution of the equation

$$\begin{aligned} T_n[K_0, \dots, K_n]z(k) &= 0; \qquad k \in \{0, \dots, N-1\} \\ z(i) - z(N+i) &= 0; \qquad i = 0, \dots, n-2, \\ z(n-1) - z(N+n-1) &= 1. \end{aligned}$$

Clearly to obtain the expression of this solution we only need to solve a linear $n \times n$ linear algebraic system.

Finally, in Section 6, we apply the results proved in the previous sections to first and second order equations.

Throughout the paper, for each $p \geq 0$ given, if $x = \{x_0, \dots, x_p\}$ and $y = \{y_0, \dots, y_p\} \in \mathbb{R}^{p+1}$ are such that $x_k \leq y_k$ $(x_k \geq y_k)$ for all $k \in \{0, \dots, p\}$, we shall denote $x \leq y$ $(x \geq y)$ on $\{0, \dots, p\}$ and

$$[x, y] = \left\{z = \{z_0, \dots, z_p\} \in \mathbb{R}^{p+1} : x_k \leq z_k \leq y_k, k \in \{0, \dots, p\}\right\}.$$

We will denote $x < y$ in $\{0, \dots, p\}$ ($x > y$ in $\{0, \dots, p\}$) if $x \leq y$ in $\{0, \dots, p\}$ ($x \geq y$ in $\{0, \dots, p\}$) and there exists $k \in \{0, \dots, p\}$ such that $x_k < y_k$ $(x_k > y_k)$.

Furthermore, we shall denote $I = \{0, \dots, N-1\}$ and $J = \{0, \dots, N+n-1\}$.

2. MONOTONE METHOD

In this section we present a constructive method to warrant the existence of extremal solutions of the following nth order periodic boundary value difference problem

$$(Q_n)\begin{cases} \Delta^n u(k) + \sum_{j=1}^{n-1} N_j \Delta^j u(k) = g(k, u(k), \Delta u(k), \dots, \Delta^n u(k)); \\ \quad k \in I, \\ \Delta^i u(0) - \Delta^i u(N) = \nu_i, \qquad i = 0, \dots, n-1, \end{cases}$$

with $N_j \in \mathbb{R}$, $j = 1, \dots, n-1$, $\nu_i \in \mathbb{R}$, $i = 0, \dots, n-1$, $\Delta u(k) = u(k+1) - u(k)$, and $\Delta^l u(k) = \Delta(\Delta^{l-1} u(k))$, $l = 2, \dots, n$, and $k \in I$.

To do this, we translate this problem to the following more readable expression which is, obviously, equivalent for a convenient function f and some real constants λ_i,

$$(P_n)\begin{cases} u(k+n) = f(k, u(k), u(k+1), \dots, u(k+n)), & k \in I, \\ u(i) - u(N+i) = \lambda_i, \quad i = 0, \dots, n-1. \end{cases}$$

To obtain existence results for problem (P_n) we define the concept of lower and upper solutions as follows.

DEFINITION 2.1. A real sequence $\alpha = \{\alpha_0, \alpha_1, \dots, \alpha_{N+n-1}\}$ is said to be a lower solution for problem (P_n) if

$$\alpha_{k+n} \le f(k, \alpha_k, \alpha_{k+1}, \dots, \alpha_{k+n}), \qquad k \in I,$$
$$\alpha_i - \alpha_{N+i} = \lambda_i, \qquad i = 0, \dots, n-2,$$
$$\alpha_{n-1} - \alpha_{N+n-1} \le \lambda_{n-1}.$$

The concept of upper solution for (P_n) is defined in an analogous way; it suffices to reverse the inequalities in the previous definition.

Now, we consider the following preliminary condition on f:

(H) There exists $\{M_0, \dots, M_n\} \in \mathbb{R}^{n+1}$ such that

$$f(k, x_0, \dots, x_n) + \sum_{i=0}^{n} M_i x_i \le f(k, y_0, \dots, y_n) + \sum_{i=0}^{n} M_i y_i,$$

for all $k \in I$ and $\alpha_{k+i} \le x_i \le y_i \le \beta_{k+i}$, $i = 0, \dots, n$.

Before proving the main result of this work, we introduce the concept of an inverse positive operator:

DEFINITION 2.2. Let $K_0, \dots, K_n \in \mathbb{R}$ be fixed, such that $1 + K_n > 0$ and $1 + \sum_{i=0}^{n} K_i > 0$. We say that the operator

$$T_n[K_0, \dots, K_n]u(k) \equiv u(k+n) + \sum_{i=0}^{n} K_i u(k+i)$$

is inverse positive on

$$W_N^n = \{u \in \mathbb{R}^{N+n};\ u(i) = u(N+i),\ i = 0, \dots, n-2;\ u(n-1) \ge u(N+n-1)\},$$

if the two following properties hold:

(1) There exists $T_n[K_0, \dots, K_n]^{-1}$ on

$$\Omega_N^n = \{u \in \mathbb{R}^{N+n};\ u(i) = u(N+i),\ i = 0, \dots, n-1\}.$$

(2) If $u \in W_N^n$ is such that $T_n[K_0, \ldots, K_n]u \geq 0$ on I then $u \geq 0$ on J.

Remark 2.1. If there exists not $T_n[K_0, \ldots, K_n]^{-1}$ on Ω_N^n then there exists $v \in \Omega_N^n$, $v \not\equiv 0$ in J, such that $v \in \mathrm{Ker}(T_n[K_0, \ldots, K_n])$. Then we have that $\lambda v \in \mathrm{Ker}(T_n[K_0, \ldots, K_n])$ for all $\lambda \in \mathbb{R}$. In consequence, if condition (2) in Definition 2.2 holds then condition (1) is also verified.

We use this definition for the convenience of the reader and for understanding the fundamental importance that the existence of $T_n[K_0, \ldots, K_n]^{-1}$ represents in the development of the theory exposed in this paper.

Remark 2.2. Given $C_1 < 0$, let $u \equiv \{C_1, \ldots, C_1\} \in \mathbb{R}^{N+n}$. Since we have that $T_n[K_0, \ldots, K_n]u \geq 0$ on I when $1 + \sum_{i=0}^n K_i \leq 0$, it is clear that T_n will never be an inverse positive operator in W_N^n in this case. It is for this that the condition $1 + \sum_{i=0}^n K_i > 0$ is not restrictive.

To see that $1 + K_n > 0$ is not restrictive too it suffices to take some $C_2 < 0$ and $v \equiv \{0, \ldots, 0, C_2\} \in \mathbb{R}^{N+n}$. Clearly, $v \in W_N^n$, $T_n[K_0, \ldots, K_n]v \geq 0$ on I when $1 + K_n < 0$, and v takes negative values on J.

Remark 2.3. Clearly, there exists $T_n[K_0, \ldots, K_n]^{-1}$ on Ω_N^n if and only if for all $\mu_0, \ldots, \mu_{n-1} \in \mathbb{R}$ there exists $T_n[K_0, \ldots, K_n]^{-1}$ on

$$W_N^n[\mu_0, \ldots, \mu_{n-1}] = \left\{u \in \mathbb{R}^{N+n};\, u(i) - u(N+i) = \mu_i,\, i = 0, \ldots, n-1\right\}.$$

Now, we are in a position to prove the following existence result

THEOREM 2.1. *Suppose that there exist $\alpha \leq \beta$ lower and upper solutions of problem (P_n) and that f is a continuous function satisfying condition* (H) *for some $M_0, \ldots, M_n \in \mathbb{R}$ such that the operator $T_n[M_0, \ldots, M_n]$ is inverse positive on W_N^n. Then there exist two monotone sequences in $\mathbb{R}^{N+n}$, $\{a_m\}$ and $\{b_m\}$ with $a_0 = \alpha$ and $b_0 = \beta$, which converge pointwise to the extremal solutions of problem (P_n) in $[\alpha, \beta]$.*

Proof. For each $\eta \in [\alpha, \beta]$, we consider the linear problem

$$(P_\eta)\begin{cases} T_n[M_0, \ldots, M_n]u(k) = f(k, \eta(k), \eta(k+1), \ldots, \eta(k+n)) \\ \qquad\qquad + \displaystyle\sum_{i=0}^n M_i \eta(k+i), \qquad k \in I, \\ u(i) - u(N+i) = \lambda_i, \qquad i = 0, \ldots, n-1. \end{cases}$$

Since $T_n[M_0, \ldots, M_n]$ is inverse positive on W_N^n, we know that (P_η) admits a unique solution u for each η given (see Remark 2.3).

Condition (H) implies that $T_n[M_0, \ldots, M_n](u - \alpha) \geq 0$ on I. Now, since $u - \alpha \in W_N^n$ and $T_n[M_0, \ldots, M_n]$ is inverse positive on W_N^n, we conclude that $u \geq \alpha$ on J. Similarly we prove that $u \leq \beta$ on J.

Now, let u_i, $i = 1, 2$, be the unique solutions of problem (P_{η_i}), with $\eta_1 \leq \eta_2$ on J. We know that $T_n[M_0, \ldots, M_n](u_2 - u_1) \geq 0$ on I. Clearly, $u_2 - u_1 \in W_N^n$ and then $u_2 \geq u_1$ on J.

The sequences $\{a_m\}$ and $\{b_m\}$ are obtained by recurrence: $a_0 = \alpha$, $b_0 = \beta$, and a_m and b_m are given as the unique solutions of $(P_{a_{m-1}})$ and $(P_{b_{m-1}})$, respectively.

Obviously, $\{a_m(k)\}$ and $\{b_m(k)\}$ are two monotone and bounded sequences for each $k \in J$; as a consequence there exist $\varphi(k) = \lim_{m \to \infty} a_m(k)$ and $\psi(k) = \lim_{m \to \infty} b_m(k)$, $k \in J$. It is not difficult to see that φ and ψ are the extremal solutions of (P_n) in $[\alpha, \beta]$. ∎

In the following result, we prove that the previous theorem is, in some sense, optimal.

THEOREM 2.2. *The assertion proved in Theorem* 2.1 *is optimal in the sense that for all real sequences* $\{M_0, \ldots, M_n\}$ *such that* $1 + \sum_{i=0}^n M_i > 0$ *and* $1 + M_n > 0$, *for which there exists* $T_n[M_0, \ldots, M_n]^{-1}$ *on* Ω_N^n, *but* $T_n[M_0, \ldots, M_n]$ *is not inverse positive on* W_N^n, *we can find a function f and real sequences* α *and* β *satisfying the assumptions of Theorem* 2.1 *and for which problem* (P_n) *has no solution in* $[\alpha, \beta]$.

Proof. The assertions of the theorem imply that there exists a real sequence $\sigma = \{\sigma(0), \ldots, \sigma(N-1)\} \in \mathbb{R}^N$, $\sigma(k) \geq 0$ for all $k \in I$ and $\gamma \geq 0$ such that the problem

$$u(k+n) = \sigma(k) - \sum_{i=0}^{n} M_i u(k+i)$$

$$\equiv f(k, u(k), \ldots, u(k+n)), \qquad k \in I, \tag{2.1}$$

$$u(i) - u(N+i) = 0, \qquad i = 0, \ldots, n-2, \tag{2.2}$$

$$u(n-1) - u(N+n-1) = \gamma, \tag{2.3}$$

has a unique solution u, which is not nonnegative on J. Thus, $C = \min_{j \in J} u(j) < 0$.

Clearly, $\alpha \in \{0, \ldots, 0\} \in \mathbb{R}^{N+n}$ is a lower solution of (2.1)–(2.3).

Consider $\beta(k) = u(k) - C$, $k \in J$. Then

$$\beta(k+n) + \sum_{i=0}^{n} M_i \beta(k+i)$$

$$= \sigma(k) - C\left(1 + \sum_{i=0}^{n} M_i\right) \geq \sigma(k), \qquad \text{for all } k \in I.$$

Thus, β is an upper solution of (2.1)–(2.3).

However, this problem has no solution in $[\alpha, \beta]$. ∎

3. EXISTENCE UNDER WEAKER ASSUMPTIONS

In this section we obtain existence results for problem (P_n) when f is a function verifying weaker conditions than were imposed in Theorem 2.1; i.e., f is a continuous function that does not satisfy condition (H) or f is a discontinuous function satisfying condition (H). In the first case, it is not possible to assure existence of extremal solutions in the sector $[\alpha, \beta]$ of problem (P_n); however, the given result is optimal. In the second case the existence of extremal solutions is not given via the monotone method.

To develop the first case we consider the following hypothesis:

(J) There exist $\{M_0, \dots, M_n\} \in \mathbb{R}^{n+1}$ such that

$$f(k, \alpha_k, \dots, \alpha_{k+n}) + \sum_{i=0}^{n} M_i \alpha_{k+i} \le f(k, x_0, \dots, x_n) + \sum_{i=0}^{n} M_i x_i$$
$$\le f(k, \beta_k, \dots, \beta_{k+n}) + \sum_{i=0}^{n} M_i \beta_{k+i}$$

for all $k \in I$ and $\alpha_{k+i} \le x_i \le \beta_{k+i}$, $i = 0, \dots, n$.

The result obtained is the following.

THEOREM 3.1. *Suppose that there exist $\alpha \le \beta$ lower and upper solutions of problem (P_n) and that f is a continuous function that satisfies condition* (J) *for some $M_0, \dots, M_n \in \mathbb{R}$ such that $T_n[M_0, \dots, M_n]$ is inverse positive on W_N^n. Then problem (P_n) has at least one solution in $[\alpha, \beta]$.*

Proof. Consider the modified problem

$$u(k+n) + \sum_{i=0}^{n} M_i u(k+i)$$
$$= f(k, p(k, u(k)), \dots, p(k+n, u(k+n)))$$
$$+ \sum_{i=0}^{n} M_i p(k+i, u(k+i)), \qquad k \in I,$$
$$u(i) - u(N+i) = \lambda_i, \qquad i = 0, \dots, n-1,$$

where $p(k, r) = \max\{\alpha(k), \min\{r, \beta(k)\}\}$ for all $k \in I$ and $r \in \mathbb{R}$.

Now, there exists a Green function $G(k, j)$ depending on $M_0, \dots, M_n$ and a function $H(k)$ which depends on $M_0, \dots, M_n$ and $\lambda_0, \dots, \lambda_{n-1}$, such that

$$u(k) = \sum_{j=0}^{N-1} G(k, j) Pu(j) + H(k) \equiv Fu(k), \qquad k \in J,$$

with $Pu(k) = f(k, p(k, u(k)), \dots, p(k+n, u(k+n))) + \sum_{i=0}^{n} M_i p(k+i, u(k+i))$.

By definition of P, we know that $F\colon \mathbb{R}^{N+n} \to \mathbb{R}^{N+n}$ is a continuous and bounded function. Thus, the Brower fixed point theorem implies the existence of a solution of the modified problem.

Let w be one such solution and for every $k \in I$, we have

$$\begin{aligned} T_n[M_0,\dots,M_n]w(k) &= f(k, p(k,w(k)),\dots, p(k+n, w(k+n))) \\ &\quad + \sum_{i=0}^{n} M_i p(k+i, w(k+i)) \\ &\geq f(k, \alpha(k),\dots,\alpha(k+n)) + \sum_{i=0}^{n} M_i \alpha(k+i) \\ &\geq T_n[M_0,\dots,M_n]\alpha(k). \end{aligned}$$

Taking into account the fact that $w - \alpha \in W_N^n$, the inverse positive character of T_n permits us to conclude that $w \geq \alpha$ in J.

Analogously $w \leq \beta$ in J and the proof is complete. ∎

Remark 3.1. One can prove an equivalent result to Theorem 2.2 in which the optimality of this result is exposed.

Now, we give an existence result for discontinuous functions as follows:

THEOREM 3.2. *Suppose that there exist α and β lower and upper solutions of problem (P_n) and f is a discontinuous function satisfying* (H) *for a real sequence $\{M_0,\dots,M_n\}$ such that $T_n[M_0,\dots,M_n]$ is inverse positive on W_N^n. Then problem (P_n) has extremal solutions in $[\alpha, \beta]$.*

Proof. For each $\eta \in [\alpha, \beta]$, we define $G(\eta)$ as the unique solution of problem (P_η).

As in the proof of Theorem 2.1, we prove that G is a nondecreasing function such that $G([\alpha, \beta]) \subset [\alpha, \beta]$.

Then [10, Theorem 1.2.2] assures the existence of extremal fixed points in $[\alpha, \beta]$ of function G. By construction, such fixed points are the extremal solutions of (P_n) in $[\alpha, \beta]$. ∎

3.1. *The Reverse Order Case*

All the results given in the paper relative to problem (P_n) and $\alpha \leq \beta$ can be developed for this problem when $\alpha \geq \beta$.

For this we say that for a real sequence $\{K_0,\dots,K_n\}$ such that $1 + K_n > 0$ and $1 + \sum_{i=0}^{n} K_i < 0$, the operator $T_n[K_0,\dots,K_n]$ is inverse negative in W_N^n if there exists $T_n[K_0,\dots,K_n]^{-1}$ in Ω_N^n and for all $u \in W_N^n$ such that $T_n[K_0,\dots,K_n]u \geq 0$ on I we have that $u \leq 0$ on J.

Thus, Theorems 2.1, 2.2, 3.1, and 3.2 remain valid if we replace in the enunciates $\alpha \leq \beta$ by $\alpha \geq \beta$, inverse positive by inverse negative, condition (H) by (IH), and condition (J) by (IJ), where

(IH) there exist $\{M_0, \ldots, M_n\} \in \mathbb{R}^{n+1}$ such that

$$f(k, x_0, \ldots, x_n) + \sum_{i=0}^{n} M_i x_i \leq f(k, y_0, \ldots, y_n) + \sum_{i=0}^{n} M_i y_i,$$

for all $k \in I$ and $\beta_{k+i} \leq y_i \leq x_i \leq \alpha_{k+i}$, $i = 0, \ldots, n$,

(IJ) there exist $\{M_0, \ldots, M_n\} \in \mathbb{R}^{n+1}$ such that

$$\begin{aligned} f(k, \alpha_k, \ldots, \alpha_{k+n}) &+ \sum_{i=0}^{n} M_i \alpha_{k+i} \\ &\leq f(k, x_0, \ldots, x_n) + \sum_{i=0}^{n} M_i x_i \\ &\leq f(k, \beta_k, \ldots, \beta_{k+n}) + \sum_{i=0}^{n} M_i \beta_{k+i}, \end{aligned}$$

for all $k \in I$ and $\beta_{k+i} \leq x_i \leq \alpha_{k+i}$, $i = 0, \ldots, n$.

4. STRUCTURE OF THE $K_0, \ldots, K_n$ VALUES SET

In this section we study the structure of the set of the values $K_0, \ldots, K_n \in \mathbb{R}$, for which $T_n[K_0, \ldots, K_n]$ is inverse positive on W_N^n. The result obtained is the following:

THEOREM 4.1. *Let $K = \{K_0, \ldots, K_n\}$ such that $1 + K_n > 0$ and $1 + \sum_{i=1}^{n} K_i > 0$. Let $L = \{L_0, \ldots, L_n\}$, such that $L > K$. Suppose that $T_n[K_0, \ldots, K_n]$ is not inverse positive on W_N^n. Then the operator $T_n[L_0, \ldots, L_n]$ is not inverse positive on W_N^n.*

Proof. Suppose that there exists $T_n[K_0, \ldots, K_n]^{-1}$ on Ω_N^n. In consequence we know that there exists $\sigma = \{\sigma(0), \ldots, \sigma(N-1)\} \in \mathbb{R}^N$, $\sigma(k) \geq 0$ for all $k \in I$ and $\gamma \geq 0$, such that problem (2.1)–(2.3) has a unique solution $u \ngeq 0$ in J. As we have seen in Theorem 2.2, $\alpha = \{0, \ldots, 0\}$ is a lower solution and $\beta = u - \min_{k \in J}\{u(k)\}$ is an upper solution of problem (2.1)–(2.3).

Now, for f defined in Eq. (2.1), since

$$f(k, x_0, \ldots, x_n) + \sum_{i=0}^{n} L_i x_i = \sum_{i=0}^{n} (L_i - K_i) x_i + \sigma(k),$$

we have that it is a continuous function satisfying condition (H) for every $L \geq K$.

In consequence, if $T_n[L_0, \ldots, L_n]$ is inverse positive on W_N^n for some $L > K$, we are in the hypothesis of Theorem 2.1 and we conclude that there exists at least one solution of problem (2.1)–(2.3) in $[\alpha, \beta]$, which is not true.

If there does not exist $T_n[K_0, \ldots, K_n]^{-1}$ in Ω_N^n then there is $\bar{\sigma} = \{\bar{\sigma}(0), \ldots, \bar{\sigma}(N-1)\}$ such that problem

$$u(k+n) = \bar{\sigma}(k) - \sum_{i=0}^{n} M_i u(k+i), \qquad k \in I \tag{4.1}$$

$$u(i) - u(N+i) = 0, \qquad i = 0, \ldots, n-1, \tag{4.2}$$

does not have a unique solution.

Now, we consider the equivalent first order system,

$$x(k+1) = Ax(k) + b(k), \qquad k \in I,\ x(0) = x(N), \tag{4.3}$$

with

$$A = \left(\begin{array}{c|c} 0 & I_{n-1} \\ \hline -T_0 & -T_1, \ldots, -T_{n-1} \end{array}\right); \qquad b(k) = \begin{pmatrix} 0 \\ \vdots \\ \dfrac{\bar{\sigma}(k)}{1 + K_n} \end{pmatrix},$$

I_{n-1} the $(n-1) \times (n-1)$ identity matrix and $T_i = K_i/(1 + K_n)$.

We know that (see [8])

$$x(k) = A^k x(0) + \sum_{j=0}^{k-1} A^{k-j-1} b(j) \qquad \text{for all } k \in I.$$

Thus, $x(0) = x(N)$ if and only if

$$\left(I_n - A^N\right) x(0) = \sum_{j=0}^{N-1} A^{N-j-1} b. \tag{4.4}$$

Now, let $I_n - A^N = (a_{i,j})_{i,j\in\{1,\ldots,n\}}$. Since there does not exist $(I_n - A^N)^{-1}$ we have two possibilities:

(a) There exists $i_0 \in \{2,\ldots,n\}$ and $\delta_k \in \mathbb{R}$, $k = 1,\ldots,i_0 - 1$ with some $\delta_k \neq 0$, such that $a_{i_0,j} = \sum_{k=1}^{i_0-1} \delta_k a_{k,j}$ for all $j \in \{1,\ldots,n\}$.

(b) There exists $i_0 \in \{1,\ldots,n\}$ such that $a_{i_0,j} = 0$ for all $j \in \{1,\ldots,n\}$.

Suppose case (a).

For $l \in \{1,\ldots,N-1\}$ we denote $A^l = (b_{i,j}^{(l)})_{i,j\in\{1,\ldots,n\}}$.

A necessary condition to assure the existence of solution of (4.4) is given by

$$\sum_{j=0}^{N-1}\left(b_{i_0,n}^{(N-j-1)} - \sum_{k=1}^{i_0-1}\delta_k b_{k,n}^{(N-j-1)}\right)\bar{\sigma}(j) = 0.$$

Suppose that

$$b_{i_0,n}^{(N-j-1)} = \sum_{k=1}^{i_0-1}\delta_k b_{k,n}^{(N-j-1)}, \qquad \text{for all } j \in I. \tag{4.5}$$

Clearly $y(j) = (b_{1,n}^{(N-j-1)},\ldots,b_{i_0-1,n}^{(N-j-1)})^t$ satisfies the equation

$$\begin{aligned} y(j-1) &= Dy(j), \qquad j \in \{1,\ldots,N-1\},\\ y(N-1) &= 0, \end{aligned}$$

with

$$D = \left(\begin{array}{c|c} 0 & I_{n_0-2} \\ \hline \delta_1 & \delta_2,\ldots,\delta_{i_0-1} \end{array}\right).$$

This problem has a unique solution given by $y(j) = D^{N-j-1}y(N-1) = 0$, for all $j \in I$; that is, $b_{k,n}^{(N-j-1)} = 0$, for all $k \in \{0,\ldots,i_0-1\}$ and $j \in I$. Now, Eq. (4.5) gives us that

$$b_{i_0,n}^{(N-j-1)} = 0, \qquad \text{for all } j \in I.$$

By recurrence, we conclude

$$0 = b_{i_0,n}^{(n-i_0)} = \cdots = b_{n-1,n}^{(1)} = 1,$$

which is a contradiction.

If case (b) holds, we know that a necessary condition for such a $\overline{\sigma}$ to exist is that

$$\sum_{j=0}^{N-1} b_{i_{0,n}}^{(N-j-1)} \overline{\sigma}(j) = 0.$$

Now, if

$$b_{i_0,n}^{(N-j-1)} = 0 \qquad \text{for all } j \in I, \tag{4.6}$$

we attain a contradiction as in the previous case.

In consequence, we have that (4.5) and (4.6) do not hold. Then there exists $\overline{\sigma}$ such that problem (4.1)–(4.2) has no solution. Using the fact that there exist $C_1 < 0 < C_2$ such that

$$C_1\left(1 + \sum_{i=0}^{n} K_i\right) \leq \overline{\sigma}(k) \leq C_2\left(1 + \sum_{i=0}^{n} K_i\right), \qquad \text{for all } k \in I,$$

we know that $\alpha = \{C_1, \ldots, C_1\}$ is a lower solution and $\beta = \{C_2, \ldots, C_2\}$ is an upper solution of problem (4.1)–(4.2). Reasoning as in the first part of the proof we conclude the result. ∎

Remark 4.1. If we replace $L > K$ by $L < K$, this theorem is true for inverse negative operators such that $1 + K_n > 0$ and $1 + \sum_{i=0}^{n} K_i < 0$.

5. EXPRESSION OF THE GREEN FUNCTION

As we have seen in Theorems 2.1, 2.2, 3.1, and 3.2 and in Remark 3.1, the existence of solution of problem (P_n) is translated to the study of the values $K_0, \ldots, K_n$ for which operator $T_n[K_0, \ldots, K_n]$ is inverse positive on W_N^n.

To do this, we present here a formula to obtain the expression of the Green function $G(k, j)$ associated to the operator $T_n[K_0, \ldots, K_n]^{-1}$. Such a function is obtained solving a $n \times n$ linear algebraic system. As we will see, this expression is given for the case $K_n = 0$; obviously the general case is translated to this one dividing by $1 + K_n$.

THEOREM 5.1. *Let $K_0, \ldots, K_{n-1} \in \mathbb{R}$ be fixed. Suppose that for all $\gamma_i \in \mathbb{R}$, $i = 0, \ldots, n-1$ and for all $\sigma \in \mathbb{R}^N$ there exists a unique solution of the problem*

$$u(k+n) + \sum_{i=0}^{n-1} K_i u(k+i) = \sigma(k), \qquad k \in I, \tag{5.1}$$

$$u(i) - u(N+i) = \gamma_i, \qquad i = 0, \ldots, n-1. \tag{5.2}$$

This solution u is given by the expression

$$u(k) = x(k) + \sum_{i=0}^{n-1} p_i(k)\gamma_{n-i-1}, \qquad \text{for all } k \in J, \tag{5.3}$$

where

$$x(k) = \begin{cases} \displaystyle\sum_{j=0}^{k-1} z(k-j-1)\sigma(j) + \sum_{j=k}^{N-1} z(N+k-j-1)\sigma(j), \\ \quad \text{if } k \in I \\ \displaystyle\sum_{j=0}^{k-1-N} z(k-j-1-N)\sigma(j) + \sum_{j=k-N}^{N-1} z(k-j-1)\sigma(j), \\ \quad \text{if } k \in J \setminus I, \end{cases}$$

$$p_i(k) = z(k+i) + \sum_{j=n-i}^{n-1} K_j z(k+j-n+i),$$

and z is the unique solution of the problem

$$z(k+n) + \sum_{i=0}^{n-1} K_i z(k+i) = 0, \qquad k \geq 0, \tag{5.4}$$

$$z(i) - z(N+i) = 0, \qquad i = 0, \ldots, n-2, \tag{5.5}$$

$$z(n-1) - z(N+n-1) = 1. \tag{5.6}$$

Proof. Let $k \in \{0, \ldots, N-n-1\}$. Using the boundary conditions imposed on function z, we obtain that

$$\begin{aligned} x(k+n) &+ \sum_{i=0}^{n-1} K_i x(k+i) \\ &= \sum_{j=0}^{k-1} \left(z(k+n-j-1) + \sum_{i=0}^{n-1} K_i z(k+i-j-1) \right) \sigma(j) \\ &\quad + \sum_{j=k+1}^{N-1} \left(z(N+k+n-j-1) + \sum_{i=0}^{n-1} K_i z(N+k+i-j-1) \right) \\ &\quad \times \sigma(j) \\ &\quad + \left(z(n-1) + \sum_{i=1}^{n-1} K_i z(i-1) + K_0 z(N-1) \right) \sigma(k). \end{aligned}$$

Now, since z satisfies (5.4), using the boundary conditions of function z again, we conclude that

$$\begin{aligned} x(k+n) &+ \sum_{i=0}^{n-1} K_i x(k+i) \\ &= \left(1 + z(N+n-1) + \sum_{j=0}^{n-1} K_j z(N+j-1)\right)\sigma(k) = \sigma(k). \end{aligned}$$

Now, suppose that $k \in \{N-n-1, \dots, N-1\}$. Let $l \in \{0, \dots, n-1\}$ be such that $k + n = N + l$; thus

$$\begin{aligned} x(k+n) &+ \sum_{i=0}^{n-1} K_i x(k+i) \\ &= \sum_{j=0}^{l-1} z(l-j-1)\sigma(j) + \sum_{j=l}^{N-1} z(N+l-j-1)\sigma(j) \\ &\quad + \sum_{i=1}^{l} K_{n-i}\left(\sum_{j=0}^{l-i-1} z(l-i-j-1)\sigma(j)\right. \\ &\qquad\qquad \left. + \sum_{j=l-i}^{N-1} z(N+l-i-j-1)\sigma(j)\right) \\ &\quad + \sum_{i=l+1}^{n} K_{n-i}\left(\sum_{j=0}^{N+l-i-1} z(N+l-i-j-1)\sigma(j)\right. \\ &\qquad\qquad \left. + \sum_{j=N+l-i}^{N-1} z(2N+l-i-j-1)\sigma(j)\right). \end{aligned}$$

Ordering this expression and using that z satisfies the boundary condition (5.5), we have that it is equals

$$\begin{aligned} &\sum_{j=0}^{N+l-n-1}\left(z(N+l-j-1)\sigma(j) + \sum_{i=1}^{n} K_{n-i} z(N+l-j-i-1)\sigma(j)\right) \\ &\quad + \left(z(n-1) + \sum_{i=1}^{n-1} K_{n-i} z(n-i-1) + K_0 z(N-1)\right)\sigma(k) \\ &\quad + \sum_{j=N+l-n+1}^{N-1}\left(z(2N+l-j-1)\sigma(j)\right. \\ &\qquad\qquad \left. + \sum_{i=1}^{n} K_{n-i} z(2N+l-j-i-1)\sigma(j)\right). \end{aligned}$$

Since z satisfies Eq. (5.4), from the boundary conditions it is derived that this last expression equals $\sigma(k)$.

To verify that

$$x(i) = x(N+i) \qquad \text{for all } i \in \{0, \ldots, n-1\}$$

is immediate.

Obviously, p_i verifies Eq. (5.4) for all $i \in \{0, \ldots, n-1\}$. By definition, we have that $p_0(k) = z(k)$ and then p_0 satisfies (5.4)–(5.6). In consequence, using that $p_i(k) = p_{i-1}(k+1) + K_{n-i} z(k)$ for all $i \in \{1, \ldots, n-1\}$ and $k \geq 0$, by recurrence, we conclude that p_i satisfies Eq. (5.4) and the conditions $p_i(n-1-i) - p_i(N+n-1-i) = 1$ and $p_i(n-1-j) - p_i(N+n-1-j) = 0$ for all $j \neq i$.

Hence, the function

$$u(k) = x(k) + \sum_{i=0}^{n-1} p_i(k)\gamma_{n-i-1}, \qquad k \in J,$$

is the unique solution of problem (5.1)–(5.2). ∎

COROLLARY 5.1. *Let z be the unique solution of $T_n[K_0/(1+K_n), K_{n-1}/(1+K_n), \ldots, 0]z(k) = 0$, $k \in I$ satisfying the boundary conditions* (5.5)–(5.6). *Then, if $1 + K_n > 0$ and $1 + \sum_{i=0}^{n} K_i > 0$ $(1 + \sum_{i=0}^{n} K_i < 0)$ then we have that operator $T_n[K_0, \ldots, K_n]$ is inverse positive (inverse negative) on W_N^n if and only if $z(k) \geq 0$ on J $(z(k) \leq 0$ on $J)$.*

6. APPLICATIONS TO FIRST AND SECOND ORDER PROBLEMS

In this section we obtain the expression of function z (and then of the Green function) for some particular first and second order problems. We study the values of the constants for which operators T_1 and T_2 are inverse positive and inverse negative on W_N^1 and W_N^2, respectively.

6.1. *First Order Equations*

It is clear that the unique solution of the equation

$$z(k+1) - \lambda z(k) = 0, \qquad k \geq 0, \tag{6.1}$$

$$z(0) - z(N) = 1, \tag{6.2}$$

is given by the expression $z(k) = \lambda^k/(1-\lambda^N)$ for all $\lambda \neq 0$ and by $z(0) = 1$ and $z(k) = 0$ for $k > 0$ when $\lambda = 0$. This is negative in J for all

$\lambda > 1$ and nonnegative in J for $\lambda \in [0,1)$. In the other cases z takes negative and positive values on J.

For this, we consider the explicit problem (which appears in Euler's discretization method of a first order differential equation)

$$\Delta u(k) = f(k, u(k)), k \in I; \qquad u(0) - u(N) = \gamma_0.$$

We have that $T_n[M_0 - 1, 0]$ is inverse positive on W_N^1 if and only if $0 < M_0 \leq 1$, and inverse negative for all $M_0 < 0$. Thus, we can, by virtue of Theorem 2.1 and Subsection 3.1, assure the existence and approximation of extremal solutions between a given lower solution α and an upper solution β if $f(k, x) + x$ is a nondecreasing function in $[\alpha(k), \beta(k)]$, $k \in I$ when $\alpha \leq \beta$ and, when $\alpha \geq \beta$ holds, if $f(k, x) - M_0 x$ is nonincreasing in $[\beta(k), \alpha(k)]$ for some $M_0 > 0$.

Now the implicit case is considered,

$$\Delta u(k) = f(k, u(k+1)), k \in I; \qquad u(0) - u(N) = \gamma_0.$$

We know that for all $M_1 > 0$ the operator $T_1[-1, M_1]$ is inverse positive on W_N^1 and inverse negative for $M_1 \in (-1, 0)$. In consequence if $\alpha \leq \beta$ and $f(k, x) + M_1 x$ is nondecreasing in $[\alpha(k), \beta(k)]$ for some $M_1 > 0$ or $\alpha \geq \beta$ and $f(k, x) - M_1 x$ is a nonincreasing function in $[\beta(k), \alpha(k)]$, $k \in I$, for some $M_1 \in (0, 1)$, then this problem has extremal solutions lying between α and β.

6.2. *Second Order Equations*

Now, the unique solution of (5.4)–(5.6) for $n = 2$ is given by the following expressions, depending on the circumstances:

(1) The characteristic polynomial has two real roots $\lambda_1 > \lambda_2$,

$$z(k) = \frac{1}{\lambda_1 - \lambda_2} \left(\frac{\lambda_1^k}{1 - \lambda_1^N} - \frac{\lambda_2^k}{1 - \lambda_2^N} \right).$$

In this case we have that z is nonnegative on J when $0 \leq \lambda_2 < \lambda_1 < 1$ or if $1 < \lambda_2 < \lambda_1$. On the other hand, if $0 \leq \lambda_2 < 1 < \lambda_1$ the function z has only negative values. In all the other cases z changes sign in J.

(2) The characteristic polynomial has a double real root $\lambda_1 = \lambda_2$,

$$z(k) = \frac{\lambda_1^{k-1}\left((N-k)\lambda_1^N + k\right)}{\left(1 - \lambda_1^N\right)^2},$$

if $\lambda_1 \neq 0$, and $z \equiv \{0, 1, 0, \dots, 0\}$ when $\lambda_1 = 0$. Thus z is nonnegative on J when $\lambda_1 \geq 0$, $\lambda_1 \neq 1$. In the other cases function z changes sign in J.

(3) The characteristic polynomial has complex roots $r\cos\theta \pm ir\sin\theta$. Here $r > 0$ and $\theta \in (0, \pi)$,

$$z(k) = \frac{r^{k-1}\left(r^N \sin(N-k)\theta + \sin k\theta\right)}{\sin\theta\left[\left(r^N - \cos N\theta\right)^2 + \sin^2 N\theta\right]}.$$

This function is nonnegative on J if and only if $\theta \in (0, \pi/N]$ and $\sin(N+1)\theta \geq r^N \sin\theta$.

For this, if we consider the problem

$$\Delta^2 u(k) = f(k, u(k)),\ k \in I; \qquad u(i) - u(N+i) = \gamma_i,\ i = 0, 1, \quad (6.3)$$

we know that operator $T_2[M_0 + 1, -2, 0]$ is inverse negative on W_N^2 if and only if $M_0 \in [-1, 0)$ and inverse positive if and only if $M_0 \in (0, \overline{M_0}]$, with $\overline{M_0} = \min\{\tan^2 \frac{\pi}{N}, \widetilde{M_0}\}$ and $\widetilde{M_0}$ the first positive zero of the expression

$$\sin\left((N+1)\arctan\sqrt{M_0}\right) = \sqrt{M_0}\left(\sqrt{1 + M_0}\right)^{N-1}.$$

In consequence if $\alpha \leq \beta$ and $f(k, x) + \overline{M_0}x$ is nondecreasing in $[\alpha(k), \beta(k)]$, $k \in I$, or $\alpha \geq \beta$ and $f(k, x) - x$ is a nonincreasing function in $[\beta(k), \alpha(k)]$, $k \in I$, then this problem has extremal solutions lying between α and β.

Now we consider

$$\Delta^2 u(k) = f(k, u(k+1)),\ k \in I; \qquad u(i) - u(N+i) = \gamma_i, \qquad i = 0, 1. \quad (6.4)$$

The operator $T_2[1, M_1 - 2, 0]$ is inverse negative if and only if $M_1 < 0$ and inverse positive if and only if $M_1 \in [0, \overline{M_1}]$, with $\overline{M_1} = \min\{2(1 - \cos\frac{\pi}{N}), \widetilde{M_1}\}$ and $\widetilde{M_1}$ the first positive zero of the expression

$$2\sin\left((N+1)\arctan\frac{\sqrt{4M_1 - M_1^2}}{2 - M_1}\right) = \sqrt{4M_1 - M_1^2}.$$

For this, $\alpha \leq \beta$ and f satisfying condition (H) for some $M_1 \in [0, \overline{M_1}]$ or $\alpha \geq \beta$ and f verifying condition (IH) for some $M_1 < 0$ assure the existence of extremal solutions lying between α and β.

Finally, we consider problem

$$\Delta^2 u(k) = f(k, u(k+2)),\ k \in I; \qquad u(i) - u(N+i) = \gamma_i,\ i = 0, 1. \quad (6.5)$$

We conclude that operator $T_2[1, -2, M_2]$ is inverse negative in W_N^2 if and only if $-1 < M_2 < 0$ and inverse positive if and only if $M_2 \in (0, \overline{M_0}]$ ($\overline{M_0}$ defined above). In consequence if $\alpha \leq \beta$ and $f(k, x) + \overline{M_0}x$ is nondecreasing in $[\alpha(k), \beta(k)]$, $k \in I$ or $\alpha \geq \beta$, and f verifies condition (IH) for some $M_2 \in (-1, 0)$, then we know that this problem has extremal solutions lying between α and β.

6.3. *Final Remarks*

The boundary conditions imposed in the definition of the lower solution are equivalent to

$$\Delta^i\alpha(0) - \Delta^i\alpha(N) = \lambda_i, \qquad i = 0, \dots, n-2,$$
$$\Delta^{n-1}\alpha(0) - \Delta^{n-1}\alpha(N) \leq \lambda_{n-1}.$$

We choose this definition by analogies with the classical definition of such functions for differential equations [5].

As we have seen, this definition forces us to study maximum and anti-maximum principles of the operator T_n in the space W_N^n, which can be rewritten as

$$W_N^n = \{u \in \mathbb{R}^{N+n}; \Delta^i u(0) = \Delta^i u(N), i = 0, \dots, n-2; \\ \Delta^{n-1}u(0) \geq \Delta^{n-1}u(N)\},$$

in the same way that in differential equations we study the inverse positive or inverse negative character of the corresponding linear operator in the space

$$W_N^n = \{u \in W^{n,1}[0, 2\pi]; u^{(i)}(0) = u^{(i)}(2\pi), i = 0, \dots, n-2; \\ u^{(n-1)}(0) \geq u^{(n-1)}(2\pi)\}.$$

Despite this, we could choose a more restrictive definition of lower and upper solutions, imposing the equalities in all the boundary conditions; that is, $\Delta^i\alpha(0) - \Delta^i\alpha(N) = \lambda_i$, $i = 0, \dots, n-1$, and the same for β.

Obviously, to develop the theory exposed in this work in this new situation, we must study the values of $M_0, \dots, M_n$ for which the operator $T_n[M_0, \dots, M_n]$ is inverse positive or inverse negative in Ω_N^n. A similar remark is valid also for differential equations, but in that case, as one can see in [5], the values of $M_0, \dots, M_n$ do not experiment any variation.

However, this is not true in our problem for $n \geq 2$ and represents a substantial difference between differential and difference boundary problems. To see this, since for all $u \in \Omega_N^n$ the inhomogeneous part of (5.3) vanishes, we only need to study function x. But, in the definition of such a function we see that function z takes values only in I, and we can rewrite Corollary 5.1 as "operator $T_n[M_0, \ldots, M_n]$ is inverse positive (inverse negative) in Ω_N^n if and only if $z \geq 0$ ($z \leq 0$) on I."

For instance, studying the operators treated in the previous section we have:

Operator $T_2[M_0 + 1, -2, 0]$ is inverse negative on Ω_N^2 if and only if $M_0 \in [-1, 0)$ and inverse positive if and only if $M_0 \in (0, \tan^2 \frac{\pi}{N}]$.

Operator $T_2[1, M_1 - 2, 0]$ is inverse negative on Ω_N^2 if and only if $M_1 < 0$ and inverse positive if and only if $M_1 \in [0, 2(1 - \cos \frac{\pi}{N})]$.

Operator $T_2[1, -2, M_2]$ is inverse negative on Ω_N^2 if and only if $M_2 \in (-1, 0)$ and inverse positive if and only if $M_2 \in (0, \tan^2 \frac{\pi}{N}]$.

In consequence, we extend the range of values M_0, M_1, and M_2 such that problems (6.3), (6.4), and (6.5) have extremal solutions in the presence of a lower solution and an upper solution in this new sense.

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