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Loewner Chains and the Roper–Suffridge Extension Operator

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Let f be a locally univalent function on the unit disc and let $\alpha \in [0, \frac{1}{2}]$. We consider the family of operators extending f to a holomorphic map from the unit ball B in $\mathbf{C}^n$ to $\mathbf{C}^n$ given by $\Phi_{n,\alpha}(f)(z) = (f(z_1), z'(f'(z_1))^\alpha)$, where $z' = (z_2, \dots, z_n)$. When $\alpha = \frac{1}{2}$ we obtain the Roper–Suffridge extension operator. We show that if $f \in S$ then $\Phi_{n,\alpha}(f)$ can be imbedded in a Loewner chain. Our proof shows that if $f \in S^*$ then $\Phi_{n,\alpha}(f)$ is starlike, and if $f \in \hat{S}_\beta$ with $|\beta| < \frac{\pi}{2}$ then $\Phi_{n,\alpha}(f)$ is a spirallike map of type β. In particular we obtain a new proof that the Roper–Suffridge operator preserves starlikeness. We also obtain the radius of starlikeness of $\Phi_{n,\alpha}(S)$ and the radius of convexity of $\Phi_{n,1/2}(S)$. We show that if f is a normalized univalent Bloch function on U then $\Phi_{n,\alpha}(f)$ is a Bloch mapping on B. Finally we show that if f belongs to a class of univalent functions which satisfy growth and distortion results, then $\Phi_{n,\alpha}(f)$ satisfies related growth and covering results.

Key Words: starlikeness; spirallikeness; convexity; Loewner chains; radius of starlikeness; radius of convexity.

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1. INTRODUCTION AND PRELIMINARIES

Let $\mathbf{C}^n$ denote the space of n complex variables $z = (z_1, \ldots, z_n)$ with the Euclidean inner product $\langle z, w \rangle = \sum_{j=1}^{n} z_j \overline{w}_j$ and the Euclidean norm $\|z\| = \langle z, z \rangle^{1/2}$, $z \in \mathbf{C}^n$.

Let $z' = (z_2, \ldots, z_n)$ so that $z = (z_1, z')$. Let $B_r = \{z \in \mathbf{C}^n\colon \|z\| < r\}$ and let $B = B_1$ denote the unit ball in $\mathbf{C}^n$. In the case of one variable B_r is denoted by U_r and the unit disc U_1 by U. If G is an open subset in $\mathbf{C}^n$, let $H(G)$ be the set of holomorphic mappings from G into $\mathbf{C}^n$. If $f \in H(B_r)$, $0 < r \le 1$, we say that f is normalized if $f(0) = 0$ and $Df(0) = I$. Let $S(B_r)$ denote the class of normalized univalent mappings in $H(B_r)$. A mapping $f \in S(B_r)$ will be called convex (respectively starlike) if its image is a convex (respectively starlike with respect to the origin) set in $\mathbf{C}^n$. The classes of normalized convex (respectively starlike) maps on B_r are denoted by $K(B_r)$ (respectively $S^*(B_r)$). In the case of one variable, the sets $S(U)$, $S^*(U)$, and $K(U)$ are denoted by S, S^*, and K.

Another interesting subclass of $S(B_r)$ is the class $S^0(B_r)$ consisting of those univalent mappings which can be imbedded in Loewner chains (see [10, 13]). We recall that a mapping $F\colon B_r \times [0, \infty) \to \mathbf{C}^n$, $0 < r \le 1$, is a Loewner chain if $F(\cdot, t)$ is univalent on B_r, $F(0, t) = 0$, $DF(0, t) = e^t I$ for all $t \ge 0$, and

$$F(z, s) \prec F(z, t), \qquad z \in B_r, \qquad 0 \le s \le t < +\infty,$$

where the symbol $\prec$ means the usual subordination. Thus $F \in S^0(B_r)$ if there exists a Loewner chain $F(z, t)$ such that $F(z) = F(z, 0)$, $z \in B_r$. It is well known that in the case of one variable $S^0(U) = S$; however, in $\mathbf{C}^n$, $n \ge 2$, $S^0(B) \subsetneqq S(B)$ (see [10]).

Certain subclasses of $S(B)$ can be characterized in terms of Loewner chains. In particular, f is starlike iff $f(z, t) = e^t f(z)$, $z \in B$, $t \ge 0$, is a Loewner chain (see [14]). Starlikeness also has an analytic characterization due to Matsuno [12] and Suffridge [20]: a locally univalent map $f\colon B \to \mathbf{C}^n$ such that $f(0) = 0$ is starlike iff

$$\operatorname{Re}\left\langle [Df(z)]^{-1} f(z), z \right\rangle > 0 \qquad z \in B \setminus \{0\}.$$

Another such class is the following, introduced in [7]:

DEFINITION 1.1. Let $f \in S(B)$ and $\beta \in \mathbf{R}$, $|\beta| < \frac{\pi}{2}$. We say that f is a spirallike mapping of type β if the spiral $\exp(-e^{-i\beta} t) f(z)$ $(t \ge 0)$ is contained in $f(B)$ for any $z \in B$.

Let $\hat{S}_\beta(B)$ denote the set of spirallike mappings of type β on B. In the case of one variable this class is well known and is denoted by $\hat{S}_\beta$.

Let $f \in H(B)$ and $\beta \in \mathbf{R}$, $|\beta| < \frac{\pi}{2}$, and let

$$f(z,t) = e^{(1-ia)t} f(e^{iat} z), \qquad z \in B, \qquad t \geq 0, \tag{1.1}$$

where $a = \tan\beta$. Hamada and Kohr [7, 8] showed that if $f\colon B \to \mathbf{C}^n$ is a normalized locally univalent map, then the map $f(z,t)$ given by (1.1) is a Loewner chain if and only if f is a spirallike mapping of type β.

Again there is an analytic characterization, also due to Hamada and Kohr [7]: if $f\colon B \to \mathbf{C}^n$ is a normalized locally univalent map, then f is spirallike of type β iff

$$\operatorname{Re}\left\langle e^{-i\beta}[Df(z)]^{-1} f(z), z\right\rangle > 0, \qquad z \in B \setminus \{0\}.$$

We note that a spirallike map of type 0 is a starlike map.

There is a more general notion of spirallikeness in several variables—spirallikeness with respect to a linear operator—which was considered by Gurganus [6] and Suffridge [22]. However, such maps need not belong to $S^0(B)$ [9].

In order to generate mappings in $S^0(B)$, we will use the following result due to Pfaltzgraff [13].

LEMMA 1.2. *Let $f_t(z) = f(z,t) = e^t z + \cdots$ be a mapping from $B_r \times [0,\infty)$ into $\mathbf{C}^n$ such that $f_t(z) \in H(B_r)$ for each $t \geq 0$, and such that $f(z,t)$ is a locally absolutely continuous function of t locally uniformly with respect to $z \in B_r$, where $0 < r \leq 1$.*

Let $h(z,t)\colon B \times [0,\infty) \to \mathbf{C}^n$ satisfy the following conditions:

(i) $h(0,t) = 0$, $Dh(0,t) = I$, $\operatorname{Re}\langle h(z,t), z\rangle > 0$, $z \in B \setminus \{0\}$, $t \geq 0$;

(ii) *for each $z \in B$, $h(z,t)$ is a measurable function of t on $0 \leq t < +\infty$;*

(iii) *for each $T > 0$ and $r \in (0,1)$ there exists a number $K = K(r,T)$ such that*

$$\|h(z,t)\| \leq K(r,T), \qquad \|z\| \leq r, \qquad 0 \leq t \leq T.$$

Suppose that

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)h(z,t) \qquad \text{a.e. } t \geq 0,$$

and for all $z \in B_r$, and suppose there exists a sequence $\{t_m\}$, $t_m > 0$, increasing to ∞ such that

$$\lim_{m\to\infty} e^{-t_m} f(z,t_m) = G(z),$$

locally uniformly in B_r. Then $f(z,t)$ is a Loewner chain and for each $t \geq 0$, $f(z,t)$ can be extended to a map that is univalent on B.

One further class of mappings we shall consider is the class of Bloch mappings. A mapping $f \in H(B)$ is called a Bloch mapping if the family

$$\mathscr{F}_f = \{g: g(z) = f(\varphi(z)) - f(\varphi(0)) \text{ for some } \varphi \in \text{Aut } B\}$$

is a normal family, where Aut B denotes the set of biholomorphic automorphisms of the unit ball B. Letting

$$m(f) = \sup\{(1 - \|z\|^2)\|Df(z)\|: z \in B\},$$

this condition is equivalent to $m(f) < \infty$.

Let $\mathscr{B}_0$ denote the subclass of S consisting of functions with Bloch seminorm 1, i.e., such that

$$\sup_{z \in U} (1 - |z|^2)|f'(z)| = f'(0) = 1.$$

Let $\mathscr{B}_\infty$ denote the set of locally univalent functions on U with Bloch seminorm 1, normalized so that $f(0) = f'(0) - 1 = 0$.

The Roper–Suffridge extension operator is defined for normalized locally univalent functions on U by

$$\Phi_n(f)(z) = F(z) = \left(f(z_1), \sqrt{f'(z_1)}\, z'\right), \tag{1.2}$$

where the branch of the square root is chosen such that $\sqrt{f'(0)} = 1$.

Roper and Suffridge [17] obtained the beautiful result that if $f \in K$ then $\Phi_n(f)$ is convex on B, and in [3] it was shown that if $f \in S^*$ then $\Phi_n(f)$ is starlike.

In this paper we introduce the operators

$$\Phi_{n,\alpha}(f)(z) = F_\alpha(z) = \left(f(z_1), (f'(z_1))^\alpha z'\right), \qquad z \in B, \tag{1.3}$$

where $\alpha \in [0, \frac{1}{2}]$ and f is a locally univalent function in U, normalized by $f(0) = f'(0) - 1 = 0$. We choose the branch of the power function such that

$$(f'(z_1))^\alpha|_{z_1=0} = 1.$$

Of course when $\alpha = \frac{1}{2}$ we obtain the Roper–Suffridge operator.

We obtain a number of extension results which are valid for $\alpha \in [0, \frac{1}{2}]$: If $f \in S$ then $\Phi_{n,\alpha}(f) \in S^0(B)$; if $f \in S^*$ then $\Phi_{n,\alpha}(f) \in S^*(B)$; if $f \in \hat{S}_\beta$ where $|\beta| < \frac{\pi}{2}$ then $\Phi_{n,\alpha}(f) \in \hat{S}_\beta(B)$; and if $f \in \mathscr{B}_0$ then $\Phi_{n,\alpha}(f)$ is a

Bloch mapping. We also show that $\Phi_{n,\alpha}$ preserves growth and covering results. In addition we obtain the radius of starlikeness of $\Phi_{n,\alpha}(S)$ and the radius of convexity of $\Phi_n(S)$. We give a conjecture and an open problem concerning the radius of starlikeness and convexity of $S^0(B)$. Also we will see that in dimension greater than one the radius of convexity of $S^*(B)$ is strictly less than $2-\sqrt{3}$.

In [4] two of the present authors considered another one-parameter family of extension operators from S to $S(B)$. For a particular value of the parameter one obtains an operator used by Pfaltzgraff and Suffridge [15] to construct starlike mappings of B.

Thus the dependence of extension operators from S to $S(B)$ on parameters appears to be an interesting subject. However we have not yet been able to show that there is any perturbation of the Roper–Suffridge operator which has the convexity-preserving property.

2. LOEWNER CHAINS ASSOCIATED WITH THE OPERATOR $\Phi_{n,\alpha}$

We begin this section with the following main result.

THEOREM 2.1. *Suppose that* $f \in S$ *and* $\alpha \in [0, \frac{1}{2}]$. *Then* $F_\alpha = \Phi_{n,\alpha}(f) \in S^0(B)$.

Proof. It suffices to give the proof in the case $n = 2$. Since $f \in S$, there exists a Loewner chain $f(z_1, t)$ such that $f(z_1) = f(z_1, 0)$, $z_1 \in U$. Let $F_\alpha(z,t)$ be the map defined by

$$F_\alpha(z,t) = \big(f(z_1,t), e^{(1-\alpha)t} z_2 (f'(z_1,t))^\alpha\big),$$

$$z = (z_1, z_2) \in B, \qquad t \geq 0. \quad (2.1)$$

We prove that $F_\alpha(z,t)$ is a Loewner chain.

Since $f(z_1,t)$ is a Loewner chain in U, it is well known that $f(z_1, \cdot)$ is a locally absolutely continuous function in $[0,\infty)$ for each $z_1 \in U$, and for each $r \in (0,1)$ there exists $K_0 = K_0(r) > 0$ such that

$$|f(z_1,t)| \leq K_0 e^t, \qquad |z_1| \leq r, \qquad t \geq 0.$$

Also there exists a function $p(z_1,t)$ that is holomorphic on U and measurable in $t \geq 0$, with $p(0,t) = 1$, $\mathrm{Re}\, p(z_1,t) > 0$ for $z_1 \in U$, $0 \leq t < +\infty$, and such that

$$\frac{\partial f}{\partial t}(z_1,t) = z_1 f'(z_1,t) p(z_1,t), \qquad \text{a.e. } t \geq 0, \quad (2.2)$$

and for all $z_1 \in U$ (see [16, Theorem 6.2]).

Now it is obvious to see that $F_\alpha(\cdot, t) \in H(B)$, $F_\alpha(0,t) = 0$, $DF_\alpha(0,t) = e^t I$, and also $F_\alpha(z,t)$ satisfies the absolute continuity hypothesis of Lemma 1.2. From (2.1) we obtain

$$\frac{\partial F_\alpha}{\partial t}(z,t) = \left(\frac{\partial f}{\partial t}(z_1,t), z_2\left((1-\alpha)e^{(1-\alpha)t}\left(f'(z_1,t)\right)^\alpha \right.\right.$$
$$\left.\left. + e^{(1-\alpha)t}\frac{\partial}{\partial t}\left(f'(z_1,t)\right)^\alpha \right)\right).$$

Since $f(z_1,t)$ is a locally absolutely continuous function in $[0,\infty)$, it follows that for almost all $t \geq 0$ we have

$$\begin{aligned}\frac{\partial f'}{\partial t}(z_1,t) &= \frac{\partial}{\partial z_1}\left(\frac{\partial f}{\partial t}(z_1,t)\right) = \frac{\partial}{\partial z_1}\left(z_1 f'(z_1,t)p(z_1,t)\right)\\ &= f'(z_1,t)p(z_1,t) + z_1 f''(z_1,t)p(z_1,t)\\ &\quad + z_1 f'(z_1,t)p'(z_1,t),\end{aligned}$$

where we have used (2.2) and the fact that the order of differentiation can be changed.

Thus

$$\frac{\partial F_\alpha}{\partial t}(z,t) = \left(z_1 f'(z_1,t)p(z_1,t), z_2 e^{(1-\alpha)t}\left(f'(z_1,t)\right)^\alpha \right.$$
$$\left. \times\left(1-\alpha+\alpha p(z_1,t) + \alpha\frac{z_1 f''(z_1,t)}{f'(z_1,t)}p(z_1,t) + \alpha z_1 p'(z_1,t)\right)\right),$$

a.e. $t \geq 0$, and for all $z = (z_1, z_2) \in B$.

A straightforward computation now yields

$$\begin{aligned}&\left[DF_\alpha(z,t)\right]^{-1}\frac{\partial F_\alpha}{\partial t}(z,t)\\ &\quad = \left(z_1 p(z_1,t), (1-\alpha)z_2 + \alpha z_2 p(z_1,t) + \alpha z_1 z_2 p'(z_1,t)\right),\end{aligned}$$

a.e. $t \geq 0$ and for all $z \in B$. Thus,

$$\frac{\partial F_\alpha}{\partial t}(z,t) = DF_\alpha(z,t)h(z,t), \qquad \text{a.e. } t \geq 0$$

for all $z \in B$, where

$$h(z,t) = \left(z_1 p(z_1,t), (1-\alpha)z_2 + \alpha z_2 p(z_1,t) + \alpha z_1 z_2 p'(z_1,t)\right),$$

for $z = (z_1, z_2) \in B$ and $t \geq 0$.

Clearly, $h(\cdot, t) \in H(B)$, $h(0, t) = 0$, $Dh(0, t) = I$, and

$$\begin{aligned}\operatorname{Re}\langle h(z,t), z\rangle &= |z_1|^2 \operatorname{Re} p(z_1,t) + (1-\alpha)|z_2|^2 \\ &\quad + \alpha|z_2|^2 \operatorname{Re} p(z_1,t) + \alpha|z_2|^2 \operatorname{Re}\big(z_1 p'(z_1,t)\big), \\ &\qquad z \in B, \qquad t \geq 0. \qquad (2.3)\end{aligned}$$

Next we may assume that $z = (z_1, z_2)$, $z_2 \neq 0$, because the case $z = (z_1, 0)$ is easily handled. Also we can suppose that $p(\cdot, t)$ is holomorphic on $\overline{U}$, for otherwise we can use a limiting procedure to reduce to this situation.

Applying the minimum principle for harmonic functions, it suffices to prove that

$$\operatorname{Re}\langle h(z,t), z\rangle \geq 0, \qquad z = (z_1, z_2) \in \mathbf{C}^2, \qquad |z_1|^2 + |z_2|^2 = 1,$$

$z \neq (z_1, 0)$, $t \geq 0$.

Since $p(0, t) = 1$ and $\operatorname{Re} p(z_1, t) > 0$, $z_1 \in U$, $t \geq 0$, we may write

$$p(z_1,t) = \frac{1+\varphi(z_1,t)}{1-\varphi(z_1,t)},$$

where $\varphi(z_1, t)$ is a Schwarz function. Using the Schwarz–Pick lemma on the unit disc, we deduce that

$$\begin{aligned}|p'(z_1,t)| &= \left|\frac{2\varphi'(z_1,t)}{(1-\varphi(z_1,t))^2}\right| \leq \frac{2}{|1-\varphi(z_1,t)|^2} \cdot \frac{1-|\varphi(z_1,t)|^2}{1-|z_1|^2} \\ &= \frac{2}{1-|z_1|^2} \operatorname{Re} p(z_1,t).\end{aligned}$$

Therefore,

$$\operatorname{Re}\big(z_1 p'(z_1,t)\big) \geq -\frac{2|z_1|}{1-|z_1|^2} \operatorname{Re} p(z_1,t), \qquad z_1 \in U, \qquad t \geq 0.$$

Hence using the relation (2.3), the fact that $\alpha \in [0, \frac{1}{2}]$, and the above inequality, we obtain

$$\begin{aligned}\operatorname{Re}\langle h(z,t), z\rangle &\geq (1-\alpha)\big(1-|z_1|^2\big) \\ &\quad + \operatorname{Re} p(z_1,t)\big((1-\alpha)|z_1|^2 - 2\alpha|z_1| + \alpha\big) > 0,\end{aligned}$$

for $z = (z_1, z_2) \in \mathbf{C}^2$, $|z_1|^2 + |z_2|^2 = 1$, $z_2 \neq 0$, and $t \geq 0$. Thus $h(\cdot, t)$ satisfies the assumption (i) from Lemma 1.2, for all $t \geq 0$.

On the other hand it is obvious to see that the mapping h satisfies the measurability condition (ii) from Lemma 1.2. Moreover, because $p(0,t) = 1$ and $\operatorname{Re} p(z_1,t) > 0$, $z_1 \in U$, $t \geq 0$, we obtain

$$\frac{1-r}{1+r} \leq |p(z_1,t)| \leq \frac{1+r}{1-r}, \qquad |z_1| \leq r, \qquad t \geq 0.$$

Hence

$$|z_1 p'(z_1,t)| \leq \frac{2|z_1|}{1-|z_1|^2} \operatorname{Re} p(z_1,t) \leq \frac{2r}{(1-r)^2}, \qquad |z_1| \leq r, \qquad t \geq 0.$$

It follows that for each $r \in (0,1)$, there exists a positive constant $K = K(r)$ such that

$$\|h(z,t)\| \leq K(r), \qquad \|z\| \leq r, \qquad t \geq 0.$$

Thus the relation (iii) from Lemma 1.2 is also satisfied.

Finally, since $f(\cdot,t)$ is locally uniformly bounded on U for each $t \geq 0$, there exists a sequence $\{t_m\}$, $t_m > 0$, increasing to ∞, such that

$$\lim_{m\to\infty} e^{-t_m} f(z_1,t_m) = g(z_1)$$

locally uniformly in U. Therefore we obtain

$$\lim_{m\to\infty} e^{-t_m} F_\alpha(z,t_m) = \left(g(z_1), z_2 (g'(z_1))^\alpha\right) = G_\alpha(z)$$

locally uniformly in B.

Since all assumptions of Lemma 1.2 are satisfied, we conclude that $F_\alpha(z,t)$ is a Loewner chain. But the initial element of this chain is F_α, so $F_\alpha \in S^0(B)$. This completes the proof.

A direct consequence of the above theorem is the following result. We remark that for the case of the Roper–Suffridge operator ($\alpha = \frac{1}{2}$), this result was recently obtained in [3], using the analytical characterization of starlikeness due to Matsuno and Suffridge.

COROLLARY 2.2. *Let* $f \in S^*$ *and* $\alpha \in [0, \frac{1}{2}]$. *Then* $F_\alpha = \Phi_{n,\alpha}(f) \in S^*(B)$.

Proof. We recall that $f \in S^*$ iff $f(z_1,t) = e^t f(z_1)$, $z_1 \in U$, $t \geq 0$, is a Loewner chain. Hence, taking into account the relation (2.1) and the proof of Theorem 2.1, we deduce that $F_\alpha(z,t) = e^t F_\alpha(z)$, $z \in B$, $t \geq 0$, is also a Loewner chain. This implies $F_\alpha \in S^*(B)$, as claimed.

Another consequence of Theorem 2.1 is given in the following

COROLLARY 2.3. *Let* $f \in \hat{S}_\beta$, *where* $\beta \in \mathbf{R}$, $|\beta| < \frac{\pi}{2}$, *and let* $F_\alpha = \Phi_{n,\alpha}(f)$. *Then* $F_\alpha \in \hat{S}_\beta(B)$.

Proof. Since $f \in \hat{S}_\beta$, the following is a Loewner chain

$$f(z_1, t) = e^{(1-ia)t} f\left(e^{iat} z_1\right), \qquad z_1 \in U, \qquad t \geq 0,$$

where $a = \tan\beta$ (see [16, Theorem 6.6]). A short computation shows that

$$F_\alpha(z, t) = e^{(1-ia)t} F_\alpha(e^{iat} z),$$

where $F_\alpha(z,t)$ is given by (2.1). In view of the proof of Theorem 2.1, we conclude that this map is a Loewner chain; hence $F_\alpha \in \hat{S}_\beta(B)$ too.

We have therefore established that the Roper–Suffridge extension operator preserves spirallikeness of type β.

Remark 2.4. As for the preservation of convexity under the operators $\Phi_{n,\alpha}$, we know that $\Phi_{n,1/2}(K) \subseteq K(B)$ and $\Phi_{n,0}(K) \not\subset K(B)$ (see [17]). Using arguments similar to those in the proof of [17, Theorem 2], we can show that the operator $\Phi_{n,\alpha}$ does not preserve convexity for $n \geq 2$ and $\alpha \in (0, 1/2)$. For this purpose it suffices to consider the function $f: U \to \mathbf{C}$ given by $f(z_1) = \frac{1}{2}\log\frac{1+z_1}{1-z_1}$.

3. RADIUS OF STARLIKENESS AND RADIUS OF CONVEXITY

Let $\mathscr{F}$ be a non-empty subset of $S(B)$. Then we let

$$r^*(\mathscr{F}) = \sup\{r: f \text{ is starlike on } B_r, f \in \mathscr{F}\}$$

and

$$r_c(\mathscr{F}) = \sup\{r: f \text{ is convex on } B_r, f \in \mathscr{F}\},$$

denote the radius of starlikeness and radius of convexity of $\mathscr{F}$, respectively.

Shi [19] showed that the radius of convexity for $S^*(B)$ is strictly positive and also that there exists a positive radius of convexity for the set of normalized locally uniformly bounded maps on B.

In the following we deduce the values of $r^*(\Phi_{n,\alpha}(S))$, $\alpha \in [0, \frac{1}{2}]$, and $r_c(\Phi_n(S))$, when $n \geq 2$. We begin with the following observation:

Remark 3.1. It is obvious that if f: $U \to \mathbf{C}$ is a locally univalent function on U, normalized by $f(0) = f'(0) - 1 = 0$, and if for some $\alpha \in [0, \frac{1}{2}]$ and some $r \in (0,1)$ $\Phi_{n,\alpha}(f) \in S(B_r)$, then $f \in S(U_r)$. Also if $\Phi_{n,\alpha}(f) \in S^*(B_r)$ (resp. $K(B_r)$), then $f \in S^*(U_r)$ (resp. $K(U_r)$) too.

On the other hand, if $f \in S(U_r)$, $0 < r \leq 1$, then, using the result of Theorem 2.1, we deduce that $\Phi_{n,\alpha}(f) \in S^0(B_r)$, for all $\alpha \in [0, \frac{1}{2}]$.

We can now prove

THEOREM 3.2. *$r^*(\Phi_{n,\alpha}(S)) = \tanh \frac{\pi}{4}$, for all $\alpha \in [0, \frac{1}{2}]$.*

Proof. It is well known that if $f \in S$ then f is starlike in U_r, where $r = \tanh \frac{\pi}{4}$. In fact this positive number is the radius of starlikeness for the class S (see for example [16]). Hence

$$\operatorname{Re} \frac{z_1 f'(z_1)}{f(z_1)} > 0, \qquad |z_1| < r,$$

and this quantity can be negative if $|z_1| > r$.

Now let $F_\alpha = \Phi_{n,\alpha}(f)$. Taking into account the result of Corollary 2.2 and using Remark 3.1, we deduce that $F_\alpha \in S^*(B_r)$ and furthermore that F_α may not be starlike in any ball B_{r_1} with $r_1 > r$. Therefore $r = \tanh \frac{\pi}{4}$ is the biggest radius for which each $F_\alpha \in \Phi_{n,\alpha}(S)$ becomes starlike in B_r. This completes the proof.

Since $\Phi_{n,\alpha}(S) \subseteq S^0(B)$ for $\alpha \in [0, \frac{1}{2}]$, we must have $r^*(S^0(B)) \leq r^*(\Phi_{n,\alpha}(S)) = \tanh \frac{\pi}{4}$, for $n \geq 2$. Hence Theorem 3.2 leads to the following

Conjecture 3.3. $r^*(S^0(B)) = \tanh \frac{\pi}{4}$.

With similar reasoning to that in the proof of Theorem 3.2, we obtain the following result concerning the radius of convexity of $\Phi_n(S)$.

THEOREM 3.4. *$r_c(\Phi_n(S)) = r_c(\Phi_n(S^*)) = 2 - \sqrt{3}$.*

Proof. Let $F \in \Phi_n(S^*)$ (or $F \in \Phi_n(S)$). Then $F = \Phi_n(f)$, where $f \in S^*$ (or $f \in S$). It is well known that $f \in K(U_r)$ where $r = 2 - \sqrt{3}$, and this number is the radius of convexity for S^* (or for S) (see for example [16]). Hence

$$\operatorname{Re}\left[\frac{z_1 f''(z_1)}{f'(z_1)} + 1\right] > 0, \qquad |z_1| < r,$$

and this quantity can be negative if $|z_1| > r$.

Now if $g \in K(U_\rho)$, $0 < \rho \leq 1$, then it is clear that if we set $g_\rho(\zeta) = \frac{1}{\rho} g(\rho\zeta)$, $\zeta \in U$ then $g_\rho \in K$. Hence from [17, Theorem 1] (see also [3, Theorem 2.1]), we deduce that $\Phi_n(g_\rho) \in K(B)$. This gives $\Phi_n(g) \in K(B_\rho)$, because it is obvious to see that

$$\Phi_n(g_\rho)(z) = \frac{1}{\rho}\Phi_n(g)(\rho z), \qquad z \in B.$$

Therefore, using the above argument, we conclude that $\Phi_n(f) \in K(B_r)$, where $r = 2 - \sqrt{3}$.

Taking into account Remark 3.1, we deduce that F may not be convex in any ball B_{r_1}, with $r_1 > r$. Therefore $r_c(\Phi_n(S^*)) = r_c(\Phi_n(S)) = 2 - \sqrt{3}$. This completes the proof.

Because $\Phi_n(S) \subseteq S^0(B)$, $\Phi_n(S^*) \subseteq S^*(B)$, we conclude from Theorem 3.4 that

$$r_c(S^0(B)) \le r_c(S^*(B)) \le 2 - \sqrt{3}.$$

We are grateful to Ted Suffridge, who suggested the following example which shows that in $\mathbf{C}^n$, $n \ge 2$, the radius of convexity of $S^*(B)$ is strictly less than $2 - \sqrt{3}$. Thus, it remains an open problem to find this radius in several complex variables.

EXAMPLE 3.5. Let $n = 2$, and let $f\colon B \to \mathbf{C}^2$ be given by

$$f(z) = (z_1 + az_2^2, z_2), \qquad z = (z_1, z_2) \in B,$$

where $a \in \mathbf{C}$, $|a| = 3\sqrt{3}/2$. Then $f \in S^*(B)$, and $f \in K(B_r)$ where $r = 1/3\sqrt{3}$. However, f is not convex in any ball of radius greater than r.

Proof. Since $|a| = 3\sqrt{3}/2$, we deduce from [18, Example 5] that f is starlike on B. Next we show that f is convex on B_r, using a similar argument to that in the proof of [18, Example 7].

Taking into account the necessary and sufficient condition of convexity given in [21, Theorems 4 and 5], we have to show that

$$\operatorname{Re}\left\langle [Df(z)]^{-1}(f(z) - f(u)), z\right\rangle \ge 0, \qquad \|u\| \le \|z\| < r.$$

A straightforward computation yields

$$\begin{aligned}
&\operatorname{Re}\left\langle [Df(z)]^{-1}(f(z) - f(u)), z\right\rangle \\
&\quad = \operatorname{Re}\left\{|z_1|^2 + |z_2|^2 - u_1\bar{z}_1 - u_2\bar{z}_2 - a\bar{z}_1(z_2 - u_2)^2\right\} \\
&\quad = \|z\|^2 - \operatorname{Re}\langle z, u\rangle - \operatorname{Re}\left\{a\bar{z}_1(z_2 - u_2)^2\right\} \\
&\quad \ge \|z\|^2 - \operatorname{Re}\langle z, u\rangle - |a|\,|z_1|\,|z_2 - u_2|^2
\end{aligned}$$

$$
\begin{aligned}
&= \|z\|^2(1 - |a|\,|z_1|) - \mathrm{Re}\langle z, u\rangle(1 - 2|a|\,|z_1|) \\
&\quad - |a|\,|z_1|\big(\|u\|^2 - |z_1 - u_1|^2\big) \\
&\geq \|z\|^2(1 - |a|\,|z_1|) - \mathrm{Re}\langle z, u\rangle(1 - 2|a|\,|z_1|) \\
&\quad - |a|\,|z_1|\big(\|z\|^2 - |z_1 - u_1|^2\big) \\
&= (\|z\|^2 - \mathrm{Re}\langle z, u\rangle)(1 - 2|a|\,|z_1|) + |a|\,|z_1|\,|z_1 - u_1|^2 \geq 0,
\end{aligned}
$$

for all $z = (z_1, z_2) \in B_r$, $u = (u_1, u_2) \in B_r$, $\|u\| \leq \|z\|$, when $|a| = 3\sqrt{3}/2$ and $r = 1/3\sqrt{3}$. Therefore f is convex on B_r.

On the other hand, f is not convex in any ball B_{r_1} with $r_1 > 1/3\sqrt{3}$. Indeed, let $z = (z_1, z_2)$ and $u = (u_1, u_2)$, where $z_1 = u_1$, $z_2 = -u_2 \in \mathbf{R}\setminus\{0\}$, $|z_1| > 1/3\sqrt{3}$, and $\mathrm{Re}\{a\bar{z}_1\} > \frac{1}{2}$. Hence $\|z\| > 1/3\sqrt{3}$ and

$$
\begin{aligned}
&\mathrm{Re}\Big\langle [Df(z)]^{-1}(f(z) - f(u)), z\Big\rangle \\
&\qquad = \|z\|^2 - \mathrm{Re}\langle z, u\rangle - \mathrm{Re}\Big\{a\bar{z}_1(z_2 - u_2)^2\Big\} \\
&\qquad = \|z\|^2 - \mathrm{Re}\{z_1\bar{z}_1 - z_2\bar{z}_2\} - 4z_2^2\,\mathrm{Re}\{a\bar{z}_1\} \\
&\qquad = 2z_2^2\{1 - 2\,\mathrm{Re}(a\bar{z}_1)\} < 0.
\end{aligned}
$$

Thus f is not convex in any ball of radius greater than $r = 1/3\sqrt{3}$. This completes the proof.

Open Problem 3.6. Find $r_c(S^*(B))$ and $r_c(S^0(B))$, where B is the unit ball of $\mathbf{C}^n$, $n \geq 2$.

Next we make the following observation:

Remark 3.7. There is no radius of convexity for the class of normalized starlike mappings on the unit polydisk P of $\mathbf{C}^n$, $n \geq 2$.

Proof. Let $F: P \to \mathbf{C}^n$ be a locally univalent map, $F(0) = 0$, $DF(0) = I$. According to Suffridge's characterization of convexity (see [20, Theorem 3]), F is convex if and only if F has the representation

$$
F(z) = T\big(f_1(z_1), \ldots, f_n(z_n)\big), \qquad z = (z_1, \ldots, z_n) \in P,
$$

where T is a non-singular $n \times n$ matrix and f_j are univalent convex functions of one variable. Since $DF(0) = I$, T must be a diagonal matrix and after absorbing constants, we may assume that $T = I$ and $f_j'(0) = 1$, $j = 1, \dots, n$. Hence F must be of the form

$$F(z) = (g_1(z_1), \dots, g_n(z_n)), \tag{3.1}$$

where the g_j are normalized convex functions in the unit disc.

Now let

$$F(z) = \left(\frac{z_1}{(1-z_1)^2}, \frac{z_2}{(1-z_1)^2}, \dots, \frac{z_n}{(1-z_1)^2} \right),$$
$$z = (z_1, \dots, z_n) \in P.$$

Then F is normalized locally univalent on P, and a short computation shows that

$$w(z) = [DF(z)]^{-1} F(z) = \left(z_1 \frac{1-z_1}{1+z_1}, z_2 \frac{1-z_1}{1+z_1}, \dots, z_n \frac{1-z_1}{1+z_1} \right),$$

for $z = (z_1, \dots, z_n) \in P$.

Hence

$$\operatorname{Re} \frac{w_j(z)}{z_j} = \operatorname{Re} \frac{1-z_1}{1+z_1} > 0, \qquad \|z\| = |z_j|, \qquad 1 \le j \le n.$$

Applying [20, Theorem 1], we conclude that F is starlike on P.

On the other hand, it is obvious to see that F does not admit a decomposition as in (3.1); hence F is not convex in rP for any $r \in (0,1)$. This completes the proof.

4. THE EXTENSION OF UNIVALENT BLOCH FUNCTIONS TO BLOCH MAPPINGS

In this section we show that the same set of parameter values ($\alpha \in [0, \frac{1}{2}]$) arises when one considers whether normalized univalent Bloch functions are extended to Bloch mappings by $\Phi_{n,\alpha}$.

THEOREM 4.1. *If $f \in \mathscr{B}_0$ and $\alpha \in [0, \frac{1}{2}]$, then $F_\alpha = \Phi_{n,\alpha}(f)$ is a Bloch mapping.*

Proof. We need to show that $m(F_\alpha) < \infty$. It suffices to give the proof when $n = 2$.

Since

$$DF_\alpha(z)u = \Big(u_1 f'(z_1), \alpha z_2 u_1 (f'(z_1))^{\alpha-1} f''(z_1) + u_2 (f'(z_1))^{\alpha}\Big),$$

for all $z = (z_1, z_2) \in B$ and $u = (u_1, u_2) \in \mathbf{C}^2$, we obtain the relations

$$\begin{aligned}
\| DF_\alpha(z)u \|^2 &= |u_1|^2 |f'(z_1)|^2 + |f'(z_1)|^{2\alpha} \left| \alpha z_2 u_1 \frac{f''(z_1)}{f'(z_1)} + u_2 \right|^2 \\
&\le |f'(z_1)|^2 + |f'(z_1)|^{2\alpha} \left[\alpha^2 \big(1 - |z_1|^2\big) \left| \frac{f''(z_1)}{f'(z_1)} \right|^2 \right. \\
&\qquad\qquad\qquad \left. + 1 + 2\alpha \left| \frac{f''(z_1)}{f'(z_1)} \right| \right] \\
&\le |f'(z_1)|^2 + \frac{|f'(z_1)|^{2\alpha}}{1 - |z_1|^2} \\
&\quad \times \Big((4\alpha^2 - 1)|z_1|^2 + 4\alpha(1 + 4\alpha)|z_1| + 16\alpha^2 + 8\alpha + 1 \Big) \\
&\le |f'(z_1)|^2 + \frac{|f'(z_1)|^{2\alpha}}{1 - |z_1|^2} (32\alpha^2 + 12\alpha + 1),
\end{aligned}$$

for all $z = (z_1, z_2) \in B$ and $u \in \mathbf{C}^2$, $\|u\| = 1$. In the above relations we have used the fact that $\alpha \in [0, \frac{1}{2}]$ and $f \in S$; hence f satisfies the well known inequality

$$\left| \frac{1 - |z_1|^2}{2} \cdot \frac{f''(z_1)}{f'(z_1)} - \bar{z}_1 \right| \le 2,$$

for all $z_1 \in U$.

Now, since $f \in \mathscr{B}_0$, we have

$$|f'(z_1)| \le \frac{1}{1 - |z_1|^2}, \qquad z_1 \in U,$$

and since $\alpha \in [0, \frac{1}{2}]$, we deduce

$$\begin{aligned}
&(1 - \|z\|^2)^2 \| DF_\alpha(z) \|^2 \\
&\qquad \le \big(1 - |z_1|^2\big)^2 |f'(z_1)|^2 + \big(1 - |z_1|^2\big)^{2\alpha} |f'(z_1)|^{2\alpha} (32\alpha^2 + 12\alpha + 1) \\
&\qquad \le 16,
\end{aligned}$$

for all $z \in B$, which means $m(F_\alpha) \le 4$. Thus F_α is a Bloch mapping, as claimed. This completes the proof.

We note that for the case of the Roper–Suffridge operator ($\alpha = \frac{1}{2}$), the above result was recently obtained in [3].

Remark 4.2. Liu and Minda showed that if $f \in \mathscr{B}_\infty$ then f is univalent in U_{r_0} where $r_0 = \sqrt{\frac{\pi}{4+\pi}} \simeq 0.6633$ [11, Theorem 1]. From Theorem 2.1 and Remark 3.1 it follows that $\Phi_{n,\alpha}(f) \in S^0(B_{r_0})$. However, we do not know that $\Phi_{n,\alpha}(f)$ is a Bloch mapping. (Of course if $f \in \mathscr{B}_0$ then $\Phi_{n,\alpha}(f) \in S^0(B)$.)

5. GROWTH AND COVERING THEOREMS FOR FAMILIES OF THE FORM $\Phi_{n,\alpha}(\mathscr{F})$

In this section we prove the following growth and covering results for families $\Phi_{n,\alpha}(\mathscr{F})$, where $\alpha \in [0, \frac{1}{2}]$ and $\mathscr{F}$ is a subfamily of S whose members satisfy growth and distortion results.

THEOREM 5.1. *Suppose that $\mathscr{F}$ is a subfamily of S such that all $f \in \mathscr{F}$ satisfy*

$$\varphi(r) \le |f(z_1)| \le \psi(r), \qquad |z_1| = r \tag{5.1}$$

$$\varphi'(r) \le |f'(z_1)| \le \psi'(r), \qquad |z_1| = r, \tag{5.2}$$

where

$$\varphi, \psi \text{ are twice differentiable on } [0,1), \tag{5.3}$$

$$\varphi(0) = \varphi'(0) - 1 = 0, \qquad \varphi'(r) \ge 0, \qquad \varphi''(r) \le 0 \qquad \text{on } [0,1); \tag{5.4}$$

$$\psi(0) = \psi'(0) - 1 = 0, \qquad \psi'(r) \ge 0, \qquad \psi''(r) \ge 0 \qquad \text{on } [0,1). \tag{5.5}$$

If $F_\alpha \in \Phi_{n,\alpha}(\mathscr{F})$, then

$$\varphi(r) \le \|F_\alpha(z)\| \le \psi(r), \qquad \|z\| = r. \tag{5.6}$$

Furthermore, if for some $f \in \mathscr{F}$ the lower (respectively upper) estimate in (5.1) *is sharp at $z_1 \in U$, then the lower (respectively upper) estimate in* (5.6) *is sharp for $\Phi_{n,\alpha}(f)$ at $(z_1, 0, \ldots, 0)$.*

To prove this we need the following lemma

LEMMA 5.2. *Suppose that* φ *and* ψ *are functions which satisfy the conditions* (5.3)–(5.5) *of Theorem* 5.1 *and* $\alpha \in [0, \frac{1}{2}]$. *Then for fixed* $r \in [0, 1)$,

the minimum of $(\varphi(t))^2 + (r^2 - t^2)(\varphi'(t))^{2\alpha}$ *for* $t \in [0, r]$ *occurs when* $t = r$;

the maximum of $(\psi(t))^2 + (r^2 - t^2)(\psi'(t))^{2\alpha}$ *for* $t \in [0, r]$ *occurs when* $t = r$.

Proof. We consider the sign of the first derivative on $(0, r]$, taking account of the fact that $\alpha \in [0, \frac{1}{2}]$ and the relations (5.3)–(5.5).

Proof of Theorem 5.1. Let $\|z\| = r$. Taking into account the result of Lemma 5.2, it is not difficult to obtain the lower and the upper estimates for

$$|f(z_1)|^2 + \|z'\|^2 |f'(z_1)|^{2\alpha} = |f(z_1)|^2 + \left(r^2 - |z_1|^2\right)|f'(z_1)|^{2\alpha}.$$

As a direct consequence of Theorem 5.1 we obtain the following growth result (compare with [2, Corollary 2.3; 4, Corollary 2.3]).

COROLLARY 5.3. *If* $f \in S$, *then*

$$\frac{r}{(1+r)^2} \leq \|\Phi_{n,\alpha}(f)(z)\| \leq \frac{r}{(1-r)^2}, \qquad \|z\| = r. \tag{5.7}$$

If $f \in K$, *then*

$$\frac{r}{1+r} < \|\Phi_{n,\alpha}(f)(z)\| \leq \frac{r}{1-r}, \qquad \|z\| = r. \tag{5.8}$$

If $f \in K$ *and* $f''(0) = 0, \dots, f^{(k)}(0) = 0$, *then*

$$\int_0^r \frac{dt}{(1+t^k)^{2/k}} \leq \|\Phi_{n,\alpha}(f)(z)\| \leq \int_0^r \frac{dt}{(1-t^k)^{2/k}}, \qquad \|z\| = r. \tag{5.9}$$

If $f \in \mathscr{B}_0$, *then*

$$\frac{1}{2}\left(1 - \exp\left(-\frac{2r}{1-r}\right)\right) \leq \|\Phi_{n,\alpha}(f)(z)\| \leq \frac{1}{2}\log\frac{1+r}{1-r}, \qquad \|z\| = r. \tag{5.10}$$

All of these estimates are sharp except for the lower estimate in (5.10).

Finally we give the following covering theorem for the class $\Phi_{n,\alpha}(\mathscr{F})$.

THEOREM 5.4. *Suppose that* $\alpha \in [0, \frac{1}{2}]$ *and that the family* $\mathcal{F} \subset S$ *and the functions* φ *and* ψ *satisfy the hypotheses of Theorem* 5.1. *Then for all* $f \in \mathcal{F}$, *the image of* $\Phi_{n,\alpha}(f)$ *contains the ball* B_ρ, *where* $\rho = \lim_{r \nearrow 1} \varphi(r)$.

Proof. The existence of ρ follows from the fact that φ is a bounded increasing function on $[0, 1)$. This can be proved using (5.4) and the fact that $\varphi(r)/r$ is decreasing on $[0, 1)$. On the other hand, since $\Phi_{n,\alpha}(f)$ is an open mapping, $\Phi_{n,\alpha}(f)(B) \supset B_\rho$, as claimed. This completes the proof.

COROLLARY 5.5. *If* $f \in S$, *then* $\Phi_{n,\alpha}(f)(B) \supseteq B_{1/4}$.
If $f \in K$, *then* $\Phi_{n,\alpha}(f)(B) \supseteq B_{1/2}$.
If $f \in K$ *and* $f''(0) = 0, \ldots, f^{(k)}(0) = 0$, *then*

$$\Phi_{n,\alpha}(f)(B) \supseteq B_{r_k}, \qquad r_k = \int_0^1 \frac{dt}{(1 + t^k)^{1/k}}.$$

If $f \in \mathcal{B}_0$, *then* $\Phi_{n,\alpha}(f)(B) \supseteq B_{1/2}$.
All results are sharp except for the last.

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