Available at WWW.MATHEMATICSWEB.ORG
POWERED BY SCIENCE DIRECT

J. Math. Anal. Appl. 277 (2003) 670–680

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

On existence of oscillatory solutions of second order Emden–Fowler equations

C.H. Ou and James S.W. Wong *

Department of Mathematics, City University of Hong Kong, China

Received 27 June 2002

Submitted by Steven G. Krantz

Abstract

We study the second order Emden–Fowler equation

$$y''(t) + a(x)|y|^{\gamma}\operatorname{sgn} y = 0, \quad \gamma > 0, \tag{E}$$

where $a(x)$ is a positive and absolutely continuous function on $(0,\infty)$. Let $\phi(x) = a(x)x^{(\gamma+3)/2}$, $\gamma \neq 1$, and bounded away from zero. We prove the following theorem. *If* $\phi'_{-}(x) \in L^1(0,\infty)$ *where* $\phi'_{-}(x) = -\min(\phi'(x), 0)$, *then Eq.* (E) *has oscillatory solutions*. In particular, this result embodies earlier results by Jasny, Kurzweil, Heidel and Hinton, Chiou, and Erbe and Muldowney.

Keywords: Oscillation; Nonlinear; Second order; Ordinary differential equation

1. We consider the second order Emden–Fowler differential equation

$$y''(t) + a(x)\left|y(x)\right|^{\gamma}\operatorname{sgn} y = 0, \quad \gamma > 0, \tag{1}$$

on $(0,\infty)$, where $a(x)$ is a positive and absolutely continuous function on $(0,\infty)$. Under these conditions, it is known that every solution of (1) can be continued to the right throughout the entire interval $(0,\infty)$, see Heidel [3]. A solution $y(x)$ is said to be oscillatory if it has arbitrarily large zeros, i.e., for any $x_0 \in (0,\infty)$ there exists a $x_1 > x_0$ such that $y(x_1) = 0$. Otherwise, the solution $y(x)$ is said to be nonoscillatory, i.e., it has only finite number of zeros. Equation (1) is called superlinear if $\gamma > 1$ and is called sublinear if $0 < \gamma < 1$.

* Corresponding author.
E-mail addresses: 50002591@plink.cityu.edu.hk (C.H. Ou), jsww@chinneyhonkwok.com (J.S.W. Wong).

doi:10.1016/S0022-247X(02)00617-0

The purpose of this paper is to prove a general result for Eq. (1) on the existence of oscillatory solutions which embodies all previous results. Moreover, the proof used is substantively different from those of the earlier results. Denote $\phi(x) = a(x)x^{(\gamma+3)/2}$ and assume that $\phi(x)$ is bounded away from zero at infinity, i.e.,

$$\phi(x) \geqslant k > 0, \quad x \geqslant x_0 > 0, \tag{2}$$

where x_0 depends on the function $\phi(x)$.

Theorem 1. *Let $\gamma \neq 1$. If $\phi(x)$ satisfies* (2) *and $\phi'_-(x) \in L^1(0,\infty)$ where $\phi'_-(x) = -\min(\phi'(x), 0)$, then Eq.* (1) *has oscillatory solutions.*

As corollaries to the above theorem, we have

Corollary 1. *Let $\gamma \neq 1$. If $\phi(x)$ satisfies* (2) *and, in addition, $\phi^+(x) \in L^1(0,\infty)$ where $\phi^+(x) = \max(\phi(x), 0)$, then Eq.* (1) *has oscillatory solutions.*

Corollary 2. *Let $\gamma \neq 1$. If $\phi(x)$ satisfies* (2) *and is monotone in x on (x_0,∞) for some x_0 as given in* (2)*, then Eq.* (1) *has oscillatory solutions.*

As a consequence of Corollary 2, we can deduce the following results:

Theorem A (Jasny [5], Kurzweil [7]). *Let $\gamma > 1$. If $\phi(x)$ is nondecreasing in x, then Eq.* (1) *has oscillatory solutions.*

Theorem B (Heidel and Hinton [4], Chiou [1]). *Let $0 < \gamma < 1$. If $\phi(x)$ is nondecreasing in x, then Eq.* (1) *has oscillatory solutions.*

Theorem C (Erbe and Muldowney [2]). *Let $\gamma > 1$. If $\phi(x)$ is nonincreasing in x and bounded away from zero, then Eq.* (1) *has oscillatory solutions.*

Theorem D (Chiou [1]). *Let $0 < \gamma < 1$. If $\phi(x)$ is nonincreasing in x and bounded away from zero, then Eq.* (1) *has oscillatory solutions.*

2. In this section, we present the proof of Theorem 1 and its corollaries.

Proof of Theorem 1. Let $y(x)$ be a solution of (1) satisfying the initial conditions

$$y(x_0) = 0, \qquad y'(x_0) = x_0^{-1/2}c, \tag{3}$$

where c is a constant to be chosen later. Introduce the "oscillation invariant" transformation

$$t = \log x, \qquad w(t) = x^{-1/2}y(x) \tag{4}$$

which transforms the original equation (1) into

$$\ddot{w} - \frac{1}{4}w + f(t)|w|^\gamma \operatorname{sgn} w = 0, \tag{5}$$

where dot denotes differentiation with respect t and $f(t) = a(x)x^{(\gamma+3)/2} = \phi(x)$. Clearly the transformation (4) preserves the oscillatory nature between the solutions of (1) and that of (5). Introduce the energy function

$$G\big(w(t)\big) = \frac{\dot{w}^2}{2} + \frac{f(t)}{1+\gamma}|w|^{\gamma+1} - \frac{1}{8}w^2 \tag{6}$$

which satisfies

$$\frac{dG(w(t))}{dt} = \frac{\dot{f}(t)}{1+\gamma}|w|^{\gamma+1}. \tag{7}$$

For the superlinear case when $\gamma > 1$, using Young's inequality on the term $\frac{1}{8}w^2$ in (6) gives

$$\begin{aligned}
\frac{1}{8}w^2 &= \frac{1}{4}\big(f^{2/(\gamma+1)}\big)\left(\frac{1}{2}f^{-2/(1+\gamma)}\right)\\
&\leqslant \frac{1}{2(\gamma+1)}f|w|^{\gamma+1} + \frac{\gamma-1}{4(\gamma+1)}\left(\frac{1}{2}f^{-2/(\gamma+1)}\right)^{(\gamma+1)/(\gamma-1)}\\
&\leqslant \frac{1}{2(\gamma+1)}f|w|^{\gamma+1} + \frac{\gamma-1}{4(\gamma+1)}\left(\frac{1}{2}k^{-2/(\gamma+1)}\right)^{(\gamma+1)/(\gamma-1)},
\end{aligned}$$

so the energy function $G(w(t))$ satisfies

$$\begin{aligned}
G\big(w(t)\big) &\geqslant \frac{f(t)}{1+\gamma}|w|^{\gamma+1} - \frac{1}{8}w^2 \geqslant \frac{f(t)}{2(1+\gamma)}|w|^{\gamma+1} - K_\gamma\\
&\geqslant \frac{k}{2(1+\gamma)}|w|^{\gamma+1} - K_\gamma
\end{aligned}$$

where

$$K_\gamma = \frac{\gamma-1}{4(\gamma+1)}\left(\frac{1}{2}\right)^{(1+\gamma)/(\gamma-1)} k^{2/(1-\gamma)}.$$

From above inequality it is easy to choose suitable constants m_1 and m_2 such that

$$\frac{1}{1+\gamma}|w|^{\gamma+1} \leqslant m_1 G\big(w(t)\big) + m_2. \tag{8}$$

Since $\phi(x)$ can be decomposed as

$$\phi(x) = \phi(0) + \int_0^x \phi'_+(x)\,dx - \int_0^x \phi'_-(x)\,dx \tag{9}$$

we can easily observe by (2) and $\phi'_-(x) \in L^1(0,\infty)$ that $\phi(x)$ is bounded below if and only if

$$\int_0^\infty \phi'_+(x)\,dx < \infty. \tag{10}$$

Consider first the case when (10) is true, i.e., there exists a constant K such that

$$\left|\phi(x)\right| \leqslant K, \quad x > 0. \tag{11}$$

We shall show that if c and x_0 in (3) are chosen sufficiently large, the solution $y(x)$ of (1) is oscillatory. Suppose to the contrary that $y(x)$ is nonoscillatory and we may without loss of generality assume that $y(x) > 0$ for $x > X_0 \geqslant x_0$, hence by (1) $y''(x) < 0$ for $x > X_0$. It is then obvious that $y'(x)$ cannot be oscillatory. Thus there must exist $X_1 > X_0$ so that $y'(x)$ is of constant sign on $[X_1, \infty)$. If $y'(x) < 0$, then $y(x) \to -\infty$ contradicting that $y(x) > 0$. Hence $y'(x) > 0$, and we claim that $\lim_{t\to\infty} y'(x) = 0$. Suppose that $\lim_{x\to\infty} y'(x) = c_0 > 0$, then $y(x) > c_1 x$ on $[X_1, \infty)$ for some $c_1 > 0$. Integrating (1) from X_1 to x, we observe by (1) and $\gamma > 1$

$$\begin{aligned} y'(x) &= y'(X_1) - \int_{X_1}^{x} a(s) y^{\gamma}(s)\, ds \leqslant y'(X_1) - c_1^{\gamma} \int_{X_1}^{x} a(s) s^{\gamma}\, ds \\ &\leqslant y'(X_1) - k c_1^{\gamma} \int_{X_1}^{x} s^{-1+(\gamma/2-1/2)}\, ds \to -\infty \end{aligned} \tag{12}$$

which is impossible; hence $c_0 = 0$. Using $\lim_{t\to\infty} y'(x) = 0$, we first integrate (1) from x to ∞ to obtain

$$y'(x) = \int_{x}^{\infty} a(s) y^{\gamma}(s)\, ds. \tag{13}$$

Now we integrate (13) from X_1 to x to obtain

$$y(x) = y(X_1) + \int_{X_1}^{x} dv \int_{v}^{\infty} a(s) y^{\gamma}(s)\, ds. \tag{14}$$

Since $y'(x) > 0$ on $[X_1, \infty)$, we can estimate (14) from below as

$$\begin{aligned} y(x) &\geqslant \int_{X_1}^{x} dv \int_{x}^{\infty} a(s) y^{\gamma}(s)\, ds \geqslant (x - X_1) y^{\gamma}(x) \int_{x}^{\infty} a(s)\, ds \\ &\geqslant (x - X_1) y^{\gamma}(x) \int_{x}^{\infty} k s^{-(\gamma+3)/2}\, ds = \frac{2k}{\gamma+1} (x - X_1) y^{\gamma}(x) x^{-(1+\gamma)/2}. \end{aligned} \tag{15}$$

For $x \geqslant 2X_1$, we have $x - X_1 \geqslant x/2$, so (15) yields

$$y(x) \leqslant K_1 x^{1/2}, \tag{16}$$

where

$$K_1 = \left(\frac{1+\gamma}{k}\right)^{1/(\gamma-1)}.$$

Using (1), (11) and the fact $\lim_{x\to\infty} y'(x) = 0$, we can estimate (13) as

$$\left|y'(x)\right| = \left|\int_x^\infty a(s)y^\gamma(s)\,ds\right| \leqslant K(K_1)^\gamma \int_x^\infty s^{-3/2}\,ds = 2K(K_1)^\gamma x^{-1/2}. \tag{17}$$

By (4), we note that

$$\begin{cases} w(t) = x^{-1/2}y(x), \\ \dot{w}(t) = x^{1/2}y'(x) - \frac{1}{2}x^{-1/2}y(x). \end{cases} \tag{18}$$

Using (16) and (17) in (18) we conclude that

$$w^{(j)}(t) = O(1)$$

as $t \to \infty$ for $j = 0, 1$. Let C_0 be the uniform bound for $w(t)$ and $\dot{w}(t)$, i.e.,

$$\left|w^{(j)}(t)\right| \leqslant C_0$$

for sufficiently large t, where C_0 depends on γ, k and K but is independent of c. It is thus easy to see from (6) that $\limsup_{t\to\infty} G(w(t)) = M_0$ and is finite, where M_0 is independent of c.

Noting that $\dot{f}(t) \leqslant \dot{f}_+(t)$, we insert (8) into (7) to obtain

$$\frac{dG(w(t))}{dt} \leqslant \frac{\dot{f}_+(t)}{1+\gamma}\left|w(t)\right|^{1+\gamma} \leqslant \dot{f}_+(t)\bigl(m_1 G\bigl(w(t)\bigr) + m_2\bigr). \tag{19}$$

Upon integration it follows from (19) that

$$\begin{aligned} G\bigl(w(t)\bigr) &\leqslant \left\{G\bigl(w(t_0)\bigr) + m_2\int_{t_0}^t \dot{f}_+(t)\,dt\right\}\exp\left(m_1\int_{t_0}^t \dot{f}_+(t)\,dt\right) \\ &\leqslant \left\{\frac{c^2}{2} + m_2\int_{t_0}^t \dot{f}_+(t)\,dt\right\}\exp\left(m_1\int_{t_0}^t \dot{f}_+(t)\,dt\right), \end{aligned} \tag{20}$$

where $t_0 = \log x_0$. Turning back to (8), and by (10), we have the estimate

$$\begin{aligned} \frac{1}{1+\gamma}|w|^{\gamma+1} &\leqslant m_1 G\bigl(w(t)\bigr) + m_2 \\ &\leqslant m_1\left\{\frac{c^2}{2} + m_2\int_{t_0}^\infty \dot{f}_+(t)\,dt\right\}\exp\left(m_1\int_{t_0}^\infty \dot{f}_+(t)\,dt\right) + m_2 \\ &= l_1c^2 + l_2, \end{aligned} \tag{21}$$

where

$$l_1 = \frac{m_1}{2}\exp\left(m_1\int_{t_0}^\infty \dot{f}_+(t)\,dt\right)$$

and

$$l_2 = m_1 m_2 \int_{t_0}^{\infty} \dot{f}_+(t)\,dt \exp\left(m_1 \int_{t_0}^{\infty} \dot{f}_+(t)\,dt\right) + m_2.$$

Integrating (7) from t_0 to T yields

$$-G\big(w(T)\big) + \int_{t_0}^{T} \frac{\dot{f}}{1+\gamma} |w|^{1+\gamma}\,dt = -G\big(w(t_0)\big) = -\frac{c^2}{2}. \tag{22}$$

Using (21), we can estimate the left hand side of (22) from below, where T is chosen sufficiently large so that $G(w(T)) \leqslant M_0 + 1$:

$$\begin{aligned} -G\big(w(T)\big) + \int_{t_0}^{T} \frac{\dot{f}}{1+\gamma} |w|^{1+\gamma}\,dt &\geqslant -\limsup G\big(w(t)\big) - \int_{t_0}^{\infty} \frac{\dot{f}_-}{1+\gamma} |w|^{\gamma+1}\,dt \\ &\geqslant -M_0 - 1 - (l_1 c^2 + l_2) \int_{t_0}^{\infty} \dot{f}_-(t)\,dt \end{aligned}$$

which contradicts the right hand side of (22) if we choose c and t_0 sufficiently large such that

$$l_1 \int_{t_0}^{\infty} \dot{f}_-(t)\,dt < \frac{1}{4}$$

and

$$\frac{c^2}{4} > M_0 + 1 + l_2 \int_{t_0}^{\infty} \dot{f}_-(t)\,dt.$$

For the remaining case when $\int_0^{\infty} \phi'_+(x)\,dx = \infty$, we note by (9) and $\phi'_-(x) \in L^1(0,\infty)$ that there always exists a $X_2 > X_1 \geqslant X_0$ such that

$$4\phi_0(x) \geqslant \phi(x) \geqslant \phi_0(x) \tag{23}$$

for any $x > X_2$, where

$$\phi_0(x) = \frac{1}{2}\phi(0) + \frac{1}{2} \int_0^x \phi'_+(x)\,dx.$$

Using the same argument as in (12) we again deduce that $\lim_{x\to\infty} y'(x) = 0$. Hence (13) remains valid. Now integrate (13) from X_2 to x to obtain

$$y(x) \geqslant \int_{X_2}^{x} dv \int_{x}^{\infty} a(s) y^{\gamma}(s)\,ds \geqslant (x - X_2) y^{\gamma}(x) \int_{x}^{\infty} a(s)\,ds$$

$$\geqslant (x - X_2) y^\gamma(x) \int_x^\infty \phi_0(x) s^{-(\gamma+3)/2}\, ds$$
$$= \frac{2\phi_0(x)}{\gamma+1}(x - X_2) y^\gamma(x) x^{-(1+\gamma)/2}. \tag{24}$$

For $x \geqslant 2X_2$, it then follows from (24) that

$$y(x) \leqslant K_2(x) x^{1/2}, \tag{25}$$

where

$$K_2(x) = \left(\frac{1+\gamma}{\phi_0(x)}\right)^{1/(\gamma-1)}.$$

On the other hand, since $\phi_0'(x) \geqslant 0$, we find by (25)

$$\left|y'(x)\right| = \left|\int_x^\infty a(s) y^\gamma(s)\, ds\right| = \left|\int_x^\infty \phi(s) y^\gamma(s) s^{-(\gamma+3)/2}\, ds\right|$$
$$\leqslant \left|\int_x^\infty 4\phi_0(s)\left(\frac{1+\gamma}{\phi_0(x)}\right)^{1/(\gamma-1)} s^{-3/2}\, ds\right|$$
$$\leqslant 8(1+\gamma)^{\gamma/(\gamma-1)}\left(\frac{1}{\phi_0(x)}\right)^{1/(\gamma-1)} x^{-1/2}. \tag{26}$$

Using (18) and the fact that $\lim_{x\to\infty}\phi(x) = \infty$, we can deduce from (25) and (26) that

$$w^{(j)}(t) = o(1) \tag{27}$$

as $t \to \infty$ for $j = 0, 1$. Furthermore, by (25) we also have

$$\frac{f(t)}{1+\gamma}\left|w(t)\right|^{1+\gamma} = \frac{\phi(x)}{1+\gamma}\left(K_2(x)\right)^{1+\gamma} \leqslant \frac{4\phi_0(x)}{1+\gamma}\left(\frac{1+\gamma}{\phi_0(x)}\right)^{(1+\gamma)/(\gamma-1)} \to 0 \tag{28}$$

as $x \to \infty$. By (27) and (28) it follows from (6) that

$$\lim_{t\to\infty} G\left(w(t)\right) = 0. \tag{29}$$

But on the other hand, using (7) and (8) we have

$$\frac{dG(w(t))}{dt} \geqslant -\dot{f}_-(t)\left(m_1 G\left(w(t)\right) + m_2\right). \tag{30}$$

An integration of (30) gives

$$G\left(w(t)\right) \geqslant \left\{G\left(w(t_0)\right) - m_2 \int_{t_0}^t \dot{f}_-(t)\, dt\right\} \exp\left(-m_1 \int_{t_0}^t \dot{f}_-(t)\, dt\right)$$
$$\geqslant \left\{\frac{c^2}{2} - m_2 \int_{t_0}^t \dot{f}_-(t)\, dt\right\} \exp\left(-m_1 \int_{t_0}^t \dot{f}_-(t)\, dt\right)$$

which is bounded away from zero if we choose c sufficiently large for fixed t_0. This contradicts (29) and this completes the proof of Theorem 1 for the superlinear case.

For the sublinear case when $0 < \gamma < 1$, again let $y(x)$ be a solution of (1) satisfying (3). We shall prove that $y(x)$ is oscillatory by choosing x_0 sufficiently large and c sufficiently small. If (11) holds, we choose c and x_0 such that

$$c^{1-\gamma} < \min\left(1, \frac{1-\gamma}{2(1+\gamma)} k^{2/(1-\gamma)}\right) \tag{31}$$

and

$$K^{(1+\gamma)/(1-\gamma)} \int_{t_0}^{\infty} \dot{f}_- \, dt \leqslant \frac{k^{2/(1-\gamma)}}{2} \leqslant \frac{1}{2} f^{2/(1-\gamma)}(t_0), \tag{32}$$

where $t_0 = \log x_0$. Upon these choices we claim first that

$$w(t) < cf(t)^{1/(1-\gamma)} \tag{33}$$

for any $t \geqslant t_0$. Assume to contrary that there exists a point t_1 such that

$$w(t_1) = cf(t_1)^{1/(1-\gamma)} \tag{34}$$

and

$$w(t) < cf(t)^{1/(1-\gamma)} \tag{35}$$

for $t_0 \leqslant t < t_1$. Then by (31) and (34) we have from (6)

$$G\big(w(t_1)\big) \geqslant \frac{f(t_1)}{1+\gamma} \big|w(t_1)\big|^{1+\gamma} - \frac{1}{8} w^2(t_1) > \frac{7-\gamma}{8(1+\gamma)} c^{1+\gamma} f^{2/(1-\gamma)}(t_1). \tag{36}$$

But on the other hand, for any $t \in [t_0, t_1]$, it follows from (7) and (35)

$$\frac{dG(w(t))}{dt} \leqslant \frac{\dot{f}_+}{1+\gamma} \big|w(t)\big|^{\gamma+1} \leqslant \frac{c^{1+\gamma}}{1+\gamma} \dot{f}_+ f^{(1+\gamma)/(1-\gamma)}(t). \tag{37}$$

An integration of (37) yields

$$\begin{aligned} G\big(w(t)\big) \leqslant{} & G\big(w(t_0)\big) + \frac{1-\gamma}{2(1+\gamma)} c^{1+\gamma} \big(f^{2/(1-\gamma)}(t) - f^{2/(1-\gamma)}(t_0)\big) \\ & + \int_{t_0}^{t} \frac{c^{1+\gamma}}{1+\gamma} \left(\frac{1-\gamma}{2}\right) \dot{f}_-(t) f^{(1+\gamma)/(1-\gamma)} \, dt. \end{aligned} \tag{38}$$

Using (31) and (32), we have from (3) and (6)

$$G\big(w(t_0)\big) = \frac{c^2}{2} \leqslant \frac{1-\gamma}{4(1+\gamma)} c^{\gamma+1} f^{2/(1-\gamma)}(t_0) \tag{39}$$

and

$$\int_{t_0}^{\infty} \frac{1-\gamma}{2(1+\gamma)} \dot{f}_-(t) f^{(1+\gamma)/(1-\gamma)} \, dt \leqslant \frac{1-\gamma}{4(1+\gamma)} \; f^{2/(1-\gamma)}(t_0). \tag{40}$$

Substituting both (39) and (40) into (38) gives

$$G\big(w(t)\big) \leqslant \frac{1-\gamma}{2(1+\gamma)} c^{1+\gamma} f^{2/(1-\gamma)}(t). \tag{41}$$

In particular, we have

$$G\big(w(t_1)\big) \leqslant \frac{1-\gamma}{2(1+\gamma)} c^{1+\gamma} f^{2/(1-\gamma)}(t_1) < \frac{7-\gamma}{8(1+\gamma)} c^{1+\gamma} f^{2/(1-\gamma)}(t_1)$$

which contradicts (36), and this proves that $w(t) < cf(t)^{1/(1-\gamma)}$ for any $t \geqslant t_0$.

If (11) fails, i.e., $\lim_{x\to\infty} \phi(x) = \lim_{t\to\infty} f(t) = \infty$, we note by (23) there exists $T > 0$ such that

$$f_0(t) \leqslant f(t) \leqslant 4 f_0(t), \quad t \geqslant T,$$

where $f_0(t) = \phi_0(x)$, $\dot{f}_0(t) \geqslant 0$ and $\lim_{t\to\infty} f_0(t) = \infty$ by (23). Now we choose c and $t_0 \geqslant T$ such that (31) holds and

$$4^{(1+\gamma)/(1-\gamma)} \int_{t_0}^{\infty} \frac{1}{1+\gamma}\left(\frac{1-\gamma}{2}\right) \dot{f}_-(t)\, dt < \frac{3}{8}. \tag{42}$$

We again claim that (33) holds. Otherwise there exists a point t_1 so that all of (34)–(36) are true. But from (38) we have by (31), (39) and (42)

$$\begin{aligned}
G\big(w(t)\big) &\leqslant \frac{c^2}{2} + \frac{1-\gamma}{2(1+\gamma)} c^{1+\gamma}\big(f^{2/(1-\gamma)}(t) - f^{2/(1-\gamma)}(t_0)\big) \\
&\quad + 4^{(1+\gamma)/(1-\gamma)} f_0^{(1+\gamma)/(1-\gamma)}(t) \int_{t_0}^{t} \frac{c^{1+\gamma}}{1+\gamma}\left(\frac{1-\gamma}{2}\right) \dot{f}_-(t)\, dt \\
&\leqslant \frac{c^2}{2} + \frac{1-\gamma}{2(1+\gamma)} c^{1+\gamma}\big(f^{2/(1-\gamma)}(t) - f^{2/(1-\gamma)}(t_0)\big) \\
&\quad + 4^{(1+\gamma)/(1-\gamma)} f^{(1+\gamma)/(1-\gamma)}(t) \int_{t_0}^{t} \frac{c^{1+\gamma}}{1+\gamma}\left(\frac{1-\gamma}{2}\right) \dot{f}_-(t)\, dt \\
&\leqslant \frac{1-\gamma}{2(1+\gamma)} c^{1+\gamma} f^{2/(1-\gamma)}(t) \\
&\quad + 4^{(1+\gamma)/(1-\gamma)} f^{2/(1-\gamma)} \int_{t_0}^{t} \frac{c^{1+\gamma}}{1+\gamma}\left(\frac{1-\gamma}{2}\right) \dot{f}_-(t)\, dt \\
&< \frac{7-\gamma}{8(1+\gamma)} c^{1+\gamma} f^{2/(1-\gamma)}(t)
\end{aligned}$$

which contradicts (36) at the point $t = t_1$. This again proves that $w(t) < cf(t)^{1/(1-\gamma)}$ for any $t \geqslant t_0$ when $\lim_{t\to\infty} f(t) = \infty$. Now $y(x)$ satisfies the linear equation $z'' + a(x)|y(x)|^{\gamma-1} z = 0$. Observe that

$$a(x)\big|y(x)\big|^{\gamma-1} = a(x) x^{(\gamma-1)/2}\big|w(t)\big|^{\gamma-1} > x^{-2} c^{\gamma-1}.$$

Since $c^{1-\gamma} < 1$, so $c^{\gamma-1} > 1$ and $z'' + x^{-2}c^{\gamma-1}z = 0$ is oscillatory. Therefore $y(x)$ must be oscillatory. This completes the proof of Theorem 1. $\square$

Proof of Corollary 1. We note that for $x \geqslant x_0 > 0$

$$k - \phi(x_0) \leqslant \phi(x) - \phi(x_0) \leqslant \int_{x_0}^{x} \phi'_+(x)\,dx - \int_{x_0}^{x} \phi'_-(x)\,dx. \tag{43}$$

Let $M_1 = \int_{x_0}^{\infty} \phi'_+(x)\,dx$. It is easy to see by (2) that

$$\int_{x_0}^{\infty} \phi'_-(x)\,dx \leqslant M_1 + \phi(x_0).$$

We can now apply Theorem 1 to obtain the desired conclusion. $\square$

Proof of Corollary 2. Let $\phi(x)$ be nonincreasing. Then $\phi'_+(x) \equiv 0$, so the conclusion follows from Corollary 1. Next consider the case when $\phi(x)$ is nondecreasing, so $\phi'_-(x) \equiv 0$. The conclusion follows from Theorem 1. $\square$

3. In this section, we give examples and remarks which relate our theorem to that of earlier results.

Remark 1. Theorem 1 is clearly false when $\gamma = 1$ as can be seen in the case of the Euler equation $y''(x) + \frac{1}{4}x^{-2}y = 0$.

Remark 2. Take $a(x) = x^{-(\gamma+3)/2}\{2 + \sin x/x^2\}$. Here $\phi(x)$ satisfies the assumptions of Theorem 1 or Corollary 1 since $\phi'(x) = \{\cos x/x^2 - 2\sin x/x^3\} \in L^1[x_0, \infty)$ for any $x_0 > 0$. Note that for any positive integer n

$$\phi(n\pi) = \phi\big((n+1)\pi\big) = 2 \quad \text{and} \quad \phi\left((2n \pm 1)\frac{\pi}{2}\right) = 2 \pm \frac{4}{(2n+1)^2\pi^2},$$

so $\phi(x)$ is not monotone and none of Theorems A, B, C and D is applicable.

Remark 3. Consider $\phi(x) = \phi_1(x) + \phi_2(x)$, where $\phi_1(x) = x^{-2}\sin x$ and $\phi_2(x)$ is defined by

$$\phi_2(x) = \begin{cases} \frac{1}{\pi}x - n, & 2n\pi \leqslant x \leqslant (2n+1)\pi, \\ n+1, & 2(n+1)\pi \leqslant x \leqslant (2n+2)\pi, \end{cases}$$

where n is any positive integer. Here $\lim_{x\to\infty}\phi(x) = \infty$, $\phi'_+(x) \notin L^1(0,\infty)$, $\phi(x)$ is not monotone but $\phi'_-(x) \in L^1(0,\infty)$, so the equation

$$y''(x) + x^{-(\gamma+3)/2}\phi(x)|y|^{\gamma}\operatorname{sgn} y = 0, \quad \gamma > 0,\ \gamma \neq 1,$$

has oscillatory solutions.

Remark 4. The condition that $\phi(x)$ be bounded away from zero cannot be removed. Let $\delta > 0$ and $\psi(x) = \phi(x)x^{\delta}$. It is known that if $\psi(x)$ is nonincreasing then Eq. (1) is nonoscillatory, i.e., it does not have oscillatory solutions. See Kiguradze [6] for $\gamma > 1$ and Wong [8] for $0 < \gamma < 1$.

It is also interesting to note that the main Theorem 1 complements a recent result of the second author, namely,

Theorem E (Wong [9]). *Let $\psi(x) = a(x)x^{(\gamma+3)/2+\delta}$, where $\delta > 0$ and $\gamma \neq 1$. If $\psi(x) \geqslant k > 0$ for $x \geqslant x_0 > 0$ and $\psi'_+(x) \in L^1(0,\infty)$ where $\psi'_+(x) = \max(\psi'(x), 0)$, then Eq.* (1) *is nonoscillatory.*

Note that the assumptions on $\psi(x)$ in Theorem E imply in fact that $\psi'(x) \in L^1(0,\infty)$.

References

[1] K.L. Chiou, The existence of oscillatory solutions for the equation $y'' + q(t)y^{\gamma} = 0$, $0 < \gamma < 1$, Proc. Amer. Math. Soc. 35 (1972) 120–122.
[2] L.H. Erbe, J.S. Muldowney, On the existence of oscillatory solutions to nonlinear differential equations, Ann. Mat. Pure Appl. 109 (1976) 23–38.
[3] J.W. Heidel, Uniqueness, continuation and nonoscillation for a second order nonlinear differential equation, Pacific J. Math. 32 (1970) 715–721.
[4] J.W. Heidel, D.B. Hinton, Existence of oscillatory solutions for a nonlinear differential equation, SIAM J. Math. Anal. 3 (1972) 344–351.
[5] M. Jansy, On the existence of an oscillatory solution of nonlinear differential equation of second order $y''(x) + f(x)y^{2n-1} = 0$, $f(x) > 0$, Căsopis Pěst Mat. 85 (1960) 73–83, in Russian.
[6] I.T. Kiguradze, A note on the oscillation of solution of equation $u'' + a(t)|u|^n \operatorname{sgn} u = 0$, Căsopis Pěst Mat. 92 (1967) 343–350, in Russian.
[7] J. Kurzweil, A note on oscillatory solutions of the equation $y'' + f(x)y^{2n-1} = 0$, Căsopis Pěst Mat. 85 (1960) 357–358, in Russian.
[8] J.S.W. Wong, A nonoscillation theorem for sublinear Emden–Fowler equation, Anal. Appl., to appear.
[9] J.S.W. Wong, A nonoscillation theorem for Emden–Fowler equations, J. Math. Anal. Appl. (2003), in press.