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Generalized binomial expansion on Lorentz cones ☆

Hongming Ding

Department of Mathematics and Mathematical Computer Science, University of St. Louis, St. Louis, MO 63103, USA

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Abstract

We study some properties of generalized binomial coefficients for symmetric cones and we obtain a generalized binomial expansion formula for Lorentz cones.

Keywords: Jordan algebra; Symmetric cone; Lorentz cone; Spherical polynomial; Spherical function; Binomial expansion

1. Introduction

Let Ω be an irreducible symmetric cone and let J be the associated Jordan algebra. Denote the rank of Ω by r and the dimension of J by n. Fix a Jordan frame $\{e_1, \ldots, e_r\}$ in J and define the following subspaces: $V_j = \{x \in J\colon\ e_j \circ x = x\}$ and $V_{ij} = \{x \in J\colon\ e_i \circ x = (1/2)x$ and $e_j \circ x = (1/2)x\}$, where $\circ$ is the Jordan product in J. Then $V_j = \mathbb{R}e_j$ for $j = 1, \ldots, r$ are 1-dimensional subalgebras of J, while the subspaces V_{ij} of J for $i, j = 1, \ldots, r$ with $i \neq j$ all have a common dimension d. It follows that $n = r + (d/2)r(r-1)$ and

$$\frac{n}{r} = 1 + \frac{d}{2}(r-1). \tag{1.1}$$

For $j = 1, \ldots, r$, let $E_j = e_1 + \cdots + e_j$, and set $J_j = \{x \in J\colon\ E_j \circ x = x\}$. Then J_j is a subalgebra of J of rank j. Denote by P_j the orthogonal projection of J onto J_j and define

$$\Delta_j(x) = \delta_j(P_j x) \tag{1.2}$$

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E-mail address: dingh@slu.edu.

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for $x \in J$, where δ_j denotes the Koecher norm, or the determinant function for J_j. Then Δ_j is a polynomial on J that is homogeneous of degree j. We call $\Delta_j(x)$ the jth principal minor of x.

Let $\lambda_1, \dots, \lambda_r$ be complex numbers, and define the function Δ_λ on J by

$$\Delta_\lambda(x) = \Delta(x)^{\lambda_r} \prod_{j=1}^{r-1} \Delta_j(x)^{\lambda_j - \lambda_{j+1}}, \tag{1.3}$$

where $\Delta(x) = \Delta_r(x)$ is the Koecher norm function on J. The function Δ_λ is the generalized power function on J. In particular, when $\lambda_j = m_j$ are integers for all $j = 1, \dots, r$ and $m_1 \geqslant \cdots \geqslant m_r \geqslant 0$, $\mathbf{m} = (m_1, \dots, m_r)$ is called a partition, and we write $\mathbf{m} \geqslant 0$. The length $|\mathbf{m}|$ of $\mathbf{m}$ is defined by $|\mathbf{m}| = m_1 + \cdots + m_r$, and $\Delta_{\mathbf{m}}$ becomes a polynomial function on J which is homogeneous of degree $|\mathbf{m}|$. Let $G(\Omega)$ denote the automorphism group of Ω and let G be the connected component of the identity in $G(\Omega)$. Then G acts transitively on $\Omega \cong G/K$, where K is the stability group of identity element e in Ω. In fact, K is actually a maximal compact subgroup of G. For each partition $\mathbf{m}$, the spherical polynomial of weight $\mathbf{m}$ on Ω may be defined by

$$\Phi_{\mathbf{m}}(x) = \int_K \Delta_{\mathbf{m}}(k \cdot x)\, dk, \tag{1.4}$$

where dk is the Haar measure on K. The algebra of all K-invariant polynomials on J, denoted by $P(J)^K$, decomposes as

$$P(J)^K = \sum_{\mathbf{m}} \bigoplus \mathbb{C}\Phi_{\mathbf{m}}.$$

Let $\rho = (\rho_1, \dots, \rho_r)$ be an r-tuple given by $\rho_j = (d/4)(2j - r - 1)$ for $j = 1, \dots, r$, and define the spherical function ϕ_λ on Ω for $\lambda \in \mathbb{C}^r$ by

$$\phi_\lambda(x) = \int_K \Delta_{\lambda+\rho}(k \cdot x)\, dk. \tag{1.5}$$

It is clear from (1.4) and (1.5) that $\Phi_{\mathbf{m}}$ and ϕ_λ are K-invariant and for any partition $\mathbf{m}$, $\Phi_{\mathbf{m}} = \phi_{\mathbf{m}-\rho}$. Moreover, $\phi_\lambda = \phi_{\lambda'}$ if and only if there exists a permutation w such that $\lambda' = w\lambda$.

For a fixed Jordan frame $\{e_1, \dots, e_r\}$, any x in J can be written as

$$x = k \cdot a, \quad k \in K,\ a = \sum_{j=1}^{r} a_j e_j, \tag{1.6}$$

where $a_1, \dots, a_r$ are called the eigenvalues of x, and we may assume $a_1 \geqslant \cdots \geqslant a_r$ for the uniqueness. If $x \in \Omega$, then all $a_1, \dots, a_r$ are positive. Since the functions $\Phi_{\mathbf{m}}$ and ϕ_λ are K-invariant, they depend only on the eigenvalues $a_1, \dots, a_r$ of $x \in \Omega$, and we may write $\Phi_{\mathbf{m}}(x) = \Phi_{\mathbf{m}}(a_1, \dots, a_r)$. Let $J^{\mathbb{C}}$ be the complexification of J. Then every element $z \in J^{\mathbb{C}}$ has a spectral decomposition

$$z = u \cdot (a_1 e_1 + \cdots + a_r e_r), \tag{1.7}$$

where $u \in U$, a compact subgroup of $GL(J^{\mathbb{C}})$, $a_1 \geqslant \cdots \geqslant a_r \geqslant 0$. $|z| = a_1$ is called the spectral norm of $z \in J^{\mathbb{C}}$. The open unit ball D of $J^{\mathbb{C}}$ is defined by

$$D = \{z \in J^{\mathbb{C}}\colon |z| < 1\}. \tag{1.8}$$

The generalized binomial expansion on the symmetric cone Ω is [4, p. 343]

$$\Phi_{\mathbf{m}'}(e+x) = \sum_{\mathbf{m} \geqslant 0} \binom{\mathbf{m}'}{\mathbf{m}} \Phi_{\mathbf{m}}(x), \tag{1.9}$$

where $\mathbf{m}'$ is a partition. Here, the binomial coefficients $\binom{\mathbf{m}'}{\mathbf{m}}$ on symmetric cones Ω are generalizations of the usual binomial coefficients

$$\binom{m}{n} = \frac{m!}{n!(m-n)!}$$

which arises in the expansion

$$(1+x)^m = \sum_{n=0}^{m} \binom{m}{n} x^n \tag{1.10}$$

on the real line. It is known [5,12] that $\binom{\mathbf{m}'}{\mathbf{m}} = 0$ unless $\mathbf{m} \leqslant \mathbf{m}'$; i.e., if $\mathbf{m}' = (m_1', \ldots, m_r')$ and $\mathbf{m} = (m_1, \ldots, m_r)$, then $m_j \leqslant m_j'$ for $j = 1, \ldots, r$. Therefore, the sum in (1.9) has only finitely many terms $\Phi_{\mathbf{m}}(x)$ for which $\mathbf{m} \leqslant \mathbf{m}'$.

Because of importance in analysis and other mathematical areas, generalized binomial coefficients have brought attentions to mathematicians; e.g., they are discussed in [5–9,12]. By the spherical Taylor formula [4, XII.1],

$$\begin{aligned} \binom{\mathbf{m}'}{\mathbf{m}} &= d_{\mathbf{m}} \frac{1}{(n/r)_{\mathbf{m}}} \Phi_{\mathbf{m}}\left(\frac{\partial}{\partial x}\right) \Phi_{\mathbf{m}'}(e+x)\Big|_{x=0} \\ &= d_{\mathbf{m}} \frac{1}{(n/r)_{\mathbf{m}}} \Phi_{\mathbf{m}}\left(\frac{\partial}{\partial x}\right) \Phi_{\mathbf{m}}'(x)\Big|_{x=e}, \end{aligned} \tag{1.11}$$

where $d_{\mathbf{m}}$ is the dimension of the space $\mathcal{P}_{\mathbf{m}}$ of polynomials on J generated by $\Delta_{\mathbf{m}}$. In [3, Lemma 4.4], we generalized the binomial coefficient $\binom{\mathbf{m}'}{\mathbf{m}}$ for a partition $\mathbf{m}'$ to $\binom{\lambda}{\mathbf{m}}$ for $\lambda \in \mathbb{C}^r$. In [3, Proposition 4.5], we gave an integral expression for this binomial coefficient (cf. (2.6) below). However, still very little is known about the explicit form of these coefficients.

By classification, there are four families of classical irreducible symmetric cones $\Pi_r(\mathbb{R})$, $\Pi(\mathbb{C})$, $\Pi_r(\mathbb{H})$, the cones of all $r \times r$ positive definite matrices over $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$, the Lorentz cones Λ_n, and an exceptional cone $\Pi_3(\mathbb{O})$ [4]. In [1], we found an explicit binomial expansion formula for symmetric cones $\Pi_r(\mathbb{C})$ and computed the spherical transform of $\phi_{-\lambda}(I_r + x)$ for $\Pi_r(\mathbb{C})$, where I_r is the $r \times r$ identity matrix. In this paper, we study some properties of the binomial coefficients for symmetric cones Ω and we obtain an explicit binomial expansion for Lorentz cones Λ_n. These results would be interesting for analysis on symmetric cones, in particular, for analysis on Lorentz cones.

2. Properties of the binomial coefficients for symmetric cones

Let $\lambda = (\lambda_1, \dots, \lambda_r) \in \mathbb{C}^r$, $\Delta_\lambda(x)$ be the generalized power function defined by (1.3), $d_* x = \Delta(x)^{-n/r}\, dx$, where dx is the Lebesgue measure on Ω, and let $\operatorname{tr} x$ be the trace of x. The gamma function Γ_Ω for the cone Ω is defined by

$$\Gamma_\Omega(\lambda) = \int_\Omega e^{-\operatorname{tr} x} \Delta_\lambda(x)\, d_* x \tag{2.1}$$

whenever the integral converges absolutely. By [4, Theorem VII.1.1], the integral in (2.1) converges absolutely if and only if $\operatorname{Re} \lambda_j > (j-1)d/2$ for $j = 1, 2, \dots, r$. In this range, $\Gamma_\Omega(\lambda)$ can be calculated by

$$\Gamma_\Omega(\lambda) = (2\pi)^{(n-r)/2} \prod_{j=1}^{r} \Gamma\left(\lambda_j - (j-1)\frac{d}{2}\right), \tag{2.2}$$

and (2.2) defines the meromorphic continuation Γ_Ω to all of $\mathbb{C}^r$. Note that $1/\Gamma_\Omega(\lambda)$ is an entire function on $\mathbb{C}^r$.

Recall that for $\alpha \in \mathbb{C}$ and nonnegative integer j, the classical Pochhammer symbol $(\alpha)_j$ is defined by

$$(\alpha)_j = \frac{\Gamma(\alpha + j)}{\Gamma(\alpha)} = \alpha(\alpha+1)\dots(\alpha+j-1). \tag{2.3}$$

For $\lambda \in \mathbb{C}^r$ and $\mathbf{m}$, any partition the Pochhammer symbol $[\lambda]_{\mathbf{m}}$ for Ω is defined by

$$[\lambda]_{\mathbf{m}} = \frac{\Gamma_\Omega(\lambda + \mathbf{m})}{\Gamma_\Omega(\lambda)}. \tag{2.4}$$

It follows from (2.2) that

$$[\lambda]_{\mathbf{m}} = \prod_{j=1}^{r} \left(\lambda_j - (j-1)\frac{d}{2}\right)_{m_j}.$$

For $\mathbf{m}' = (m'_1, \dots, m'_r) \in \mathbb{C}^r$, a partition $\mathbf{m} = (m_1, \dots, m_r)$, and a positive integer k, define $\mathbf{m}' + k = (m'_1 + k, \dots, m'_r + k)$ and $\mathbf{m} + k = (m_1 + k, \dots, m_r + k)$. Then we have

Theorem 2.1.

$$\binom{\mathbf{m}' + k}{\mathbf{m} + k} = (-1)^{kr} \binom{\mathbf{m}'}{\mathbf{m}} \frac{[-\mathbf{m}' - k + 2\rho]_k}{[n/r + \mathbf{m}]_k}. \tag{2.5}$$

Proof. By [3, Proposition 4.5],

$$\binom{\mathbf{m}'}{\mathbf{m}} = \frac{(-1)^{|\mathbf{m}|}}{|\mathbf{m}|!\,\Gamma_\Omega(-\mathbf{m}' + 2\rho)} \int_\Omega e^{-\operatorname{tr} x} \frac{d_{\mathbf{m}} |\mathbf{m}|!}{[n/r]_{\mathbf{m}}} \Phi_{\mathbf{m}}(x) \phi_{\mathbf{m}' - \rho}(x^{-1})\, d_* x \tag{2.6}$$

for $\mathrm{Re}(m'_j - \rho_j) < (d/4)(r-1)$, $j = 1, \ldots, r$. Similarly, if $\mathrm{Re}(m'_j + k - \rho_j) < (d/4) \times (r-1)$, $j = 1, \ldots, r$, then

$$\binom{\mathbf{m}'+k}{\mathbf{m}+k} = \frac{(-1)^{|\mathbf{m}+k|}}{\Gamma_\Omega(-\mathbf{m}'-k+2\rho)} \times \int_\Omega e^{-\operatorname{tr} x} \frac{d_{\mathbf{m}+k}}{[n/r]_{\mathbf{m}+k}} \Phi_{\mathbf{m}+k}(x) \phi_{\mathbf{m}'+k-\rho}(x^{-1})\, d_* x. \tag{2.7}$$

It is easy to see that $|\mathbf{m}+k| = |\mathbf{m}| + kr$. Since $\Delta_{\mathbf{m}+k}(x) = \Delta_{\mathbf{m}}(x)(\Delta(x))^k$, and $\Delta(x)$ is invariant under the subgroup K,

$$\Phi_{\mathbf{m}+k}(x) = \Phi_{\mathbf{m}}(x)\big(\Delta(x)\big)^k. \tag{2.8}$$

By the same reason,

$$\phi_{\mathbf{m}'+k-\rho}(x^{-1}) = \phi_{\mathbf{m}'-\rho}(x^{-1})\big(\Delta(x^{-1})\big)^k = \frac{\phi_{\mathbf{m}'-\rho}(x^{-1})}{(\Delta(x))^k}. \tag{2.9}$$

By a formula for $d_{\mathbf{m}}$ [4, p. 315], $d_{\mathbf{m}+k} = d_{\mathbf{m}}$. It follows from (2.6)–(2.9) that

$$\binom{\mathbf{m}'+k}{\mathbf{m}+k} = \frac{(-1)^{kr}\Gamma_\Omega(-\mathbf{m}'+2\rho)[n/r]_{\mathbf{m}}}{\Gamma_\Omega(-\mathbf{m}'-k+2\rho)[n/r]_{\mathbf{m}+k}} \binom{\mathbf{m}'}{\mathbf{m}}. \tag{2.10}$$

By (2.10), (2.4), and a calculation, (2.5) holds for $\mathrm{Re}(m'_j + k - \rho_j) < (d/4)(r-1)$, $j = 1, \ldots, r$. Since $\binom{\mathbf{m}'}{\mathbf{m}}$ is a polynomial in $\mathbf{m}' \in \mathbb{C}^r$ [3, Lemma 4.4], (2.5) holds for all $\mathbf{m}' \in \mathbb{C}^r$. $\square$

We now fix d, and let $\mathbf{m}' = (m'_1, \ldots, m'_r) \in \mathbb{C}^r$ and $\mathbf{m} = (m_1, \ldots, m_r)$ be a partition.

Theorem 2.2. *The binomial coefficients satisfy*

$$\binom{(m'_1, \ldots, m'_r, 0)}{(m_1, \ldots, m_r, 0)} = \binom{\mathbf{m}'}{\mathbf{m}}$$

and

$$\binom{(m'_1, \ldots, m'_r, 0)}{(m_1, \ldots, m_r, m_{r+1})} = 0$$

if $m_{r+1} \neq 0$.

Proof. Let Ω be an irreducible symmetric cone of rank $r+1$ and let $x_1, \ldots, x_r, x_{r+1}$ be eigenvalues of $x \in \Omega$ defined by (1.6). By [10, Theorem 5.3],

$$\begin{aligned} &\Phi_{(m'_1,\ldots,m'_r,0)}(x_1, \ldots, x_r, x_{r+1}) \\ &\quad = \frac{\Gamma(d(r+1)/2)}{(\Gamma(d/2))^{r+1}} \prod_{i<j} (x_i - x_j)^{1-d} \int_{x_{r+1}}^{x_r} \ldots \int_{x_2}^{x_1} \Phi_{\mathbf{m}'}(\mu_1, \ldots, \mu_r) \\ &\qquad \times \prod_{i=1}^{r} \prod_{j=1}^{r+1} |\mu_i - x_j|^{d/2-1} \prod_{i<p} (\mu_i - \mu_p)\, d\mu_1 \ldots d\mu_r \end{aligned} \tag{2.11}$$

for a partition $\mathbf{m}' = (m'_1, \ldots, m'_r)$. We rewrite the r-tuple ρ as ρ^r; i.e., $\rho^r = (\rho^r_1, \ldots, \rho^r_r)$, where $\rho^r_j = (d/4)(2j - r - 1)$ for all $j = 1, \ldots, r$, and we introduce an $(r+1)$-tuple ρ^{r+1}; i.e., $\rho^{r+1} = (\rho^{r+1}_1, \ldots, \rho^{r+1}_r, \rho^{r+1}_{r+1})$, where $\rho^{r+1}_j = (d/4)(2j - r - 2)$. Since $\Phi_{\mathbf{m}} = \phi_{\mathbf{m}-\rho}$, we may rewrite (2.11) as

$$\begin{aligned}
&\phi_{(m'_1-\rho^{r+1}_1,\ldots,m'_r-\rho^{r+1}_r,-\rho^{r+1}_{r+1})}(x_1, \ldots, x_r, x_{r+1}) \\
&\quad = \frac{\Gamma(d(r+1)/2)}{(\Gamma(d/2))^{r+1}} \prod_{i<j} (x_i - x_j)^{1-d} \int_{x_{r+1}}^{x_r} \cdots \int_{x_2}^{x_1} \phi_{\mathbf{m}'-\rho^r}(\mu_1, \ldots, \mu_r) \\
&\qquad \times \prod_{i=1}^{r} \prod_{j=1}^{r+1} |\mu_i - x_j|^{d/2-1} \prod_{i<p} (\mu_i - \mu_p)\, d\mu_1 \ldots d\mu_r. \qquad (2.12)
\end{aligned}$$

By the generalized Carlson's theorem [3, Corollary 3.11], (2.12) holds for all $\mathbf{m}' \in \mathbb{C}^r$. (The proof of this fact is simple and is almost identical to the proof of [1, Lemma 2].) It follows that

$$\begin{aligned}
&\phi_{(m'_1-\rho^{r+1}_1,\ldots,m'_r-\rho^{r+1}_r,-\rho^{r+1}_{r+1})}(1 + x_1, \ldots, 1 + x_r, 1 + x_{r+1}) \\
&\quad = \frac{\Gamma(d(r+1)/2)}{(\Gamma(d/2))^{r+1}} \prod_{i<j} \bigl(1 + x_i - (1 + x_j)\bigr)^{1-d} \int_{1+x_{r+1}}^{1+x_r} \cdots \int_{1+x_2}^{1+x_1} \phi_{\mathbf{m}'-\rho^r}(\mu_1, \ldots, \mu_r) \\
&\qquad \times \prod_{i=1}^{r} \prod_{j=1}^{r+1} \bigl|\mu_i - (1 + x_j)\bigr|^{d/2-1} \prod_{i<p} (\mu_i - \mu_p)\, d\mu_1 \ldots d\mu_r \\
&\quad = \frac{\Gamma(d(r+1)/2)}{(\Gamma(d/2))^{r+1}} \prod_{i<j} (x_i - x_j)^{1-d} \int_{x_{r+1}}^{x_r} \cdots \int_{x_2}^{x_1} \phi_{\mathbf{m}'-\rho^r}(1 + t_1, \ldots, 1 + t_r) \\
&\qquad \times \prod_{i=1}^{r} \prod_{j=1}^{r+1} |t_i - x_j|^{d/2-1} \prod_{i<p} (t_i - t_p)\, dt_1 \ldots dt_r. \qquad (2.13)
\end{aligned}$$

Theorem 2.2 follows now from (2.13). □

3. Binomial expansion for Lorentz cones

The only cone of rank 1 is $\mathbb{R}^+$ and $\Phi_m(x) = x^m$. If $m' \in \mathbb{C}$, then

$$(1 + x)^{m'} = \sum_{m=0}^{\infty} \binom{m'}{m} x^m = \sum_{m=0}^{\infty} \frac{m'(m'-1)\ldots(m'-m+1)}{m!} x^m \qquad (3.1)$$

holds in the interval $-1 < x < 1$. In particular, if m' is a positive integer, then the series (3.1) has only finitely many terms and it becomes (1.10).

It is known [4] that the rank of a Lorentz cone Λ_n is 2, the dimension is n, $d = n - 2$, $\rho_1 = -d/4$, and $\rho_2 = d/4$. The associated Jordan algebra with Λ_n is $J = \mathbb{R}^n$, and the complexification $J^{\mathbb{C}}$ of J is $\mathbb{C}^n$. As a special case of (1.7), every element $z \in J^{\mathbb{C}}$ has a spectral decomposition

$$z = u \cdot (a_1 e_1 + a_2 e_2), \tag{3.2}$$

where $\{e_1, e_2\}$ is a fixed Jordan frame, $u \in U = \mathrm{SO}(n) \times \mathrm{SO}(2)$, $a_1 \geqslant a_2 \geqslant 0$. By [4, Theorem XII.1.1],

$$\left|\Phi_{\mathbf{m}}(z)\right| \leqslant a_1^{m_1} a_2^{m_2} \tag{3.3}$$

for z given by (3.2).

By Theorem 2.2, if m_1' is a positive integer, then

$$\Phi_{(m_1',0)}(1 + x_1, 1 + x_2) = \sum_{m_1=0}^{m_1'} \binom{m_1'}{m_1} \Phi_{(m_1,0)}(x_1, x_2). \tag{3.4}$$

More generally,

$$\phi_{(m_1'+d/4,-d/4)}(1 + x_1, 1 + x_2) = \sum_{m_1=0}^{\infty} \binom{m_1'}{m_1} \Phi_{(m_1,0)}(x_1, x_2), \tag{3.5}$$

where $m_1' \in \mathbb{C}$, x_1 and x_2 are eigenvalues of $x \in D$. The expansions (3.4) and (3.5) are the binomial expansions for the special index (m_1', m_2'), where $m_2' = 0$. For the general case $m_2' \neq 0$, we have

Theorem 3.1. *For $(m_1', m_2') \in \mathbb{C}^2$ and a nonnegative integer m_1, the binomial coefficients*

$$\binom{(m_1', m_2')}{(m_1, 0)} = \sum_{k=0}^{m_1} \binom{m_1' - m_2'}{k} 2^{m_1-k} \binom{m_2'}{m_1 - k} \prod_{i=k}^{m_1-1} \frac{d+i}{d+2i}. \tag{3.6}$$

Proof. By (1.5) and (1.3),

$$\begin{aligned} &\phi_{(m_1'+d/4,m_2'-d/4)}(1 + x_1, 1 + x_2) \\ &\quad = \phi_{(m_1'-m_2'+d/4,-d/4)}(1 + x_1, 1 + x_2)\big(\Delta(e + x)\big)^{m_2'}, \end{aligned} \tag{3.7}$$

where $x_1 \geqslant x_2 \geqslant 0$ are two eigenvalues of $x \in \Lambda_n$. By [4, Proposition XI.5.2],

$$\Delta(e + x) = 1 + 2\Phi_{(1,0)}(x) + \Phi_{(1,1)}(x).$$

By (3.3), $|2\Phi_{(1,0)}(x) + \Phi_{(1,1)}(x)| < 1$ in a neighborhood D' of 0 in $J^{\mathbb{C}}$. By the binomial formula on $\mathbb{R}$,

$$\begin{aligned} \big(\Delta(e + x)\big)^{m_2'} &= \big(1 + 2\Phi_{(1,0)}(x) + \Phi_{(1,1)}(x)\big)^{m_2'} \\ &= \sum_{j=0}^{\infty} \binom{m_2'}{j} 2^j \big(\Phi_{(1,0)}(x)\big)^j + \text{terms involving } \Phi_{(1,1)}(x) \end{aligned} \tag{3.8}$$

for $x \in D'$. By (3.5),

$$\phi_{(m_1'-m_2'+d/4,-d/4)}(1 + x_1, 1 + x_2) = \sum_{k \geqslant 0} \binom{m_1' - m_2'}{k} \Phi_{(k,0)}(x) \tag{3.9}$$

for $x \in D$. We now multiply (3.8) with (3.9) and observe that the product of any term involving $\Phi_{(1,1)}(x)$ in (3.8) with any term in (3.9) is a linear combination of such spherical polynomials $\Phi_{(m_1,m_2)}(x)$ for which $m_2 \neq 0$. Hence, to study the binomial coefficient $\binom{(m_1',m_2')}{(m_1,0)}$, we need only to multiply

$$\sum_{j=0}^{\infty} \binom{m_2'}{j} 2^j \big(\Phi_{(1,0)}(x)\big)^j$$

with (3.9).

By the recurrence formula [11],

$$\Phi_{(1,0)}(x)\Phi_{(k,0)}(x) = \frac{d+k}{d+2k}\Phi_{(k+1,0)}(x) + \frac{k}{d+2k}\Phi_{(k,1)}(x)$$

for a positive integer k. By mathematical induction,

$$\big(\Phi_{(1,0)}(x)\big)^j \Phi_{(k,0)}(x) = \Phi_{(k+j,0)}(x) \prod_{i=k}^{k+j-1} \frac{d+i}{d+2i} + \text{terms involving } \Phi_{(m_1,m_2)}(x) \text{ for which } m_2 > 0. \tag{3.10}$$

To compute $\binom{(m_1',m_2')}{(m_1,0)}$, we multiply

$$\sum_{j=0}^{\infty} \binom{m_2'}{j} 2^j \big(\Phi_{(1,0)}(x)\big)^j$$

with (3.9) and apply (3.10). Adding all terms with $k + j = m_1$ up and noticing $j = m_1 - k$, we prove (3.6). □

We are now ready to compute all binomial coefficients $\binom{(m_1',m_2')}{(m_1,m_2)}$ and derive the binomial formula for the Lorentz cones Λ_n.

By (2.5),

$$\begin{aligned}\binom{(m_1',m_2')}{(m_1,m_2)} &= \binom{(m_1'-m_2,m_2'-m_2)}{(m_1-m_2,0)} \frac{[-\mathbf{m}'+2\rho]_{m_2}}{[n/2+(m_1-m_2,0)]_{m_2}} \\ &= \binom{(m_1'-m_2,m_2'-m_2)}{(m_1-m_2,0)} \frac{(-m_1'-d/2)_{m_2}(-m_2')_{m_2}}{(m_1-m_2+d/2+1)_{m_2} m_2!}.\end{aligned} \tag{3.11}$$

By (3.6) and (3.11),

$$\begin{aligned}\binom{(m_1',m_2')}{(m_1,m_2)} = {} & \frac{(-m_1'-d/2)_{m_2}(-m_2')_{m_2}}{(m_1-m_2+d/2+1)_{m_2} m_2!} \\ & \times \sum_{k=0}^{m_1-m_2} \binom{m_1'-m_2'}{k} 2^{m_1-m_2-k} \binom{m_2'-m_2}{m_1-m_2-k} \\ & \times \prod_{i=k}^{m_1-m_2-1} \frac{d+i}{d+2i}.\end{aligned} \tag{3.12}$$

Theorem 3.2 (Binomial formula for Lorentz cones).

$$\phi_{(m_1'+d/4,m_2'-d/4)}(e+x)=\sum_{\mathbf{m}\geqslant 0}\binom{(m_1',m_2')}{(m_1,m_2)}\Phi_{\mathbf{m}}(x), \tag{3.13}$$

where the sum is over all partitions $\mathbf{m}=(m_1,m_2)$ *and the coefficients*

$$\binom{\mathbf{m}'}{\mathbf{m}}=\binom{(m_1',m_2')}{(m_1,m_2)}$$

are given by (3.12)*. Moreover, the series in* (3.13) *converges in* D *defined by* (3.2)*, and defines an extension of* $\phi_{(m_1'+d/4,m_2'-d/4)}(e+x)$ *to* D*. If* $\mathbf{m}'=(m_1',m_2')$ *is a partition, then the series is finite.*

Proof. If $\mathbf{m}'=(m_1',m_2')$ is a partition, then (3.13) is just (1.9), the coefficients $\binom{\mathbf{m}'}{\mathbf{m}}$ are given by (3.12), and it is known that the series (3.13) is finite.

It follows from (3.12) that the coefficients

$$\binom{\mathbf{m}'}{\mathbf{m}}=\binom{(m_1',m_2')}{(m_1,m_2)}$$

have polynomial growth in $\mathbf{m}$. In the spectral decomposition (3.2) of $z\in D$, $a_2\leqslant a_1<1$. By (3.3), the series (3.13) converges in D. By (1.11), the both sides of (3.13) have the same derivatives at 0, therefore, being analytic, they are equal in $\Lambda_n\cap D$. Moreover, the series in (3.13) defines an extension of $\phi_{(m_1'+d/4,m_2'-d/4)}(e+x)$ to D. $\square$

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