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Asymptotic solutions and error estimates for linear systems of difference and differential equations

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Abstract

Classical results concerning the asymptotic behavior solutions of systems of linear differential or difference equations lead to formulas containing factors that are asymptotically constant, i.e., $k + o(1)$ as t tends to infinity. Here we are interested in more precise information about the $o(1)$ terms, specifically how they depend precisely on certain perturbation terms in the equation. Results along these lines were given by Gel'fond and Kubenskaya for scalar difference equations and we will both extend and generalize one of them as well as provide some corresponding results for differential equations.
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1. Introduction

We consider linear systems of difference equations with complex valued entries

$$x(k+1) = A(k)x(k), \quad k \geqslant k_0,$$

which can be put to L-diagonal form

$$x(k+1) = \big[\Lambda(k) + R(k)\big]x(k), \quad k \geqslant k_0, \tag{1}$$

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where $\Lambda(k) = \operatorname{diag}\{\lambda_1(k), \ldots, \lambda_n(k)\}$ and $R(k)$ is a small perturbation in a sense to be made precise. We will always assume that $A(k)$ is invertible for all k sufficiently large.

A goal of an asymptotic analysis of solutions is to represent a fundamental matrix of (1) in the form

$$X(k) = \big[I + E(k)\big] \prod_{l=k_0}^{k-1} \tilde{\Lambda}(l), \tag{2}$$

where $\tilde{\Lambda}(l)$ is an explicitly constructed diagonal matrix whose main terms come from $\Lambda(l)$, and $E(k)$ is a matrix with the property $E(k) = \mathrm{o}(1)$ as $k \to +\infty$. (Here we use the Hardy–Littlewood symbols O(1) and o(1) with their usual interpretation [3, pp. 4, 10].)

The representation (2) reduces the problem of determining the asymptotic behavior of solutions of say n-dimensional systems (1) to that of the n scalar (first order) equations $x_i(k+1) = \tilde{\lambda}_i(k)x_i(k)$. To obtain explicit asymptotic formulas for solutions of such equations, one can use standard techniques such as taking logarithms and using Euler–Maclaurin type results for sums.

The error terms $E(k)$ in (2) depend upon the perturbation terms and our purpose is to find quantitative estimates which show this dependence. This problem has been considered by Gel'fond and Kubenskaya [7] for asymptotically constant nth order scalar difference equations. It was also investigated by Li [13] for asymptotically constant systems, who obtained similar results under somewhat weaker assumptions.

Our results for systems of difference equations are given in Section 2 and also apply to not necessarily asymptotically constant systems. More specifically, we investigate the case that $R(\cdot)$ in (1) is an l^1-perturbation in Section 2.1 and study l^p-perturbations with $1 < p \leqslant 2$ in Section 2.2. In Section 3, we take a closer look at the asymptotically constant case and also at applications to scalar linear perturbed difference equations. We will see in this section that our results for systems give an improvement of known results in the scalar case. Finally, in Section 4 we will work within the framework of asymptotic integration to establish analogous results for perturbed systems of differential equations.

2. Error estimates for difference equations

2.1. l^1-perturbations

The methods we will be using to find estimates of the term $E(k)$ in (2) are inherited from the discrete analogue of the fundamental theorem of Levinson as established in Benzaid and Lutz [1, Lemma 2.1]. This theorem, requiring a dichotomy condition on the leading matrix $\Lambda(k)$ and a growth condition on the perturbation $R(k)$, just implies that $E(k) = \mathrm{o}(1)$ as $k \to \infty$ in (2). In order to obtain estimates of the size of E in terms of the perturbation R, we will strengthen Levinson's dichotomy condition. For the sake of comparison, we begin by recalling this discrete version of Levinson's result:

Theorem 1. *Let $\Lambda(k) = \operatorname{diag}\{\lambda_1(k), \ldots, \lambda_n(k)\}$ for $k \geqslant k_0$. Assume that*

$$\lambda_i(k) \neq 0 \quad \textit{for all } 1 \leqslant i \leqslant n \textit{ and } k \geqslant k_0. \tag{3}$$

Let $R(k)$ be an $n \times n$ matrix defined for all $k \geqslant k_0$, satisfying

$$\sum_{k=k_0}^{\infty} \frac{\|R(k)\|}{|\lambda_i(k)|} < \infty \quad \text{for all } 1 \leqslant i \leqslant n. \tag{4}$$

Suppose that $\Lambda(k)$ satisfies the following dichotomy condition: *For each index pair (i, j), $i \neq j$, there exist constants K_m $(m = 1, 2)$ such that either*

$$\prod_{k_0}^{k} \left| \frac{\lambda_j(l)}{\lambda_i(l)} \right| \to 0 \quad \text{as } k \to \infty \quad \text{and} \quad \prod_{k_1}^{k_2} \left| \frac{\lambda_j(l)}{\lambda_i(l)} \right| \leqslant K_1 \quad \forall k_0 \leqslant k_1 \leqslant k_2, \tag{5}$$

or

$$\prod_{k_1}^{k_2} \left| \frac{\lambda_j(l)}{\lambda_i(l)} \right| \geqslant K_2 \quad \forall k_0 \leqslant k_1 \leqslant k_2. \tag{6}$$

Then the linear system (1) *has a fundamental matrix satisfying, as $k \to \infty$,*

$$X(k) = \big[I + \mathrm{o}(1)\big] \prod_{k_0}^{k-1} \Lambda(l).$$

Remark 1. We mention, for completeness, that Rapoport [14] also derived the result of Theorem 1 under the somewhat stronger condition that for each pair (i, j)

$$\text{either} \quad \left| \frac{\lambda_i(k)}{\lambda_j(k)} \right| \geqslant 1 \quad \forall k \geqslant k_0 \quad \text{or} \quad \left| \frac{\lambda_i(k)}{\lambda_j(k)} \right| \leqslant 1 \quad \forall k \geqslant k_0.$$

See also Benzaid and Lutz [1, p. 202] for other sufficient conditions for Theorem 1 that are easily verified in applications.

The dichotomy conditions of Theorem 1 appear to be too weak to provide an estimate of the error term o(1) in terms of the perturbation $R(\cdot)$. In particular, we replace (5) by (7) in the following result. We wish to emphasize that the leading matrix $\Lambda(k)$ is not required to be constant.

Theorem 2. *Let $\Lambda(k) = \operatorname{diag}\{\lambda_1(k), \ldots, \lambda_n(k)\}$ for $k \geqslant k_0$ satisfying* (3). *Suppose that $\Lambda(k)$ satisfies the following dichotomy condition*: *For every fixed $1 \leqslant i \leqslant n$, assume that for all $1 \leqslant j \leqslant n$ there exist constants $K > 0$ and $q \in (0, 1)$ such that either*

$$\prod_{k_1}^{k_2} \left| \frac{\lambda_j(l)}{\lambda_i(l)} \right| \leqslant K q^{k_2 - k_1} \quad \forall k_0 \leqslant k_1 \leqslant k_2, \tag{7}$$

or that (6) *holds. Let $R(k)$ be an $n \times n$ matrix defined for all $k \geqslant k_0$, satisfying* (4) *for all $1 \leqslant i \leqslant n$. Then* (1) *has n linearly independent solution vectors satisfying, as $k \to \infty$,*

$$x_i(k) = \left[e_i + \mathrm{O}\left(\sum_{l=k_0}^{k-1} q^{k-l} \frac{\|R(l)\|}{|\lambda_i(l)|} \right) + \mathrm{O}\left(\sum_{l=k}^{\infty} \frac{\|R(l)\|}{|\lambda_i(l)|} \right) \right] \prod_{l=k_0}^{k-1} \lambda_i(l), \tag{8}$$

for $1 \leqslant i \leqslant n$. Here e_i is the ith euclidean unit vector.

Proof. We proceed as Benzaid and Lutz [1], who established the following difference equation analogue of a classical result by Coppel [5, Chapter 4] for ordinary differential systems: if a linear system of difference equations having an ordinary dichotomy on $\mathbb{N}$ is perturbed by an absolutely summable perturbation, then there is a one-to-one bicontinuous mapping between the bounded solutions of the unperturbed and perturbed system. We make, for fixed $1 \leqslant i \leqslant n$, the preliminary change of variables

$$x(k) = w(k) \prod_{l=k_0}^{k-1} \lambda_i(l). \tag{9}$$

Then (1) implies that

$$w(k+1) = \frac{1}{\lambda_i(k)}\big[A(k) + R(k)\big]w(k). \tag{10}$$

In the unperturbed system $y(k+1) = A(k)y(k)$ we make the same preliminary transformation $y(k) = z(k)\prod_{l=k_0}^{k-1}\lambda_i(l)$, i.e.,

$$z(k+1) = \frac{A(k)}{\lambda_i(k)}z(k). \tag{11}$$

Let $P_i = \operatorname{diag}\{p_{i1}, \ldots, p_{in}\}$, where

$$p_{ij} = \begin{cases} 1 & \text{if } (i,j) \text{ satisfies (7)}, \\ 0 & \text{if } (i,j) \text{ satisfies (6)}. \end{cases}$$

Let $Q_i = I - P_i$. Then there exists a constant $m > 0$ such that $\|Z(k)P_iZ^{-1}(l+1)\| \leqslant mq^{k-l} \leqslant m$ for all $k_0 \leqslant l < k$ and $\|Z(k)Q_iZ^{-1}(l+1)\| \leqslant m$ for all $l \geqslant k \geqslant k_0$. Let k_1 be sufficiently large such that $m\sum_{k_1}^{\infty}\|R(l)\|/|\lambda_i(l)| < 1$. One can show as in [1] that the operator T defined by

$$(Tw)(k) = \sum_{l=k_1}^{k-1} Z(k)P_iZ^{-1}(l+1)\frac{R(l)}{\lambda_i(l)}w(l) - \sum_{l=k}^{\infty} Z(k)Q_iZ^{-1}(l+1)\frac{R(l)}{\lambda_i(l)}w(l)$$

is a contraction in the Banach space l_n^{∞} of bounded n-dimensional vector sequences with norm $\|w\| = \sup_{k\geqslant k_1}\|w(k)\|$. Since $z_i(k) = e_i$ is a bounded solution of (11), the operator equation $w = e_i + Tw$ has a unique solution $w_i \in l_n^{\infty}$, which in turn can be shown to be a solution of (10). By the boundedness of w_i, it follows that there exist nonnegative constants M_{im} $(m = 1, 2)$ such that for $k \geqslant k_1$

$$\big|w_i(k) - e_i\big| = \big|(Tw_i)(k)\big| \leqslant M_{i1}\sum_{l=k_1}^{k-1} q^{k-l}\frac{\|R(l)\|}{|\lambda_i(l)|} + M_{i2}\sum_{l=k}^{\infty}\frac{\|R(l)\|}{|\lambda_i(l)|}. \tag{12}$$

Hence, using $k_1 \geqslant k_0$, it follows that

$$w_i(k) = e_i + \mathrm{O}\left(\sum_{l=k_0}^{k-1} q^{k-l}\frac{\|R(l)\|}{|\lambda_i(l)|}\right) + \mathrm{O}\left(\sum_{l=k}^{\infty}\frac{\|R(l)\|}{|\lambda_i(l)|}\right),$$

and (9) then implies (8) for this fixed value of i. Repeating this process for all $1 \leqslant i \leqslant n$ establishes the theorem. □

Remark 2. For a proof that $\sum_{l=k_0}^{k-1} q^{k-l}\|R(l)\|/|\lambda_i(l)| \to 0$ as $k \to \infty$, see, e.g., [6, Lemma 3.1].

Remark 3. In Theorem 2, the dichotomy conditions (7) and (6) could also have been stated in the following equivalent way using "exponential-ordinary dichotomies": For every fixed $1 \leqslant i \leqslant n$, let Z_i denote the fundamental matrix of (11) satisfying $Z(k_0) = I$, and assume that there exists a projection P_i and constants $K \geqslant 1$ and $0 < q < 1$ such that

$$\begin{aligned}
&\left\| Z_i(k) P_i Z_i^{-1}(l) \right\| \leqslant K q^{k-l}, \quad k \geqslant l \geqslant k_0,\\
&\left\| Z_i(k)(I - P_i) Z_i^{-1}(l) \right\| \leqslant K, \quad l \geqslant k \geqslant k_0.
\end{aligned}$$

In (8) we see that the remainder has two components. A more uniform result can be obtained by requiring in addition to the assumptions of Theorem 2 that the perturbation $R(\cdot)$ does not decrease too fast.

Theorem 3. *Suppose that $\Lambda(k)$ and $R(k)$ satisfy all conditions of Theorem* 2*, and let $q \in (0,1)$ be defined as in Theorem* 2*. Let $\phi_i(k) \geqslant \|R(k)\|/|\lambda_i(k)|$ for $k \geqslant k_0$ and $1 \leqslant i \leqslant n$ such that*

$$\sum_{k=k_0}^{\infty} \phi_i(k) < \infty \quad \text{for all } 1 \leqslant i \leqslant n. \tag{13}$$

For every fixed $1 \leqslant i \leqslant n$, if (7) *is satisfied by at least one j, assume that there exists $1 \leqslant b < 1/q$ such that the sequence $\phi_i(k)b^k$ is nondecreasing. (That is, assume in this case that*

$$\phi_i(k_1)b^{k_1} \leqslant \phi_i(k_2)b^{k_2} \quad \text{for all } k_0 \leqslant k_1 \leqslant k_2.) \tag{14}$$

Then (1) *has n linearly independent solution vectors satisfying*

$$x_i(k) = \left[e_i + \mathrm{O}\left(\sum_{l=k}^{\infty} \phi_i(l) \right) \right] \prod_{l=k_0}^{k-1} \lambda_i(l) \quad \text{as } k \to \infty,\ 1 \leqslant i \leqslant n. \tag{15}$$

Proof. We follow the proof of Theorem 2 and consider (12). For a fixed value of i, if all (i, j) satisfy (6), then $P_i = 0$, $M_{i1} = 0$ and

$$\left| w_i(k) - e_i \right| = \mathrm{O}\left(\sum_{l=k}^{\infty} \frac{\|R(l)\|}{|\lambda_i(l)|} \right) = \mathrm{O}\left(\sum_{l=k}^{\infty} \phi_i(l) \right),$$

and thus (9) implies that (1) has a solution of the form (15). Otherwise, $\phi_i(l) \leqslant b^{k-l}\phi_i(k)$ for $k \geqslant l \geqslant k_0$ by (14), and thus it follows from (12) that for $k \to \infty$,

$$\left| w_i(k) - e_i \right| \leqslant M_{i1}\phi_i(k) \sum_{l=k_1}^{k-1} (qb)^{k-l} + M_{i2} \sum_{l=k}^{\infty} \phi_i(l) = \mathrm{O}\left(\sum_{l=k}^{\infty} \phi_i(l) \right),$$

since $0 < qb < 1$. It follows from (9) that (1) has, for this fixed value of i, a solution of the form (15). Repeating this process for all $1 \leqslant i \leqslant n$ establishes the theorem. □

Sufficient for Theorem 3 is also the following Hartman–Wintner type [1, Corollary 3.4] dichotomy condition:

Corollary 4. *Let* $\Lambda(k) = \operatorname{diag}\{\lambda_1(k), \ldots, \lambda_n(k)\}$. *Assume that there exists* δ *with* $0 < \delta < 1$ *such that* $|\lambda_i(k)| \geqslant \delta$ *for* $k \geqslant k_0$ *for all* $1 \leqslant i \leqslant n$ *and that for each pair* (i, j) *with* $i \neq j$ *either*

$$\left|\frac{\lambda_j(k)}{\lambda_i(k)}\right| \leqslant 1 - \delta \tag{16}$$

or

$$\left|\frac{\lambda_j(k)}{\lambda_i(k)}\right| \geqslant 1 + \delta, \tag{17}$$

for all $k \geqslant k_0$. *Let* $R(k)$ *be an* $n \times n$ *matrix defined for all* $k \geqslant k_0$, *and suppose that there exists a scalar sequence* $\phi(k) \geqslant \|R(k)\|$ *for* $k \geqslant k_0$ *such that*

$$\sum_{k=k_0}^{\infty} \phi(k) < \infty.$$

Also assume that there exists $1 \leqslant b < 1/(1-\delta)$ *such that* (14) *holds with* $\phi_i(k)$ *replaced by* $\phi(k)$. *Then* (1) *has a fundamental solution matrix satisfying*

$$X(k) = \left[I + \mathrm{O}\left(\sum_{l=k}^{\infty} \phi(l)\right)\right] \prod_{l=k_0}^{k-1} \Lambda(l) \quad \text{as } k \to \infty. \tag{18}$$

Proof. Fix $1 \leqslant i \leqslant n$. Now (16) implies (7) with $q = 1 - \delta$, hence $0 < q < 1$. Since $|\lambda_i(k)| \geqslant \delta$, (12) can be rewritten as

$$\left|w_i(k) - e_i\right| \leqslant \frac{M_{i1}}{\delta} \sum_{l=k_1}^{k-1} (1-\delta)^{k-l} \phi(l) + \frac{M_{i2}}{\delta} \sum_{l=k}^{\infty} \left\|\phi(l)\right\|.$$

By (14), $\sum_{l=k_1}^{k-1} (1-\delta)^{k-l} \phi(l) \leqslant \phi(k) \sum_{l=k_1}^{k-1} \{b(1-\delta)\}^{k-l} = \mathrm{O}(\phi(k))$ since $0 < b(1-\delta) < 1$. Thus $|w_i(k) - e_i| = \mathrm{O}(\sum_{l=k}^{\infty} \phi(l))$, and (15) follows (with ϕ_i replaced by ϕ) which implies (18) by repeating the process for all $1 \leqslant i \leqslant n$. □

Note that (at least for the nontrivial case $n \geqslant 2$), (16) is always satisfied for some (i, j), and therefore (14) is always required for $n \geqslant 2$ in Corollary 4.

Remark 4. Note that we required in Corollary 4 that $\lambda_i(k)$ all stay bounded away from zero which allowed us to work with one majorant ϕ of the $\|R\|$ instead of n majorants ϕ_i of $\|R\|/|\lambda_i|$. However, if for *some fixed* value of i, $\lambda_i(k) \to \infty$ as $k \to \infty$ and if (16) holds for all $j \neq i$ for this i, then one might be interested in finding *one* solution (instead of n linearly independent solutions) of the form (15). Choosing then $\phi = \phi_i \geqslant \|R\|/|\lambda_i|$ and assuming that ϕ_i satisfies (14) for some $1 \leqslant b < 1/(1-\delta)$, one sees that there is *one* solution of the form $x(k) = [e_i + \mathrm{O}(\sum_{l=k}^{\infty} \phi_i(l))] \prod_{l=k_0}^{k-1} \lambda_i(l)$, as $k \to \infty$, whose error term $\mathrm{O}(\sum_{l=k}^{\infty} \phi_i(l))$ is smaller than $\mathrm{O}(\sum_{l=k}^{\infty} \|R(l)\|)$ due to the increasing nature of λ_i.

Example 1. Consider (1) of the form

$$x(k+1) = \left[\begin{pmatrix} 3+\cos(k\pi) & 0 \\ 0 & 3 \end{pmatrix} + R(k) \right] x(k). \tag{19}$$

Suppose that the perturbation R satisfies $\|R(k)\| \leqslant c/k^p$ for some positive constant c and some $p > 1$. Choose $\phi_1(k) = c/(2k^p)$ and $\phi_2(k) = c/(3k^p)$. Now $\prod_{k_1}^{k_2} |\lambda_1(l)|/|\lambda_2(l)|$ satisfies (7) with $q = \sqrt{8}/3$, $\prod_{k_1}^{k_2} |\lambda_2(l)|/|\lambda_1(l)|$ satisfies (6) (and so do, trivially, $\prod_{k_1}^{k_2} |\lambda_m(l)|/|\lambda_m(l)|$ for $m = 1, 2$). Then $\phi_2(k)/\phi_2(k+1) = (1 + 1/k)^p \leqslant b := (1 + 3/\sqrt{8})/2$ for all k sufficiently large and, by Theorem 3, (19) has a fundamental matrix satisfying as $k \to \infty$

$$X(k) = \left[I + \mathrm{O}\left(\frac{1}{k^{p-1}} \right) \right] \prod_{l=k_0}^{k-1} \Lambda(l).$$

Remark 5. This example also illustrates why it is advantageous to work with the majorants $\phi_i(k) \geqslant \|R(k)\|/|\lambda_i(k)|$ in Theorem 3 instead of considering $\|R(k)\|/|\lambda_i(k)|$ directly. Recall that we need to establish $\phi_i(k) \leqslant b\phi_i(k+1)$ for an appropriate constant b. First, if one only has an upper bound for the perturbation, it is not possible to find an upper bound for quotients of the form

$$\frac{\|R(k)\|}{\|R(k+1)\|} \frac{|\lambda_i(k+1)|}{|\lambda_i(k)|}.$$

Secondly, even if the perturbation R is known explicitly, oscillations in the λ_i might make such an estimate unattainable. Loosely speaking, the majorants ϕ_i have a "smoothing effect."

2.2. l^p-perturbations with $1 < p \leqslant 2$

While the discrete version of Levinson's theorem considered l^1-perturbations R in (1), the discrete version of the theorem of Hartman–Wintner was concerned with l^p-perturbations for some $1 < p \leqslant 2$ [1, Corollary 3.4]. This was made feasible by strengthening the dichotomy condition on Λ to (16) and (17). We will not quote this well-known result, but state another result that could be called an "averaged" Hartman–Wintner theorem, since it requires weaker averaged conditions of the form (21) and (22) instead of the pointwise conditions (16) and (17). Results using those weaker conditions for differential systems follow as a special case from a more general result published recently by Hsieh and Xie [11]. To the authors' knowledge, such an averaged result for systems of difference equations has not been published, and we include it here for the convenience of the reader.

Theorem 5. *Let $\Lambda(k) = \mathrm{diag}\{\lambda_1(k), \ldots, \lambda_n(k)\}$ for $k \geqslant k_0$ and assume that*

$$\left|\lambda_i(k)\right| \geqslant \delta > 0 \quad \text{for } 1 \leqslant i \leqslant n \text{ and } k \geqslant k_0. \tag{20}$$

Assume that $\Lambda(k)$ satisfies the following dichotomy condition: *There exist constants $K > 0$ and $q \in (0, 1)$ such that for each index pair (i, j), $i \neq j$,*

$$\text{either} \quad \prod_{k_1}^{k_2-1}\left|\frac{\lambda_j(l)}{\lambda_i(l)}\right| \leqslant Kq^{k_2-k_1} \quad \forall k_0 \leqslant k_1 \leqslant k_2, \tag{21}$$

$$\text{or} \quad \prod_{k_1}^{k_2-1}\left|\frac{\lambda_i(l)}{\lambda_j(l)}\right| \leqslant Kq^{k_2-k_1} \quad \forall k_0 \leqslant k_1 \leqslant k_2. \tag{22}$$

Let $\|V(l)\| \in l^p[k_0,\infty)$ *for some* $1 < p \leqslant 2$. *Then*

$$x(k+1) = \big[\Lambda(k) + V(k)\big]x(k) \tag{23}$$

has a fundamental solution matrix satisfying as $k \to \infty$

$$X(k) = \big[I + \mathrm{o}(1)\big]\prod_{l=k_0}^{k-1}\big[\Lambda(l) + \operatorname{diag} V(l)\big]. \tag{24}$$

Proof. The proof is based on the so-called Q-transformation which was first introduced for differential equations by Harris and Lutz [8,9] and later on modified for difference equations by Benzaid and Lutz [1]. Since those methods have been well-established, we will just mention a few major steps here which will be used later on for the error estimates. We want to emphasize that the proof established in [1] holds without any modifications for the weaker hypotheses of Theorem 5.

We set $x(k) = [I + Q(k)]w(k)$, where $\operatorname{diag} Q(k) = 0$ and the off-diagonal entries of Q will be chosen as appropriate solutions of

$$V(k) - \operatorname{diag} V(k) + \Lambda(k)Q(k) - Q(k+1)\Lambda(k) = 0. \tag{25}$$

Then (23) implies that

$$w(k+1) = \big\{\Lambda(k) + \operatorname{diag} V(k) + \hat{R}(k)\big\}w(k) = \big\{\hat{\Lambda}(k) + \hat{R}(k)\big\}w(k), \tag{26}$$

where

$$\hat{R}(k) = \big[I + Q(k+1)\big]^{-1}\big[V(k)Q(k) - Q(k+1)\operatorname{diag} V(k)\big]. \tag{27}$$

If (21) holds for the ordered pair (i, j), $i \neq j$, define

$$q_{ij}(k) = -\sum_{m=k}^{\infty}\frac{v_{ij}(m)}{\lambda_j(m)}\prod_{l=k}^{m}\frac{\lambda_j(l)}{\lambda_i(l)}. \tag{28}$$

If (22) holds for the ordered pair (i, j), $i \neq j$, define

$$q_{ij}(k) = \sum_{m=k_0}^{k-1}\frac{v_{ij}(m)}{\lambda_i(m)}\prod_{l=m}^{k-1}\frac{\lambda_i(l)}{\lambda_j(l)}. \tag{29}$$

In exactly the same fashion as in [1], one can show that q_{ij} are solutions of (25), $q_{ij}(k) \to 0$ as $k \to \infty$ and $q_{ij} \in l^p$, $1 < p \leqslant 2$. Hence $\hat{R}(\cdot) \in l^1$ in (27). After showing that $\Lambda + \operatorname{diag} V$ satisfies the dichotomy conditions (5) and (6), one applies Theorem 1 to (26) to finish the proof. □

Thus it is known that $Q(\cdot)$ is in the same l^p-class as the perturbation $R(\cdot)$. However, our goal is to find an estimate of the term $o(1)$ in (24) and to that end we need a more precise estimate on the entries $q_{ij}(k)$ which is made possible by the following lemma. Note that $s(k) = ck^{-p}$ $(p \geqslant 1)$ satisfies the hypotheses of the lemma.

Lemma 6. *Let $\Lambda(k)$ satisfy the conditions of Theorem* 5*, and let q be defined as in Theorem* 5*. Assume that the sequence $\{s(k)\}_{k=k_0}^{\infty}$ satisfies $0 < s(k+1) \leqslant s(k)$ for all $k \geqslant k_0$, and $s(k) \leqslant \beta s(k+1)$ for all $k \geqslant k_0$ and for some $\beta \in [1, 1/q)$. If* (21) *holds for the ordered pair (i, j), then*

$$\left| \sum_{m=k}^{\infty} \frac{s(m)}{\lambda_j(m)} \prod_{l=k}^{m} \frac{\lambda_j(l)}{\lambda_i(l)} \right| \leqslant \frac{qK}{\delta(1-q)} s(k). \tag{30}$$

If (22) *holds for the ordered pair (i, j), then*

$$\left| \sum_{m=k_0}^{k-1} \frac{s(m)}{\lambda_i(m)} \prod_{l=m}^{k-1} \frac{\lambda_i(l)}{\lambda_j(l)} \right| \leqslant \frac{K}{\delta(1-q\beta)} s(k). \tag{31}$$

Proof. To establish (30), note that (20), (21) and $s(k+1) \leqslant s(k)$ for all $k \geqslant k_0$ imply that

$$\left| \sum_{m=k}^{\infty} \frac{s(m)}{\lambda_j(m)} \left[\prod_{l=k}^{m} \frac{\lambda_j(l)}{\lambda_i(l)} \right] \right| \leqslant \frac{Kq}{\delta} \sum_{m=k}^{\infty} s(m) q^{m-k} \leqslant \frac{Kq}{\delta(1-q)} s(k).$$

(31) follows straightforward from (20), (22) and the hypothesis that $s(k) \leqslant \beta s(k+1)$ for all $k \geqslant k_0$. □

Remark 6. The condition $s(k) \leqslant \beta s(k+1)$ is necessary to show that the left hand side in (31) is of order $O(s(k))$. For example, let $q = 0.5$ and $s(n) = e^{-n}$: Then $\sum_{m=0}^{k-1} (1/2)^{k-m} e^{-m} \to 0$ as $k \to \infty$, but is not $O(e^{-k})$.

The following theorem utilizes Lemma 6 to give an estimate for the error term $o(1)$ in the case of an l^p-perturbation with $1 < p \leqslant 2$:

Theorem 7. *Assume that $\Lambda(k) = \operatorname{diag}\{\lambda_1(k), \ldots, \lambda_n(k)\}$ satisfies the conditions of Theorem* 5*, and let q be defined as in Theorem* 5*. Let $V(k)$ be an $n \times n$ matrix defined for all $k \geqslant k_0$. Assume that there exists a scalar sequence $\psi(k)$ defined for all $k \geqslant k_0$ such that $\|V(k)\| \leqslant \psi(k)$ for all $k \geqslant k_0$, $0 < \psi(k+1) \leqslant \psi(k)$, $\psi(k) \in l^p[k_0, \infty)$ for some $1 < p \leqslant 2$, and, moreover, $\psi(k) < \gamma \psi(k+1)$ for some $1 \leqslant \gamma < 1/\sqrt{q}$. Then* (23) *has a fundamental solution matrix satisfying as $k \to \infty$*

$$X(k) = \left[I + O\big(\psi(k)\big) + O\left(\sum_{l=k}^{\infty} \psi^2(l) \right) \right] \prod_{k_0}^{k-1} \big[\Lambda(l) + \operatorname{diag} V(l) \big]. \tag{32}$$

Proof. In (23), we make the same Q-transformation $x(k) = [I + Q(k)]w(k)$ as in Theorem 5. That is, $\operatorname{diag} Q = 0$, the off-diagonal elements are defined by (28) and (29), and

$Q(k) \to 0$ as $k \to \infty$. Then (23) implies (26) to which we want to apply Theorem 3. We first claim that $\|\hat{R}\|/|\hat{\lambda}_i(k)|$ have majorants ϕ_i that are l^1-perturbation as required in (13). Replacing $s(k)$ and β by $\psi_{ij}(k)$ and γ, respectively, and noting that $1 \leqslant \gamma \leqslant 1/\sqrt{q} < 1/q$, (28), (29), and Lemma 6 imply for $B = \max\{qK/\delta(1-q), K/\delta(1-q\gamma)\}$ that

$$\left|q_{ij}(k)\right| \leqslant B\psi(k) \quad \text{for all } k \geqslant k_0,\ 1 \leqslant i \neq j \leqslant n.$$

Hence

$$\left\|Q(k)\right\| \leqslant \hat{B}\psi(k) \quad \text{for some } \hat{B} > 0 \text{ and all } k \geqslant k_0. \tag{33}$$

Since we assumed that $\psi(k+1) \leqslant \psi(k)$, it also follows that $\|Q(k+1)\| \leqslant \hat{B}\psi(k)$ and therefore by (27)

$$\left\|\hat{R}(k)\right\| \leqslant M\psi^2(k).$$

Here M is some finite positive constant depending on B and the particular norm chosen. Since $|\lambda_i(k)| \geqslant \delta > 0$ for $1 \leqslant i \leqslant n$ and $\|V(k)\| \in l^p$, one can see that $|\hat{\lambda}_i(k)| = |\lambda_i(k) + v_{ii}(k)| > \delta/2$ for all k sufficiently large. Choose

$$\phi_i(k) = \phi(k) = \frac{M\psi^2(k)}{\delta/2}, \quad 1 \leqslant i \leqslant n.$$

Then for $1 \leqslant i \leqslant n$, $\phi(k) \geqslant \|\hat{R}(k)\|/|\hat{\lambda}_i(k)\|$ for k sufficiently large and $\sum_{k=0}^{\infty}\phi(k) < \infty$.

Moreover,

$$\frac{\phi(k)}{\phi(k+1)} = \frac{\psi^2(k)}{\psi^2(k+1)} \leqslant \gamma^2 \quad \text{for } k \text{ sufficiently large,}$$

and hence (14) is satisfied with $1 \leqslant b = \gamma^2 < 1/q$.

Finally, we claim that $\hat{\Lambda}(k) = \Lambda(k) + \operatorname{diag} V(k)$ satisfies (for k sufficiently large) the dichotomy conditions (6) and (7) of Theorem 3, where q in (6) is replaced by a slightly larger number $\hat{q} \in (0,1)$ such that $1 \leqslant b = \gamma^2 < 1/\hat{q}$. More precisely, we define

$$\hat{q} = \frac{1}{2}\left(\frac{1}{\gamma^2} + q\right),$$

which satisfies these conditions. Since $v_{ij}(k) \to 0$ as $k \to \infty$ and $|\lambda_i(k)| \geqslant \delta > 0$, it follows that there exists an integer $\hat{k} \geqslant k_0$ such that

$$\left|\frac{1 + \frac{v_{jj}(k)}{\lambda_j(k)}}{1 + \frac{v_{ii}(k)}{\lambda_i(k)}}\right| < \frac{\hat{q}}{q} \quad \forall k \geqslant \hat{k},\ \forall 1 \leqslant i, j \leqslant n.$$

Then, if the index pair (i, j) satisfies (21), one can see that for $k_2 \geqslant k_1 \geqslant \hat{k}$

$$\prod_{k_1}^{k_2-1}\left|\frac{\hat{\lambda}_j(l)}{\hat{\lambda}_i(l)}\right| = \prod_{k_1}^{k_2-1}\left|\frac{\lambda_j(l)}{\lambda_i(l)}\right| \prod_{k_1}^{k_2-1}\left|\frac{1 + v_{jj}(l)/\lambda_j(l)}{1 + v_{ii}(l)/\lambda_i(l)}\right| \leqslant K\hat{q}^{k_2-k_1},$$

i.e., (7) is satisfies with q being replaced by $\hat{q}$. Similar computations show that (6) is satisfied for $k_2 \geqslant k_1 \geqslant \hat{k}$ if (i, j) satisfies (22). By making the constants K, K_2 larger, if necessary, (6) and (7) are satisfied for all $k_0 \leqslant k_1 \leqslant k_2$.

By Theorem 3, (26) has n linearly independent solution vectors

$$w_i(k) = \left[e_i + \mathrm{O}\left(\sum_{l=k}^{\infty} \phi(l)\right)\right] \prod_{l=k_0}^{k-1} \hat{\lambda}_i(l) = \left[e_i + \mathrm{O}\left(\sum_{l=k}^{\infty} \psi^2(l)\right)\right] \prod_{l=k_0}^{k-1} \hat{\lambda}_i(l)$$

as $k \to \infty$, $1 \leqslant i \leqslant n$. This and (33) imply that (23) has a fundamental solution matrix

$$X(k) = \big[I + \mathrm{O}\big(\psi(k)\big)\big]\left[I + \mathrm{O}\left(\sum_{l=k}^{\infty} \psi^2(l)\right)\right] \prod_{k_0}^{k-1} \big[\Lambda(l) + \operatorname{diag} V(l)\big]$$

as $k \to \infty$ which is of form (32). □

Example 2. Consider (23) of the form

$$x(k+1) = \left[\begin{pmatrix} 3 + \cos(k\pi) & 0 \\ 0 & 3 \end{pmatrix} + V(k)\right] x(k). \tag{34}$$

Then $\Lambda(k)$ satisfies all assumptions of Theorem 7, in particular, (21) and (22) hold with $q = \sqrt{8}/3$. Suppose that the perturbation V satisfies $\|V(k)\| \leqslant c/k = \psi(k)$ for some positive constant c. Choosing, for example, $\gamma = 1.02 \in [1, 1/\sqrt{q})$, then $\phi(k)$ satisfies the conditions of Theorem 7 for k sufficiently large and all $p \in (1, 2]$. Observe that

$$\left[I + \mathrm{O}\big(\psi(k)\big) + \mathrm{O}\left(\sum_{l=k}^{\infty} \psi^2(l)\right)\right] = I + \mathrm{O}\left(\frac{1}{k}\right),$$

and therefore (34) has, as $k \to \infty$, a fundamental matrix

$$X(k) = \left[I + \mathrm{O}\left(\frac{1}{k}\right)\right] \prod_{l=k_0}^{k-1} \big[\Lambda(l) + \operatorname{diag} V(l)\big].$$

3. Asymptotically constant systems and reduction to scalar equations

Many difference equations encountered in applications are scalar equations. Historically, scalar equations were treated first and their solutions often have special properties. In this section, we will apply our results for systems to scalar equations that are asymptotically constant, and in some cases we—somewhat surprisingly—obtain better results from the systems approach.

For that purpose, we first apply Theorem 3 to asymptotically constant systems, which provides the following corollary:

Corollary 8. *Consider*

$$x(k+1) = \big[A + R(k)\big]x(k), \tag{35}$$

where A is a constant, invertible and diagonalizable $n \times n$ matrix, say $A \simeq \Lambda = \operatorname{diag}\{\lambda_1, \ldots, \lambda_n\}$, $\lambda_i \neq 0$, $1 \leqslant i \leqslant n$. Suppose that there is a sequence $\phi(k) \geqslant \|R(k)\|$

for $k \geqslant k_0$ such that $\sum_{k=k_0}^{\infty} \phi(k) < \infty$. If there exists at least one pair (i, j) such that $|\lambda_j|/|\lambda_i| < 1$, define $q \in (0, 1)$ by

$$q = \max\{|\lambda_j|/|\lambda_i|\colon\ 1 \leqslant i, j \leqslant n \text{ such that } |\lambda_j|/|\lambda_i| < 1\}$$

and, moreover, assume that there exists $1 \leqslant b < 1/q$ such that (14) *holds* (*with ϕ_i replaced by ϕ*). *Then* (35) *has a fundamental solution matrix satisfying*

$$X(k) = P\left[I + \mathrm{O}\left(\sum_{l=k}^{\infty} \phi(l)\right)\right]\Lambda^k \quad \text{as } k \to \infty, \tag{36}$$

where P is a matrix consisting of eigenvectors of A.

Remark 7. Note that (36) implies that

$$X(k)P^{-1} = \left[I + \mathrm{O}\left(\sum_{l=k}^{\infty} \phi(l)\right)\right]A^k \quad \text{as } k \to \infty,$$

another representation for a fundamental solution.

We want to apply Corollary 8 to the linear homogeneous scalar equation

$$x(k+n) + p_1(k)x(k+n-1) + \cdots + p_n(k)x(k) = 0 \quad \text{for } k \geqslant k_0, \tag{37}$$

where

$$p_\nu(k) = a_\nu + r_\nu(k), \quad a_n \neq 0,$$

and

$$\sum_{k=k_0}^{\infty} \left|r_\nu(k)\right| < \infty \quad \text{for } 1 \leqslant \nu \leqslant n.$$

Then (37) is equivalent to the system

$$z(k+1) = C(k)z(k), \tag{38}$$

where the companion matrix $C(k) = C + R(k)$ is asymptotically constant. Note that C is diagonalizable if and only if the limiting characteristic polynomial

$$\lambda^n + a_1\lambda^{n-1} + \cdots + a_n = (\lambda - \lambda_1)\ldots(\lambda - \lambda_n), \tag{39}$$

has distinct roots $\{\lambda_1, \ldots, \lambda_n\}$. In this case, one may use the Vandermonde matrix

$$P = \begin{pmatrix} 1 & \ldots & 1 \\ \lambda_1 & \ldots & \lambda_n \\ \vdots & & \vdots \\ \lambda_1^{n-1} & \ldots & \lambda_n^{n-1} \end{pmatrix},$$

to diagonalize C since $P^{-1}CP = \Lambda = \mathrm{diag}\{\lambda_1, \ldots, \lambda_n\}$. If there exits a sequence $\phi(k)$ satisfying the hypotheses of Corollary 8, then (38) has a fundamental matrix of the form

$$Z(k) = P\left[I + \mathrm{O}\left(\sum_{l=k}^{\infty} \phi(l)\right)\right]\Lambda^k \quad \text{as } k \to \infty.$$

Considering the first elements in each column yields the following:

Corollary 9. *Consider* (37) *and assume that the roots* λ_ν *of the limiting equation* (39) *are distinct and nonzero. Suppose that there is a sequence* $\phi(k) \geqslant |r_\nu(k)|$ *for all* $1 \leqslant \nu \leqslant n$ *for* $k \geqslant k_0$ *such that* $\sum_{k=k_0}^{\infty} \phi(k) < \infty$. *If there exists at least one pair* (i, j) *such that* $|\lambda_j|/|\lambda_i| < 1$, *define* $q \in (0, 1)$ *by*

$$q = \max\{|\lambda_j|/|\lambda_i|\colon\ 1 \leqslant i, j \leqslant n \text{ such that } |\lambda_j|/|\lambda_i| < 1\},$$

and, moreover, assume that there exists $1 \leqslant b < 1/q$ *such that* (14) *holds* (*with* ϕ_i *replaced by* ϕ). *Then* (37) *has* n *linearly independent solutions such that*

$$x_\nu(k) = \lambda_\nu^k \left[1 + \mathrm{O}\left(\sum_{l=k}^{\infty} \phi(l)\right)\right] \quad \text{as } k \to \infty,\ 1 \leqslant \nu \leqslant n. \tag{40}$$

In [7], Gel'fond and Kubenskaya considered scalar equations of the form (37). In a corollary to their result, they considered the special case when the perturbation is summable and derived (40) under stronger conditions on the roots of the limiting equation and the majorant ϕ defined above. Specifically, they required the roots of the limiting equation (39) to be nonzero and to have *distinct moduli*, say $0 < |\lambda_1| < |\lambda_2| < \cdots < |\lambda_n|$. Recall that we just required those roots to be nonzero and distinct. They also assumed that the majorant satisfies the condition $0 < \phi(k+1) \leqslant \phi(k)$, $\lim_{k\to\infty} \phi(k+1)/\phi(k) = 1$, and $\sum_{k=k_0}^{\infty} \phi(k) < \infty$. These conditions imply that, given $\epsilon > 0$, $\phi(k) \leqslant \phi(k+1)/(1-\epsilon)$ for all k sufficiently large which implies our condition (14) for sufficiently large k and ϵ sufficiently small. Note that $\phi(k) = (2q/(q+1))^k$ satisfies the hypotheses of Corollary 9, but not the conditions of Gel'fond and Kubenskaya. As stated in Coffman [4], (40) remains true if $\lim_{k\to\infty} \phi(k+1)/\phi(k) = 1$ is replaced by the weaker statement

$$\liminf_{k\to\infty} \frac{\phi(k+1)}{\phi(k)} > \max\{|\lambda_j|/|\lambda_i|\colon\ 1 \leqslant i, j \leqslant n \text{ such that } |\lambda_j|/|\lambda_i| < 1\}.$$

It is not hard to show that Coffman's conditions is still stronger than ours because of the stricter assumption on the roots of the limiting equations and his additional assumption that $\phi(k+1) \leqslant \phi(k)$.

It is interesting to observe that in some cases results such as Gel'fond and Kubenskaya for scalar equations can be used to obtain similar results for systems. This follows by transforming a system $y(k+1) = A(k)y(k)$ into a system $z(k+1) = C(k)z(k)$ whose coefficient matrix is a companion matrix of the form

$$C(k) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ c_1(k) & c_2(k) & \dots & \dots & c_n(k) \end{pmatrix},$$

and noting that the first component of a vector solution then satisfies an equation of the form (37). A transformation $T(k)y(k) = z(k)$ can be constructed with the aid of a so-called "cyclic row vector" $t_1(k)$ and defining $t_{\nu+1}(k) = t_\nu(k)A(k)$ for $1 \leqslant \nu \leqslant n-1$.

If $\{t_1(k), \ldots, t_n(k)\}$ are linearly independent, say for $k \geqslant k_0$, then the matrix

$$T(k) = \begin{pmatrix} t_1(k) \\ \vdots \\ t_n(k) \end{pmatrix}$$

can be used. If $A(k) = A + o(1)$ and A has distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ one may always choose $t_1(k) = (1, 1, \ldots, 1)$, and it follows that the set is linearly independent for all t sufficiently large, using the Vandermonde and continuity of the determinant.

On the other hand, results about scalar equations are not necessarily simpler to prove and in fact we have shown above that the systems approach can lead to better theorems even for scalar equations. Also, in case $A(k) = \Lambda(k) + R(k)$, where $\Lambda(k)$ is not asymptotically constant, it would be difficult to find a cyclic vector and control the influence of the resulting transformation on the perturbations.

4. Error estimates for differential equations

In this section, we consider the perturbed linear differential system

$$x' = \big[\Lambda(t) + R(t)\big]x, \quad t \geqslant t_0, \tag{41}$$

where $\Lambda(t) = \operatorname{diag}\{\lambda_1(t), \ldots, \lambda_n(t)\}$ and $R(\cdot)$ is an $n \times n$ perturbation which is small in a sense to be made precise.

Systems of the form (41) are a continuous analogue of discrete systems of the form (1) treated above; historically, the study of the asymptotic behavior of solutions of (41) has pre-dated the study in the discrete case. Recently, it has been shown that many features of the asymptotic theory of solutions of both types of systems can be treated from a common perspective in a unified approach (see [2]), but we will not take that point of view here. The central idea in the more general theory lies in a generalization of a fundamental idea due to Levinson for differential equations [12], which we now state as

Theorem 10. *Let $\Lambda(t) = \operatorname{diag}\{\lambda_1(t), \ldots, \lambda_n(t)\}$ be continuous for $t \geqslant t_0$ and assume for each index pair $i \neq j$ that either*

$$\begin{cases} \int_{t_0}^{t} \operatorname{Re}\{\lambda_j(\tau) - \lambda_i(\tau)\}\, d\tau \to -\infty \quad \text{as } t \to \infty, \quad \text{and} \\ \int_{s}^{t} \operatorname{Re}\{\lambda_j(\tau) - \lambda_i(\tau)\}\, d\tau < K \quad \forall t_0 \leqslant s \leqslant t, \end{cases} \tag{42}$$

or

$$\int_{s}^{t} \operatorname{Re}\big\{\lambda_j(\tau) - \lambda_i(\tau)\big\}\, d\tau > -K \quad \forall t_0 \leqslant s \leqslant t.$$

Furthermore, assume that $R(t)$ is continuous for $t \geqslant t_0$ and $R(t) \in L^1[t_0, \infty)$. Then the linear differential system (41) *has a fundamental matrix satisfying as $t \to \infty$*

$$X(t) = \big[I + o(1)\big]e^{\int^t \Lambda(s)\, ds}.$$

As for difference equations, results are made possible by carefully balancing the dichotomy condition on the elements of Λ and the size of the perturbation R. Furthermore, to obtain estimates of the error term $o(1)$ it appears that one needs to impose somewhat stronger conditions. Our results and proofs for difference systems in Section 2 by and large carry over to differential systems. In order to avoid repetition, we choose not to reproduce every single result of Section 2, but we feel that it is important to state the main results and to indicate their proofs.

We begin with a continuous analogue to Theorem 3 which gives us an estimate of the error term in terms of this absolutely integrable perturbation. In comparison with Levinson's theorem, we need to strengthen the dichotomy condition (42) and, moreover, require that the perturbation $R(\cdot)$ does not decrease too fast.

Theorem 11. *Let $\Lambda(t) = \operatorname{diag}\{\lambda_1(t), \ldots, \lambda_n(t)\}$ be continuous for $t \geqslant t_0$. Let $i \in \{1, \ldots, n\}$ be fixed and assume that for all $1 \leqslant j \leqslant n$, there exist constants $K_l > 0$ $(l = 1, 2)$ and $\alpha > 0$ such that either*

$$\exp\left\{\operatorname{Re}\int_s^t \big[\lambda_j(\tau) - \lambda_i(\tau)\big]\,d\tau\right\} \leqslant K_1 e^{-\alpha(t-s)} \quad \text{for } t \geqslant s \geqslant t_0, \tag{43}$$

or

$$\exp\left\{\operatorname{Re}\int_s^t \big[\lambda_j(\tau) - \lambda_i(\tau)\big]\,d\tau\right\} \geqslant K_2 \quad \text{for } t \geqslant s \geqslant t_0. \tag{44}$$

Assume that $R(t)$ is an $n \times n$ matrix continuous for $t \geqslant t_0$, and suppose that there exists $\phi(t) \geqslant \|R(t)\|$ for $t \geqslant t_0$ such that

$$\int_{t_0}^{\infty} \phi(t)\,dt < \infty.$$

In addition, if (43) *is satisfied for at least one j, assume that there exists $\beta \in [0, \alpha)$ such that*

$$\phi(t_1)e^{\beta t_1} \leqslant \phi(t_2)e^{\beta t_2} \quad \text{for all } t_0 \leqslant t_1 \leqslant t_2. \tag{45}$$

Then (41) *has a solution vector satisfying as $t \to \infty$*

$$x_i(t) = \left[e_i + \mathrm{O}\left(\int_t^{\infty} \phi(\tau)\,d\tau\right)\right]\exp\left\{\int_{t_0}^t \lambda_i(s)\,ds\right\}.$$

Proof. Let $x(t) = \exp[\int_{t_0}^t \lambda_i(\tau)\,d\tau]w$. Then (41) implies that

$$w' = \big[\Lambda(t) - \lambda_i(t)I + R(t)\big]w. \tag{46}$$

Similarly, in the unperturbed system $y' = \Lambda(t)y$ we make the same preliminary transformation $y(t) = \exp[\int_{t_0}^t \lambda_i(\tau)\,d\tau]z$ leading to

$$z' = \big[\Lambda(t) - \lambda_i(t)I\big]z. \tag{47}$$

We define a projection matrix $P = \operatorname{diag}\{p_1, \ldots, p_n\}$, where

$$p_j = \begin{cases} 1 & \text{if } (i, j) \text{ satisfies (43)}, \\ 0 & \text{if } (i, j) \text{ satisfies (44)}. \end{cases}$$

Since (43) and (44) imply that (47) possesses an ordinary dichotomy and since $\|R\|$ is integrable, a well-known result by Coppel [5, p. 35] yields that

$$(Tw)(t) = \int_{t_1}^{t} Z(t)PZ^{-1}(\tau)R(\tau)w(\tau)\,d\tau - \int_{t}^{\infty} Z(t)[I - P]Z^{-1}(\tau)R(\tau)\omega(\tau)\,d\tau$$

is a contraction in the Banach of bounded functions equipped with the supremum norm $\|w\| = \sup_{t \geqslant t_1} |w(t)|$ for t_1 sufficiently large. Hence

$$w = z + Tw$$

establishes a one-to-one correspondence between bounded solutions of (46) and (47). Now $z_i(t) = e_i$ is a bounded solution of (47), and hence there exist nonnegative constants L_l $(l = 1, 2)$ such that for $t \geqslant t_1$,

$$\big|w_i(t) - e_i\big| = \big|(Tw_i)(t)\big| \leqslant L_1 \int_{t_1}^{t} e^{-\alpha(t-\tau)}\phi(\tau)\,d\tau + L_2 \int_{t}^{\infty} \phi(\tau)\,d\tau.$$

If, for this given value of i, all ordered pairs (i, j) satisfy (44), then $P = 0$, $L_1 = 0$, and

$$\big|w_i(t) - e_i\big| = \mathrm{O}\left(\int_{t}^{\infty} \phi(\tau)\,d\tau\right),$$

and therefore (41) has a solution of the form

$$x_i(t) = \left[e_i + \mathrm{O}\left(\int_{t}^{\infty} \phi(\tau)\,d\tau\right)\right] \exp\left\{\int_{t_0}^{t} \lambda_i(s)\,ds\right\} \quad \text{as } t \to \infty. \tag{48}$$

If there exists at least one value of j such that (i, j) satisfies (43), then by (45)

$$\int_{t_1}^{t} e^{-\alpha(t-\tau)}\phi(\tau)\,d\tau \leqslant \phi(t) \int_{t_1}^{t} e^{(\beta-\alpha)(t-\tau)}\,d\tau \leqslant \frac{\phi(t)}{\alpha - \beta}.$$

On the other hand, we can assume without loss of generality that $\beta > 0$ in (45). Then

$$\int_{t}^{\infty} \phi(\tau)\,d\tau \geqslant \phi(t) \int_{t}^{\infty} e^{-\beta(\tau-t)}\,d\tau = \frac{\phi(t)}{\beta}.$$

Combining these two observations yields that

$$\int_{t_1}^{t} e^{-\alpha(t-\tau)}\phi(\tau)\,d\tau \leqslant \frac{\phi(t)}{\alpha-\beta} \leqslant \frac{\beta}{\alpha-\beta}\int_{t}^{\infty}\phi(\tau)\,d\tau,$$

and hence (41) has again a solution of the form (48). □

We now consider continuous $L^p[t_0,\infty)$-perturbations with $1 < p \leqslant 2$. Hartman and Wintner [10] were the first to study systems of the form

$$x' = \big[\Lambda(t) + V(t)\big]x, \tag{49}$$

where

$$\int_{t_0}^{\infty} \big\|V(t)\big\|^p\,dt < \infty \quad \text{for some } 1 < p \leqslant 2. \tag{50}$$

Assuming that the continuous diagonal matrix $\Lambda(t)$ satisfies the dichotomy condition that for each index pair $i \neq j$,

$$\big|\mathrm{Re}\big\{\lambda_j(t) - \lambda_i(t)\big\}\big| \geqslant \delta > 0 \quad \forall t \geqslant t_0, \tag{51}$$

they showed that (49) has a fundamental matrix of the form

$$X(t) = \big[I + \mathrm{o}(1)\big]e^{\int^t \{\Lambda(s)+\mathrm{diag}\,V(s)\}\,ds} \quad \text{as } t \to \infty. \tag{52}$$

Harris and Lutz [8,9] simplified the original proof of this Hartman–Wintner result significantly by introducing the so-called "Q-transformation," an explicit change of the dependent variable, that allowed them to reduce a perturbed systems (49) satisfying the Hartman–Wintner conditions to an equation of the form (41) satisfying the hypotheses of Levinson's Theorem 10. This Q-transformation has become a standard tool in asymptotic integration.

One of the results in Hsieh and Xie [11] was to weaken the pointwise condition (51) to an averaged dichotomy condition. More specifically, they assumed that there exist two positive constants K and α such that for each pair of indices i and j $(1 \leqslant i \neq j \leqslant n)$

$$\exp\left\{\int_{s}^{t} \mathrm{Re}\big[\lambda_i(u) - \lambda_j(u)\big]\,du\right\} \leqslant K e^{-\alpha(t-s)} \quad \text{for } t_0 \leqslant s \leqslant t \text{ and } i < j$$

and

$$\exp\left\{\int_{s}^{t} \mathrm{Re}\big[\lambda_i(u) - \lambda_j(u)\big]\,du\right\} \leqslant K e^{-\alpha(s-t)} \quad \text{for } s \geqslant t \geqslant t_0 \text{ and } i > j.$$

While the proof by Harris and Lutz using the Q-transformation would still apply without any changes, Hsieh and Xie instead utilized a block-diagonalization theorem of Sibuya to show that (52) still holds under those weaker averaged conditions.

To derive an estimate for the error term $\mathrm{o}(1)$ in the following theorem, we will use the same dichotomy conditions (only we do not require this specific ordering of the $\{\lambda_1(t),\ldots,\lambda_n(t)\}$), but we need to impose additional assumptions on the perturbation V.

Theorem 12. *Let* $\Lambda(t) = \operatorname{diag}\{\lambda_1(t), \ldots, \lambda_n(t)\}$ *be continuous for* $t \geqslant t_0$ *satisfying the following dichotomy condition*: *There exist positive constants* K *and* α *such that for each index pair* (i, j), $i \neq j$, *either*

$$\exp\left\{\int_s^t \operatorname{Re}\big[\lambda_i(\tau) - \lambda_j(\tau)\big]\,d\tau\right\} \leqslant K e^{-\alpha(t-s)} \quad \text{for } t \geqslant s \geqslant t_0, \tag{53}$$

or

$$\exp\left\{\int_s^t \operatorname{Re}\big[\lambda_i(\tau) - \lambda_j(\tau)\big]\,d\tau\right\} \leqslant K e^{-\alpha(s-t)} \quad \text{for } s \geqslant t \geqslant t_0. \tag{54}$$

Let $V(\cdot)$ *be an* $n \times n$ *matrix defined and continuous for* $t \geqslant t_0$. *Assume that there exists a scalar function* $\psi(t)$ *defined for* $t \geqslant t_0$ *such that* $\|V(t)\| \leqslant \psi(t)$ *for all* $t \geqslant t_0$, $\int_{t_0}^{\infty} \psi^p(t)\,dt < \infty$ *for some* $1 < p \leqslant 2$,

$$0 < \psi(t+\eta) \leqslant \psi(t) \quad \text{for } t \geqslant t_0 \text{ and all } \eta \geqslant 0, \tag{55}$$

and

$$\psi(t_1)e^{\gamma t_1} \leqslant \psi(t_2)e^{\gamma t_2} \quad \text{for some } 0 \leqslant \gamma < \frac{\alpha}{2} \text{ and } \forall t_0 \leqslant t_1 \leqslant t_2. \tag{56}$$

Then (49) *has a fundamental solution matrix satisfying as* $t \to \infty$

$$X(t) = \left[I + \mathrm{O}\big(\psi(t)\big) + \mathrm{O}\left(\int_t^{\infty} \psi^2(\tau)\,d\tau\right)\right] e^{\int^t \{\Lambda(s) + \operatorname{diag} V(s)\}\,ds}. \tag{57}$$

Proof. In (49), we make the Q-transformation $x(t) = [I + Q(t)]w(t)$, where $\operatorname{diag} Q(t) = 0$ and $Q' = \Lambda Q - Q\Lambda + V - \operatorname{diag} V$, i.e.,

$$q'_{ij} = (\lambda_i - \lambda_j)q_{ij} + v_{ij} \quad \text{for } 1 \leqslant i \neq j \leqslant n. \tag{58}$$

If (53) holds for the ordered pair (i, j), we choose as solution of (58)

$$q_{ij}(t) = \int_{t_0}^{t} e^{\int_s^t [\lambda_i(\tau) - \lambda_j(\tau)]\,d\tau} v_{ij}(s)\,ds,$$

and it follows by (53) and (56) that

$$\big|q_{ij}(t)\big| \leqslant K \int_{t_0}^{t} e^{-\alpha(t-s)}\psi(s)\,ds \leqslant K\psi(t) \int_{t_0}^{t} e^{(\gamma-\alpha)(t-s)}\,ds \leqslant \frac{K}{\alpha - \gamma}\psi(t).$$

If (54) holds for the ordered pair (i, j), we choose as solution of (58)

$$q_{ij}(t) = -\int_{t}^{\infty} e^{\int_s^t [\lambda_i(\tau) - \lambda_j(\tau)]\,d\tau} v_{ij}(s)\,ds,$$

and it follows by (54) and (55) that

$$\left|q_{ij}(t)\right| \leqslant K\psi(t)\int_t^\infty e^{-\alpha(s-t)}\,ds = \frac{K}{\alpha}\psi(t).$$

Thus we have shown that $\|Q(t)\| = \mathrm{O}(\psi(t))$. Using Hölder's inequality, one can show as in [9, p. 576] that $q_{ij}(t) = \mathrm{o}(1)$ as $t \to \infty$, and therefore $I + Q(t)$ is invertible for t sufficiently large. Then (49) implies for t sufficiently large that

$$w' = \Big\{\underbrace{\Lambda + \operatorname{diag} V}_{\hat{\Lambda}} + (I+Q)^{-1}[VQ - Q\operatorname{diag} V]\Big\}w = \big\{\hat{\Lambda} + \hat{R}(t)\big\}w. \tag{59}$$

Both $\|V(t)\|$ and $\|Q(t)\|$ are of order $\mathrm{O}(\psi(t))$ and hence there exists a positive constant M such that

$$\big\|\hat{R}(t)\big\| \leqslant M\psi^2(t) =: \phi(t) \quad \text{for all } t \text{ sufficiently large.}$$

We want to apply Theorem 11 to (59). Note that $\hat{R}$ and ϕ satisfy the hypotheses of Theorem 11. In particular, (45) holds with $\beta = 2\gamma$ due to (56), and $\phi \in L^{p/2}[t_0,\infty)$ and bounded implies that $\phi \in L^1[t_0,\infty)$. Moreover, (54) and (53) imply that $\Lambda + \operatorname{diag} V$ satisfies the dichotomy conditions (43) and (44) of Theorem 11, respectively, where α in (43) is replaced by $\hat{\alpha} = (2\gamma+\alpha)/2$ (thus $2\gamma < \hat{\alpha} < \alpha$), and for appropriately chosen positive constants K_1 and K_2 in (43) and (44). This can be seen by noting that Hölder's inequality implies for an L_p-function $v(t)$ that for $t \geqslant s$

$$\int_s^t \left|v(\tau)\right| d\tau \leqslant \left(\int_s^t \left|v(\tau)\right|^p d\tau\right)^{1/p} (t-s)^{1-1/p} \quad \text{and} \quad 1 - 1/p \in (0, 1/2],$$

which can be absorbed by changing α to the smaller $\hat{\alpha}$ in (43) and adjusting the values of the positive constants K_1 and K_2, if necessary. Therefore, by Theorem 11 (applied to all $1 \leqslant i \leqslant n$), (59) has a fundamental solution matrix satisfying as $t \to \infty$

$$W(t) = \left[I + \mathrm{O}\left(\int_t^\infty \psi^2(\tau)\,d\tau\right)\right]\exp\left\{\int_{t_0}^t \big[\Lambda(s) + \operatorname{diag} V(s)\big]\,ds\right\}.$$

Recalling that $Q(t) = \mathrm{O}(\psi(t))$, one sees that (49) has a fundamental matrix of the form (57). □

Example. Consider for $t \geqslant 4$

$$x' = \left[\begin{pmatrix} 3 + 2\cos t & 0 \\ 0 & 2 \end{pmatrix} + V(t)\right]x, \tag{60}$$

where $\|V(t)\| \leqslant c/t =: \psi(t)$. Then Λ satisfies the dichotomy conditions (53) and (54) of Theorem 12 with $\alpha = 1$ and $K = e^4$. Moreover, ψ satisfies conditions (55) and (56) of that theorem with say $\gamma = 1/4$ for $t \geqslant t_0 = 4$ and $\int_4^\infty \psi^p(t)\,dt < \infty$ for all $1 < p \leqslant 2$. Thus (60) has a fundamental matrix, as $t \to \infty$, of the form

$$X(t) = \left[I + \mathrm{O}\left(\frac{1}{t}\right) + \mathrm{O}\left(\int_t^\infty \frac{1}{\tau^2}\, d\tau \right) \right] e^{\int^t \{\Lambda(s) + \operatorname{diag} V(s)\}\, ds}$$
$$= \left[I + \mathrm{O}\left(\frac{1}{t}\right) \right] e^{\int^t \{\Lambda(s) + \operatorname{diag} V(s)\}\, ds}.$$

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