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A complete classification of bifurcation diagrams of a Dirichlet problem with concave–convex nonlinearities [☆]

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Dedicated to Professor Hwai-Chiuan Wang on his 65th birthday

Abstract

We study the bifurcation diagrams of positive solutions of the two point boundary value problem

$$\begin{cases} u''(x) + f_\lambda(u(x)) = 0, & -1 < x < 1, \\ u(-1) = u(1) = 0, \end{cases}$$

where $f_\lambda(u) = \lambda g(u) + h(u)$, $g, h \in C[0, \infty) \cap C^2(0, \infty)$, and $\lambda > 0$ is a bifurcation parameter. We assume that functions g and h satisfy hypotheses (H1)–(H3). Under hypotheses (H1)–(H3), we give a complete classification of bifurcation diagrams, and we prove that, on the $(\lambda, \|u\|_\infty)$ -plane, each bifurcation diagram consists of exactly one curve which is either a monotone curve or has exactly one turning point where the curve turns to the left. Hence the problem has at most two positive solutions for each $\lambda > 0$. More precisely, we prove the exact multiplicity of positive solutions.

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Keywords: Bifurcation diagram; Positive solution; Exact multiplicity; Solution curve; Concave–convex nonlinearity; Time map

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1. Introduction

We study the bifurcation diagrams of positive solutions of the two point boundary value problem

$$\begin{cases} u''(x) + f_\lambda(u(x)) = 0, & -1 < x < 1, \\ u(-1) = u(1) = 0, \end{cases} \quad (1.1)$$

where $f_\lambda(u) = \lambda g(u) + h(u)$, $g, h \in C[0, \infty) \cap C^2(0, \infty)$, and $\lambda > 0$ is a bifurcation parameter. We assume that functions g and h satisfy the following hypotheses:

- (H1) $g(0) = h(0) = 0$ and $g(u), h(u) > 0$ for $u > 0$,
- (H2) $g''(u) < 0$ and $h''(u) > 0$ for $u > 0$, and there exist numbers $0 < a < 1$, $b_1, b_2, c > 0$ such that $|g'(u)| \leq b_1 u^{a-1}$, $|g''(u)| \leq b_2 u^{a-2}$ for $u \in (0, c)$.
- (H3) The positive functions $-(h(u) - uh'(u))/(g(u) - ug'(u))$ and $-h''(u)/g''(u)$ are both strictly increasing on $(0, \infty)$.

(Note that (H1) and (H2) imply $g(u) - ug'(u) > 0$ and $h(u) - uh'(u) < 0$ for $u > 0$.)

Under (H1)–(H3), we give a complete classification of bifurcation diagrams of (1.1), and we prove that, on the $(\lambda, \|u\|_\infty)$ -plane, each bifurcation diagram consists of exactly one curve which is either a monotone curve or has exactly one turning point where the curve turns to the left; see Fig. 1. Hence (1.1) has at most two positive solutions for each $\lambda > 0$. More precisely, we prove exact multiplicity of positive solutions.

This research is motivated by a well-known paper by Ambrosetti et al. [2] in which they studied the combined effects of *concave and convex nonlinearities* to the exact structure of the solution curve for the elliptic boundary value problem

$$\begin{cases} \Delta u + f_\lambda(u) = 0 & \text{in } \Omega \ (\Omega \subset \mathbf{R}^N, \ N \geq 1), \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Specifically, they took $f_\lambda(u) = \lambda u^q + u^p$ and studied

$$\begin{cases} \Delta u + \lambda u^q + u^p = 0 & \text{in } \Omega \ (\Omega \subset \mathbf{R}^N, \ N \geq 1), \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.2)$$

where $0 < q < 1 < p$, and Ω is a general bounded domain with smooth boundary $\partial\Omega$. They obtained the existence of two positive solutions of (1.2) for small $\lambda > 0$ by using sub- and supersolutions and variational arguments. If $N \geq 1$ and Ω is a ball in \mathbf{R}^N , and $0 < q < 1 < p \leq N^*$ (note that $N^* = (N+2)/(N-2)$ for $N \geq 3$ and $N^* = \infty$ for $N = 1, 2$), they conjectured that there exists a constant $\lambda^* > 0$ such that (1.2) has exactly two positive solutions for $0 < \lambda < \lambda^*$, exactly one positive solution for $\lambda = \lambda^*$, and no positive solution for $\lambda > \lambda^*$; see [2, Fig. 1].

When $N = 1$ and $\Omega = (-1, 1)$, Addou et al. [1] and Sanchez and Ubilla [12] independently proved exact multiplicity of positive solutions for a more general k -Laplacian problem

$$\begin{cases} (\varphi_k(u'(x)))' + \lambda u^q + u^p = 0, & -1 < x < 1, \\ u(-1) = u(1) = 0, \end{cases} \quad (1.3)$$

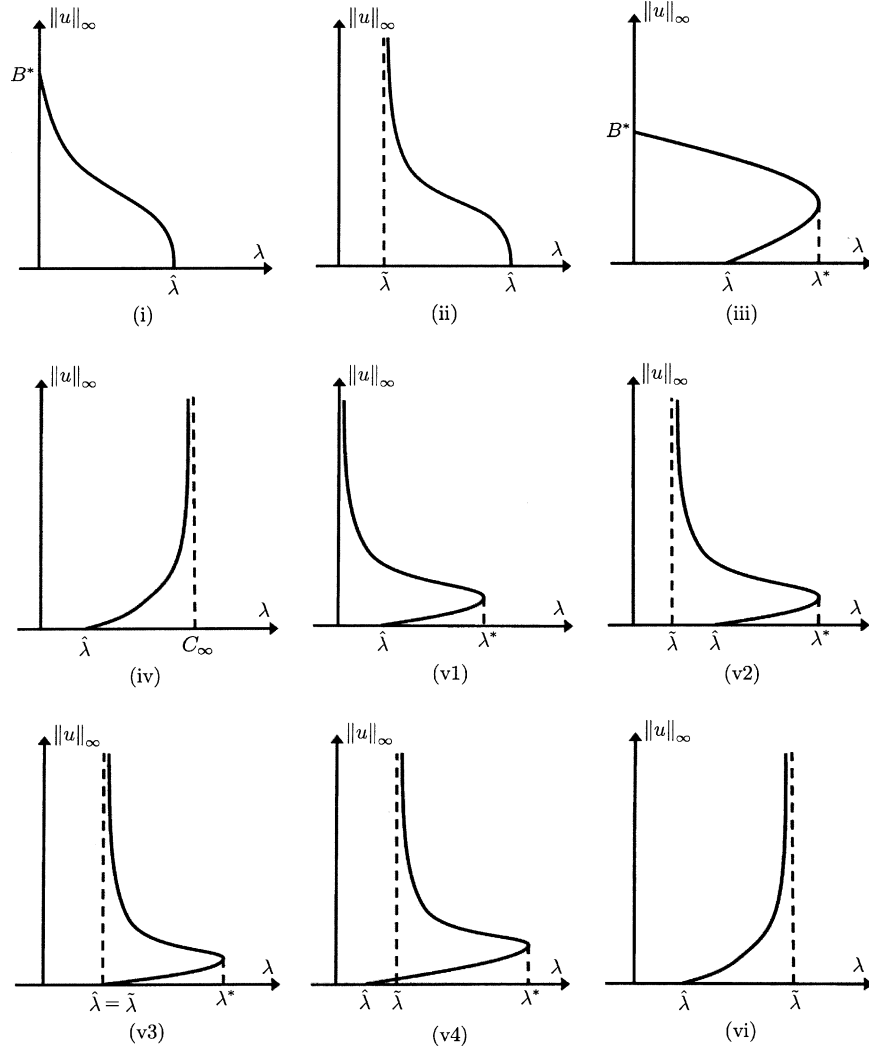


Fig. 1. Completely classified bifurcation diagrams of (1.1) under (H1)–(H3).

where $k > 1$, $\varphi_k(y) = |y|^{k-2}y$ and $(\varphi_k(u'))'$ is the one-dimensional k -Laplacian. For $0 < q < k - 1 < p$, they proved the existence of some $\lambda^* > 0$ such that (1.3) has exactly two positive solutions for $0 < \lambda < \lambda^*$, exactly one positive solution for $\lambda = \lambda^*$, and no positive solution for $\lambda > \lambda^*$. Very recently, for $N \geq 3$ and $\Omega = B_1$ is the unit ball in \mathbf{R}^N , Tang [15] proved the same exact multiplicity result for (1.2) where $0 < q < 1 < p \leq N^*$. By using the transformation

$$w = \lambda^{-1/(p-q)}u, \quad R = \lambda^{(p-1)/(2(p-q))}, \quad y = Rx, \quad (1.4)$$

he converted (1.2) into

$$\begin{cases} \Delta w(y) + w^q + w^p = 0 & \text{in } B_R, \\ w = 0 & \text{on } \partial B_R, \end{cases} \quad (1.5)$$

where B_R is a ball of radius $R > 0$ in \mathbf{R}^N . He studied (1.5) by an ODE method since any solution of (1.5) in B_R is necessary radial by the well-known symmetry result of Gidas–Ni–Nirenberg [4]. Then he converted the results obtained for (1.5), to the counterpart for (1.2). However, we note that the technique of using transformation (1.4) does *not* work to study problem (1.1) for *general* nonlinearity $f_\lambda = \lambda g(u) + h(u)$.

Recently, the authors [16] studied the exact structure of positive solutions of the problem

$$\begin{cases} u''(x) + \lambda \sum_{i=1}^m a_i u^{q_i} + \sum_{j=1}^n b_j u^{p_j} = 0, & -1 < x < 1, \\ u(-1) = u(1) = 0, \end{cases} \quad (1.6)$$

where $\lambda > 0$ is a bifurcation parameter and $f_\lambda(u) = \lambda \sum_{i=1}^m a_i u^{q_i} + \sum_{j=1}^n b_j u^{p_j}$ satisfies

$$\begin{cases} 0 < q_1 < q_2 < \cdots < q_m < 1 \leq p_1 < p_2 < \cdots < p_n, & m, n \geq 1, \quad p_n > 1, \\ a_i > 0 & \text{for } i = 1, 2, \dots, m \quad \text{and} \quad b_j > 0 \quad \text{for } j = 1, 2, \dots, n, \\ \text{and either } p_1 > 1 \text{ or } b_1 < \frac{\pi^2}{4}. \end{cases} \quad (1.7)$$

(Note that $\pi^2/4$ is the first eigenvalue of the operator $-d^2/dx^2$ on $(-1, 1)$ with zero Dirichlet boundary conditions.) The authors of [16, Theorem 2.2] mainly proved

Theorem 1.1 (See Fig. 1(iii) with $\hat{\lambda} = 0$). *Consider (1.6) where $f_\lambda = \lambda \sum_{i=1}^m a_i u^{q_i} + \sum_{j=1}^n b_j u^{p_j}$ satisfies (1.7). Then*

- (i) *There exists $\lambda^* > 0$ such that (1.6) has exactly two positive solutions u_λ, v_λ with $u_\lambda < v_\lambda$ for $0 < \lambda < \lambda^*$, exactly one positive solution u_{λ^*} for $\lambda = \lambda^*$, and no positive solution for $\lambda > \lambda^*$. Moreover, if we denote $u_{\lambda^*} = v_{\lambda^*}$ when $\lambda = \lambda^*$, then for $0 < \lambda_1 < \lambda_2 \leq \lambda^*$, the solutions of (1.6) satisfy $\|u_{\lambda_1}\|_\infty < \|u_{\lambda_2}\|_\infty$ and $\|v_{\lambda_1}\|_\infty > \|v_{\lambda_2}\|_\infty$.*
- (ii) *Let u be a positive solution of (1.6). Then there exists a unique positive number B^* satisfying $T_0(B^*) = 1$, where T_0 is defined in (1.8) below, such that $\|u\|_\infty < B^*$. In particular, if $n = 1$,*

$$B^* = \left[\frac{\sqrt{\pi(p_1+1)}\Gamma\left(\frac{p_1+2}{p_1+1}\right)}{\sqrt{2b_1}\Gamma\left(\frac{p_1+3}{2p_1+2}\right)} \right]^{2/(p_1-1)}.$$

- (iii) *Suppose u_λ, v_λ are the two positive solutions of (1.6) with $u_\lambda < v_\lambda$. Then $\|u_\lambda\|_\infty < \|u_{\lambda^*}\|_\infty < \|v_\lambda\|_\infty$, $\lim_{\lambda \rightarrow 0^+} \|u_\lambda\|_\infty = 0$ and $\lim_{\lambda \rightarrow 0^+} \|v_\lambda\|_\infty = B^*$.*
- (iv) *If $p_1 = 1$, for fixed a_i, b_j, q_i, p_j ($1 \leq i \leq m$ and $2 \leq j \leq n$), positive numbers $\lambda^* = \lambda^*(b_1)$ and $B^* = B^*(b_1)$ are both strictly decreasing in $b_1 \in (0, \pi^2/4)$. In addition, $\lim_{b_1 \rightarrow (\pi^2/4)^-} \lambda^*(b_1) = 0 = \lim_{b_1 \rightarrow (\pi^2/4)^-} B^*(b_1)$.*

We note that, very recently, the authors of [17, Theorem 2.2] extended all the results in Theorem 1.1 for more general k -Laplacian problems, which generalized and improved the results of Addou et al. [1] and Sanchez and Ubilla [12].

Now, for (1.1) with $f_\lambda(u) = \lambda g(u) + h(u)$ satisfying (H1)–(H3), we first define

$$m_0^g = \lim_{u \rightarrow 0^+} g(u)/u \in [0, \infty], \quad m_\infty^g = \lim_{u \rightarrow \infty} g(u)/u \in [0, \infty],$$

$$m_0^h = \lim_{u \rightarrow 0^+} h(u)/u \in [0, \infty], \quad m_\infty^h = \lim_{u \rightarrow \infty} h(u)/u \in [0, \infty].$$

Remark 1. Hypotheses (H1) and (H2) imply that $g(u)/u$ is strictly decreasing in $u > 0$ and $h(u)/u$ is strictly increasing in $u > 0$. So $0 < m_0^g, m_\infty^h \leq \infty$ and $0 \leq m_\infty^g, m_0^h < \infty$.

Remark 2. For (1.6), it is easy to check that nonlinearity $f_\lambda(u) = \lambda g(u) + h(u) = \lambda \sum_{i=1}^m a_i u^{q_i} + \sum_{j=1}^n b_j u^{p_j}$ satisfying (1.7) satisfies (H1)–(H3) and is a concave–convex function on $(0, \infty)$ for each fixed $\lambda > 0$. While in (1.1), for general nonlinearity $f_\lambda(u) = \lambda g(u) + h(u)$ satisfying (H1)–(H3), since $f_\lambda''(u) = \lambda g''(u) + h''(u)$, it is easy to see that

- (i) $f_\lambda(u)$ is a convex function on $(0, \infty)$ for each fixed $\lambda \in (0, D_0]$,
- (ii) $f_\lambda(u)$ is a concave–convex function on $(0, \infty)$ for each fixed $\lambda \in (D_0, D_\infty)$,
- (iii) $f_\lambda(u)$ is a concave function on $(0, \infty)$ for each fixed $\lambda \in [D_\infty, \infty)$, where

$$D_0 := - \lim_{u \rightarrow 0^+} \frac{h''(u)}{g''(u)} \in [0, \infty], \quad D_\infty := - \lim_{u \rightarrow \infty} \frac{h''(u)}{g''(u)} \in [0, \infty].$$

(Note that the strictly increasing hypothesis of $-h''(u)/g''(u)$ in (H3) is essentially used here to get the results in (i)–(iii). Also note that, if $f_\lambda(u) = \lambda g(u) + h(u)$ satisfies (H1)–(H3), then $D_0 \in [0, \infty)$ and $D_\infty \in (0, \infty]$.) In addition, for $f_\lambda(u) = \lambda g(u) + h(u) = \lambda \sum_{i=1}^m a_i u^{q_i} + \sum_{j=1}^n b_j u^{p_j}$ satisfying (1.7) satisfies $m_0^g = \infty$, $m_\infty^g = 0$, $0 \leq m_0^h < \infty$, and $m_\infty^h = \infty$. While in (1.1), for general nonlinearity $f_\lambda(u) = \lambda g(u) + h(u)$ satisfying (H1)–(H3), we allow $0 < m_0^g, m_\infty^h \leq \infty$ and $0 \leq m_\infty^g, m_0^h < \infty$ by Remark 1. These facts make the analysis of the bifurcation diagrams of (1.1) for general nonlinearity $f_\lambda(u) = \lambda g(u) + h(u)$ becomes more difficult. These facts also make the occurrence of additional types of bifurcation diagrams of (1.1), different from the one (Fig. 1(iii) with $\hat{\lambda} = 0$) obtained in Theorem 1.1; see Theorem 2.1 stated in Section 2 and Figs. 1–3.

In this paper, for $\lambda \geq 0$, we assume that $f_\lambda(u) = \lambda g(u) + h(u)$, $g, h \in C[0, \infty) \cap C^2(0, \infty)$, and we define

$$F_\lambda(u) = \int_0^u f_\lambda(t) dt, \quad G(u) = \int_0^u g(t) dt, \quad H(u) = \int_0^u h(t) dt,$$

$$T_\lambda(\alpha) = \frac{1}{\sqrt{2}} \int_0^\alpha [F_\lambda(\alpha) - F_\lambda(u)]^{-1/2} du \quad \text{for } 0 < \alpha < \infty, \quad (1.8)$$

$$\theta_{f_\lambda}(u) = 2F_\lambda(u) - u f_\lambda(u), \quad \theta_g(u) = 2G(u) - u g(u),$$

$$\theta_h(u) = 2H(u) - u h(u), \quad (1.9)$$

$$C_0 = - \lim_{u \rightarrow 0^+} \frac{\theta_h'(u)}{\theta_g'(u)} = - \lim_{u \rightarrow 0^+} \frac{h(u) - u h'(u)}{g(u) - u g'(u)} \in [0, \infty],$$

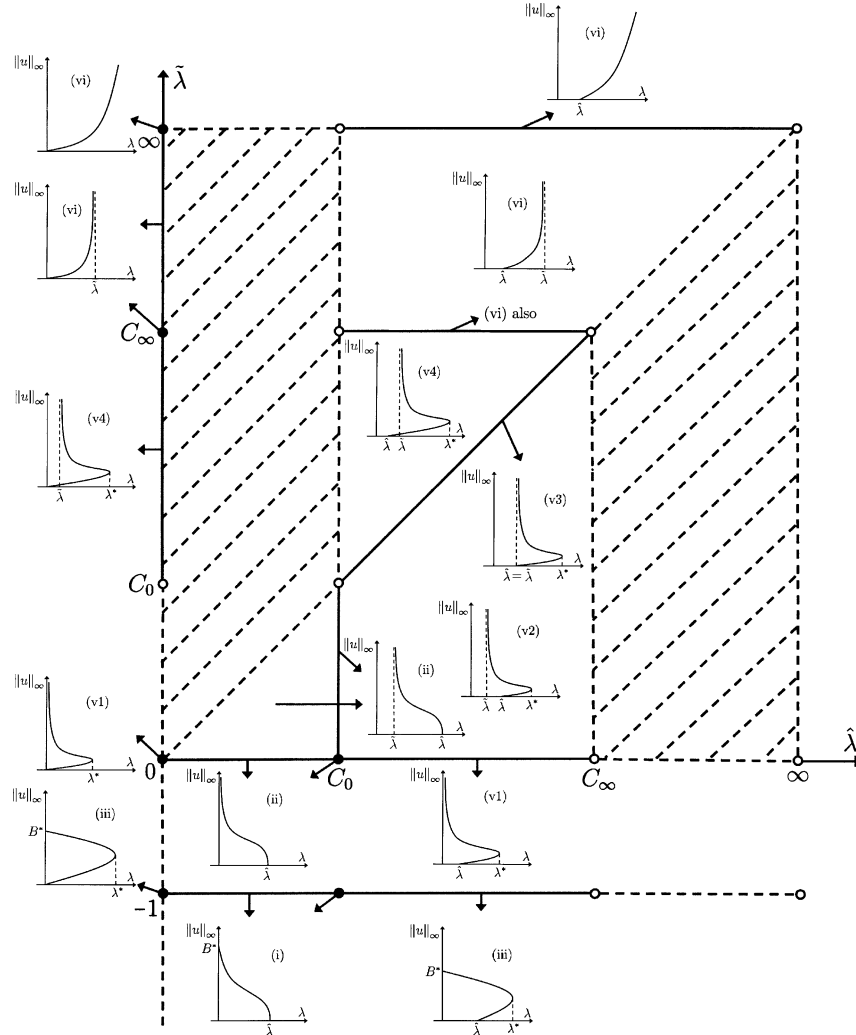


Fig. 2. Completely classified bifurcation diagrams of (1.1) drawn on the $(\hat{\lambda}, \tilde{\lambda})$ -plane when $0 \leq m_0^h < \pi^2/4$ and $0 < m_\infty^g < \infty$. Slash-lined regions represent regions of nonexistence of positive solutions of (1.1) under (H1)–(H3); see Appendix A for the proof of nonexistence results on some regions.

$$C_\infty = - \lim_{u \rightarrow \infty} \frac{\theta'_h(u)}{\theta'_g(u)} = - \lim_{u \rightarrow \infty} \frac{h(u) - uh'(u)}{g(u) - ug'(u)} \in [0, \infty],$$

$$\hat{\lambda} = \begin{cases} \frac{\pi^2/4 - m_0^h}{m_0^g} \in [0, \infty] & \text{if } 0 \leq m_0^h < \frac{\pi^2}{4} \text{ and } 0 \leq m_0^g \leq \infty, \\ 0 & \text{if } m_0^h = \frac{\pi^2}{4}, \\ -1 & \text{if } m_0^h > \frac{\pi^2}{4}, \end{cases}$$

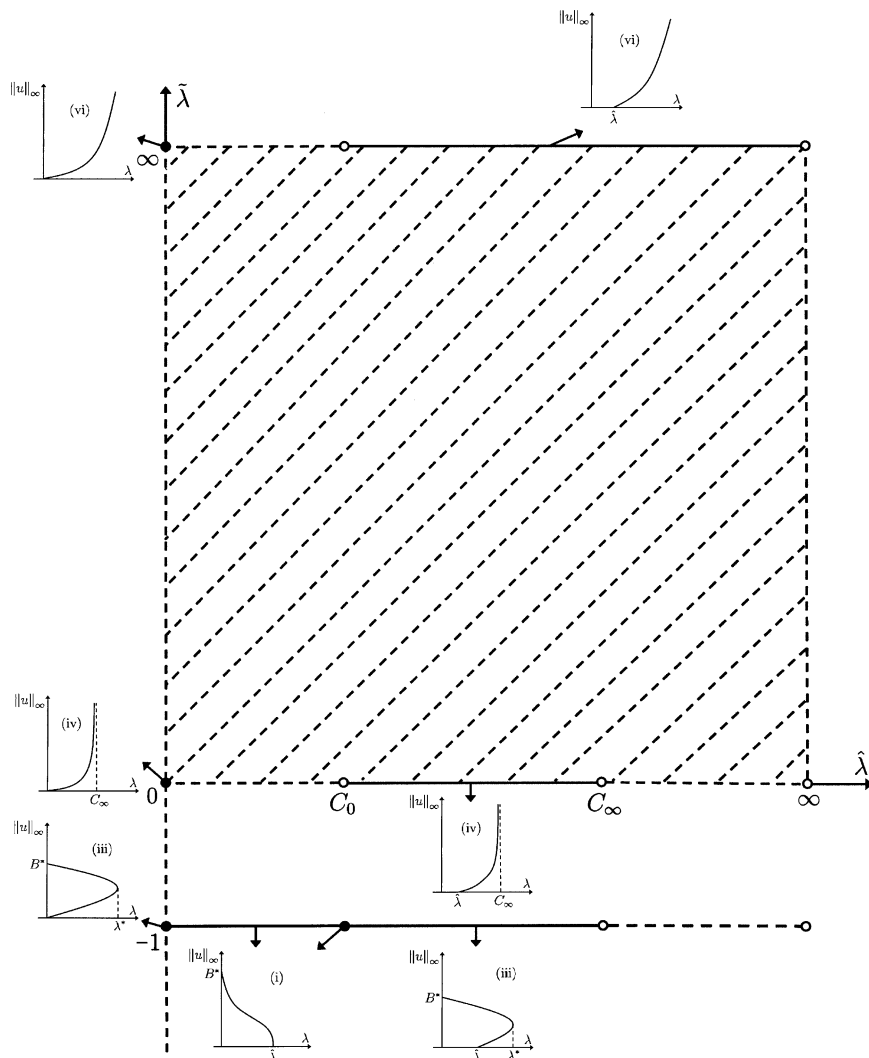


Fig. 3. Completely classified bifurcation diagrams of (1.1) drawn on the $(\hat{\lambda}, \tilde{\lambda})$ -plane when $0 \leq m_0^h < \pi^2/4$ and $m_\infty^g = 0$. Slash-lined regions represent regions of nonexistence of positive solutions of (1.1) under (H1)–(H3); see Appendix A for the proof of nonexistence results on some regions.

$$\tilde{\lambda} = \begin{cases} \frac{\pi^2/4 - m_\infty^h}{m_\infty^g} \in [0, \infty] & \text{if } 0 \leq m_\infty^h < \frac{\pi^2}{4} \text{ and } 0 \leq m_\infty^g \leq \infty, \\ 0 & \text{if } m_\infty^h = \frac{\pi^2}{4}, \\ -1 & \text{if } m_\infty^h > \frac{\pi^2}{4}. \end{cases}$$

Remark 3. If $f_\lambda(u) = \lambda g(u) + h(u)$ satisfies (H1)–(H3), then $C_0 \in [0, \infty)$ and $C_\infty \in (0, \infty]$.

Remark 4. By [8, Theorems 2.7 and 2.10], for (1.1), it can be easily proved that

- (i) If $0 < \hat{\lambda} < \infty$, then $\hat{\lambda}$ is the unique positive number such that $\lim_{\alpha \rightarrow 0^+} T_{\hat{\lambda}}(\alpha) = 1$. In addition, by Remark 1, $m_0^g > 0$, and hence $\hat{\lambda} < \infty$ by the definition of $\hat{\lambda}$.
- (ii) If $0 < \tilde{\lambda} < \infty$, then $\tilde{\lambda}$ is the unique positive number such that $\lim_{\alpha \rightarrow \infty} T_{\tilde{\lambda}}(\alpha) = 1$.

In Theorem 2.1 stated in Section 2, for $f_{\lambda}(u) = \lambda g(u) + h(u)$ satisfying (H1)–(H3), we give a complete classification of bifurcation diagrams of (1.1) as follows (see Figs. 1–3).

Case 1: $\pi^2/4 \leq m_0^h < \infty$. In this case, there exists no positive solution for (1.1).

Case 2: $0 \leq m_0^h < \pi^2/4$ and $0 < \hat{\lambda} \leq C_0$. In this case, we classify all possible subcases as follows:

- (i) $\tilde{\lambda} < 0$ (see Fig. 1(i)),
- (ii) $\tilde{\lambda} \geq 0$ (see Fig. 1(ii)).

Case 3: $0 \leq m_0^h < \pi^2/4$ and either $\hat{\lambda} = 0$ or $C_0 < \hat{\lambda} < \infty$. In this case, we classify all possible subcases as follows:

- (iii) $\tilde{\lambda} < 0$ (see Fig. 1(iii)),
- (iv) $\tilde{\lambda} = 0$ and $m_{\infty}^g = 0$ (see Fig. 1(iv)),
- (v1) $\tilde{\lambda} = 0$ and $0 < m_{\infty}^g < \infty$ (see Fig. 1(v1)),
- (v2) $0 < \tilde{\lambda} < C_{\infty}$ and $\hat{\lambda} > \tilde{\lambda}$ (see Fig. 1(v2)),
- (v3) $0 < \tilde{\lambda} < C_{\infty}$ and $\hat{\lambda} = \tilde{\lambda}$ (see Fig. 1(v3)),
- (v4) $0 < \tilde{\lambda} < C_{\infty}$ and $\hat{\lambda} < \tilde{\lambda}$ (see Fig. 1(v4)),
- (vi) $C_{\infty} \leq \tilde{\lambda} \leq \infty$ (see Fig. 1(vi)).

Finally, we note that it is interesting to study *all* possible bifurcation diagrams of problem (1.1) simply under hypotheses (H1) and (H2). To this, only very partial results are obtained and further research is needed. However, it is important to note that, problem (1.1) may have *more than two* positive solutions for some nonlinearities g, h satisfying (H1) and (H2) and for some $\lambda > 0$. For example, take

$$g(u) = \frac{u + u^{3/2}}{1 + 9u^{1/2}} \quad \text{and} \quad h(u) = \frac{u + 2u^2}{1 + u},$$

which satisfy (H1) and (H2). A numerical simulation for solving the equation $T_{\lambda=4}(\alpha) = 1$ (see (4.1) stated below) shows that, for $\lambda = 4$, problem (1.1) has *three* positive solutions u^1, u^2, u^3 with $\alpha = \|u^1\|_{\infty} \approx 0.12$, $\alpha = \|u^2\|_{\infty} \approx 10.14$, $\alpha = \|u^3\|_{\infty} \approx 237.84$, respectively; see Fig. 4(a). Actually, numerical simulations show that the bifurcation curve of (1.1) on the $(\lambda, \|u\|_{\infty})$ -plane is S-shaped; that is, the bifurcation curve has at least two turning points; see Fig. 4(b).

The paper is organized as follows. Section 2 contains the statement of Theorem 2.1 and corresponding examples. Section 3 contains the lemmas needed to prove Theorem 2.1. Section 4 contains the proof of Theorem 2.1. Appendix A contains the proof of nonexistence of positive solutions of (1.1) in slash-lined regions in Figs. 2 and 3.

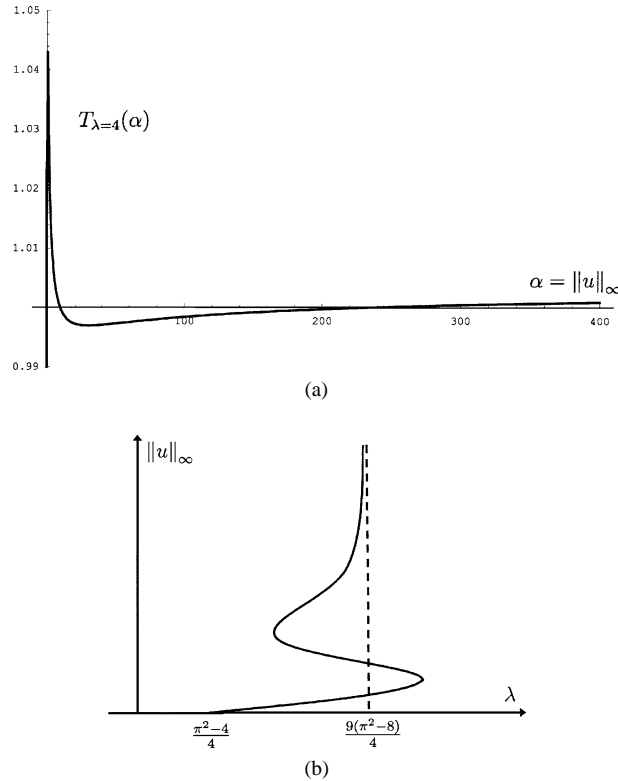


Fig. 4. (a) Numerical simulation of $T_{\lambda=4}(\alpha)$: $f_4 = 4 \frac{u+u^{3/2}}{1+9u^{1/2}} + \frac{u+2u^2}{1+u}$ for $\alpha \in (10^{-6}, 400)$. Note that $\lim_{\alpha \rightarrow 0^+} T_{\lambda=4}(\alpha) = \frac{\pi}{2\sqrt{5}} \approx 0.702 < 1$ and $\lim_{\alpha \rightarrow \infty} T_{\lambda=4}(\alpha) = \frac{3\pi}{2\sqrt{22}} \approx 1.005 > 1$. (b) Conjectured S-shaped bifurcation diagram of (1.1) for $f_{\lambda} = \lambda \frac{u+u^{3/2}}{1+9u^{1/2}} + \frac{u+2u^2}{1+u}$, $\lambda > 0$.

2. Main result

The main result in this paper is next Theorem 2.1 in which we study the exact multiplicity of positive solutions of (1.1) for all cases classified in Section 1.

Theorem 2.1 (See Figs. 1–3). *Consider (1.1) where $f_{\lambda}(u) = \lambda g(u) + h(u)$, $g, h \in C[0, \infty) \cap C^2(0, \infty)$, and g, h satisfy (H1)–(H3). Then*

- (I) *If $\pi^2/4 \leq m_0^h < \infty$, then (1.1) has no positive solution for all $\lambda > 0$.*
- (II) *Suppose that $0 \leq m_0^h < \pi^2/4$ and $0 < \hat{\lambda} \leq C_0$.*
 - (i) *(See Fig. 1(i).) If $\tilde{\lambda} < 0$, then (1.1) has exactly one positive solution u_{λ} for $0 < \lambda < \hat{\lambda}$, and no positive solution for $\lambda \geq \hat{\lambda}$. Moreover,*
 - (a) *For $0 < \lambda_1 < \lambda_2 < \hat{\lambda}$, $\|u_{\lambda_1}\|_{\infty} > \|u_{\lambda_2}\|_{\infty}$.*
 - (b) *Let u be a positive solution of (1.1). Then there exists a unique positive number B^* satisfying $T_0(B^*) = 1$ such that $\|u\|_{\infty} < B^*$.*

- (c) $\lim_{\lambda \rightarrow 0^+} \|u_\lambda\|_\infty = B^*$ and $\lim_{\lambda \rightarrow \hat{\lambda}^-} \|u_\lambda\|_\infty = 0$.
- (ii) (See Fig. 1(ii).) If $\tilde{\lambda} \geq 0$, then $\tilde{\lambda} < \hat{\lambda}$, and (1.1) has exactly one positive solution u_λ for $\tilde{\lambda} < \lambda < \hat{\lambda}$, and no positive solution for $0 < \lambda \leq \tilde{\lambda}$ and $\lambda \geq \hat{\lambda}$. Moreover,
- (a) For $\tilde{\lambda} < \lambda_1 < \lambda_2 < \hat{\lambda}$, $\|u_{\lambda_1}\|_\infty > \|u_{\lambda_2}\|_\infty$.
- (b) $\lim_{\lambda \rightarrow \tilde{\lambda}^+} \|u_\lambda\|_\infty = \infty$ and $\lim_{\lambda \rightarrow \hat{\lambda}^-} \|u_\lambda\|_\infty = 0$.
- (III) Suppose that $0 \leq m_0^h < \pi^2/4$ and either $\hat{\lambda} = 0$ or $C_0 < \hat{\lambda} < \infty$.
- (iii) (See Fig. 1(iii).) If $\tilde{\lambda} < 0$, then there exists a positive number $\lambda^* > \hat{\lambda}$ such that (1.1) has exactly two positive solutions u_λ, v_λ with $u_\lambda < v_\lambda$ for $\hat{\lambda} < \lambda < \lambda^*$, exactly one positive solution v_λ for $0 < \lambda \leq \hat{\lambda}$ and $\lambda = \lambda^*$, and no positive solution for $\lambda > \lambda^*$. Moreover, if we denote $u_{\lambda^*} = v_{\lambda^*}$ when $\lambda = \lambda^*$, then
- (a) For $\hat{\lambda} < \lambda_1 < \lambda_2 \leq \lambda^*$, $\|u_{\lambda_1}\|_\infty < \|u_{\lambda_2}\|_\infty$.
- (b) For $0 < \lambda_1 < \lambda_2 \leq \lambda^*$, $\|v_{\lambda_1}\|_\infty > \|v_{\lambda_2}\|_\infty$.
- (c) Let u be a positive solution of (1.1). Then there exists a unique positive number B^* satisfying $T_0(B^*) = 1$ such that $\|u\|_\infty < B^*$.
- (d) $\lim_{\lambda \rightarrow \hat{\lambda}^+} \|u_\lambda\|_\infty = 0$ and $\lim_{\lambda \rightarrow 0^+} \|v_\lambda\|_\infty = B^*$.
- (iv) (See Fig. 1(iv).) If $\tilde{\lambda} = 0$ and $m_\infty^g = 0$, then (1.1) has exactly one positive solution u_λ for $\hat{\lambda} < \lambda < C_\infty$ ($\leq \infty$), and no positive solution for $0 < \lambda \leq \hat{\lambda}$ and $\lambda \geq C_\infty$. Moreover,
- (a) For $\hat{\lambda} < \lambda_1 < \lambda_2 < C_\infty$, $\|u_{\lambda_1}\|_\infty < \|u_{\lambda_2}\|_\infty$.
- (b) $\lim_{\lambda \rightarrow \hat{\lambda}^+} \|u_\lambda\|_\infty = 0$ and $\lim_{\lambda \rightarrow C_\infty^-} \|u_\lambda\|_\infty = \infty$.
- (v) (See Fig. 1(v1) for $\tilde{\lambda} = 0$ and $0 < m_\infty^g < \infty$; see Fig. 1(v2) for $0 < \tilde{\lambda} < C_\infty$ and $\hat{\lambda} > \tilde{\lambda}$; see Fig. 1(v3) for $0 < \tilde{\lambda} < C_\infty$ and $\hat{\lambda} = \tilde{\lambda}$; see Fig. 1(v4) for $0 < \tilde{\lambda} < C_\infty$ and $\hat{\lambda} < \tilde{\lambda}$.) If either ($\tilde{\lambda} = 0$ and $0 < m_\infty^g < \infty$) or $0 < \tilde{\lambda} < C_\infty$, then there exists a positive number $\lambda^* > \max\{\hat{\lambda}, \tilde{\lambda}\}$ such that (1.1) has exactly two positive solutions u_λ, v_λ with $u_\lambda < v_\lambda$ for $\max\{\hat{\lambda}, \tilde{\lambda}\} < \lambda < \lambda^*$, exactly one positive solution u_λ (if $\hat{\lambda} \leq \tilde{\lambda}$) or v_λ (if $\hat{\lambda} > \tilde{\lambda}$) for $(\min\{\hat{\lambda}, \tilde{\lambda}\} < \lambda \leq \max\{\hat{\lambda}, \tilde{\lambda}\})$ and $\lambda = \lambda^*$, and no positive solution for $0 < \lambda \leq \min\{\hat{\lambda}, \tilde{\lambda}\}$ and $\lambda > \lambda^*$. Moreover, if we denote $u_{\lambda^*} = v_{\lambda^*}$ when $\lambda = \lambda^*$, then
- (a) For $\hat{\lambda} < \lambda_1 < \lambda_2 \leq \lambda^*$, $\|u_{\lambda_1}\|_\infty < \|u_{\lambda_2}\|_\infty$.
- (b) For $\tilde{\lambda} < \lambda_1 < \lambda_2 \leq \lambda^*$, $\|v_{\lambda_1}\|_\infty > \|v_{\lambda_2}\|_\infty$.
- (c) $\lim_{\lambda \rightarrow \hat{\lambda}^+} \|u_\lambda\|_\infty = 0$ and $\lim_{\lambda \rightarrow \tilde{\lambda}^+} \|v_\lambda\|_\infty = \infty$.
- (vi) (See Fig. 1(vi).) If $C_\infty \leq \tilde{\lambda} \leq \infty$, then (1.1) has exactly one positive solution u_λ for $\hat{\lambda} < \lambda < \tilde{\lambda}$, and no positive solution for $0 < \lambda \leq \hat{\lambda}$ and $\lambda \geq \tilde{\lambda}$. Moreover,
- (a) For $\hat{\lambda} < \lambda_1 < \lambda_2 < \tilde{\lambda}$, $\|u_{\lambda_1}\|_\infty < \|u_{\lambda_2}\|_\infty$.
- (b) $\lim_{\lambda \rightarrow \hat{\lambda}^+} \|u_\lambda\|_\infty = 0$ and $\lim_{\lambda \rightarrow \tilde{\lambda}^-} \|u_\lambda\|_\infty = \infty$.

We then give corresponding examples of Theorem 2.1(i)–(vi) as follows.

Example of Theorem 2.1(i). Take

$$g(u) = \frac{u + u^2}{1 + 2u} \quad \text{and} \quad h(u) = 2u + u^2$$

with $m_0^h = 2 < \pi^2/4$, $0 < \hat{\lambda} = (\pi^2 - 8)/4 < 1 = C_0$, and $\tilde{\lambda} = -1$.

Example of Theorem 2.1(ii). Take

$$g(u) = \frac{2u + 2u^2}{2 + 3u} \quad \text{and} \quad h(u) = \frac{4u + \pi^2 u^2}{4 + 4u}$$

with $m_0^h = 1 < \pi^2/4$, $0 < \hat{\lambda} = (\pi^2 - 4)/4 < (\pi^2 - 4)/2 = C_0$, and $\tilde{\lambda} = 0$.

Example of Theorem 2.1(iii). Take

$$g(u) = u^{1/2} \quad \text{and} \quad h(u) = u^2$$

with $m_0^h = 0 < \pi^2/4$, $\hat{\lambda} = 0$, and $\tilde{\lambda} = -1$.

Example of Theorem 2.1(iv). Take

$$g(u) = \frac{u}{1 + 2u} \quad \text{and} \quad h(u) = \frac{4u + \pi^2 u^2}{4 + 4u}$$

with $m_0^h = 1 < \pi^2/4$, $C_0 = (\pi^2 - 4)/8 < (\pi^2 - 4)/4 = \hat{\lambda} < C_\infty = (\pi^2 - 4)/2$, $\tilde{\lambda} = 0$, and $m_\infty^g = 0$.

Examples of Theorem 2.1(v). (v1) Take

$$g(u) = \frac{u + 2u^2}{1 + 4u} \quad \text{and} \quad h(u) = \frac{4u + \pi^2 u^2}{4 + 4u}$$

with $m_0^h = 1 < \pi^2/4$, $C_0 = (\pi^2 - 4)/8 < (\pi^2 - 4)/4 = \hat{\lambda}$, $\tilde{\lambda} = 0$, and $m_\infty^g = 1/2 > 0$.

(v2) Take

$$g(u) = \frac{u + u^2}{1 + 2u} \quad \text{and} \quad h(u) = \frac{u + 2u^2}{1 + u}$$

with $m_0^h = 1 < \pi^2/4$, $C_0 = 1 < (\pi^2 - 4)/4 = \hat{\lambda}$, $0 < \tilde{\lambda} = (\pi^2 - 8)/2 < 4 = C_\infty$, and $\hat{\lambda} = (\pi^2 - 4)/4 > (\pi^2 - 8)/2 = \tilde{\lambda}$.

(v3) Take

$$g(u) = \frac{(\pi^2 - 4)u + 4(\pi^2 - 8)u^2}{4 + 16u} \quad \text{and} \quad h(u) = \frac{u + 2u^2}{1 + u}$$

with $m_0^h = 1 < \pi^2/4$, and $1/4 = C_0 < \hat{\lambda} = 1 = \tilde{\lambda} < C_\infty = 4$.

(v4) Take

$$g(u) = \frac{u + u^2}{1 + 4u} \quad \text{and} \quad h(u) = \frac{u + 2u^2}{1 + u}$$

with $m_0^h = 1 < \pi^2/4$, and $1/3 = C_0 < \hat{\lambda} = (\pi^2 - 4)/4 < \pi^2 - 8 = \tilde{\lambda} < C_\infty = 16/3$.

Example of Theorem 2.1(vi). Take

$$g(u) = \frac{u}{1 + 2u} \quad \text{and} \quad h(u) = \frac{u + 2u^2}{1 + u}$$

with $m_0^h = 1 < \pi^2/4$, $C_0 = 1/2 < (\pi^2 - 4)/4 = \hat{\lambda}$, and $C_\infty = 2 < \infty = \tilde{\lambda}$.

3. Lemmas

To prove Theorem 2.1, we modify the time map techniques applied to prove Theorem 1.1. We need the following Lemmas 3.1–3.3. Consider the two point boundary value problem

$$\begin{cases} u''(x) + \mu f(u) = 0, & -1 < x < 1, \\ u(-1) = u(1) = 0, \end{cases} \quad (3.1)$$

where $\mu > 0$ is a bifurcation parameter. Assume that $f \in C[0, \infty) \cap C^2(0, \infty)$. Let $F(u) = \int_0^u f(t) dt$ and $\theta(u) = 2F(u) - uf(u)$. The time map formula for problem (3.1) takes the form as follows:

$$\sqrt{\mu} = \frac{1}{\sqrt{2}} \int_0^\alpha [F(\alpha) - F(u)]^{-1/2} du := T(\alpha) \quad \text{for } 0 < \alpha < \infty; \quad (3.2)$$

see [8, Eq. (2.4)]. Positive solutions u of (3.1) correspond to $\|u\|_\infty = \alpha$ and $T(\alpha) = \sqrt{\mu}$. Thus to study the number of positive solutions of (3.1) is equivalent to study the shape of the time map $T(\alpha)$ on $(0, \infty)$. The following key lemma is mainly due to Korman and Shi [7, Theorem 3] after a generalization by Shi [14]. We are grateful to Shi for providing the proof of the key lemma that we need.

Lemma 3.1. *Consider (3.1). Suppose that $f \in C[0, \infty) \cap C^2(0, \infty)$ satisfies $f(0) = 0$, $f(u) > 0$ for $u > 0$, and assume that for some $\gamma > \beta > 0$ we have*

- (i) $\theta(\gamma) \leq 0$,
- (ii) $\theta'(u) = f(u) - uf'(u) < 0$ for $u > \gamma$,
- (iii) $f''(u) < 0$ for $0 < u < \beta$, $f''(u) > 0$ for $\beta < u < \gamma$,
- (iv) *there exist numbers $0 < a < 1$, $b_1, b_2, c > 0$ such that $|f'(u)| \leq b_1 u^{a-1}$, $|f''(u)| \leq b_2 u^{a-2}$ for $u \in (0, c)$.*

If we set $0 < \lim_{u \rightarrow 0^+} f(u)/u := m_0 \leq \infty$ and $0 < \lim_{u \rightarrow \infty} f(u)/u := m_\infty \leq \infty$, then

$$\lim_{\alpha \rightarrow 0^+} T(\alpha) = \frac{\pi}{2\sqrt{m_0}} \geq 0, \quad \lim_{\alpha \rightarrow \infty} T(\alpha) = \frac{\pi}{2\sqrt{m_\infty}} \geq 0,$$

and $T(\alpha)$ has exactly one critical point, a maximum, on $(0, \infty)$.

Proof. For any positive solution $u(x)$ of (3.1), $u \in C_0^1[-1, 1]$ from the Hölder estimates and $[u'(x)]^2 = 2F(u(0)) - 2F(u(x))$. Also since $|u'(\pm 1)| > 0$, then there exist constants $k_1, k_2 > 0$ such that $k_1 d(x) < u(x) < k_2 d(x)$ for a solution u of (3.1), where $d(x) = 1 - |x|$; see [5, p. 778]. Therefore assumptions (H.1)–(H.3) in [5, p. 778] are satisfied because of assumption (iv).

Following the approach in [5, Section 3], we define $G: C_0^1[-1, 1] \rightarrow C_0^1[-1, 1]$ to be the Green operator; that is, $G(v) = u$ if $-u'' = v$, $u(-1) = u(1) = 0$. Then (3.1) becomes

$$F(\mu, u) := u - \mu G(f(u)) = 0. \quad (3.3)$$

From [5, Theorem 3.1] and assumption (iv), F is C^2 with respect to u , and

$$\begin{aligned}\frac{\partial F}{\partial u}v &= v - \mu G(f'(u)v), \\ \frac{\partial^2 F}{\partial u^2}(v_1, v_2) &= -\mu G(f''(u)v_1v_2)\end{aligned}$$

for $v, v_1, v_2 \in C_0^1[-1, 1]$. Thus the solution curve $(\mu(\alpha), u(\alpha))$ is C^2 with respect to $\alpha = \|u\|_\infty$. By differentiating Eq. (3.3) with respect to α , we obtain

$$\begin{aligned}u_\alpha - \mu G(f'(u)u_\alpha) - \mu_\alpha G(f(u)) &= 0, \\ u_{\alpha\alpha} - \mu G(f'(u)u_{\alpha\alpha}) - 2\mu_\alpha G(f'(u)u_\alpha) - \mu_{\alpha\alpha} G(f(u)) - \mu G(f''(u)u_\alpha^2) &= 0.\end{aligned}$$

Notice that $f'(u)u_\alpha, f''(u)u_\alpha^2 \in C_0^1[-1, 1]$ from assumption (iv). At a critical point $\alpha = \alpha_0$, $\mu_\alpha(\alpha_0) = 0$, thus we have

$$\begin{aligned}u_\alpha - \mu G(f'(u)u_\alpha) &= 0, \\ u_{\alpha\alpha} - \mu G(f'(u)u_{\alpha\alpha}) - \mu_{\alpha\alpha} G(f(u)) - \mu G(f''(u)u_\alpha^2) &= 0.\end{aligned}$$

In particular, u_α and $u_{\alpha\alpha}$ are, respectively, weak solutions of

$$\begin{aligned}u_\alpha'' + \mu f'(u)u_\alpha &= 0, \quad u_\alpha(-1) = u_\alpha(1) = 0, \\ u_{\alpha\alpha}'' + \mu f'(u)u_{\alpha\alpha} + \mu_{\alpha\alpha} f(u) + \mu f''(u)u_\alpha^2 &= 0, \quad u_{\alpha\alpha}(-1) = u_{\alpha\alpha}(1) = 0.\end{aligned}\quad (3.4)$$

Therefore, from the definition of weak solutions, we have

$$-\int_{-1}^1 u_\alpha' u_{\alpha\alpha}' dx + \int_{-1}^1 \mu f'(u)u_\alpha u_{\alpha\alpha} dx = 0, \quad (3.5)$$

$$\begin{aligned}-\int_{-1}^1 u_\alpha' u_{\alpha\alpha}' dx + \int_{-1}^1 \mu f'(u)u_\alpha u_{\alpha\alpha} dx \\ + \int_{-1}^1 [\mu_{\alpha\alpha} f(u) + \mu f''(u)u_\alpha^2]u_\alpha dx &= 0.\end{aligned}\quad (3.6)$$

From (3.5) and (3.6), we obtain

$$\mu_{\alpha\alpha}(\alpha_0) = -\mu(\alpha_0) \frac{\int_{-1}^1 f''(u)w^3 dx}{\int_{-1}^1 f(u)w dx}, \quad (3.7)$$

where $w(x) = u_\alpha(\alpha_0, x)$ is the weak solution of (3.4). From the uniqueness of the solutions of ordinary differential equations, both $u(x)$ and $w(x)$ are even functions, thus (3.7) can also be written as

$$\mu_{\alpha\alpha}(\alpha_0) = -\mu(\alpha_0) \frac{\int_0^1 f''(u)w^3 dx}{\int_0^1 f(u)w dx}. \quad (3.8)$$

Note that (3.8) is well known when $f \in C^2[0, \infty)$; see [3,9,10,13]. Also from the uniqueness of the solutions of ordinary differential equations, $w \in C^2[0, 1) \cap C_0^1[0, 1]$, and w satisfies

$$w'' + \mu f'(u)w = 0, \quad x \in (0, 1), \quad w'(0) = w(1) = 0.$$

From Sturm's comparison lemma, w has no zeros on $(0, 1)$, and $w(0) = 1$, thus $w(x) > 0$ for $x \in [0, 1)$. On the other hand, $u' \in C^2[0, 1) \cap C[0, 1]$, and it satisfies

$$(u')'' + \mu f'(u)u' = 0, \quad x \in (0, 1), \quad u'(0) = 0, \quad u'(x) < 0, \quad x \in (0, 1]. \quad (3.9)$$

From Sturm's comparison lemma, w and $-u'$ has at most one intersection point on $(0, 1)$, and since $w(0) > 0$, $w(1) = 0$, $-u'(0) = 0$, $-u'(1) > 0$, then the graphs of w and $-u'$ intersects exactly once on $(0, 1)$. Same is true for kw for $k > 0$ and $-u'$, so for any $x_0 \in (0, 1)$, we can choose some $\bar{k} > 0$ such that

$$\bar{k}w(x) > -u'(x), \quad x \in [0, x_0], \quad \bar{k}w(x) < -u'(x), \quad x \in [x_0, 1]. \quad (3.10)$$

Because of assumption (iii) and $u'(x) < 0$ for $x \in (0, 1]$, there exists $x_0 \in (0, 1)$ such that $u(x_0) = \beta$, and $u(x) > \beta$ for $x \in [0, \beta)$, $u(x) < \beta$ for $x \in (\beta, 1]$. From assumption (iii) and (3.10), we have

$$\bar{k}^2 \int_0^1 f''(u)w^3 dx > \int_0^1 f''(u)(u')^2 w dx.$$

Note that the latter integral is convergent since $|f''(u(x))| \leq C|x - 1|^{a-2}$, $|w(x)| \leq C|x - 1|$ for some constant $C > 0$ when $x \rightarrow 1^-$. Since w' satisfies

$$(w')'' + \mu f'(u)w' + \mu f''(u)u'w = 0, \quad x \in (0, 1), \quad w'(0) = 0, \quad (3.11)$$

then by (3.9) and (3.11), for any $y \in (0, 1)$, we have

$$\begin{aligned} -\mu f(u(y))w'(y) + \mu f'(u(y))w(y)u'(y) &= u''(y)w'(y) - w''(y)u'(y) \\ &= \mu \int_0^y f''(u)(u')^2 w dx. \end{aligned}$$

As $y \rightarrow 1^-$, the limit of the left hand side is 0 since $|f'(u(y))w(y)| \leq C|y - 1|^{a-1}$. $|y - 1| \rightarrow 0$. Therefore

$$\bar{k}^2 \int_0^1 f''(u)w^3 dx > \int_0^1 f''(u)(u')^2 w dx = 0.$$

On the other hand, $\int_0^1 f(u)w dx > 0$ since $f > 0$ and $w > 0$. Hence $\mu''(\alpha_0) (= \mu_{\alpha\alpha}(\alpha_0)) < 0$ for any critical point $\alpha = \alpha_0$. In particular, it implies that $\mu(\alpha)$ has at most one critical point on $(0, \infty)$.

From the time-map formula (3.2), it is easy to see that $\mu'(\alpha) > 0$ for small $\alpha > 0$, $\mu'(\alpha) < 0$ for all $\alpha \geq \gamma$ from (i) and (ii); see details in [6] or [7, Theorem 3]. Therefore

$\mu(\alpha)$ has exactly one critical point on $(0, \gamma)$, which is a maximum since $\mu''(\alpha_0) < 0$ at the critical point $\alpha = \alpha_0$. This completes the proof of Lemma 3.1. \square

In next two lemmas we study some properties of time map $T_\lambda(\alpha)$ defined in (1.8) for (1.1).

Lemma 3.2. Consider (1.1) where $f_\lambda(u) = \lambda g(u) + h(u)$, $g, h \in C[0, \infty) \cap C^2(0, \infty)$, and g, h satisfy (H1). Then, for each fixed $\alpha > 0$, $T_\lambda(\alpha)$ is a continuous function of $\lambda \geq 0$ and $\lim_{\lambda \rightarrow \infty} T_\lambda(\alpha) = 0$.

Proof. First, for each fixed $\alpha > 0$, by (1.8),

$$\begin{aligned} T_\lambda(\alpha) &= \frac{1}{\sqrt{2}} \int_0^\alpha [F_\lambda(\alpha) - F_\lambda(u)]^{-1/2} du \\ &= \frac{1}{\sqrt{2}} \alpha \int_0^{1/2} [F_\lambda(\alpha) - F_\lambda(\alpha v)]^{-1/2} dv + \frac{1}{\sqrt{2}} \alpha \int_{1/2}^1 [F_\lambda(\alpha) - F_\lambda(\alpha v)]^{-1/2} dv \\ &= I(\lambda) + J(\lambda), \end{aligned}$$

where

$$I(\lambda) := \frac{1}{\sqrt{2}} \alpha \int_0^{1/2} [F_\lambda(\alpha) - F_\lambda(\alpha v)]^{-1/2} dv$$

and

$$J(\lambda) := \frac{1}{\sqrt{2}} \alpha \int_{1/2}^1 [F_\lambda(\alpha) - F_\lambda(\alpha v)]^{-1/2} dv.$$

We define

$$J_n(\lambda) = \frac{1}{\sqrt{2}} \alpha \int_{1/2}^{1-1/n} [F_\lambda(\alpha) - F_\lambda(\alpha v)]^{-1/2} dv.$$

For each fixed $\lambda_0 \geq 0$, $J_n(\lambda)$ is continuous on $[0, \lambda_0]$ and

$$\begin{aligned} |J_n(\lambda) - J(\lambda)| &= \frac{1}{\sqrt{2}} \alpha \int_{1-1/n}^1 [F_\lambda(\alpha) - F_\lambda(\alpha v)]^{-1/2} dv \\ &= \frac{1}{\sqrt{2}} \alpha \int_{1-1/n}^1 \left[\int_{\alpha v}^\alpha f_\lambda(s) ds \right]^{-1/2} dv \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\sqrt{2}} \alpha \int_{1-1/n}^1 \left[\int_{\alpha v}^{\alpha} h(s) ds \right]^{-1/2} dv \\
&\leq \frac{1}{\sqrt{2}} \alpha \int_{1-1/n}^1 [\rho \alpha (1-v)]^{-1/2} dv, \quad \text{where } \rho = \min_{u \in [\alpha/2, \alpha]} h(u), \\
&= \sqrt{\frac{\alpha}{2\rho}} [-2(1-v)^{1/2}] \Big|_{1-1/n}^1 \\
&= \sqrt{\frac{2\alpha}{\rho n}} \rightarrow 0 \quad \text{as } n \rightarrow \infty
\end{aligned}$$

for $\lambda \in [0, \lambda_0]$. So $J_n(\lambda)$ is uniformly converges to $J(\lambda)$ on $[0, \lambda_0]$. Then $J(\lambda)$ is continuous on $[0, \lambda_0]$, and since $I(\lambda)$ is continuous on $[0, \lambda_0]$. Hence $T_\lambda(\alpha)$ is a continuous function of $\lambda \geq 0$.

Next, for each fixed $\alpha > 0$,

$$\begin{aligned}
T_\lambda(\alpha) &= \frac{1}{\sqrt{2}} \int_0^\alpha [F_\lambda(\alpha) - F_\lambda(u)]^{-1/2} du \\
&= \frac{1}{\sqrt{2}} \alpha \int_0^1 [F_\lambda(\alpha) - F_\lambda(\alpha v)]^{-1/2} dv \\
&= \frac{1}{\sqrt{2\lambda}} \alpha \int_0^{1/2} \left[\int_{\alpha v}^{\alpha} \frac{1}{\lambda} f_\lambda(s) ds \right]^{-1/2} dv + \frac{1}{\sqrt{2\lambda}} \alpha \int_{1/2}^1 \left[\int_{\alpha v}^{\alpha} \frac{1}{\lambda} f_\lambda(s) ds \right]^{-1/2} dv \\
&= K_1(\lambda) + K_2(\lambda),
\end{aligned}$$

where

$$K_1(\lambda) := \frac{1}{\sqrt{2\lambda}} \alpha \int_0^{1/2} \left[\int_{\alpha v}^{\alpha} \frac{1}{\lambda} f_\lambda(s) ds \right]^{-1/2} dv$$

and

$$K_2(\lambda) := \frac{1}{\sqrt{2\lambda}} \alpha \int_{1/2}^1 \left[\int_{\alpha v}^{\alpha} \frac{1}{\lambda} f_\lambda(s) ds \right]^{-1/2} dv.$$

Since

$$K_1(\lambda) = \frac{1}{\sqrt{2\lambda}} \alpha \int_0^{1/2} \left[\int_{\alpha v}^{\alpha} \frac{1}{\lambda} f_\lambda(s) ds \right]^{-1/2} dv$$

$$\begin{aligned}
&\leq \frac{1}{\sqrt{2\lambda}} \alpha \int_0^{1/2} \left[\int_{\alpha v}^{\alpha} g(s) ds \right]^{-1/2} dv \\
&\leq \frac{1}{\sqrt{2\lambda}} \alpha \int_0^{1/2} \left[\int_{\alpha/2}^{\alpha} g(s) ds \right]^{-1/2} dv \\
&= \frac{1}{\sqrt{8\lambda}} \alpha \left[\int_{\alpha/2}^{\alpha} g(s) ds \right]^{-1/2} \rightarrow 0 \quad \text{as } \lambda \rightarrow \infty
\end{aligned}$$

since $[\int_{\alpha/2}^{\alpha} g(s) ds]^{-1/2} < \infty$, and

$$\begin{aligned}
K_2(\lambda) &= \frac{1}{\sqrt{2\lambda}} \alpha \int_{1/2}^1 \left[\int_{\alpha v}^{\alpha} \frac{1}{\lambda} f_{\lambda}(s) ds \right]^{-1/2} dv \\
&\leq \frac{1}{\sqrt{2\lambda}} \alpha \int_{1/2}^1 \left[\int_{\alpha v}^{\alpha} g(s) ds \right]^{-1/2} dv \\
&\leq \frac{1}{\sqrt{2\lambda}} \alpha \int_{1/2}^1 [\sigma \alpha (1-v)]^{-1/2} dv, \quad \text{where } \sigma = \min_{u \in [\alpha/2, \alpha]} g(u), \\
&= \sqrt{\frac{\alpha}{2\lambda\sigma}} [-2(1-v)^{1/2}] \Big|_{1/2}^1 \\
&= \sqrt{\frac{\alpha}{\lambda\sigma}} \rightarrow 0 \quad \text{as } \lambda \rightarrow \infty,
\end{aligned}$$

we obtain, for each fixed $\alpha > 0$, $\lim_{\lambda \rightarrow \infty} T_{\lambda}(\alpha) = 0$. \square

Consider (1.1) where $f_{\lambda}(u) = \lambda g(u) + h(u)$, $g, h \in C[0, \infty) \cap C^2(0, \infty)$, and g, h satisfy (H1). Let λ_1, λ_2 be two positive constants. Suppose that, for $\lambda_1 \leq \lambda \leq \lambda_2$, $T_{\lambda}(\alpha)$ has exactly one critical point, a maximum at some α_{λ}^* , on $(0, \infty)$. Then for $\lambda_1 \leq \lambda \leq \lambda_2$, let

$$M(\lambda) := T_{\lambda}(\alpha_{\lambda}^*) = \max_{\alpha \in (0, \infty)} T_{\lambda}(\alpha).$$

For $u > 0$, $f_{\lambda}(u) = \lambda g(u) + h(u)$ is strictly increasing in $\lambda > 0$ since $g(u) > 0$ for $u > 0$. This implies that, for any fixed $\alpha > 0$, $T_{\lambda}(\alpha)$ is strictly decreasing in $\lambda > 0$ by a comparison theorem of [8, Theorem 2.3]. Thus $M(\lambda)$ is strictly decreasing in $\lambda \in [\lambda_1, \lambda_2]$.

Lemma 3.3. Consider (1.1) where $f_{\lambda}(u) = \lambda g(u) + h(u)$, $g, h \in C[0, \infty) \cap C^2(0, \infty)$, and g, h satisfy (H1). Assume that there exist two positive numbers $\lambda_1 < \lambda_2$ such that

- (i) for $\lambda_1 \leq \lambda \leq \lambda_2$, $T_{\lambda}(\alpha)$ has exactly one critical point, a maximum at some α_{λ}^* , on $(0, \infty)$,

- (ii) $M(\lambda_2) < 1 < M(\lambda_1)$,
 (iii) $0 < \inf\{\alpha_\lambda^* \mid \lambda \in [\lambda_1, \lambda_2]\} \leq \sup\{\alpha_\lambda^* \mid \lambda \in [\lambda_1, \lambda_2]\} < \infty$.

Then there exists a unique number $\lambda^* \in (\lambda_1, \lambda_2)$ such that $M(\lambda^*) = 1$.

Proof. By (iii), we set positive numbers $\alpha_1 = \inf_{\lambda \in [\lambda_1, \lambda_2]} \alpha_\lambda^*$ and $\alpha_2 = \sup_{\lambda \in [\lambda_1, \lambda_2]} \alpha_\lambda^*$. If $\alpha_1 = \alpha_2$, then $\alpha_\lambda^* = \alpha_1 = \alpha_2$ for all $\lambda \in [\lambda_1, \lambda_2]$. By (ii), we have $T_{\lambda_1}(\alpha_1) = T_{\lambda_1}(\alpha_{\lambda_1}^*) > 1$ and $T_{\lambda_2}(\alpha_1) = T_{\lambda_2}(\alpha_{\lambda_2}^*) < 1$. By Lemma 3.2 and the intermediate value theorem, there exists a number $\lambda^* \in (\lambda_1, \lambda_2)$ such that $M(\lambda^*) = T_{\lambda^*}(\alpha_{\lambda^*}^*) = T_{\lambda^*}(\alpha_1) = 1$. Since $M(\lambda)$ is strictly decreasing in $\lambda \in [\lambda_1, \lambda_2]$, λ^* is unique.

While if $\alpha_1 < \alpha_2$, we first show that $M(\lambda)$ is continuous on $[\lambda_1, \lambda_2]$ as follows. By Lemma 3.2, $T_\lambda(\alpha)$ is a continuous function of $\lambda \geq 0$. Also, it is known that for any fixed $\alpha > 0$, $T_\lambda(\alpha)$ is strictly decreasing in $\lambda > 0$. So for any fixed $\lambda_0 \in [\lambda_1, \lambda_2]$, by the Dini theorem [11, p. 195], it is easy to see that

$$\lim_{\lambda \rightarrow \lambda_0} \left(\max_{\alpha \in [\alpha_1, \alpha_2]} T_\lambda(\alpha) \right) = \max_{\alpha \in [\alpha_1, \alpha_2]} T_{\lambda_0}(\alpha). \quad (3.12)$$

Since for any $\lambda \in [\lambda_1, \lambda_2]$, the maximum of $T_\lambda(\alpha)$ occurs at $\alpha_\lambda^* \in [\alpha_1, \alpha_2]$. So

$$M(\lambda) = \max_{\alpha \in (0, \infty)} T_\lambda(\alpha) = \max_{\alpha \in [\alpha_1, \alpha_2]} T_\lambda(\alpha) \quad \text{for } \lambda \in [\lambda_1, \lambda_2]. \quad (3.13)$$

By (3.12) and (3.13), $\lim_{\lambda \rightarrow \lambda_0} M(\lambda) = M(\lambda_0)$. Hence $M(\lambda)$ is continuous on $[\lambda_1, \lambda_2]$. By (ii) and the intermediate value theorem, there exists a number $\lambda^* \in (\lambda_1, \lambda_2)$ such that $M(\lambda^*) = 1$. Since $M(\lambda)$ is strictly decreasing in $\lambda \in [\lambda_1, \lambda_2]$, λ^* is unique. \square

4. Proof of Theorem 2.1

For fixed $\lambda > 0$, suppose that $u_\lambda(x)$ is a solution of (1.1) with $\|u_\lambda\|_\infty = \alpha$. We write

$$f_\lambda(u) = \lambda g(u) + h(u) = \lambda \left[g(u) + \frac{1}{\lambda} h(u) \right],$$

and recall that $F_\lambda(u) = \int_0^u f_\lambda(t) dt$. Then, by (3.2), it is easy to see that

$$\begin{aligned} \sqrt{\lambda} &= \frac{1}{\sqrt{2}} \int_0^\alpha \left[\int_u^\alpha g(s) + \frac{1}{\lambda} h(s) ds \right]^{-1/2} du \\ &= \frac{\sqrt{\lambda}}{\sqrt{2}} \int_0^\alpha \left[\int_u^\alpha \lambda g(s) + h(s) ds \right]^{-1/2} du \\ &= \frac{\sqrt{\lambda}}{\sqrt{2}} \int_0^\alpha [F_\lambda(\alpha) - F_\lambda(u)]^{-1/2} du. \end{aligned}$$

This and (1.8) imply that positive solution $u_\lambda(x)$ of (1.1) corresponds to $\|u_\lambda\|_\infty = \alpha$ and

$$T_\lambda(\alpha) = \frac{1}{\sqrt{2}} \int_0^\alpha [F_\lambda(\alpha) - F_\lambda(u)]^{-1/2} du = 1. \quad (4.1)$$

It is easy to check that (4.1) holds for any $\lambda \geq 0$. By (4.1), it is easy to compute that

$$T'_\lambda(\alpha) = \frac{1}{2\sqrt{2}} \int_0^\alpha \frac{\theta_{f_\lambda}(\alpha) - \theta_{f_\lambda}(u)}{[F_\lambda(\alpha) - F_\lambda(u)]^{3/2}} \frac{du}{\alpha}, \quad (4.2)$$

where $\theta_{f_\lambda}(u) = 2F_\lambda(u) - uf_\lambda(u)$ is defined in (1.9).

Proof of Theorem 2.1. We first prove part (I). By (H1) and (H2), $\theta'_h(u) = h(u) - uh'(u) < 0$ for $u > 0$. Hence, by (4.2), for $\lambda = 0$ and $f_0(u) = h(u)$, $T_0(\alpha)$ is a strictly decreasing function of $\alpha > 0$. Also, by [8, Theorem 2.10] and the assumption that $\pi^2/4 \leq m_0^h < \infty$, we obtain $T_0(\alpha) < \lim_{\alpha \rightarrow 0^+} T_0(\alpha) = \pi/(2\sqrt{m_0^h}) \leq 1$ for $\alpha > 0$. Now, for any $\lambda > 0$,

$$f_\lambda(u) = \lambda g(u) + h(u) > h(u) = f_0(u), \quad u > 0.$$

So $T_\lambda(\alpha) < T_0(\alpha) < 1$ for $\alpha > 0$ by a comparison theorem of [8, Theorem 2.3]. Thus, (1.1) has no positive solution for all $\lambda > 0$. So part (I) holds.

For parts (II) and (III), we study the shape and asymptotic behaviors of $T_\lambda(\alpha)$ for each fixed $\lambda \geq 0$. Similarly, for $\lambda = 0$, it is known that

(1) $T_0(\alpha)$ is a strictly decreasing function of $\alpha > 0$. In addition, $\lim_{\alpha \rightarrow 0^+} T_0(\alpha) = \pi/(2\sqrt{m_0^h}) > 1$ and $0 \leq \lim_{\alpha \rightarrow \infty} T_0(\alpha) = \pi/(2\sqrt{m_\infty^h}) < \infty$ by [8, Theorems 2.5, 2.7, 2.9, 2.10], Remark 1, and the assumption that $0 \leq m_0^h < \pi^2/4$.

We then study the shape and asymptotic behaviors of $T_\lambda(\alpha)$ for each fixed $\lambda > 0$. Recall that

$$0 \leq C_0 = - \lim_{u \rightarrow 0^+} \frac{\theta'_h(u)}{\theta'_g(u)} < C_\infty = - \lim_{u \rightarrow \infty} \frac{\theta'_h(u)}{\theta'_g(u)} \leq \infty$$

by (H3), and we write $(0, \infty) = (0, C_0] \cup (C_0, C_\infty) \cup [C_\infty, \infty)$.

For $0 < \lambda \leq C_0$, by (H1)–(H3), we have $\theta'_g(u) > 0$ and $\theta'_h(u)/\theta'_g(u) < -C_0$ for all $u > 0$. Thus,

$$\theta'_{f_\lambda}(u) = \lambda \theta'_g(u) + \theta'_h(u) = \theta'_g(u) \left[\lambda + \frac{\theta'_h(u)}{\theta'_g(u)} \right] < 0$$

for all $u > 0$. (Note that the strictly increasing hypothesis of $-(h(u) - uh'(u))/(g(u) - ug'(u))$ in (H3) is essentially used here to get the result in above.) Then $T'_\lambda(\alpha) < 0$ for all $\alpha > 0$ by (4.2). So we obtain

(2) For $0 < \lambda \leq C_0$, $T_\lambda(\alpha)$ is strictly decreasing in $\alpha > 0$. In addition, we have $0 < \lim_{\alpha \rightarrow 0^+} T_\lambda(\alpha) = \pi/(2\sqrt{\lambda m_0^g + m_0^h}) < \infty$ and $0 \leq \lim_{\alpha \rightarrow \infty} T_\lambda(\alpha) = \pi/(2\sqrt{\lambda m_\infty^g + m_\infty^h}) < \infty$ by [8, Theorems 2.5, 2.7, 2.10] and Remark 1.

For $C_0 < \lambda < C_\infty$, similarly, by above analysis for $\theta'_{f_\lambda}(u)$, there exists a number $A_\lambda > 0$ such that

$$\begin{cases} \theta'_{f_\lambda}(u) > 0 & \text{on } (0, A_\lambda), \\ \theta'_{f_\lambda}(A_\lambda) = 0, \\ \theta'_{f_\lambda}(u) < 0 & \text{on } (A_\lambda, \infty). \end{cases} \quad (4.3)$$

We then prove $(C_0, C_\infty) \subseteq (D_0, D_\infty)$ as follows. By Remark 2, if $0 < \lambda \leq D_0$, then $f''_\lambda(u) > 0$ for all $u > 0$. So $\theta'_{f_\lambda}(u) < 0$ for all $u > 0$ by the fact that $\theta_{f_\lambda}(0) = \theta'_{f_\lambda}(0) = 0$ and $\theta''_{f_\lambda}(u) = -uf''_\lambda(u)$. Similarly, if $D_\infty \leq \lambda < \infty$, then $\theta'_{f_\lambda}(u) > 0$ for all $u > 0$. So for any $C_0 < \lambda < C_\infty$, $\lambda \in (D_0, D_\infty)$ by (4.3). This and Remark 2 imply that, for $C_0 < \lambda < C_\infty$, there exists a positive number $\beta_\lambda < A_\lambda$ such that

$$\begin{cases} f''_\lambda(u) < 0 & \text{on } (0, \beta_\lambda), \\ f''_\lambda(\beta_\lambda) = 0, \\ f''_\lambda(u) > 0 & \text{on } (\beta_\lambda, \infty). \end{cases} \quad (4.4)$$

Since $\theta_{f_\lambda}(0) = \theta'_{f_\lambda}(0) = 0$, $\theta''_{f_\lambda}(u) = -uf''_\lambda(u)$, and by (4.3) and (4.4), it is easy to see that there exists a number $\gamma_\lambda > A_\lambda$ such that

$$\begin{cases} \theta_{f_\lambda}(u) > 0 & \text{on } (0, \gamma_\lambda), \\ \theta_{f_\lambda}(\gamma_\lambda) = 0, \\ \theta_{f_\lambda}(u) < 0 & \text{on } (\gamma_\lambda, \infty). \end{cases} \quad (4.5)$$

By (H1)–(H2) and (4.3)–(4.5), for $C_0 < \lambda < C_\infty$, we obtain that $f_\lambda(u)$ satisfies all assumptions of Lemma 3.1. Thus by Lemma 3.1, we obtain

(3) For $C_0 < \lambda < C_\infty$, $T_\lambda(\alpha)$ has exactly one critical point, a maximum, at some α_λ^* , on $(0, \infty)$. In addition, $0 \leq \lim_{\alpha \rightarrow 0^+} T_\lambda(\alpha) = \pi/(2\sqrt{\lambda m_0^g + m_0^h}) < \infty$ and $0 \leq \lim_{\alpha \rightarrow \infty} T_\lambda(\alpha) = \pi/(2\sqrt{\lambda m_\infty^g + m_\infty^h}) < \infty$ by [8, Theorems 2.5, 2.7, 2.9, 2.10] and Remark 1.

For $C_\infty \leq \lambda < \infty$, similarly, by above analysis for $\theta'_{f_\lambda}(u)$, we have $\theta'_{f_\lambda}(u) > 0$ for all $u > 0$. Then $T'_\lambda(\alpha) > 0$ for all $\alpha > 0$ by (4.2). So we obtain

(4) For $C_\infty \leq \lambda < \infty$, $T_\lambda(\alpha)$ is strictly increasing in $\alpha > 0$. In addition, $0 \leq \lim_{\alpha \rightarrow 0^+} T_\lambda(\alpha) = \pi/(2\sqrt{\lambda m_0^g + m_0^h}) < \infty$ and $0 < \lim_{\alpha \rightarrow \infty} T_\lambda(\alpha) = \pi/(2\sqrt{\lambda m_\infty^g + m_\infty^h}) < \infty$ by [8, Theorems 2.7, 2.9, 2.10] and Remark 1.

Secondly, for $T_\lambda(\alpha)$ with $\lambda \geq 0$, we have the following properties:

(5) For each fixed $\alpha > 0$, $T_\lambda(\alpha)$ is a continuous function of $\lambda \geq 0$, $\lim_{\lambda \rightarrow 0^+} T_\lambda(\alpha) = T_0(\alpha)$ and $\lim_{\lambda \rightarrow \infty} T_\lambda(\alpha) = 0$ by Lemma 3.2.

(6) For $0 \leq \lambda_1 < \lambda_2$, $T_{\lambda_1}(\alpha) > T_{\lambda_2}(\alpha)$ for $\alpha > 0$ by a comparison theorem of [8, Theorem 2.3] since $f_{\lambda_1}(u) = \lambda_1 g(u) + h(u) < \lambda_2 g(u) + h(u) = f_{\lambda_2}(u)$, $u > 0$.

Suppose that $M(\lambda_2) < 1 < M(\lambda_1)$ for some positive numbers $\lambda_1 < \lambda_2$ satisfying $C_0 < \lambda_1 < \lambda_2 < C_\infty$. By (H1) and (H2) and since $\theta_g(0) = 0$, functions $\theta_{f_\lambda}(u) = \lambda \theta_g(u) + \theta_h(u)$ and $\theta'_{f_\lambda}(u) = \lambda \theta'_g(u) + \theta'_h(u)$ are both strictly increasing in $\lambda > 0$. Thus positive numbers A_λ and γ_λ are both strictly increasing in $\lambda > 0$ by (4.3) and (4.5). It is clear that, for $C_0 < \lambda < C_\infty$, we have $A_\lambda < \alpha_\lambda^* < \gamma_\lambda$ by (4.2). Hence

$$\begin{aligned}
0 < A_{\lambda_1} &= \inf\{A_\lambda \mid \lambda \in [\lambda_1, \lambda_2]\} \\
&\leq \inf\{\alpha_\lambda^* \mid \lambda \in [\lambda_1, \lambda_2]\} \leq \sup\{\alpha_\lambda^* \mid \lambda \in [\lambda_1, \lambda_2]\} \\
&\leq \sup\{\gamma_\lambda \mid \lambda \in [\lambda_1, \lambda_2]\} = \gamma_{\lambda_2} < \infty.
\end{aligned}$$

Then by previous results in part (3) and by Lemma 3.3, we obtain

(7) If $M(\lambda_2) < 1 < M(\lambda_1)$ for some positive numbers $\lambda_1 < \lambda_2$ satisfying $C_0 < \lambda_1 < \lambda_2 < C_\infty$, then there exists a unique positive number $\lambda^* \in (\lambda_1, \lambda_2) \subset (C_0, C_\infty)$ such that $M(\lambda^*) = \max_{\alpha \in (0, \infty)} T_{\lambda^*}(\alpha) = 1$.

By above analysis for $T_\lambda(\alpha)$, we are in a position to prove parts (II) and (III).

Proof of part (II)(i). Since $\tilde{\lambda} < 0$, by the definition of $\tilde{\lambda}$, we obtain $m_\infty^h > \pi^2/4$. This and the assumption that $0 \leq m_0^h < \pi^2/4$ imply

$$(i)(8) \quad 0 \leq \lim_{\alpha \rightarrow \infty} T_0(\alpha) = \pi/(2\sqrt{m_\infty^h}) < 1 < \lim_{\alpha \rightarrow 0^+} T_0(\alpha) = \pi/(2\sqrt{m_0^h}) \leq \infty.$$

Since $0 < \hat{\lambda} \leq C_0$ and by properties (2) and (6), we obtain

(i)(9) For $0 < \lambda \leq \hat{\lambda}$, $T_\lambda(\alpha)$ is strictly decreasing in $\alpha > 0$, and for $\lambda > \hat{\lambda}$, $T_\lambda(\alpha) < 1$ for all $\alpha > 0$.

So properties (1)–(6), (i)(8), and (i)(9) imply immediately the uniqueness and nonexistence results in part (II)(i) and the results in parts (II)(i)(a)–(II)(i)(c).

Proof of part (II)(ii). Since $\tilde{\lambda} \geq 0$, by the definition of $\tilde{\lambda}$ and Remark 1, we obtain $0 < m_\infty^h \leq \pi^2/4$. This and the assumption that $0 \leq m_0^h < \pi^2/4$ imply

$$(ii)(8) \quad 1 < \lim_{\alpha \rightarrow 0^+} T_0(\alpha) = \pi/(2\sqrt{m_0^h}) \leq \infty \text{ and } 1 \leq \lim_{\alpha \rightarrow \infty} T_0(\alpha) = \pi/(2\sqrt{m_\infty^h}) < \infty.$$

Since $0 < \hat{\lambda} \leq C_0$, we have

$$\lim_{\alpha \rightarrow \infty} T_{\hat{\lambda}}(\alpha) = \frac{\pi}{2\sqrt{\hat{\lambda}m_\infty^g + m_\infty^h}} < 1 \leq \lim_{\alpha \rightarrow \infty} T_0(\alpha) = \frac{\pi}{2\sqrt{m_\infty^h}}$$

by (2) and (ii)(8). So by Remark 1, $0 < m_\infty^g < \infty$. This and the assumptions that $0 < m_\infty^h \leq \pi^2/4$ and $0 < \hat{\lambda} \leq C_0$ imply $\tilde{\lambda} < \hat{\lambda}$. In addition,

(ii)(9) The number $\lim_{\alpha \rightarrow \infty} T_\lambda(\alpha) = \pi/(2\sqrt{\lambda m_\infty^g + m_\infty^h})$ is strictly decreasing in $\lambda \geq 0$.

Since $0 < \hat{\lambda} \leq C_0$ and by properties (2) and (6), we obtain

(ii)(10) For $0 < \lambda \leq \hat{\lambda}$, $T_\lambda(\alpha)$ is strictly decreasing in $\alpha > 0$, and for $\lambda > \hat{\lambda}$, $T_\lambda(\alpha) < 1$ for all $\alpha > 0$.

So properties (1)–(6) and (ii)(8)–(ii)(10) imply immediately the uniqueness and nonexistence results in part (II)(ii) and the results in parts (II)(ii)(a) and (II)(ii)(b).

Proof of part (III)(iii). There are two cases to be studied. That is, $\hat{\lambda} = 0$ and $C_0 < \hat{\lambda} < \infty$. Since $\tilde{\lambda} < 0$, by the definition of $\tilde{\lambda}$, we obtain $m_\infty^h > \pi^2/4$. This and the assumption that $0 \leq m_0^h < \pi^2/4$ imply

$$(iii)(8) \quad 0 \leq \lim_{\alpha \rightarrow \infty} T_0(\alpha) = \pi/(2\sqrt{m_\infty^h}) < 1 < \lim_{\alpha \rightarrow 0^+} T_0(\alpha) = \pi/(2\sqrt{m_0^h}) \leq \infty.$$

Suppose $\hat{\lambda} = 0$. Since $0 \leq m_0^h < \pi^2/4$, by the definition of $\hat{\lambda}$, we obtain $m_0^g = \infty$. Thus,

(iii)(9) For $\lambda > 0$, $\lim_{\alpha \rightarrow 0^+} T_\lambda(\alpha) = 0$ by [8, Theorem 2.9], and hence $C_0 = 0$ by properties (2)–(4).

By above properties (1), (3)–(6), (iii)(8), and (iii)(9), there exist two positive numbers $\lambda_1 < \lambda_2$ satisfying $0 = C_0 < \lambda_1 < \lambda_2 < C_\infty$ such that $\max_{\alpha \in (0, \infty)} T_{\lambda_2}(\alpha) < 1 < \max_{\alpha \in (0, \infty)} T_{\lambda_1}(\alpha)$. So property (7) holds. Thus,

(iii)(10) For $0 < \lambda \leq \lambda^*$, $T_\lambda(\alpha)$ has exactly one critical point, a maximum, on $(0, \infty)$. In addition, for $\lambda > \lambda^*$, $T_\lambda(\alpha) < 1$ for all $\alpha > 0$.

So properties (1), (3)–(7), and (iii)(8)–(iii)(10) imply immediately the exactly multiplicity result in part (III)(iii) and the results in parts (III)(iii)(a)–(III)(iii)(d).

Suppose $C_0 < \hat{\lambda} < \infty$. By properties (3), (4), (6), and (iii)(8), we have $C_0 < \hat{\lambda} < C_\infty$. This implies that $T_{\hat{\lambda}}(\alpha)$ has exactly one critical point, a maximum, on $(0, \infty)$.

By above properties (1), (3)–(6), and (iii)(8) and since $C_0 < \hat{\lambda} < C_\infty$, there exist two positive numbers $\lambda_1 < \lambda_2$ satisfying $C_0 < \lambda_1 < \lambda_2 < C_\infty$ such that $\max_{\alpha \in (0, \infty)} T_{\lambda_2}(\alpha) < 1 < \max_{\alpha \in (0, \infty)} T_{\lambda_1}(\alpha)$. So property (7) holds. Thus,

(iii)(11) For $0 \leq C_0 < \hat{\lambda} \leq \lambda \leq \lambda^*$, $T_\lambda(\alpha)$ has exactly one critical point, a maximum, on $(0, \infty)$. In addition, for $\lambda > \lambda^*$, $T_\lambda(\alpha) < 1$ for all $\alpha > 0$.

So properties (1)–(7), (iii)(8), and (iii)(11) imply immediately the exactly multiplicity result in part (III)(iii) and the results in parts (III)(iii)(a)–(III)(iii)(d).

Proof of part (III)(iv). There are two cases to be studied. That is, $\hat{\lambda} = 0$ and $C_0 < \hat{\lambda} < \infty$. Since $\hat{\lambda} = 0$ and $m_\infty^g = 0$, by the definition of $\tilde{\lambda}$, we obtain $m_\infty^h = \pi^2/4$. This and assumptions that $0 \leq m_0^h < \pi^2/4$ and $m_\infty^g = 0$ imply

$$(iv)(8) \lim_{\alpha \rightarrow \infty} T_0(\alpha) = \pi/(2\sqrt{m_\infty^h}) = 1 < \lim_{\alpha \rightarrow 0^+} T_0(\alpha) = \pi/(2\sqrt{m_0^h}) \leq \infty \text{ and } \lim_{\alpha \rightarrow \infty} T_\lambda(\alpha) = \pi/(2\sqrt{\lambda m_\infty^g + m_\infty^h}) = 1 \text{ for } \lambda > 0.$$

Suppose $\hat{\lambda} = 0$. Since $0 \leq m_0^h < \pi^2/4$, by the definition of $\hat{\lambda}$, we obtain $m_0^g = \infty$. Thus,

(iv)(9) For $\lambda > 0$, $\lim_{\alpha \rightarrow 0^+} T_\lambda(\alpha) = 0$ by [8, Theorem 2.9], and hence $C_0 = 0$ by properties (2)–(4).

By properties (4), (6), and (iv)(8), we obtain

(iv)(10) For $\lambda \geq C_\infty$, $T_\lambda(\alpha) < 1$ for all $\alpha > 0$.

So properties (1), (3)–(6), and (iv)(8)–(iv)(10) imply immediately the uniqueness and nonexistence results in part (III)(iv) and the results in parts (III)(iv)(a) and (III)(iv)(b).

Suppose $C_0 < \hat{\lambda} < \infty$. By properties (3), (4), (6), and (iv)(8), we have $C_0 < \hat{\lambda} < C_\infty$. This implies

(iv)(11) $T_{\hat{\lambda}}(\alpha)$ has exactly one critical point, a maximum, on $(0, \infty)$.

By properties (4), (6), (iv)(8), and (iv)(11), we obtain

(iv)(12) For $0 < \lambda \leq \hat{\lambda}$, $T_\lambda(\alpha) > 1$ for all $\alpha > 0$, and for $\lambda \geq C_\infty$, $T_\lambda(\alpha) < 1$ for all $\alpha > 0$.

So properties (1)–(6), (iv)(8), (iv)(11), and (iv)(12) imply immediately the uniqueness and nonexistence results in part (III)(iv) and the results in parts (III)(iv)(a) and (III)(iv)(b).

Proof of part (III)(v). We first consider the case (v1) $\tilde{\lambda} = 0$ and $0 < m_\infty^g < \infty$. Since $\tilde{\lambda} = 0$ and $0 < m_\infty^g < \infty$, by the definition of $\tilde{\lambda}$, we obtain $m_\infty^h = \pi^2/4$. This and assumptions that $0 \leq m_0^h < \pi^2/4$ and $0 < m_\infty^g < \infty$ imply

$$(v1)(8) \lim_{\alpha \rightarrow \infty} T_0(\alpha) = \pi/(2\sqrt{m_\infty^h}) = 1 < \lim_{\alpha \rightarrow 0^+} T_0(\alpha) = \pi/(2\sqrt{m_0^h}) \leq \infty \text{ and } \text{the number } \lim_{\alpha \rightarrow \infty} T_\lambda(\alpha) = \pi/(2\sqrt{\lambda m_\infty^g + m_\infty^h}) \text{ is strictly decreasing in } \lambda \geq 0.$$

Suppose $\hat{\lambda} = 0$. Since $0 \leq m_0^h < \pi^2/4$, by the definition of $\hat{\lambda}$, we obtain $m_0^g = \infty$. Thus,

(v1)(9) For $\lambda > 0$, $\lim_{\alpha \rightarrow 0^+} T_\lambda(\alpha) = 0$ by [8, Theorem 2.9], and hence $C_0 = 0$ by properties (2)–(4).

By above properties (1), (3)–(6), (v1)(8), and (v1)(9), there exist two positive numbers $\lambda_1 < \lambda_2$ satisfying $0 = C_0 < \lambda_1 < \lambda_2 < C_\infty$ such that $\max_{\alpha \in (0, \infty)} T_{\lambda_2}(\alpha) < 1 < \max_{\alpha \in (0, \infty)} T_{\lambda_1}(\alpha)$. So property (7) holds. Thus,

(v1)(10) For $0 < \lambda \leq \lambda^*$, $T_\lambda(\alpha)$ has exactly one critical point, a maximum, on $(0, \infty)$. In addition, for $\lambda > \lambda^*$, $T_\lambda(\alpha) < 1$ for all $\alpha > 0$.

So properties (1), (3)–(7), and (v1)(8)–(v1)(10) imply immediately the exactly multiplicity result in part (III)(v) and the results in parts (III)(v)(a)–(III)(v)(c).

Suppose $C_0 < \hat{\lambda} < \infty$. By properties (3), (4), (6), and (v1)(8), we have $C_0 < \hat{\lambda} < C_\infty$. This implies that $T_{\hat{\lambda}}(\alpha)$ has exactly one critical point, a maximum, on $(0, \infty)$.

By above properties (1), (3)–(6), and (v1)(8) and since $C_0 < \hat{\lambda} < C_\infty$, there exist two positive numbers $\lambda_1 < \lambda_2$ satisfying $C_0 < \lambda_1 < \lambda_2 < C_\infty$ such that $\max_{\alpha \in (0, \infty)} T_{\lambda_2}(\alpha) < 1 < \max_{\alpha \in (0, \infty)} T_{\lambda_1}(\alpha)$. So property (7) holds. Thus,

(v1)(11) For $0 \leq C_0 < \hat{\lambda} \leq \lambda \leq \lambda^*$, $T_\lambda(\alpha)$ has exactly one critical point, a maximum, on $(0, \infty)$. In addition, for $\lambda > \lambda^*$, $T_\lambda(\alpha) < 1$ for all $\alpha > 0$.

So properties (1)–(7), (v1)(8), and (v1)(11) imply immediately the exactly multiplicity result in part (III)(v) and the results in parts (III)(v)(a)–(III)(v)(c).

Next, we consider the case (v2) $0 < \tilde{\lambda} < C_\infty$ and $\hat{\lambda} > \tilde{\lambda}$. Since $0 < \tilde{\lambda} < C_\infty$, by the definition of $\tilde{\lambda}$ and Remark 1, we obtain $0 < m_\infty^g < \infty$ and $0 < m_\infty^h < \pi^2/4$. These and the assumption that $0 \leq m_0^h < \pi^2/4$ imply

(v2)(8) $1 < \lim_{\alpha \rightarrow 0^+} T_0(\alpha) = \pi/(2\sqrt{m_0^h}) \leq \infty$ and $1 < \lim_{\alpha \rightarrow \infty} T_0(\alpha) = \pi/(2\sqrt{m_\infty^h}) < \infty$. In addition, the number $\lim_{\alpha \rightarrow \infty} T_\lambda(\alpha) = \pi/(2\sqrt{\lambda m_\infty^g + m_\infty^h})$ is strictly decreasing in $\lambda \geq 0$.

Since $0 < \tilde{\lambda} < C_\infty$ and $\hat{\lambda} > \tilde{\lambda}$, we obtain $\hat{\lambda} > 0$, and hence $C_0 < \hat{\lambda} < \infty$ by the assumption that either $\hat{\lambda} = 0$ or $C_0 < \hat{\lambda} < \infty$. Since $C_0 < \hat{\lambda} < \infty$, $\hat{\lambda} > \tilde{\lambda} > 0$, and by properties (3), (4), (6), and (v2)(8), we have $C_0 < \hat{\lambda} < C_\infty$. This implies that $T_{\hat{\lambda}}(\alpha)$ has exactly one critical point, a maximum, on $(0, \infty)$.

By above properties (1), (3)–(6), and (v2)(8) and since $C_0 < \hat{\lambda} < C_\infty$, there exist two positive numbers $\lambda_1 < \lambda_2$ satisfying $C_0 < \lambda_1 < \lambda_2 < C_\infty$ such that $\max_{\alpha \in (0, \infty)} T_{\lambda_2}(\alpha) < 1 < \max_{\alpha \in (0, \infty)} T_{\lambda_1}(\alpha)$. So property (7) holds. Thus,

(v2)(9) For $0 < \tilde{\lambda} < \hat{\lambda} \leq \lambda \leq \lambda^*$, $T_\lambda(\alpha)$ has exactly one critical point, a maximum, on $(0, \infty)$. In addition, for $\lambda > \lambda^*$, $T_\lambda(\alpha) < 1$ for all $\alpha > 0$.

So properties (1)–(7), (v2)(8), and (v2)(9) imply immediately the exactly multiplicity result in part (III)(v) and the results in parts (III)(v)(a)–(III)(v)(c).

Next, we consider the case (v3) $0 < \tilde{\lambda} < C_\infty$ and $\hat{\lambda} = \tilde{\lambda}$. Since $0 < \tilde{\lambda} < C_\infty$, by the definition of $\tilde{\lambda}$ and Remark 1, we obtain $0 < m_\infty^g < \infty$ and $0 < m_\infty^h < \pi^2/4$. These and the assumption that $0 \leq m_0^h < \pi^2/4$ imply

(v3)(8) $1 < \lim_{\alpha \rightarrow 0^+} T_0(\alpha) = \pi/(2\sqrt{m_0^h}) \leq \infty$ and $1 < \lim_{\alpha \rightarrow \infty} T_0(\alpha) = \pi/(2\sqrt{m_\infty^h}) < \infty$. In addition, the number $\lim_{\alpha \rightarrow \infty} T_\lambda(\alpha) = \pi/(2\sqrt{\lambda m_\infty^g + m_\infty^h})$ is strictly decreasing in $\lambda \geq 0$.

Since $0 < \hat{\lambda} = \tilde{\lambda} < C_\infty$ and by the assumption that either $\hat{\lambda} = 0$ or $C_0 < \hat{\lambda} < \infty$, we have $C_0 < \hat{\lambda} = \tilde{\lambda} < C_\infty$. This implies that $T_{\hat{\lambda}}(\alpha)$ has exactly one critical point, a maximum, on $(0, \infty)$.

By above properties (1), (3)–(6), and (v3)(8) and since $C_0 < \hat{\lambda} = \tilde{\lambda} < C_\infty$, there exist two positive numbers $\lambda_1 < \lambda_2$ satisfying $C_0 < \lambda_1 < \lambda_2 < C_\infty$ such that $\max_{\alpha \in (0, \infty)} T_{\lambda_2}(\alpha) < 1 < \max_{\alpha \in (0, \infty)} T_{\lambda_1}(\alpha)$. So property (7) holds. Thus,

(v3)(9) For $0 < \hat{\lambda} = \tilde{\lambda} \leq \lambda \leq \lambda^*$, $T_\lambda(\alpha)$ has exactly one critical point, a maximum, on $(0, \infty)$. In addition, for $\lambda > \lambda^*$, $T_\lambda(\alpha) < 1$ for all $\alpha > 0$.

So properties (1)–(7), (v3)(8), and (v3)(9) imply immediately the exactly multiplicity result in part (III)(v) and the results in parts (III)(v)(a)–(III)(v)(c).

Next, we consider the case (v4) $0 < \tilde{\lambda} < C_\infty$ and $\hat{\lambda} < \tilde{\lambda}$. Since $0 < \tilde{\lambda} < C_\infty$ and $\hat{\lambda} < \tilde{\lambda}$, then $\hat{\lambda} < C_\infty$. Since $0 < \tilde{\lambda} < C_\infty$, by the definition of $\tilde{\lambda}$ and Remark 1, we obtain $0 < m_\infty^g < \infty$ and $0 < m_\infty^h < \pi^2/4$. These and the assumption that $0 \leq m_0^h < \pi^2/4$ imply

(v4)(8) $1 < \lim_{\alpha \rightarrow 0^+} T_0(\alpha) = \pi/(2\sqrt{m_0^h}) \leq \infty$ and $1 < \lim_{\alpha \rightarrow \infty} T_0(\alpha) = \pi/(2\sqrt{m_\infty^h}) < \infty$. In addition, the number $\lim_{\alpha \rightarrow \infty} T_\lambda(\alpha) = \pi/(2\sqrt{\lambda m_\infty^g + m_\infty^h})$ is strictly decreasing in $\lambda \geq 0$.

Suppose $\hat{\lambda} = 0$. Since $0 \leq m_0^h < \pi^2/4$, by the definition of $\hat{\lambda}$, we obtain $m_0^g = \infty$. Thus,

(v4)(9) For $\lambda > 0$, $\lim_{\alpha \rightarrow 0^+} T_\lambda(\alpha) = 0$ by [8, Theorem 2.9], and hence $C_0 = 0$ by properties (2)–(4).

By (v4)(9) and since $0 < \tilde{\lambda} < C_\infty$, we obtain $C_0 < \tilde{\lambda} < C_\infty$.

By above properties (1), (3)–(6), (v4)(8), and (v4)(9) and since $C_0 < \tilde{\lambda} < C_\infty$, there exist two positive numbers $\lambda_1 < \lambda_2$ satisfying $0 = C_0 < \lambda_1 < \lambda_2 < C_\infty$ such that $\max_{\alpha \in (0, \infty)} T_{\lambda_2}(\alpha) < 1 < \max_{\alpha \in (0, \infty)} T_{\lambda_1}(\alpha)$. So property (7) holds. Thus,

(v4)(10) For $0 < \lambda \leq \lambda^*$, $T_\lambda(\alpha)$ has exactly one critical point, a maximum, on $(0, \infty)$. In addition, for $\lambda > \lambda^*$, $T_\lambda(\alpha) < 1$ for all $\alpha > 0$.

So properties (1), (3)–(7), and (v4)(8)–(v4)(10) imply immediately the exactly multiplicity result in part (III)(v) and the results in parts (III)(v)(a)–(III)(v)(c).

Suppose $C_0 < \hat{\lambda} < \infty$. Since $0 < \tilde{\lambda} < C_\infty$ and $\hat{\lambda} < \tilde{\lambda}$, we have $C_0 < \hat{\lambda} < \tilde{\lambda} < C_\infty$.

By above properties (1), (3)–(6), and (v4)(8) and since $C_0 < \hat{\lambda} < \tilde{\lambda} < C_\infty$, there exist two positive numbers $\lambda_1 < \lambda_2$ satisfying $C_0 < \lambda_1 < \lambda_2 < C_\infty$ such that $\max_{\alpha \in (0, \infty)} T_{\lambda_2}(\alpha) < 1 < \max_{\alpha \in (0, \infty)} T_{\lambda_1}(\alpha)$. So property (7) holds. Thus,

(v4)(11) $C_0 < \hat{\lambda} < \tilde{\lambda} < \lambda^*$ and for $\hat{\lambda} \leq \lambda \leq \lambda^*$, $T_\lambda(\alpha)$ has exactly one critical point, a maximum, on $(0, \infty)$. In addition, for $\lambda > \lambda^*$, $T_\lambda(\alpha) < 1$ for all $\alpha > 0$.

So properties (1)–(7), (v4)(8), and (v4)(11) imply immediately the exactly multiplicity result in part (III)(v) and the results in parts (III)(v)(a)–(III)(v)(c).

Proof of part (III)(vi). There are two cases to be studied. That is, $\hat{\lambda} = 0$ and $C_0 < \hat{\lambda} < \infty$. Since $C_\infty \leq \tilde{\lambda} \leq \infty$, by the definition of $\tilde{\lambda}$ and Remark 1, we obtain $0 \leq m_\infty^g < \infty$ and $0 < m_\infty^h < \pi^2/4$. These and the assumption that $0 \leq m_0^h < \pi^2/4$ imply

(vi)(8) $1 < \lim_{\alpha \rightarrow 0^+} T_0(\alpha) = \pi/(2\sqrt{m_0^h}) \leq \infty$ and $1 < \lim_{\alpha \rightarrow \infty} T_0(\alpha) = \pi/(2\sqrt{m_\infty^h}) < \infty$.

In addition, since $C_\infty \leq \tilde{\lambda} \leq \infty$, by the definition of $\tilde{\lambda}$, we obtain

(vi)(9) If $C_\infty \leq \tilde{\lambda} < \infty$, then the number $\lim_{\alpha \rightarrow \infty} T_\lambda(\alpha) = \pi / (2\sqrt{\lambda m_\infty^g + m_\infty^h})$ is strictly decreasing in $\lambda \geq 0$ and $T_{\tilde{\lambda}}(\alpha)$ is strictly increasing in $\alpha > 0$. If $\tilde{\lambda} = \infty$, then for $\lambda > 0$, $\lim_{\alpha \rightarrow \infty} T_\lambda(\alpha) = \lim_{\alpha \rightarrow \infty} T_0(\alpha)$.

Suppose $\hat{\lambda} = 0$. Since $0 \leq m_0^h < \pi^2/4$, by the definition of $\hat{\lambda}$, we obtain $m_0^g = \infty$. Then,

(vi)(10) For $\lambda > 0$, $\lim_{\alpha \rightarrow 0^+} T_\lambda(\alpha) = 0$ by [8, Theorem 2.9], and hence $C_0 = 0$ by properties (2)–(4).

So properties (1), (3)–(6), and (vi)(8)–(vi)(10) imply immediately the uniqueness and nonexistence results in part (III)(vi) and the results in parts (III)(vi)(a) and (III)(vi)(b).

Suppose $C_0 < \hat{\lambda} < \infty$. By properties (3), (4), (6), (vi)(8), and (vi)(9), we obtain

(vi)(11) For $0 < \lambda < \hat{\lambda}$, $T_\lambda(\alpha) > 1$ for all $\alpha > 0$.

So properties (1)–(6), (vi)(8), (vi)(9), and (vi)(11) imply immediately the uniqueness and nonexistence results in part (III)(vi) and the results in parts (III)(vi)(a) and (III)(vi)(b). \square

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Appendix A. Proof of nonexistence of positive solutions of (1.1) in slash-lined regions in Figs. 2 and 3

The nonexistence of positive solutions of (1.1) in slash-lined regions in Figs. 2 and 3 can be proved region by region. We simply give proofs for some regions; proofs of other regions are similar.

(i) For the region with $0 \leq m_0^h < \pi^2/4$, $0 \leq m_\infty^g < \infty$, $0 < \hat{\lambda} \leq C_0$ and $C_0 < \tilde{\lambda} < \infty$, by Remark 4 and property (2) in the proof of Theorem 2.1, we have

$$\lim_{\alpha \rightarrow \infty} T_{\tilde{\lambda}}(\alpha) = 1 = \lim_{\alpha \rightarrow 0^+} T_{\hat{\lambda}}(\alpha) > \lim_{\alpha \rightarrow \infty} T_{\hat{\lambda}}(\alpha).$$

This contradicts to property (6) in the proof of Theorem 2.1 since $0 < \hat{\lambda} \leq C_0 < \tilde{\lambda} < \infty$. Thus we get nonexistence of positive solutions of (1.1) for this region.

(ii) For the region with $0 \leq m_0^h < \pi^2/4$, $0 \leq m_\infty^g < \infty$, $0 < \hat{\lambda} \leq \tilde{\lambda} \leq C_0$, by Remark 4 and property (2) in the proof of Theorem 2.1, we have

$$\lim_{\alpha \rightarrow \infty} T_{\tilde{\lambda}}(\alpha) = 1 = \lim_{\alpha \rightarrow 0^+} T_{\hat{\lambda}}(\alpha) > \lim_{\alpha \rightarrow \infty} T_{\hat{\lambda}}(\alpha).$$

This contradicts to property (6) in the proof of Theorem 2.1 since $0 < \hat{\lambda} \leq \tilde{\lambda} \leq C_0$. Thus we get nonexistence of positive solutions of (1.1) for this region.

(iii) For the region with $0 \leq m_0^h < \pi^2/4$, $0 \leq m_\infty^g < \infty$, $C_\infty \leq \hat{\lambda} < \infty$ and $0 < \tilde{\lambda} < C_\infty$, by Remark 4 and property (4) in the proof of Theorem 2.1, we have

$$\lim_{\alpha \rightarrow \infty} T_{\tilde{\lambda}}(\alpha) = 1 = \lim_{\alpha \rightarrow 0^+} T_{\hat{\lambda}}(\alpha) < \lim_{\alpha \rightarrow \infty} T_{\hat{\lambda}}(\alpha).$$

This contradicts to property (6) in the proof of Theorem 2.1 since $0 < \tilde{\lambda} < C_\infty \leq \hat{\lambda} < \infty$. Thus we get nonexistence of positive solutions of (1.1) for this region.

(iv) For the region with $0 \leq m_0^h < \pi^2/4$, $0 \leq m_\infty^g < \infty$, $C_\infty \leq \tilde{\lambda} \leq \hat{\lambda} < \infty$, by Remark 4 and property (4) in the proof of Theorem 2.1, we have

$$\lim_{\alpha \rightarrow \infty} T_{\tilde{\lambda}}(\alpha) = 1 = \lim_{\alpha \rightarrow 0^+} T_{\hat{\lambda}}(\alpha) < \lim_{\alpha \rightarrow \infty} T_{\hat{\lambda}}(\alpha).$$

This contradicts to property (6) in the proof of Theorem 2.1 since $C_\infty \leq \tilde{\lambda} \leq \hat{\lambda} \leq \infty$. Thus we get nonexistence of positive solutions of (1.1) for this region.

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