Available online at www.sciencedirect.com
SCIENCE DIRECT®

J. Math. Anal. Appl. 301 (2005) 158–169

Journal of
MATHEMATICAL
ANALYSIS AND
APPLICATIONS

www.elsevier.com/locate/jmaa

Best constants for tensor products of Bernstein type operators ☆

Jesús de la Cal [a,*], Javier Cárcamo [b]

[a] Departamento de Matemática Aplicada y Estadística e Investigación Operativa, Facultad de Ciencia y Tecnología, Universidad del País Vasco, Apartado 644, 48080 Bilbao, Spain
[b] Departamento de Economía Aplicada V, E.U. de Estudios Empresariales, Universidad del País Vasco, Elcano 21, 48008 Bilbao, Spain

Received 29 May 2003

Available online 22 September 2004

Submitted by J. McCarthy

Abstract

For the tensor product of k copies of the same one-dimensional Bernstein-type operator L, we consider the problem of finding the best constant in preservation of the usual modulus of continuity for the l_p-norm on $\mathbb{R}^k$. Two main results are obtained: the first one gives both necessary and sufficient conditions in order that $1 + k^{1-1/p}$ is the best uniform constant for a single operator; the second one gives sufficient conditions in order that $1 + k^{1-1/p}$ is the best uniform constant for a family of operators. The general results are applied to several classical families of operators usually considered in approximation theory. Throughout the paper, probabilistic concepts and methods play an important role.

Keywords: Best constant; Bernstein-type operators; Tensor product; Gamma operators; Beta operators; Baskakov operators; Modulus of continuity; Gamma distribution; Beta distribution; Negative binomial distribution; Poisson process; Gamma process

☆ Research supported by the Spanish MCYT, Proyecto BFM2002-04163-C02-02, and by FEDER.
* Corresponding author.
E-mail addresses: mepcaagj@lg.ehu.es (J. de la Cal), mebcaurj@lg.ehu.es (J. Cárcamo).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2004.06.064

1. Introduction and main results

Let Δ_k be a (nonempty) convex subset of $\mathbb{R}^k$, and let $L^{\langle k\rangle}$ be a positive linear operator acting on a set $\mathcal{L}^{\langle k\rangle}$ of real functions on Δ_k, which assigns a real function $L^{\langle k\rangle} f$ on Δ_k to each $f \in \mathcal{L}^{\langle k\rangle}$. The problem of global smoothness preservation can be described as the problem of obtaining estimates of the form

$$\omega_p(L^{\langle k\rangle} f;\delta) \leqslant C(\delta)\omega_p(f;\delta), \quad \delta > 0,\ f \in \mathcal{L}^{\langle k\rangle}, \tag{1}$$

where $C(\delta)$ is a positive constant not depending upon f, and

$$\omega_p(f;\delta) := \sup\bigl\{\bigl|f(\mathbf{x}) - f(\mathbf{y})\bigr|:\ \mathbf{x},\mathbf{y} \in \Delta_k,\ \|\mathbf{x}-\mathbf{y}\|_p \leqslant \delta\bigr\}$$

is the usual modulus of continuity for the l_p-norm on $\mathbb{R}^k$ ($p \in [1,\infty]$). In particular, it is interesting to determine the value of the best possible constant on the right-hand side in (1). Provided that $L^{\langle k\rangle} f$ is constant whenever f is, such a best constant is obviously given by

$$C_p^{\langle k\rangle}(\delta) := \sup_{f \in \mathcal{L}_*^{\langle k\rangle}} \frac{\omega_p(L^{\langle k\rangle} f;\delta)}{\omega_p(f;\delta)}, \quad \delta > 0, \tag{2}$$

where

$$\mathcal{L}_*^{\langle k\rangle} := \bigl\{f \in \mathcal{L}^{\langle k\rangle}:\ 0 < \omega_p(f;1) < \infty\bigr\}.$$

Problems of this kind have been discussed in several works by using different approaches (see, for instance, [1–10,12] and references therein). The probabilistic approach developed in [1,2,6–9,12] has proved to be suitable and fruitful when dealing with operators of probabilistic type (also called Bernstein-type operators), that is, operators allowing for a representation of the form

$$L^{\langle k\rangle} f(\mathbf{x}) = Ef\bigl(\xi^{\langle k\rangle}(\mathbf{x})\bigr), \quad \mathbf{x} \in \Delta_k,\ f \in \mathcal{L}^{\langle k\rangle}, \tag{3}$$

where E denotes mathematical expectation, $\{\xi^{\langle k\rangle}(\mathbf{x}):\ \mathbf{x} \in \Delta_k\}$ is a stochastic process taking values in Δ_k, and $\mathcal{L}^{\langle k\rangle}$ is the set of all real functions on Δ_k for which the right-hand side in (3) makes sense.

In the present paper, we consider k-dimensional operators which are tensor products of k copies of the same one-dimensional Bernstein-type operator. More precisely, the setting is the following.

Let I be the interval $[0,1]$ or $[0,\infty)$, and let L be the Bernstein-type operator over I given by

$$Lf(x) := Ef\bigl(\xi(x)\bigr), \quad x \in I,\ f \in \mathcal{L},$$

where the I-valued stochastic process $\{\xi(x):\ x \in I\}$ is assumed to be integrable (this always holds when $I = [0,1]$). It is easy to see that the domain $\mathcal{L}$ contains all the real (measurable, when needed) functions on I such that $\omega(f;1) < \infty$.

The tensor product $L^{\langle k\rangle} := L \otimes \cdots \otimes L$ of k copies of L is the k-dimensional operator over $\Delta_k := I^k$ given by (3), where

$$\xi^{\langle k\rangle}(\mathbf{x}) := \bigl(\xi_1(x_1),\ldots,\xi_k(x_k)\bigr), \quad \mathbf{x} := (x_1,\ldots,x_k) \in I^k, \tag{4}$$

and $\{\xi_i(x)\colon x \in I\}$ $(i = 1, \dots, k)$ are k stochastically independent copies of $\{\xi(x)\colon x \in I\}$ defined on the same probability space. Note that, if $f_1, \dots, f_k \in \mathcal{L}$ and f is the function on I^k given by

$$f(\mathbf{x}) := \prod_{i=1}^{k} f_i(x_i), \quad \mathbf{x} := (x_1, \dots, x_k) \in I^k,$$

then we have, by (3), (4), and the independence assumption,

$$L^{\langle k\rangle} f(\mathbf{x}) = \prod_{i=1}^{k} L f_i(x_i), \quad \mathbf{x} := (x_1, \dots, x_k) \in I^k,$$

which is actually the distinctive feature of a tensor product operator.

The following theorem gives exact formulae and upper bounds for the best constants corresponding to these tensor products. It is a consequence of [8, Theorems 14.2 and 14.3], and generalizes some one-dimensional results early obtained in [2]. The symbol $\lceil\cdot\rceil$ denotes the ceiling function, i.e.,

$\lceil x\rceil :=$ the smallest integer not less than x.

Theorem A. *Assume that the following two conditions are fulfilled*:

(a) *L is centered, i.e., $E\xi(x) = x$, for all $x \in I$.*
(b) *The process $\{\xi(x)\colon x \in I\}$ has stationary increments, i.e., for all $0 \leqslant x < y \in I$, the random variable $\xi(y) - \xi(x)$ has the same probability distribution as $\xi(y - x) - \xi(0)$.*

Then, for all $k \geqslant 1$ and $p \in [1, \infty]$, we have

$$C_p^{\langle k\rangle}(\delta) = \sup_{\substack{\mathbf{x}\in I^k \\ \|\mathbf{x}\|_p=\delta}} E\left\lceil \frac{\|\xi^{\langle k\rangle}(\mathbf{x})\|_p}{\delta} \right\rceil \leqslant 1 + k^{1-1/p}, \tag{5}$$

where $\delta \in (0, k^{1/p}]$ or $\delta > 0$, according to $I = [0, 1]$ or $I = [0, \infty)$.

Remark 1. Assumptions (a) and (b) obviously imply that $\xi(0) = 0$ a.s. and $\xi(x) \leqslant \xi(y)$ a.s., whenever $0 \leqslant x \leqslant y \in I$. As a consequence, for $p = \infty$, the preceding formula becomes

$$C_\infty^{\langle k\rangle}(\delta) = E\left\lceil \frac{\|\xi^{\langle k\rangle}(\mathbf{d})\|_\infty}{\delta} \right\rceil, \quad 0 < \delta \in I,\ k \geqslant 1,$$

where $\mathbf{d} := (\delta, \dots, \delta)$.

Theorem A applies to many classical operators usually considered in approximation theory (see [8]). However, $C_p^{\langle k\rangle}(\cdot)$ is a quite irregular function, and the theoretical computation of the best uniform constant, i.e., $\sup_{\delta>0} C_p^{\langle k\rangle}(\delta)$, typically requires specific techniques adapted to the particular case under consideration. Moreover, the results in [6,9] show how different values can be found, even in the case $p = \infty$.

In the present paper, we find a class of operators for which the best uniform constant is just the upper bound in (5). More precisely, we show the following result.

Theorem 1. *Let L be a Bernstein-type operator satisfying the requirements in Theorem* A. *In order that we have*

$$\sup_{0<\delta\in k^{1/p}I} C_p^{\langle k\rangle}(\delta) = \lim_{\delta\downarrow 0} C_p^{\langle k\rangle}(\delta) = 1 + k^{1-1/p}, \quad k \geqslant 1,\ p \in [1,\infty],$$

it is both necessary and sufficient that the following two conditions be fulfilled:

(c) $\xi(x)/x$ *converges in probability to* 0 *as* $x \downarrow 0$.
(d) $P(\xi(x) = 0) = 0$ *for all* $0 < x \in I$.

Remark 2. In terms of the operator L, conditions (c) and (d) mean that

$$L1_{[0,ax]}(x) = P\big(\xi(x) \leqslant ax\big) \to 1 \quad (x \downarrow 0),\ a > 0,$$

and

$$L1_{\{0\}}(x) = 0 \quad \text{for all } 0 < x \in I,$$

respectively, where (here and hereafter) 1_A stands for the indicator function of the set or event A. Condition (d) is obviously fulfilled if, for each $0 < x \in I$, the distribution of $\xi(x)$ is absolutely continuous. Recalling Remark 1, it is clear that, in the setting of Theorem 1 condition (d) is equivalent to

$$P\big(\xi(x) = 0\big) \to 0 \quad (x \downarrow 0).$$

Our second main result concerns a family $\{L_t\colon t > 0\}$ of operators instead of a single operator, and gives sufficient conditions in order that $1 + k^{1-1/p}$ be the best uniform constant for the *complete* family. We denote by $C_{t,p}^{\langle k\rangle}(\delta)$ the best constant defined by (2) when $L^{\langle k\rangle}$ is replaced by $L_t^{\langle k\rangle} := L_t \otimes \cdots \otimes L_t$.

Theorem 2. *Let $\{L_t\colon t > 0\}$ be a family of Bernstein-type operators over the interval $[0,\infty)$ allowing for a representation of the form*

$$L_t f(x) = Ef\big(\xi_t(x)\big),$$

where $\{\xi_t(x)\colon x \geqslant 0,\ t > 0\}$ is a double-indexed stochastic process taking values in $[0,\infty)$. Assume that

(I) *For each $t > 0$, L_t fulfills the requirements in Theorem* A.
(II) *We have*

$$\lim_{x\uparrow\infty} \frac{\xi_t(x)}{x} = \xi_t \quad \text{a.s.},\ t > 0, \tag{6}$$

where $\{\xi_t\colon t > 0\}$ is a process fulfilling the following conditions:
(a′) $E\xi_t = 1$ *for all* $t > 0$.
(c′) ξ_t *converges in probability to* 0 *as* $t \downarrow 0$.
(d′) $P(\xi_t = 0) \to 0$ *as* $t \downarrow 0$.

Then, for all $k \geqslant 1$ and $p \in [1, \infty]$, we have

$$\sup_{t,\delta>0} C_{t,p}^{\langle k\rangle}(\delta) = \lim_{t\downarrow 0} \sup_{\delta>0} C_{t,p}^{\langle k\rangle}(\delta) = 1 + k^{1-1/p}.$$

Theorems 1 and 2 are shown in the next section. Sections 3 and 4 contain applications of these results to several classical families of Bernstein-type operators.

2. Proofs of Theorems 1 and 2

Both proofs are based upon (Theorem A and) the following lemma. We denote by φ the function given by

$$\varphi(u) := 1_{(0,1]}(u) + u1_{(1,\infty)}(u), \quad u \geqslant 0, \tag{7}$$

and we observe that

$$\varphi(u) \leqslant \lceil u \rceil \leqslant (1+u)1_{(0,\infty)}(u), \quad u \geqslant 0.$$

Lemma 1. *Let $\{\xi(u)\colon\ 0 < u \in I\}$ be an I-valued stochastic process, and, for $0 < u \in I$ and $k \geqslant 1$, let $\xi^{\langle k\rangle}(u) := (\xi_1(u), \ldots, \xi_k(u))$ be a k-dimensional random vector whose components are independent and have the same distribution as $\xi(u)$. If the process fulfills the assumptions*

(i) *$E\xi(u) = c$ (a positive constant) for all $0 < u \in I$,*
(ii) *$\xi(u)$ converges in probability to 0 as $u \downarrow 0$,*
(iii) *$P(\xi(u) = 0) \to 0$ as $u \downarrow 0$,*

then, we have

$$\lim_{u\downarrow 0} E\varphi\left(\left\|\xi^{\langle k\rangle}(u)\right\|_\infty\right) = \lim_{u\downarrow 0} E\left\lceil \left\|\xi^{\langle k\rangle}(u)\right\|_\infty \right\rceil = 1 + kc, \quad k \geqslant 1.$$

Conversely, under assumption (i), *conditions* (ii) *and* (iii) *are necessary in order that*

$$\lim_{u\downarrow 0} E\left\lceil \left\|\xi^{\langle 2\rangle}(u)\right\|_\infty \right\rceil = 1 + 2c.$$

Proof of Lemma 1. To show the first assertion, let $k \geqslant 1$ be fixed. Since we have, for all $0 < u \in I$,

$$\begin{aligned} E\varphi\left(\left\|\xi^{\langle k\rangle}(u)\right\|_\infty\right) &\leqslant E\left\lceil \left\|\xi^{\langle k\rangle}(u)\right\|_\infty \right\rceil \leqslant 1 + E\left\|\xi^{\langle k\rangle}(u)\right\|_\infty \\ &\leqslant 1 + \sum_{i=1}^{k} E\xi_i(u) = 1 + kc, \end{aligned}$$

we only need to show that

$$\liminf_{u\downarrow 0} E\varphi\left(\left\|\xi^{\langle k\rangle}(u)\right\|_\infty\right) \geqslant 1 + kc. \tag{8}$$

Using the inequality

$$\left\|\xi^{\langle k\rangle}(u)\right\|_\infty 1_{(\|\xi^{\langle k\rangle}(u)\|_\infty>1)} \geqslant \sum_{i=1}^k \xi_i(u) 1_{(\xi_i(u)>1)} \prod_{j\neq i} 1_{(\xi_j(u)\leqslant 1)}, \quad 0<u\in I,$$

and the fact that $\xi_1(u),\dots,\xi_k(u)$ are independent and have the same distribution as $\xi(u)$, we obtain, for all $0<u\in I$,

$$\begin{aligned} E\varphi\left(\left\|\xi^{\langle k\rangle}(u)\right\|_\infty\right) &\geqslant P\left(0<\left\|\xi^{\langle k\rangle}(u)\right\|_\infty\leqslant 1\right) \\ &\quad + \sum_{i=1}^k E\left[\xi_i(u)1_{(\xi_i(u)>1)}\right]\prod_{j\neq i} P\left(\xi_j(u)\leqslant 1\right) \\ &= \left[P\left(\xi(u)\leqslant 1\right)\right]^k - \left[P\left(\xi(u)=0\right)\right]^k \\ &\quad + k\left[P\left(\xi(u)\leqslant 1\right)\right]^{k-1} E\left[\xi(u)1_{(\xi(u)>1)}\right]. \end{aligned}$$

By assumption (ii), we have

$$P\left(\xi(u)\leqslant 1\right)\to 1 \quad (u\downarrow 0).$$

Assumption (ii) also implies by the bounded convergence theorem

$$E\left[\xi(u)1_{(\xi(u)\leqslant 1)}\right]\to 0 \quad (u\downarrow 0),$$

which, under assumption (i), is equivalent to

$$E\left[\xi(u)1_{(\xi(u)>1)}\right]\to c \quad (u\downarrow 0).$$

From this and assumption (iii), we conclude that inequality (8) holds true, and this finishes the proof of the first assertion. To show the second one, we start from the fact that we have, for all $0<u\in I$,

$$\begin{aligned} E\left\lceil\left\|\xi^{\langle 2\rangle}(u)\right\|_\infty\right\rceil &\leqslant P\left(\left\|\xi^{\langle 2\rangle}(u)\right\|_\infty>0\right) + E\left\|\xi^{\langle 2\rangle}(u)\right\|_\infty \\ &= 1-\left[P\left(\xi(u)=0\right)\right]^2 + 2c - E\eta(u) \leqslant 1+2c, \end{aligned}$$

where $\eta(u) := \min(\xi_1(u),\xi_2(u))$, and we have used assumption (i) together with the equality

$$\left\|\xi^{\langle 2\rangle}(u)\right\|_\infty = \xi_1(u)+\xi_2(u)-\eta(u).$$

Therefore, the hypothesis $\lim_{u\downarrow 0} E\lceil\|\xi^{\langle 2\rangle}(u)\|_\infty\rceil = 1+2c$ implies that

$$\lim_{u\downarrow 0} P\left(\xi(u)=0\right)=0 \quad \text{and} \quad \lim_{u\downarrow 0} E\eta(u)=0,$$

and, by Markov's inequality, we also have

$$\lim_{u\downarrow 0}\left[P\left(\xi(u)>z\right)\right]^2 = \lim_{u\downarrow 0} P\left(\eta(u)>z\right) \leqslant \lim_{u\downarrow 0}\frac{E\eta(u)}{z} = 0, \quad 0<z\in I.$$

Thus, the process fulfills conditions (ii) and (iii), and the proof of the lemma is complete. □

Proof of Theorem 1. The necessity part directly follows from Lemma 1 and Remark 1. To show the sufficiency, let $k \geqslant 1$ and $1 \leqslant p \leqslant \infty$ be fixed. From Theorem A, we have

$$C_p^{\langle k\rangle}(k^{1/p}\delta) \geqslant E\left\lceil \frac{\|\xi^{\langle k\rangle}(\mathbf{d})\|_p}{k^{1/p}\delta} \right\rceil \geqslant E\left\lceil \frac{\|\xi^{\langle k\rangle}(\mathbf{d})\|_\infty}{k^{1/p}\delta} \right\rceil, \quad 0 < \delta \leqslant 1,$$

where $\mathbf{d} := (\delta, \dots, \delta)$ and we have used the fact that $\|\cdot\|_p \geqslant \|\cdot\|_\infty$. By the preceding lemma, we therefore obtain that

$$\liminf_{\delta\downarrow 0} C_p^{\langle k\rangle}(\delta) \geqslant \lim_{\delta\downarrow 0} E\left\lceil \frac{\|\xi^{\langle k\rangle}(\mathbf{d})\|_\infty}{k^{1/p}\delta} \right\rceil = 1 + k^{1-1/p},$$

which together with (5) yields the conclusion. □

Proof of Theorem 2. Let $k \geqslant 1$ and $1 \leqslant p \leqslant \infty$ be fixed. We only need to show that

$$\liminf_{t\downarrow 0} \sup_{\delta>0} C_{t,p}^{\langle k\rangle}(\delta) \geqslant 1 + k^{1-1/p}.$$

By Theorem A, we can write, for all $t, \delta > 0$,

$$C_{t,p}^{\langle k\rangle}(k^{1/p}\delta) \geqslant E\left\lceil \frac{\|\xi_t^{\langle k\rangle}(\mathbf{d})\|_p}{k^{1/p}\delta} \right\rceil \geqslant E\left\lceil \frac{\|\xi_t^{\langle k\rangle}(\mathbf{d})\|_\infty}{k^{1/p}\delta} \right\rceil \geqslant E\varphi\left(\frac{\|\xi_t^{\langle k\rangle}(\mathbf{d})\|_\infty}{k^{1/p}\delta} \right), \tag{9}$$

where $\xi_t^{\langle k\rangle}(\mathbf{d}) := (\xi_{t,1}(\delta), \dots, \xi_{t,k}(\delta))$, and $\{\xi_{t,j}(x)\colon x \geqslant 0,\ t > 0\}$ $(j = 1, \dots, k)$ are k independent copies of $\{\xi_t(x)\colon x \geqslant 0,\ t > 0\}$ defined on the same probability space. From (6), we have, for all $t > 0$,

$$\lim_{\delta\uparrow\infty} \frac{\xi_{t,j}(\delta)}{\delta} = \xi_{t,j} \quad \text{a.s.},\ j = 1, \dots, k,$$

implying that

$$\lim_{\delta\uparrow\infty} \frac{\|\xi_t^{\langle k\rangle}(\mathbf{d})\|_\infty}{k^{1/p}\delta} = \frac{\|\xi_t^{\langle k\rangle}\|_\infty}{k^{1/p}} \quad \text{a.s.},$$

where $\xi_t^{\langle k\rangle} := (\xi_{t,1}, \dots, \xi_{t,k})$, and the components are independent and have the same distribution as ξ_t. Since φ is continuous on $(0, \infty)$, and $\varphi(0) = 0$, we therefore have

$$\liminf_{\delta\uparrow\infty} \varphi\left(\frac{\|\xi_t^{\langle k\rangle}(\mathbf{d})\|_\infty}{k^{1/p}\delta} \right) \geqslant \varphi\left(\frac{\|\xi_t^{\langle k\rangle}\|_\infty}{k^{1/p}} \right) \quad \text{a.s.},\ t > 0,$$

and, from (9) and Fatou's lemma, we conclude that

$$\sup_{\delta>0} C_{t,p}^{\langle k\rangle}(\delta) \geqslant \liminf_{\delta\uparrow\infty} E\varphi\left(\frac{\|\xi_t^{\langle k\rangle}(\mathbf{d})\|_\infty}{k^{1/p}\delta} \right) \geqslant E\varphi\left(\frac{\|\xi_t^{\langle k\rangle}\|_\infty}{k^{1/p}} \right), \quad t > 0.$$

By the assumptions on $\{\xi_t\colon t > 0\}$ and Lemma 1, we obtain

$$\liminf_{t\downarrow 0} \sup_{\delta>0} C_{t,p}^{\langle k\rangle}(\delta) \geqslant \lim_{t\downarrow 0} E\varphi\left(\frac{\|\xi_t^{\langle k\rangle}\|_\infty}{k^{1/p}} \right) = 1 + k^{1-1/p},$$

and the proof of the theorem is complete. □

3. Applications of Theorem 1

(A) *Beta operators*. For $t > 0$, the beta operator B_t over the interval $[0, 1]$ is defined by

$$B_t f(x) := \begin{cases} \int_0^1 f(\theta)\beta_{t,x}(\theta)\,d\theta, & 0 < x < 1, \\ f(x), & x = 0, 1, \end{cases}$$

where f is any real measurable bounded function on $[0, 1]$, and $\beta_{t,x}$ is the beta probability density with parameters tx, $t(1-x)$, i.e.,

$$\beta_{t,x}(\theta) := \frac{\theta^{tx-1}(1-\theta)^{t(1-x)-1}}{B(tx, t(1-x))} 1_{(0,1)}(\theta),$$

$B(\cdot,\cdot)$ being the Euler beta function. This operator allows for the representation

$$B_t f(x) = Ef\left(\frac{\gamma_{tx}}{\gamma_t}\right), \quad x \in [0, 1],$$

where $\{\gamma_t\colon\ t \geqslant 0\}$ is a standard gamma process, i.e., a stochastic process starting at 0, having independent stationary increments, and such that, for each $t > 0$, γ_t has the gamma distribution with density

$$g_t(\theta) := \frac{\theta^{t-1}e^{-\theta}}{\Gamma(t)} 1_{(0,\infty)}(\theta). \tag{10}$$

It is well known that, for each $t > 0$, the process $\{\gamma_{tx}/\gamma_t\colon\ 0 \leqslant x \leqslant 1\}$ fulfils conditions (a) and (b) in Theorem A. As a consequence, it was established in [8] that

$$\sup_{0<\delta\leqslant k^{1/p}} C_{t,p}^{\langle k\rangle}(\delta) \leqslant 1 + k^{1-1/p}, \quad t > 0,\ p \in [1,\infty],\ k \geqslant 1,$$

and

$$\sup_{0<\delta\leqslant k} C_{t,1}^{\langle k\rangle}(\delta) = 2, \quad t > 0,\ k \geqslant 1$$

(the one-dimensional result $\sup_{0<\delta\leqslant 1} C_{t,1}^{\langle 1\rangle}(\delta) = 2$ $(t > 0)$ was early obtained in [2]). Moreover, the process $\{\gamma_{tx}/\gamma_t\colon\ 0 \leqslant x \leqslant 1\}$ trivially fulfils condition (d) in Theorem 1, and we have, for all $a > 0$,

$$P(\gamma_{tx}/\gamma_t \leqslant ax) = \int_0^{ax} \beta_{t,x}(\theta)\,d\theta \geqslant \frac{\alpha(t,x)(ax)^{tx}}{tx B(tx, t(1-x))} \to 1 \quad (x \downarrow 0)$$

(where $\alpha(t,x) := \min(1, (1-ax)^{t(1-x)-1})$), showing that the process also fulfills condition (c). According to Theorem 1, we conclude that

$$\sup_{0<\delta\leqslant k^{1/p}} C_{t,p}^{\langle k\rangle}(\delta) = \lim_{\delta\downarrow 0} C_{t,p}^{\langle k\rangle}(\delta) = 1 + k^{1-1/p}, \quad t > 0,\ p \in [1,\infty],\ k \geqslant 1.$$

(B) *Beta-type operators over the nonnegative semi-axis*. For $t > 0$, let B_t^* be the integral operator over the interval $[0, \infty)$ defined by

$$B_t^* f(x) := \begin{cases} \int_0^\infty f(\theta)\beta_{t,x}^*(\theta)\,d\theta, & x > 0, \\ f(0), & x = 0, \end{cases}$$
$$= Ef\left(\frac{\gamma_{tx}}{\gamma'_{t+1}}\right), \tag{11}$$

where $\{\gamma_t\colon t \geqslant 0\}$ and $\{\gamma'_t\colon t \geqslant 0\}$ are two independent standard gamma processes defined on the same probability space, and $\beta_{t,x}^*$ is the beta-type probability density

$$\beta_{t,x}^*(\theta) := \frac{1}{B(tx, t+1)} \frac{\theta^{tx-1}}{(1+\theta)^{tx+t+1}} 1_{(0,\infty)}(\theta).$$

This operator is a slight modification of the "inverse beta operator" introduced in [15]. It is readily checked that, for each $t > 0$, the process $\{\gamma_{tx}/\gamma'_{t+1}\colon x \geqslant 0\}$ fulfils conditions (a), (b), and (d). Since we have, for all $a > 0$,

$$P(\gamma_{tx}/\gamma'_{t+1} \leqslant ax) = \int_0^{ax} \beta_{t,x}^*(\theta)\,d\theta \geqslant \frac{(ax)^{tx}(1+ax)^{-(tx+t+1)}}{tx B(tx, t+1)} \to 1 \quad (x \downarrow 0),$$

it also fulfils condition (c). From Theorem 1, we therefore have

$$\sup_{\delta>0} C_{t,p}^{\langle k\rangle}(\delta) = \lim_{\delta\downarrow 0} C_{t,p}^{\langle k\rangle}(\delta) = 1 + k^{1-1/p}, \quad t > 0,\ p \in [1,\infty],\ k \geqslant 1.$$

4. Applications of Theorem 2

(A) *Gamma operators*. For $t > 0$, the gamma operator G_t over the interval $[0,\infty)$ is given by

$$G_t f(x) := \int_0^\infty f(x\theta/t) g_t(\theta)\,d\theta = Ef\left(\frac{x\gamma_t}{t}\right),$$

where $\{\gamma_t\colon t \geqslant 0\}$ is a standard gamma process, and g_t is the gamma probability density given in (10). It should be observed that, for each fixed $t > 0$, the process $\{x\gamma_t/t\colon x \geqslant 0\}$ fulfills conditions (a), (b), and (d), but *not* condition (c). Thus, Theorem 1 is not applicable to G_t. We actually have

$$\sup_{\delta>0} C_{t,1}^{\langle 1\rangle}(\delta) = E\lceil \gamma_t/t \rceil < 2 = E(1 + \gamma_t/t)$$

(the first equality by Theorem A, and the strict inequality because of the fact that $P(\lceil \gamma_t/t \rceil < 1 + \gamma_t/t) = 1$). However, the process $\{\gamma_t/t\colon t > 0\}$ obviously fulfils conditions (a′) and (d′) (in Theorem 2), and it also fulfils (c′), since we have, for all $a > 0$,

$$P(\gamma_t \leqslant at) = \int_0^{at} \frac{\theta^{t-1}e^{-\theta}}{\Gamma(t)}\,d\theta \geqslant \frac{e^{-at}(at)^t}{\Gamma(t+1)} \to 1 \quad (t \downarrow 0).$$

From Theorem 2, we therefore have

$$\sup_{t,\delta>0} C_{t,p}^{\langle k\rangle}(\delta) = \lim_{t\downarrow 0} \sup_{\delta>0} C_{t,p}^{\langle k\rangle}(\delta) = 1 + k^{1-1/p}, \quad p \in [1,\infty],\ k \geqslant 1.$$

(B) *Lupaş and Müller gamma operators*. For $t > 0$, let M_t be the integral operator given by

$$M_t f(x) := \int_0^\infty f(xt/\theta) g_{t+1}(\theta)\, d\theta = Ef\left(\frac{xt}{\gamma_{t+1}}\right),$$

where γ_t and g_t are the same as in the preceding example. The approximation properties of this operator have been considered in [11] (see also [13,14]). It is readily checked that the double-indexed process $\{xt/\gamma_{t+1}\colon\ x \geqslant 0,\ t > 0\}$ fulfils the same conditions as the process involved in the preceding example. Therefore, we also have

$$\sup_{t,\delta>0} C_{t,p}^{\langle k\rangle}(\delta) = \lim_{t\downarrow 0} \sup_{\delta>0} C_{t,p}^{\langle k\rangle}(\delta) = 1 + k^{1-1/p}, \quad p \in [1,\infty],\ k \geqslant 1.$$

(C) *Baskakov operators*. For $t > 0$, the Baskakov operator H_t over the interval $[0,\infty)$ is given by

$$H_t f(x) := \sum_{k=0}^\infty f(k/t) \binom{t+k-1}{k} \frac{x^k}{(1+x)^{t+k}} = Ef\left(\frac{N(x\gamma_t)}{t}\right), \tag{12}$$

where $\{\gamma_t\colon\ t \geqslant 0\}$ is a standard gamma process, $\{N(u)\colon\ u \geqslant 0\}$ is a standard Poisson process, and these two processes are assumed to be independent and defined on the same probability space. (We recall that $\{N(u)\colon\ u \geqslant 0\}$ is a stochastic process starting at 0, having independent stationary increments, and such that, for each $u > 0$, $N(u)$ has the Poisson distribution with parameter u.) For each $t > 0$, the process $\{N(x\gamma_t)/t\colon\ x \geqslant 0\}$ fulfills conditions (a), (b), and (c), but *not* condition (d), since we have

$$P\big(N(x\gamma_t)/t = 0\big) = (1+x)^{-t}.$$

It was shown in [8] that

$$\sup_{t,\delta>0} C_{t,p}^{\langle k\rangle}(\delta) \leqslant 1 + k^{1-1/p}, \quad p \in [1,\infty],\ k \geqslant 1,$$

and

$$\sup_{t,\delta>0} C_{t,1}^{\langle k\rangle}(\delta) = 2, \quad k \geqslant 1$$

(the one-dimensional version of this equality was early obtained in [2]). On the other hand, the case $p = \infty$ was specifically considered in [9], and it was established that

$$k \leqslant \sup_{\delta>0} C_{t,\infty}^{\langle k\rangle}(\delta) \leqslant 1 + k - \min\left\{1, \binom{k}{2} \frac{t}{tk+1}\right\}, \quad t > 0,\ k \geqslant 2,$$

which entails

$$\sup_{\delta>0} C_{t,\infty}^{\langle k\rangle}(\delta) = k, \qquad k \geqslant 4,\ t \geqslant 2/k(k-3).$$

We claim that the double indexed process $\{N(x\gamma_t)/t\colon\ x \geqslant 0,\ t > 0\}$ satisfies the requirements in Theorem 2. We actually have, by the strong law of large numbers for Poisson processes,

$$\lim_{x\uparrow\infty} \frac{N(x\gamma_t)}{xt} = \frac{\gamma_t}{t} \quad \text{a.s.},\ t > 0,$$

and, as said above, the process $\{\gamma_t/t\colon\ t>0\}$ fulfils conditions (a′), (c′), and (d′). We can therefore assert that

$$\sup_{t,\delta>0} C_{t,p}^{\langle k\rangle}(\delta) = \lim_{t\downarrow 0}\sup_{\delta>0} C_{t,p}^{\langle k\rangle}(\delta) = 1 + k^{1-1/p}, \quad p\in[1,\infty],\ k\geqslant 1.$$

(D) *A two-parameter family of operators*. For $t, r > 0$, let $P_{t,r}$ be the operator over the interval $[0,\infty)$ given by

$$P_{t,r} f(x) := \sum_{k=0}^{\infty} f(k/r)\binom{t+k-1}{k}\frac{B(tx+k, t+r+1)}{B(tx, t+1)} = Ef\big(\xi_{t,r}(x)\big),$$

with

$$\xi_{t,r}(x) := \frac{N(\gamma_{tx}\gamma_r'/\gamma_{t+1}'')}{r},$$

where $\{N(u)\colon\ u\geqslant 0\}$ is a standard Poisson process, $\{\gamma_t\colon\ t\geqslant 0\}$, $\{\gamma_t'\colon\ t\geqslant 0\}$, and $\{\gamma_t''\colon\ t\geqslant 0\}$ are standard gamma processes, and these four processes are supposed to be mutually independent and defined on the same probability space. It should be observed that $P_{t,r}$ is the composition $B_t^*\circ H_r$ of the beta-type operator B_t^* given in (11) with the Baskakov operator H_r given in (12). When r is a positive integer, $P_{t,r}$ becomes a modified version of an operator introduced by Stancu [16]. It is readily seen that $P_{t,r}$ fulfils conditions (a), (b), and (c), but not (d). On the other hand, from the strong laws of large numbers for both Poisson processes and gamma processes, we have

$$\lim_{x\uparrow\infty}\frac{\xi_{t,r}(x)}{x} = \xi_{t,r} := \frac{t}{\gamma_{t+1}''}\frac{\gamma_r'}{r} \quad \text{a.s.},\ t, r>0.$$

Moreover, for each $r>0$ (respectively, $t>0$), the process $\{\xi_{t,r}\colon\ t>0\}$ (respectively, $\{\xi_{t,r}\colon\ r>0\}$) fulfills conditions (a′), (c′), and (d′). From Theorem 2, we conclude that

$$\sup_{t,\delta>0} C_{t,r,p}^{\langle k\rangle}(\delta) = \lim_{t\downarrow 0}\sup_{\delta>0} C_{t,r,p}^{\langle k\rangle}(\delta) = 1 + k^{1-1/p}, \quad r>0,\ p\in[1,\infty],\ k\geqslant 1,$$

and

$$\sup_{r,\delta>0} C_{t,r,p}^{\langle k\rangle}(\delta) = \lim_{r\downarrow 0}\sup_{\delta>0} C_{t,r,p}^{\langle k\rangle}(\delta) = 1 + k^{1-1/p}, \quad t>0,\ p\in[1,\infty],\ k\geqslant 1.$$

Acknowledgment

The authors are very grateful to the referee for his appropriate comments and valuable suggestions.

References

[1] J.A. Adell, J. de la Cal, Using stochastic processes for studying Bernstein-type operators, Rend. Circ. Mat. Palermo (2) Suppl. 33 (1993) 125–141.

[2] J.A. Adell, A. Pérez-Palomares, Best constants in preservation inequalities concerning the first modulus and Lipschitz classes for Bernstein-type operators, J. Approx. Theory 93 (1998) 128–139.

[3] G.A. Anastassiou, C. Cottin, H.H. Gonska, Global smoothness of approximating functions, Analysis 11 (1991) 43–57.
[4] G.A. Anastassiou, C. Cottin, H.H. Gonska, Global smoothness preservation by multivariate approximation operators, in: S. Baron, D. Leviatan (Eds.), Israel Mathematical Conference Proceedings, vol. IV, Bar-Ilan University, Ramat-Gan, 1991, pp. 31–44.
[5] C. Cottin, H.H. Gonska, Simultaneous approximation and global smoothness preservation, Rend. Circ. Mat. Palermo (2) Suppl. 33 (1993) 259–279.
[6] J. de la Cal, J. Cárcamo, A.M. Valle, A best constant for bivariate Bernstein and Szász–Mirakyan operators, J. Approx. Theory 123 (2003) 117–124.
[7] J. de la Cal, A.M. Valle, Best constants in global smoothness preservation inequalities for some multivariate operators, J. Approx. Theory 97 (1999) 158–180.
[8] J. de la Cal, A.M. Valle, Global smoothness preservation by multivariate Bernstein-type operators, in: G.A. Anastassiou (Ed.), Handbook on Analytic Computational Methods in Applied Mathematics, CRC Press, Boca Raton, 2000, pp. 667–707.
[9] J. de la Cal, A.M. Valle, Best constants for tensor products of Bernstein, Szász and Baskakov operators, Bull. Austral. Math. Soc. 62 (2000) 211–220.
[10] W. Kratz, U. Stadtmüller, On the uniform modulus of continuity of certain discrete approximation operators, J. Approx. Theory 54 (1988) 326–337.
[11] A. Lupaş, M. Müller, Approximationseigenschaften der Gammaoperatoren, Math. Z. 98 (1967) 208–226.
[12] J. de la Cal, J. Cárcamo, On a certain best constant for Bernstein-type operators, J. Approx. Theory 113 (2001) 189–206.
[13] V. Totik, Uniform approximation by positive operators on infinite intervals, Anal. Math. 10 (1984) 163–182.
[14] J. de la Cal, J. Cárcamo, On uniform approximation by some classical Bernstein-type operators, J. Math. Anal. Appl. 279 (2003) 625–638.
[15] J.A. Adell, J. de la Cal, M. San Miguel, Inverse beta and generalized Bleimann–Butzer–Hahn operators, J. Approx. Theory 76 (1994) 54–64.
[16] D.D. Stancu, Two classes of positive linear operators, An. Univ. Timisoara Ser. Mat. 8 (1970) 213–220.