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J. Math. Anal. Appl. 311 (2005) 495–505

*Journal of*  
MATHEMATICAL  
ANALYSIS AND  
APPLICATIONS

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## Positive solutions for boundary value problem of nonlinear fractional differential equation <sup>☆</sup>

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Received 14 October 2004

Available online 2 April 2005

Submitted by A. Cellina

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### Abstract

In this paper, we investigate the existence and multiplicity of positive solutions for nonlinear fractional differential equation boundary value problem:

$$D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad 0 < t < 1,$$

$$u(0) = u(1) = 0,$$

where  $1 < \alpha \leq 2$  is a real number,  $D_{0+}^{\alpha}$  is the standard Riemann–Liouville differentiation, and  $f : [0, 1] \times [0, \infty) \rightarrow [0, \infty)$  is continuous. By means of some fixed-point theorems on cone, some existence and multiplicity results of positive solutions are obtained. The proofs are based upon the reduction of problem considered to the equivalent Fredholm integral equation of second kind.

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*Keywords:* Fractional differential equation; Boundary value problem; Positive solution; Green's function; Fixed-point theorem

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<sup>☆</sup> This work is sponsored by the National Nature Science Foundation of China (10371006) and the Doctoral Program Foundation of Education Ministry of China (1999000722).

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### 1. Introduction

Fractional differential equations have been of great interest recently. It is caused both by the intensive development of the theory of fractional calculus itself and by the applications of such constructions in various sciences such as physics, mechanics, chemistry, engineering, etc. For details, see [4,6–8,12,14] and references therein.

It should be noted that most of papers and books on fractional calculus are devoted to the solvability of linear initial fractional differential equations in terms of special functions [11,14,15]. Recently, there are some papers deal with the existence and multiplicity of solution (or positive solution) of nonlinear initial fractional differential equation by the use of techniques of nonlinear analysis (fixed-point theorems, Leray–Schauder theory, etc.), see [1–3,5,16,17].

However, there are few papers consider the Dirichlet-type problem for linear ordinary differential equations of fractional order, see [8,13]. No contributions exist, as far as we know, concerning the existence and multiplicity of positive solutions of the following problem:

$$D_{0+}^{\alpha}u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \tag{1.1}$$

$$u(0) = u(1) = 0, \tag{1.2}$$

where  $1 < \alpha \leq 2$  is a real number,  $D_{0+}^{\alpha}$  is the standard Riemann–Liouville differentiation, and  $f : [0, 1] \times [0, \infty) \rightarrow [0, \infty)$  is continuous.

In this paper, we firstly derive the corresponding Green’s function. Consequently problem (1.1), (1.2) is deduced to a equivalent Fredholm integral equation of the second kind. Finally, by the means of some fixed-point theorems, the existence and multiplicity of positive solutions are obtained.

### 2. Background materials and preliminaries

For the convenience of the reader, we present here the necessary definitions from fractional calculus theory. These definitions can be found in the recent literature.

**Definition 2.1.** The fractional integral of order  $\alpha > 0$  of a function  $y : (0, \infty) \rightarrow R$  is given by

$$I_{0+}^{\alpha}y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} y(s) ds$$

provided the right side is pointwise defined on  $(0, \infty)$ .

**Definition 2.2.** The fractional derivative of order  $\alpha > 0$  of a continuous function  $y : (0, \infty) \rightarrow R$  is given by

$$D_{0+}^{\alpha}y(t) = \frac{1}{\Gamma(n - \alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{y(s)}{(t - s)^{\alpha-n+1}} ds,$$

where  $n = [\alpha] + 1$ , provided that the right side is pointwise defined on  $(0, \infty)$ .

**Remark 2.1.** As a basic example, we quote for  $\lambda > -1$ ,

$$D_{0+}^{\alpha} t^{\lambda} = \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda - \alpha + 1)} t^{\lambda - \alpha},$$

giving in particular  $D_{0+}^{\alpha} t^{\alpha - m} = 0, m = 1, 2, \dots, N$ , where  $N$  is the smallest integer greater than or equal to  $\alpha$ .

In fact, for  $\lambda > -1$ ,

$$\begin{aligned} D_{0+}^{\alpha} t^{\lambda} &= \frac{1}{\Gamma(n - \alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{s^{\lambda}}{(t - s)^{\alpha - n + 1}} ds \\ &= \frac{1}{\Gamma(n - \alpha)} \left(\frac{d}{dt}\right)^n t^{n - \alpha + \lambda} \int_0^1 z^{\lambda} (1 - z)^{n - \alpha - 1} dz \\ &= \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + 1 + n - \alpha)} \left(\frac{d}{dt}\right)^n t^{n - \alpha + \lambda}. \end{aligned}$$

So,

$$D_{0+}^{\alpha} t^{\alpha - m} = \frac{\Gamma(\alpha - m + 1)}{\Gamma(n - m + 1)} \left(\frac{d}{dt}\right)^n t^{n - m} = 0, \quad \text{for } m = 1, 2, \dots, N.$$

From Definition 2.2 and Remark 2.1, we then obtain

**Lemma 2.1.** *Let  $\alpha > 0$ . If we assume  $u \in C(0, 1) \cap L(0, 1)$ , then the fractional deferential equation*

$$D_{0+}^{\alpha} u(t) = 0$$

has  $u(t) = C_1 t^{\alpha - 1} + C_2 t^{\alpha - 2} + \dots + C_N t^{\alpha - N}, C_i \in R, i = 1, 2, \dots, N$ , as unique solutions.

As  $D_{0+}^{\alpha} I_{0+}^{\alpha} u = u$  for all  $u \in C(0, 1) \cap L(0, 1)$ . From Lemma 2.1 we deduce the following law of composition.

**Lemma 2.2.** *Assume that  $u \in C(0, 1) \cap L(0, 1)$  with a fractional derivative of order  $\alpha > 0$  that belongs to  $C(0, 1) \cap L(0, 1)$ . Then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + C_1 t^{\alpha - 1} + C_2 t^{\alpha - 2} + \dots + C_N t^{\alpha - N},$$

for some  $C_i \in R, i = 1, 2, \dots, N$ .

In the following, we present the Green’s function of fractional differential equation boundary value problem.

**Lemma 2.3.** Given  $y \in C[0, 1]$  and  $1 < \alpha \leq 2$ , the unique solution of

$$D_{0+}^\alpha u(t) + y(t) = 0, \quad 0 < t < 1, \tag{2.1}$$

$$u(0) = u(1) = 0, \tag{2.2}$$

is

$$u(t) = \int_0^1 G(t, s)y(s) ds,$$

where

$$G(t, s) = \begin{cases} \frac{[t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \frac{[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{2.3}$$

**Proof.** We may apply Lemma 2.2 to reduce Eq. (2.1) to an equivalent integral equation

$$u(t) = I_{0+}^\alpha y(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2},$$

for some  $C_1, C_2 \in R$ . Consequently, the general solution of Eq. (2.1) is

$$u(t) = - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds + C_1 t^{\alpha-1} + C_2 t^{\alpha-2}.$$

By (2.2), there are  $C_2 = 0, C_1 = \int_0^1 (1-s)^{\alpha-1} y(s) ds / \Gamma(\alpha)$ . Therefore, the unique solution of problem (2.1), (2.2) is

$$\begin{aligned} u(t) &= - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds + \int_0^1 \frac{(1-s)^{\alpha-1} t^{\alpha-1}}{\Gamma(\alpha)} y(s) ds \\ &= \int_0^t \frac{[t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds + \int_t^1 \frac{[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)} y(s) ds \\ &= \int_0^1 G(t, s)y(s) ds. \end{aligned}$$

The proof is complete.  $\square$

**Lemma 2.4.** The function  $G(t, s)$  defined by Eq. (2.3) satisfies the following conditions:

- (1)  $G(t, s) > 0$ , for  $t, s \in (0, 1)$ ;
- (2) There exists a positive function  $\gamma \in C(0, 1)$  such that

$$\min_{1/4 \leq t \leq 3/4} G(t, s) \geq \gamma(s) \max_{0 \leq t \leq 1} G(t, s) = \gamma(s)G(s, s), \quad \text{for } 0 < s < 1. \tag{2.4}$$

**Proof.** Observing the expression of  $G(t, s)$ , it is clear that  $G(t, s) > 0$  for  $s, t \in (0, 1)$ . In the following, we consider the existence of  $\gamma(s)$ . Firstly, for given  $s \in (0, 1)$ ,  $G(t, s)$  is decreasing with respect to  $t$  for  $s \leq t$  and increasing with respect to  $t$  for  $t \leq s$ . Consequently, setting

$$g_1(t, s) = \frac{[t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, \quad g_2(t, s) = \frac{[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)},$$

one has

$$\begin{aligned} \min_{1/4 \leq t \leq 3/4} G(t, s) &= \begin{cases} g_1(\frac{3}{4}, s), & s \in (0, 1/4], \\ \min\{g_1(\frac{3}{4}, s), g_2(\frac{1}{4}, s)\}, & s \in [1/4, 3/4], \\ g_2(\frac{1}{4}, s), & s \in [3/4, 1), \end{cases} \\ &= \begin{cases} g_1(\frac{3}{4}, s), & s \in (0, r], \\ g_2(\frac{1}{4}, s), & s \in [r, 1), \end{cases} \\ &= \begin{cases} \frac{1}{\Gamma(\alpha)} \{[\frac{3}{4}(1-s)]^{\alpha-1} - (\frac{3}{4}-s)^{\alpha-1}\}, & s \in (0, r], \\ \frac{1}{\Gamma(\alpha)} \frac{1}{4^{\alpha-1}} (1-s)^{\alpha-1}, & s \in [r, 1), \end{cases} \end{aligned}$$

where  $1/4 < r < 3/4$  is the unique solution of the equation

$$\left[\frac{3}{4}(1-s)\right]^{\alpha-1} - \left(\frac{3}{4}-s\right)^{\alpha-1} = \frac{1}{4^{\alpha-1}}(1-s)^{\alpha-1}.$$

Specially,  $r = 0.5$  if  $\alpha = 2$ ;  $r \rightarrow 0.5$  as  $\alpha \rightarrow 2$  and  $r \rightarrow 0.75$  as  $\alpha \rightarrow 1$ .

Secondly, with the use of the monotonicity of  $G(t, s)$ , we have

$$\max_{0 \leq t \leq 1} G(t, s) = G(s, s) = \frac{1}{\Gamma(\alpha)} [s(1-s)]^{\alpha-1}, \quad s \in (0, 1).$$

Thus, setting

$$\gamma(s) = \begin{cases} \frac{[\frac{3}{4}(1-s)]^{\alpha-1} - (\frac{3}{4}-s)^{\alpha-1}}{[s(1-s)]^{\alpha-1}}, & s \in (0, r], \\ \frac{1}{(4s)^{\alpha-1}}, & s \in [r, 1), \end{cases}$$

the proof is complete.  $\square$

**Remark 2.2.** Clearly,  $\gamma(s) \rightarrow 0$  when  $s \rightarrow 0$  unless that  $\alpha = 2$  ( $\inf_{0 < s < 1} \gamma(s) = 1/4$  if  $\alpha = 2$ ). Consequently, we cannot acquire a positive constant  $\gamma$  take instead of the role of positive function  $\gamma(s)$  with  $1 < \alpha < 2$  in (2.4). In our opinion, it is the key that the results obtained for fractional differential equations in this paper are weaker than that have obtained for integer-order differential equations.

**Definition 2.3.** The map  $\theta$  is said to be a nonnegative continuous concave functional on a cone  $P$  of a real Banach space  $E$  provided that  $\theta : P \rightarrow [0, \infty)$  is continuous and

$$\theta(tx + (1-t)y) \geq t\theta(x) + (1-t)\theta(y)$$

for all  $x, y \in P$  and  $0 \leq t \leq 1$ .

The following fixed-point theorems are fundamental in the proofs of our main results.

**Lemma 2.5** [9]. *Let  $E$  be a Banach space,  $P \subseteq E$  a cone, and  $\Omega_1, \Omega_2$  two bounded open balls of  $E$  centered at the origin with  $\bar{\Omega}_1 \subset \Omega_2$ . Suppose that  $\mathcal{A}: P \cap (\bar{\Omega}_2 \setminus \Omega_1) \rightarrow P$  is a completely continuous operator such that either*

- (i)  $\|\mathcal{A}x\| \leq \|x\|$ ,  $x \in P \cap \partial\Omega_1$  and  $\|\mathcal{A}x\| \geq \|x\|$ ,  $x \in P \cap \partial\Omega_2$ , or
- (ii)  $\|\mathcal{A}x\| \geq \|x\|$ ,  $x \in P \cap \partial\Omega_1$  and  $\|\mathcal{A}x\| \leq \|x\|$ ,  $x \in P \cap \partial\Omega_2$

holds. Then  $\mathcal{A}$  has a fixed point in  $P \cap (\bar{\Omega}_2 \setminus \Omega_1)$ .

**Lemma 2.6** [10]. *Let  $P$  be a cone in a real Banach space  $E$ ,  $P_c = \{x \in P \mid \|x\| \leq c\}$ ,  $\theta$  a nonnegative continuous concave functional on  $P$  such that  $\theta(x) \leq \|x\|$ , for all  $x \in \bar{P}_c$ , and  $P(\theta, b, d) = \{x \in P \mid b \leq \theta(x), \|x\| \leq d\}$ . Suppose  $\mathcal{A}: \bar{P}_c \rightarrow \bar{P}_c$  is completely continuous and there exist constants  $0 < a < b < d \leq c$  such that*

- (C1)  $\{x \in P(\theta, b, d) \mid \theta(x) > b\} \neq \emptyset$  and  $\theta(\mathcal{A}x) > b$  for  $x \in P(\theta, b, d)$ ;
- (C2)  $\|\mathcal{A}x\| < a$  for  $x \leq a$ ;
- (C3)  $\theta(\mathcal{A}x) > b$  for  $x \in P(\theta, b, c)$  with  $\|\mathcal{A}x\| > d$ .

Then  $\mathcal{A}$  has at least three fixed points  $x_1, x_2, x_3$  with

$$\|x_1\| < a, \quad b < \theta(x_2), \quad a < \|x_3\| \quad \text{with } \theta(x_3) < b.$$

**Remark 2.3.** If there holds  $d = c$ , then condition (C1) of Lemma 2.6 implies condition (C3) of Lemma 2.6.

### 3. Main results

In this section, we impose growth conditions on  $f$  which allow us to apply Lemmas 2.5 and 2.6 to establish some results of existence and multiplicity of positive solutions for problem (1.1), (1.2).

Let  $E = C[0, 1]$  be endowed with the ordering  $u \leq v$  if  $u(t) \leq v(t)$  for all  $t \in [0, 1]$ , and the maximum norm,  $|u| = \max_{0 \leq t \leq 1} |u(t)|$ . Define the cone  $P \subset E$  by

$$P = \{u \in E \mid u(t) \geq 0\}.$$

Let the nonnegative continuous concave functional  $\theta$  on the cone  $P$  be defined by

$$\theta(u) = \min_{1/4 \leq t \leq 3/4} |u(t)|.$$

**Lemma 3.1.** *Let  $T: P \rightarrow E$  be the operator defined by*

$$Tu(t) := \int_0^1 G(t, s) f(s, u(s)) ds,$$

then  $T: P \rightarrow P$  is completely continuous.

**Proof.** The operator  $T : P \rightarrow P$  is continuous in view of nonnegativeness and continuity of  $G(t, s)$  and  $f(t, u)$ .

Let  $\Omega \subset P$  be bounded, i.e., there exists a positive constant  $M > 0$  such that  $\|u\| \leq M$ , for all  $u \in \Omega$ . Let  $L = \max_{0 \leq t \leq 1, 0 \leq u \leq M} |f(t, u)| + 1$ , then, for  $u \in \Omega$ , we have

$$|Tu(t)| \leq \int_0^1 G(t, s) f(s, u(s)) ds \leq L \int_0^1 G(s, s) ds.$$

Hence,  $T(\Omega)$  is bounded.

On the other hand, given  $\epsilon > 0$ , setting

$$\delta = \frac{1}{2} \left( \frac{\Gamma(\alpha)\epsilon}{M} \right)^{\frac{1}{\alpha-1}},$$

then, for each  $u \in \Omega$ ,  $t_1, t_2 \in [0, 1]$ ,  $t_1 < t_2$ , and  $t_2 - t_1 < \delta$ , one has  $|Tu(t_2) - Tu(t_1)| < \epsilon$ . That is to say,  $T(\Omega)$  is equicontinuity.

In fact,

$$\begin{aligned} & |Tu(t_2) - Tu(t_1)| \\ &= \left| \int_0^1 G(t_2, s) f(s, u(s)) ds - \int_0^1 G(t_1, s) f(s, u(s)) ds \right| \\ &= \int_0^{t_1} [G(t_2, s) - G(t_1, s)] f(s, u(s)) ds + \int_{t_2}^1 [G(t_2, s) - G(t_1, s)] f(s, u(s)) ds \\ &\quad + \int_{t_1}^{t_2} [G(t_2, s) - G(t_1, s)] f(s, u(s)) ds \\ &< \frac{M}{\Gamma(\alpha)} \left[ \int_0^{t_1} (1-s)^{\alpha-1} (t_2^{\alpha-1} - t_1^{\alpha-1}) ds \right. \\ &\quad \left. + \int_{t_2}^1 (1-s)^{\alpha-1} (t_2^{\alpha-1} - t_1^{\alpha-1}) ds + \int_{t_1}^{t_2} (1-s)^{\alpha-1} (t_2^{\alpha-1} - t_1^{\alpha-1}) ds \right] \\ &< \frac{M}{\Gamma(\alpha)} (t_2^{\alpha-1} - t_1^{\alpha-1}). \end{aligned}$$

In the following, we divide the proof into two cases.

Case 1.  $\delta \leq t_1 < t_2 < 1$ .

$$\begin{aligned} |Tu(t_2) - Tu(t_1)| &< \frac{M}{\Gamma(\alpha)} (t_2^{\alpha-1} - t_1^{\alpha-1}) \leq \frac{M}{\Gamma(\alpha)} \frac{\alpha - 1}{\delta^{2-\alpha}} (t_2 - t_1) \\ &\leq \frac{M}{\Gamma(\alpha)} (\alpha - 1) \delta^{\alpha-1} \leq \epsilon. \end{aligned}$$

Case 2.  $0 \leq t_1 < \delta, t_2 < 2\delta$ .

$$|Tu(t_2) - Tu(t_1)| < \frac{M}{\Gamma(\alpha)}(t_2^{\alpha-1} - t_1^{\alpha-1}) \leq \frac{M}{\Gamma(\alpha)}t_2^{\alpha-1} < \frac{M}{\Gamma(\alpha)}(2\delta)^{\alpha-1} \leq \epsilon.$$

By the means of the Arzela–Ascoli theorem, we have  $T : P \rightarrow P$  is completely continuous. The proof is complete.  $\square$

Denote

$$M = \left( \int_0^1 G(s, s) ds \right)^{-1}, \quad N = \left( \int_{1/4}^{3/4} \gamma(s)G(s, s) ds \right)^{-1}.$$

**Theorem 3.1.** *Let  $f(t, u)$  is continuous on  $[0, 1] \times [0, \infty)$ . Assume that there exist two positive constants  $r_2 > r_1 > 0$  such that*

- (H1)  $f(t, u) \leq Mr_2$ , for  $(t, u) \in [0, 1] \times [0, r_2]$ ;
- (H2)  $f(t, u) \geq Nr_1$ , for  $(t, u) \in [0, 1] \times [0, r_1]$ .

Then problem (1.1), (1.2) has at least one positive solution  $u$  such that  $r_1 \leq \|u\| \leq r_2$ .

**Proof.** By Lemmas 2.3 and 3.1, we know  $T : P \rightarrow P$  is completely continuous and problem (1.1), (1.2) has a solution  $u = u(t)$  if and only if  $u$  solves the operator equation  $u = Tu$ . In order to apply Lemma 2.5, we separate the proof into the following two steps.

*Step 1.* Let  $\Omega_2 := \{u \in P \mid \|u\| < r_2\}$ . For  $u \in \partial\Omega_2$ , we have  $0 \leq u(t) \leq r_2$  for all  $t \in [0, 1]$ . It follows from (H1) that for  $t \in [0, 1]$ ,

$$\|Tu\| = \max_{0 \leq t \leq 1} \int_0^1 G(t, s) f(s, u(s)) ds \leq Mr_2 \int_0^1 G(s, s) ds = r_2 = \|u\|.$$

*Step 2.* Let  $\Omega_1 := \{u \in P \mid \|u\| < r_1\}$ . For  $u \in \partial\Omega_1$ , we have  $0 \leq u(t) \leq r_1$  for all  $t \in [0, 1]$ . By assumption (H2), for  $t \in [1/4, 3/4]$ , there is

$$\begin{aligned} Tu(t) &= \int_0^1 G(t, s) f(s, u(s)) ds \geq \int_0^1 \gamma(s)G(s, s) f(s, u(s)) ds \\ &\geq Nr_1 \int_{1/4}^{3/4} \gamma(s)G(s, s) ds = r_1 = \|u\|. \end{aligned}$$

So

$$\|Tu\| \geq \|u\|, \quad \text{for } u \in \partial\Omega_1.$$

Therefore, by (ii) of Lemma 2.5, we complete the proof.  $\square$

**Example 3.1.** Consider the problem

$$D_{0+}^{3/2}u(t) + u^2 + \frac{\sin t}{4} + 1 = 0, \quad 0 < t < 1, \tag{3.1}$$

$$u(0) = u(1) = 0. \tag{3.2}$$

A simple computation showed  $M = 4/\sqrt{\pi} \approx 2.25676$ ,  $N \approx 13.6649$ . Choosing  $r_1 = 1/14$ ,  $r_2 = 1$ , we have

$$f(t, u) = 1 + \frac{\sin t}{4} + u^2 \leq 2.2107 \leq Mr_2, \quad \text{for } (t, u) \in [0, 1] \times [0, 1],$$

$$f(t, u) = 1 + \frac{\sin t}{4} + u^2 \geq 1 \geq Nr_1, \quad \text{for } (t, u) \in [0, 1] \times [0, 1/14].$$

With the use of Theorem 3.1, problem (3.1), (3.2) has at least one solution  $u$  such that  $1/14 \leq \|u\| \leq 1$ .

**Theorem 3.2.** Suppose  $f(t, u)$  is continuous on  $[0, 1] \times [0, \infty)$  and there exist constants  $0 < a < b < c$  such that the following assumptions hold:

- (A1)  $f(t, u) < Ma$ , for  $(t, u) \in [0, 1] \times [0, a]$ ;
- (A2)  $f(t, u) \geq Nb$ , for  $(t, u) \in [1/4, 3/4] \times [b, c]$ ;
- (A3)  $f(t, u) \leq Mc$ , for  $(t, u) \in [0, 1] \times [0, c]$ .

Then, the boundary value problem (1.1), (1.2) has at least three positive solutions  $u_1, u_2$ , and  $u_3$  with

$$\begin{aligned} \max_{0 \leq t \leq 1} |u_1(t)| < a, \quad b < \min_{1/4 \leq t \leq 3/4} |u_2(t)| < \max_{0 \leq t \leq 1} |u_2(t)| \leq c, \\ a < \max_{0 \leq t \leq 1} |u_3(t)| \leq c, \quad \min_{1/4 \leq t \leq 3/4} |u_3(t)| < b. \end{aligned}$$

**Proof.** We show that all the conditions of Lemma 2.4 are satisfied.

If  $u \in \bar{P}_c$ , then  $\|u\| \leq c$ . Assumption (A3) implies  $f(t, u(t)) \leq Mc$  for  $0 \leq t \leq 1$ . Consequently,

$$\begin{aligned} \|Tu\| &= \max_{0 \leq t \leq 1} \left| \int_0^1 G(t, s) f(s, u(s)) ds \right| \leq \int_0^1 G(s, s) f(s, u(s)) ds \\ &\leq \int_0^1 G(s, s) Mc ds \leq c. \end{aligned}$$

Hence,  $T : \bar{P}_c \rightarrow \bar{P}_c$ . In the same way, if  $u \in \bar{P}_a$ , then assumption (A1) yields  $f(t, u(t)) < Ma$ ,  $0 \leq t \leq 1$ . Therefore, condition (C2) of Lemma 2.6 is satisfied.

To check condition (C1) of Lemma 2.6, we choose  $u(t) = (b + c)/2$ ,  $0 \leq t \leq 1$ . It is easy to see that  $u(t) = (b + c)/2 \in P(\theta, b, c)$ ,  $\theta(u) = \theta((b + c)/2) > b$ , consequently,  $\{u \in P(\theta, b, c) \mid \theta(u) > b\} \neq \emptyset$ . Hence, if  $u \in P(\theta, b, c)$ , then  $b \leq u(t) \leq c$  for  $1/4 \leq t \leq 3/4$ . From assumption (A2), we have  $f(t, u(t)) \geq Nb$  for  $1/4 \leq t \leq 3/4$ . So

$$\begin{aligned} \theta(Tu) &= \min_{1/4 \leq t \leq 3/4} |(Tu)(t)| \geq \int_0^1 \gamma(s)G(s, s) f(s, u(s)) ds \\ &> \int_{1/4}^{3/4} \gamma(s)G(s, s) Nb ds = b, \end{aligned}$$

i.e.,

$$\theta(Tu) > b, \quad \text{for all } u \in P(\theta, b, c).$$

This shows that condition (C1) of Lemma 2.6 is also satisfied.

By Lemma 2.6 and Remark 2.3, the boundary value problem (1.1), (1.2) has at least three positive solutions  $u_1, u_2,$  and  $u_3$  satisfying

$$\begin{aligned} \max_{0 \leq t \leq 1} |u_1(t)| < a, & \quad b < \min_{1/4 \leq t \leq 3/4} |u_2(t)|, \\ a < \max_{0 \leq t \leq 1} |u_3(t)|, & \quad \min_{1/4 \leq t \leq 3/4} |u_3(t)| < b. \end{aligned}$$

The proof is complete.  $\square$

**Example 3.3.** Consider the problem

$$D_{0+}^{3/2} u(t) + f(t, u) = 0, \quad 0 < t < 1, \tag{3.3}$$

$$u(0) = u(1) = 0, \tag{3.4}$$

where

$$f(t, u) = \begin{cases} \frac{t}{20} + 14u^2, & \text{for } u \leq 1, \\ 13 + \frac{t}{20} + u, & \text{for } u > 1. \end{cases}$$

We have  $M = 4/\sqrt{\pi} \approx 2.25676, N \approx 13.6649.$  Choosing  $a = 1/10, b = 1, c = 12,$  there hold

$$f(t, u) = \frac{t}{20} + 14u^2 \leq 0.19 \leq Ma \approx 0.225, \quad \text{for } (t, u) \in [0, 1] \times [0, 1/10],$$

$$f(t, u) = 13 + \frac{t}{20} + u \geq 14.05 \geq Nb \approx 13.7, \quad \text{for } (t, u) \in [1/4, 3/4] \times [1, 12],$$

$$f(t, u) = 13 + \frac{t}{20} + u \leq 25.05 \leq Mc \approx 27.1, \quad \text{for } (t, u) \in [0, 1] \times [0, 12].$$

With the use of Theorem 3.2, problem (3.3), (3.4) has at least three positive solutions  $u_1, u_2$  and  $u_3$  with

$$\begin{aligned} \max_{0 \leq t \leq 1} |u_1(t)| < 1/10, & \quad 1 < \min_{1/4 \leq t \leq 3/4} |u_2(t)| < \max_{0 \leq t \leq 1} |u_2(t)| \leq 12, \\ 1/10 < \max_{0 \leq t \leq 1} |u_3(t)| \leq 12, & \quad \min_{1/4 \leq t \leq 3/4} |u_3(t)| < 1. \end{aligned}$$

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