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Well-posedness for the Cauchy problem associated to the Hirota–Satsuma equation: Periodic case ☆

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Abstract

We consider a system of Korteweg–de Vries (KdV) equations coupled through nonlinear terms, called the Hirota–Satsuma system. We study the initial value problem (IVP) associated to this system in the periodic case, for given data in Sobolev spaces $H^s \times H^{s+1}$ with regularity below the one given by the conservation laws. Using the Fourier transform restriction norm method, we prove local well-posedness whenever $s > -1/2$. Also, with some restriction on the parameters of the system, we use the recent technique introduced by Colliander et al., called *I-method* and *almost conserved quantities*, to prove global well-posedness for $s > -3/14$.

Keywords: KdV equation; Cauchy problem; Well-posedness

1. Introduction

Let us consider the initial value problem (IVP) in $(x,t) \in \mathbb{R}/\lambda\mathbb{Z} \times \mathbb{R} \cong [0,\lambda] \times \mathbb{R}$,

$$\begin{cases} \partial_t u - \alpha\big(\partial_x^3 u + 6u\partial_x u\big) = 2\beta v \partial_x v, \\ \partial_t v + \partial_x^3 v + 3u\partial_x v = 0, \\ u(x,0) = \phi(x), \quad v(x,0) = \psi(x), \end{cases} \tag{1.1}$$

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where $\alpha, \beta \in \mathbb{R}$ and $u = u(x,t)$, $v = v(x,t)$ are real valued functions, periodic in the space variable, with period λ. Therefore, the initial data ϕ and ψ are also real valued and equally periodic.

This model was introduced by Hirota and Satsuma in [9] and has a structure analogous to the Korteweg–de Vries (KdV) equation. The interesting feature about it is the fact that it is a system of two nonlinearly coupled KdV equations. In particular, the nonlinear coupling in the second equation of the system (1.1) compels us to work in asymmetric Sobolev spaces $H^s \times H^{s+1}$. Many authors have studied KdV equations in recent years (see for example [3–8,10–12] and references therein). The IVP associated to (1.1) has also been widely studied in the literature. In particular, existence and stability of periodic solutions is studied by Angulo in [1,2], where it is proved that the IVP (1.1) is locally well-posed for given data (ϕ, ψ) in Sobolev spaces $H^s \times H^{s+1}$, for $s \geqslant 0$, when $\alpha = -1$ and in $H^s \times H^s$, for $s \geqslant 1$, when $\alpha \neq -1, 0$, using the techniques usually applied to the single KdV model. The following are the quantities conserved by the flow of (1.1):

$$\int \left\{ \frac{1+\alpha}{2} u_x^2 + \beta v_x^2 - (1+\alpha) u^3 - \beta u v^2 \right\} dx \tag{1.2}$$

and

$$\frac{1}{2} \int \left\{ u^2 + \frac{2}{3} \beta v^2 \right\} dx. \tag{1.3}$$

Using these conserved quantities and considering $\beta > 0$ Angulo also proved, in [1,2], that the IVP (1.1) has global solutions for given data in $H^1 \times H^1$ when $\alpha \neq 0, -1$ and in $L^2 \times H^1$ when $\alpha = -1$.

In this work we are interested in improving these results, i.e., we want to prove local and global well-posedness for initial data in Sobolev spaces with lower regularity. We intend to use the Fourier transform restriction norm method, as in [4,12], to achieve the local well-posedness result. For the global well-posedness result we follow the techniques developed by Colliander et al. in [7,8]: the so-called I operator method and almost conserved quantities. However, the coupling terms in the nonlinearity, as well as the presence of the parameters create some technical difficulties that prevent the direct application of these methods. More specifically, for the asymmetric coupling term $u\partial_x v$, in the second equation of (1.1), the bilinear estimates present in the literature (as in [7,8,12]) are not directly applicable. A different bilinear estimate is therefore required to address this term, which is done in this article by using two spaces of different regularities (see (2.9) below). However, following the same type of counterexample construction as in [12], the possibility of having a similar result for symmetric Sobolev spaces is not discarded. We are thus able to work in $H^s \times H^{s+1}$ spaces to obtain our well-posedness results. Besides, they fall within the scope of the spaces that can be handled by the two conservated quantities above, for $\alpha = -1$. It is then possible to extend the local solution to a global one, for suitable values of s, by modifying the estimates that currently exist in the literature to suit in the present situation (see Lemma 2.7).

The main results of this work read as follows:

Theorem 1.1. *The initial value problem* (1.1) *is locally well-posed for given data in the periodic Sobolev spaces* $H^s \times H^{s+1}$, *for* $s \geqslant -\frac{1}{2}$.

Theorem 1.2. *The unique solution to the initial value problem* (1.1) *with* $\alpha = -1$ *and* $\beta > 0$, *given by Theorem* 1.1, *can be extended to any interval of time* $[0, T]$, *whenever* $s > -\frac{3}{14}$.

We will use the Fourier transform restriction spaces $X_{s,b}$ introduced by Bourgain. These spaces are defined via the unitary group describing the time evolution of the solution to the associated linear problem. The presence of the coefficient α in the first equation in (1.1) suggests treating the problem in two different cases. When $\alpha = -1$, the linear part of both equations in (1.1) is the same, and therefore so is their unitary group. When $\alpha \neq -1$, the situation changes. In this case, we use rescaling in the space variable to obtain the same structure in the linear part of both equations. This change of scale naturally modifies the interval of periodicity of u and v. We are thus led to defining the $X_{s,b}$ spaces depending on the length of the interval of periodicity (see below for details).

For $\alpha = -1$, the IVP (1.1) becomes

$$\begin{cases} \partial_t u + \partial_x^3 u + 6u\partial_x u = 2\beta v\partial_x v, \\ \partial_t v + \partial_x^3 v + 3u\partial_x v = 0, \\ u(x,0) = \phi(x), \quad v(x,0) = \psi(x), \end{cases} \tag{1.4}$$

with $x \in [0, \lambda]$, $t \in \mathbb{R}$.

For $\alpha \neq 0, -1$ (without loss of generality we can suppose $\alpha < 0$), by defining $w(x,t) = u(\theta x, t)$, $p(x,t) = v(\theta x, t)$ and $q(x,t) = w(\frac{x}{\theta}, t)$, where $\theta = -\alpha^{1/3}$, the IVP (1.1) becomes

$$\begin{cases} \partial_t w + \partial_x^3 w + 6\alpha^{\frac{2}{3}} w\partial_x w + 2\beta\alpha^{-\frac{1}{3}} p\partial_x p = 0, \quad (x,t) \in \left[0, \dfrac{\lambda}{\theta}\right] \times \mathbb{R}, \\ \partial_t v + \partial_x^3 v + 3q\partial_x v = 0, \quad (x,t) \in [0,\lambda] \times \mathbb{R}, \\ w(x,0) = \phi(\theta x), \quad v(x,0) = \psi(x). \end{cases} \tag{1.5}$$

As mentioned earlier, this rescaling has yielded the same linear structure on both equations in (1.5), but the period of w has become λ/θ whereas that of v is still λ.

We now introduce some definitions and notation that will be used in this work, following [7] closely. Let f be a periodic function on the real line, with period λ. We start by defining its Fourier transform as

$$\hat{f}(k) = \int_0^\lambda e^{-2\pi i k x} f(x)\,dx, \tag{1.6}$$

for $k \in \mathbb{Z}/\lambda$, with its inversion formula being given by the Fourier series,

$$f(x) = \int e^{2\pi i k x} \hat{f}(k)(dk)_\lambda, \tag{1.7}$$

where $(dk)_\lambda$ is the normalized counting measure on $\mathbb{Z}/\lambda$,

$$\int a(k)(dk)_\lambda = \frac{1}{\lambda} \sum_{k \in \mathbb{Z}/\lambda} a(k). \tag{1.8}$$

The usual properties of the Fourier transform (actually Fourier series, in this case) naturally still hold:

$$\|f\|_{L^2([0,\lambda])} = \|\hat{f}\|_{L^2((dk)_\lambda)} \quad \text{Plancharel}, \tag{1.9}$$

$$\int_0^\lambda f(x)\overline{g(x)}\,dx = \int \hat{f}(k)\overline{\hat{g}(k)}(dk)_\lambda \quad \text{Parseval}, \tag{1.10}$$

$$\widehat{fg}(k) = \hat{f} *_\lambda \hat{g}(k) = \int \hat{f}(k-k_1)\hat{g}(k_1)(dk_1)_\lambda \quad \text{Convolution.} \tag{1.11}$$

The Sobolev space $H^s(0,\lambda)$, $s \in \mathbb{R}$, is defined by the norm

$$\|f\|_{H^s(0,\lambda)} := \big\| \langle k\rangle^s \hat{f}(k)\big\|_{L^2((dk)_\lambda)}, \tag{1.12}$$

where $\langle \cdot \rangle := 1 + |\cdot|$. In what follows, we will write H^s instead of $H^s(0,\lambda)$.

Using Duhamel's principle, the IVP associated to Hirota–Satsuma system is equivalent to the following

$$\begin{cases} u(x,t) = U(t)\phi(x) - \displaystyle\int_0^t U(t-t')f(u,v)(t')\,dt', \\ v(x,t) = U(t)\psi(x) - \displaystyle\int_0^t U(t-t')g(u,v)(t')\,dt', \end{cases} \tag{1.13}$$

where f and g are the nonlinearities of the system. $U(t)$ is the time evolution unitary group associated to the linear problem in $(x,t) \in \mathbb{R}/\lambda\mathbb{Z} \times \mathbb{R}$,

$$\partial_t u + \partial_x^3 u = 0, \quad u(x,0) = \phi(x) \tag{1.14}$$

and is given, using (1.6) and (1.7), by

$$U(t)\phi(x) = \int e^{2\pi i k x} e^{-(2\pi i k)^3 t} \hat{\phi}(k)(dk)_\lambda. \tag{1.15}$$

Using a Fourier transform also for the time variable, (1.15) can be written as

$$U(t)\phi(x) = \iint e^{2\pi i k x} e^{2\pi i \tau t} \delta\big(\tau - 4\pi^2 k^3\big)\hat{\phi}(k)(dk)_\lambda\, d\tau, \tag{1.16}$$

where $\delta(\tau)$ is the Dirac mass concentrated at the origin.

For $s, b \in \mathbb{R}$, we define the $X_{s,b}$ space for the λ-periodic Hirota–Satsuma equation with the norm,

$$\|f\|_{X_{s,b}([0,\lambda]\times\mathbb{R})} := \big\| \langle k\rangle^s \big\langle \tau - 4\pi^2 k^3\big\rangle^b \hat{f}(k,\tau)\big\|_{L^2((dk)_\lambda d\tau)}, \tag{1.17}$$

where $\hat{f}(k,\tau)$ is now the space–time Fourier transform of f.

To get a solution to the IVP associated to the periodic Hirota–Satsuma equation, we follow the scheme introduced in [4,12] in the periodic KdV context, where the authors have used the contraction mapping principle in the $X_{s,1/2}$ space. Unfortunately, this space does not ensure the continuity of the time flow of the solution. To get continuity of the solution, i.e., to control the $L_t^\infty H_x^s$ norm, we introduce the following space Y_s with norm:

$$\|f\|_{Y_s} := \|f\|_{X_{s,\frac{1}{2}}} + \big\| \langle k\rangle^s \hat{f}(k,\tau)\big\|_{L^2((dk)_\lambda)L^1(d\tau)}. \tag{1.18}$$

It is easy to see that if $u \in Y_s$ then $u \in L_t^\infty H_x^s$.

In the sequel, we also need the following space Z_s, with norm motivated by the inhomogeneous term in Duhamel's formula,

$$\|f\|_{Z_s} := \|f\|_{X_{s,-\frac{1}{2}}} + \left\| \frac{\langle k\rangle^s \hat{f}(k,\tau)}{\langle \tau - 4\pi^2 k^3\rangle}\right\|_{L^2((dk)_\lambda)L^1(d\tau)}. \tag{1.19}$$

The following embeddings hold in $X_{s,b}$ spaces:

$$s > s', \quad b > b' \quad \Rightarrow \quad X_{s,b} \subset X_{s',b'},$$

$$\|u\|_{L^4_{xt}} \leqslant c\|u\|_{X_{0,\frac{1}{3}}}, \tag{1.20}$$

$$\|u\|_{L^\infty_{xt}} \leqslant c\|u\|_{X_{\frac{1}{2}+,\frac{1}{2}+}}, \tag{1.21}$$

$$\|u\|_{L^\infty_t L^2_x} \leqslant c\|u\|_{X_{0,\frac{1}{2}+}}. \tag{1.22}$$

Throughout this work we suppose that $u(x,t)$ and $v(x,t)$ are λ-periodic functions of x with zero x-mean for all t.

This article is organized as follows. We give some preliminary estimates in Section 2. The proof of the local well-posedness result is done in Section 3. Finally in Section 4, we present the proof of the global well-posedness result.

2. Some preliminary estimates

Let $\varphi \in C_0^\infty(\mathbb{R})$ be a smooth cut-off function such that

$$\varphi(t) = \begin{cases} 1, & |t| \leqslant 1, \\ 0, & |t| \geqslant 2. \end{cases} \tag{2.1}$$

Note that, multiplication by $\varphi(t)$ is a bounded operation on the spaces $X_{s,b}$, Y_s and Z_s. Also, let us record the following duality relation between Y_s and Z_{-s}.

Lemma 2.1. *For any $s \in \mathbb{R}$ and u, v on $\mathbb{R}/\lambda\mathbb{Z} \times \mathbb{R}$ the following estimate holds*:

$$\left| \iint \chi_{[0,1]}(t) u(x,t) v(x,t)\, dx\, dt \right| \leqslant c\|u\|_{Y_s}\|v\|_{Z_{-s}}. \tag{2.2}$$

Proof. See the proof of Lemma 3.2 in [8]. $\square$

Lemma 2.2. *For any $s \in \mathbb{R}$,*

$$\left\|\varphi(t)U(t)\phi\right\|_{Y_s} \leqslant c\|\phi\|_{H^s}. \tag{2.3}$$

Proof. First observe that,

$$\widehat{\varphi U(t)}\phi(k,\tau) = \hat{\phi}(k)\hat{\varphi}\big(\tau - 4\pi^2 k^3\big). \tag{2.4}$$

Therefore,

$$\begin{aligned} \left\|\varphi(t)U(t)\phi\right\|_{X_{s,\frac{1}{2}}} &= \left\| \langle k\rangle^s \big\langle \tau - 4\pi^2 k^3\big\rangle^{\frac{1}{2}} \hat{\phi}(k)\hat{\varphi}\big(\tau - 4\pi^2 k^3\big)\right\|_{L^2((dk)_\lambda d\tau)} \\ &= \|\varphi\|_{H_t^{1/2}}\|\phi\|_{H^s}, \end{aligned} \tag{2.5}$$

and

$$\left\| \langle k\rangle^s \widehat{\varphi U(t)}\phi(k,\tau)\right\|_{L^2((dk)_\lambda)L^1(d\tau)} = \|\hat{\varphi}\|_{L^1_\tau}\|\phi\|_{H^s}. \tag{2.6}$$

Now, combining (2.5) and (2.6) in (1.18) we get the required estimate (2.3). $\square$

Lemma 2.3. *For any $s \in \mathbb{R}$,*

$$\left\| \varphi(t) \int_0^t U(t-t')F(t')\,dt' \right\|_{Y_s} \leqslant c\|F\|_{Z_s}. \tag{2.7}$$

Proof. See [7,8]. □

Lemma 2.4. *Let $u(x,t)$ and $v(x,t)$ be λ-periodic functions of x with zero x-mean for all t. Then,*

$$\left\|\varphi(t)\partial_x(uv)\right\|_{Z_{-\frac{1}{2}}} \leqslant c\lambda^{0+}\|u\|_{X_{-\frac{1}{2},\frac{1}{2}}}\|v\|_{X_{-\frac{1}{2},\frac{1}{2}}}. \tag{2.8}$$

Proof. See [7,8]. □

Lemma 2.5. *Let $u(x,t)$ and $v(x,t)$ be λ-periodic functions of x with zero x-mean for all t. Then,*

$$\left\|\varphi(t)u\partial_x v\right\|_{Z_{\frac{1}{2}}} \leqslant c\lambda^{0+}\|u\|_{X_{-\frac{1}{2},\frac{1}{2}}}\|v\|_{X_{\frac{1}{2},\frac{1}{2}}}. \tag{2.9}$$

Proof. First, using the bilinear estimate (2.8) and the fact

$$\|\partial_x v\|_{X_{s,b}} \leqslant c\|v\|_{X_{s+1,b}} \tag{2.10}$$

we have

$$\begin{aligned}\left\|\varphi(t)u\partial_x v\right\|_{X_{\frac{1}{2},-\frac{1}{2}}} \sim \left\|\varphi(t)\partial_x(u\partial_x v)\right\|_{X_{-\frac{1}{2},-\frac{1}{2}}} &\leqslant c\lambda^{0+}\|u\|_{X_{-\frac{1}{2},\frac{1}{2}}}\|\partial_x v\|_{X_{-\frac{1}{2},\frac{1}{2}}}\\ &\leqslant c\lambda^{0+}\|u\|_{X_{-\frac{1}{2},\frac{1}{2}}}\|v\|_{X_{\frac{1}{2},\frac{1}{2}}}.\end{aligned} \tag{2.11}$$

The first equivalence in the norms is valid because of the zero x-mean of the functions u and v. Now we estimate the second contribution on the definition of the Z_s norm in (1.19). Again, using (2.10), we will get the required estimate, if we prove

$$\left\| \frac{\langle k\rangle^{\frac{1}{2}}\widehat{uw}(k,\tau)}{\langle \tau - 4\pi^2k^3\rangle} \right\|_{L^2((dk)_\lambda)L^1(d\tau)} \leqslant c\lambda^{0+}\|u\|_{X_{-\frac{1}{2},\frac{1}{2}}}\|w\|_{X_{-\frac{1}{2},\frac{1}{2}}}. \tag{2.12}$$

But the proof of (2.12) is given in [7, p. 734, (7.42)]. This concludes the proof of the lemma. □

For the proof of the global well-posedness theorem we still need two more results. The first one is analogous to (2.9).

Lemma 2.6. *Let $u(x,t)$ and $v(x,t)$ be λ-periodic functions of x with zero x-mean for all t. Then for all $1/2 \leqslant s \leqslant 1$ we have*

$$\|u\partial_x v\|_{Z_s} \lesssim \lambda^{0+}\|u\|_{Y_{s-1}}\|v\|_{Y_s}. \tag{2.13}$$

Proof. See the proof of Corollary 2 in [8]. □

The following result will be crucial to get almost conserved quantities. The difference in the regularity of the product functions (our case will be u and v) makes the proof of this result a bit technical, but important. It follows the same ideas as the proof of the main Theorem 3 in [8].

Lemma 2.7. *Let $u(x,t)$ and $v(x,t)$ be λ-periodic functions of x, then*

$$\|uv\|_{Y_{-1}} \lesssim \|u\|_{Y_0}\|v\|_{Y_1}. \tag{2.14}$$

Proof. We can suppose that the Fourier transforms of u and v are positive. Recall that, the norm on Y_{-1} has two components:

$$\|uv\|_{Y_{-1}} := \|uv\|_{X_{-1,\frac{1}{2}}} + \big\|\langle k\rangle^{-1}\widehat{uv}(k,\tau)\big\|_{L_k^2 L_\tau^1}.$$

We start by the second component. Using Minkowski and Cauchy–Schwarz inequalities we get

$$\begin{aligned}
\big\|\langle k\rangle^{-1}\widehat{uv}(k,\tau)\big\|_{L_k^2 L_\tau^1} &= \left\|\frac{1}{\langle k\rangle}\int \hat{u}(k-k_1,\tau-\tau_1)\hat{v}(k_1,\tau_1)\,dk_1 d\tau_1\right\|_{L_k^2 L_\tau^1} \\
&\lesssim \left\|\int \hat{u}(k-k_1,\tau-\tau_1)\hat{v}(k_1,\tau_1)\,dk_1\,d\tau_1\right\|_{L_k^2 L_\tau^1} \\
&\lesssim \int \|\hat{u}\|_{L_k^2 L_\tau^1}\big|\hat{v}(k_1,\tau_1)\big|\,dk_1\,d\tau_1 \\
&= \|\hat{u}\|_{L_k^2 L_\tau^1}\int \frac{1}{\langle k_1\rangle}\langle k_1\rangle\big|\hat{v}(k_1,\tau_1)\big|\,dk_1\,d\tau_1 \\
&\lesssim \|\hat{u}\|_{L_k^2 L_\tau^1}\int \frac{1}{\langle k_1\rangle}\big\|\langle k_1\rangle\hat{v}(k_1,\tau_1)\big\|_{L_{\tau_1}^1}\,dk_1 \\
&\lesssim \|\hat{u}\|_{L_k^2 L_\tau^1}\big\|\langle k_1\rangle\hat{v}(k_1,\tau_1)\big\|_{L_{k_1}^2 L_{\tau_1}^1} \\
&\lesssim \|u\|_{Y_0}\|v\|_{Y_1}.
\end{aligned} \tag{2.15}$$

For the first component we will follow the scheme in [8]. We claim:

$$\|uv\|_{X_{-1,\frac{1}{2}}} \lesssim \|u\|_{Y_0}\|v\|_{Y_1}. \tag{2.16}$$

Note that,

$$\|uv\|_{X_{-1,\frac{1}{2}}} = \left\|\frac{\langle\tau-k^3\rangle^{\frac{1}{2}}}{\langle k\rangle}\int_{\substack{k=k_1+k_2\\ \tau=\tau_1+\tau_2}} \hat{u}(k_1,\tau_1)\hat{v}(k_2,\tau_2)\right\|_{L_{k,\tau}^2}. \tag{2.17}$$

We divide the region of integration into three parts:

$$\begin{aligned}
&A\colon\quad \big\langle\tau-k^3\big\rangle \leqslant 1000\big\langle\tau_1-k_1^3\big\rangle,\\
&B\colon\quad \big\langle\tau-k^3\big\rangle \leqslant 1000\big\langle\tau_2-k_2^3\big\rangle,\quad \text{and}\\
&C\colon\quad (A\cup B)^c.
\end{aligned}$$

In region A, we have

$$\|uv\|_{X_{-1,\frac{1}{2}}} \lesssim \left\|\frac{1}{\langle k\rangle}\int_{\substack{k=k_1+k_2\\ \tau=\tau_1+\tau_2}} \big\langle\tau_1-k_1^3\big\rangle^{\frac{1}{2}}\hat{u}(k_1,\tau_1)\hat{v}(k_2,\tau_2)\right\|_{L_{k,\tau}^2}. \tag{2.18}$$

We intend to show that the RHS of (2.18) can be controlled by

$$\big\|\langle\tau_1-k_1^3\rangle^{\frac{1}{2}}\hat{u}(k_1,\tau_1)\big\|_{L^2_{k_1,\tau_1}}\|v\|_{Y_0}=\|u\|_{X_{0,\frac{1}{2}}}\|v\|_{Y_0}. \tag{2.19}$$

But, calling $\hat{f}=\langle\tau_1-k_1^3\rangle^{1/2}\hat{u}$ and $\hat{g}=\hat{v}$, this is equivalent to showing that

$$\left\|\frac{1}{\langle k\rangle}\int\limits_{\substack{k=k_1+k_2\\ \tau=\tau_1+\tau_2}}\hat{f}(k_1,\tau_1)\hat{g}(k_2,\tau_2)\right\|_{L^2_{k,\tau}}\lesssim\|f\|_{X_{0,0}}\|g\|_{Y_0}. \tag{2.20}$$

Now, using Plancharel, Hölder and the definition of Y_s we have

$$\begin{aligned}\left\|\frac{1}{\langle k\rangle}\int\limits_{\substack{k=k_1+k_2\\ \tau=\tau_1+\tau_2}}\hat{f}(k_1,\tau_1)\hat{g}(k_2,\tau_2)\right\|_{L^2_{k,\tau}}&=\left\|\frac{1}{\langle k\rangle}\widehat{fg}(k,\tau)\right\|_{L^2_{k,\tau}}\\ &\lesssim\|fg\|_{L^2_tL^1_x}\lesssim\|f\|_{L^2_tL^2_x}\|g\|_{L^\infty_tL^2_x}\\ &=\|\hat{f}\|_{L^2_{k,\tau}}\|g\|_{L^\infty_tH^0_x}\lesssim\|f\|_{X_{0,0}}\|g\|_{Y_0},\end{aligned} \tag{2.21}$$

as required.

In the region B, we can exchange the role of u and v and use the same argument as in the region A to get the bound $\|v\|_{X_{0,1/2}}\|u\|_{Y_0}$, from which the claim follows.

We are thus left with the region C. In this region we have

$$\langle\tau-k^3\rangle>1000\langle\tau_i-k_i^3\rangle,\quad i=1,2. \tag{2.22}$$

Using (2.22) it is easy to see that

$$\langle\tau-k^3\rangle\sim\big|k^3-\big(k_1^3+k_2^3\big)\big|\sim|k_1||k_2||k_1+k_2|. \tag{2.23}$$

Therefore, in this case we need to estimate

$$\left\|\int\limits_{\substack{k=k_1+k_2\\ \tau=\tau_1+\tau_2}}\frac{|k|^{\frac{1}{2}}|k_1|^{\frac{1}{2}}|k_2|^{\frac{1}{2}}}{\langle k\rangle\langle k_2\rangle}\hat{u}(k_1,\tau_1)\langle k_2\rangle\hat{v}(k_2,\tau_2)\right\|_{L^2_{k,\tau}}. \tag{2.24}$$

We will show that this last term is bounded by

$$\big\|\langle\tau_1-k_1^3\rangle^{\frac{1}{2}}\hat{u}(k_1,\tau_1)\big\|_{L^2_{k,\tau}}\big\|\langle k_2\rangle\langle\tau_2-k_2^3\rangle^{\frac{1}{2}}\hat{v}(k_2,\tau_2)\big\|_{L^2_{k,\tau}}=\|u\|_{X_{0,\frac{1}{2}}}\|v\|_{X_{1,\frac{1}{2}}}. \tag{2.25}$$

But again, calling $\hat{f}=\hat{u}$ and $\hat{g}=\langle k_2\rangle\hat{v}(k_2,\tau_2)$, this is equivalent to showing

$$\left\|\int\limits_{\substack{k=k_1+k_2\\ \tau=\tau_1+\tau_2}}\frac{|k|^{\frac{1}{2}}|k_1|^{\frac{1}{2}}|k_2|^{\frac{1}{2}}}{\langle k\rangle\langle k_2\rangle}\hat{f}(k_1,\tau_1)\hat{g}(k_2,\tau_2)\right\|_{L^2_{k,\tau}}\lesssim\|f\|_{X_{0,\frac{1}{2}}}\|g\|_{X_{0,\frac{1}{2}}}. \tag{2.26}$$

Now, we move to show that the multiplier in the first part of (2.26) is bounded. Note that,

$$\frac{|k|^{\frac{1}{2}}|k_1|^{\frac{1}{2}}|k_2|^{\frac{1}{2}}}{\langle k\rangle\langle k_2\rangle}\leqslant\frac{|k_1|^{\frac{1}{2}}}{\langle k_1+k_2\rangle^{1/2}\langle k_2\rangle^{1/2}}.$$

It is easy to see that the only nontrivial case happens when k_1 and k_2 have opposite signs. Otherwise, this term is obviously bounded. We divide into two cases:

(i) $|k_1| \leqslant 2|k_2|$ and
(ii) $|k_1| > 2|k_2|$.

In case (i),

$$\frac{|k_1|^{\frac{1}{2}}}{\langle k_1+k_2\rangle^{1/2}\langle k_2\rangle^{1/2}} \leqslant \frac{\sqrt{2}|k_2|^{\frac{1}{2}}}{\langle k_1+k_2\rangle^{1/2}\langle k_2\rangle^{1/2}} \leqslant \sqrt{2}.$$

In case (ii), we have $|k_1+k_2| \geqslant |k_1| - |k_2| \geqslant \frac{1}{2}|k_1|$. Therefore,

$$\frac{|k_1|^{\frac{1}{2}}}{\langle k_1+k_2\rangle^{1/2}\langle k_2\rangle^{1/2}} \leqslant \frac{\sqrt{2}|k_2|^{\frac{1}{2}}}{\langle k_1\rangle^{1/2}\langle k_2\rangle^{1/2}} \leqslant \sqrt{2}.$$

So, using Plancharel, Hölder and embedding (1.20), we can estimate the LHS of (2.26) as

$$\begin{aligned}
&\left\| \int\limits_{\substack{k=k_1+k_2\\ \tau=\tau_1+\tau_2}} \frac{|k|^{\frac{1}{2}}|k_1|^{\frac{1}{2}}|k_2|^{\frac{1}{2}}}{\langle k\rangle\langle k_2\rangle} \hat{f}(k_1,\tau_1)\hat{g}(k_2,\tau_2)\right\|_{L^2_{k,\tau}} \\
&\quad\lesssim \left\| \int\limits_{\substack{k=k_1+k_2\\ \tau=\tau_1+\tau_2}} \hat{f}(k_1,\tau_1)\hat{g}(k_2,\tau_2)\right\|_{L^2_{k,\tau}} = \big\|\widehat{fg}(k,\tau)\big\|_{L^2_{k,\tau}} = \|fg\|_{L^2_{x,t}} \leqslant \|f\|_{L^4_{x,t}}\|g\|_{L^4_{x,t}} \\
&\quad\leqslant \|f\|_{X_{0,\frac{1}{3}}}\|g\|_{X_{0,\frac{1}{3}}} \leqslant \|f\|_{X_{0,\frac{1}{2}}}\|g\|_{X_{0,\frac{1}{2}}}
\end{aligned} \tag{2.27}$$

as required. □

Remark 2.8. It is not clear if the result in the above lemma is sharp. But following the method that we presented in the proof above, the statement cannot be improved neither by increasing the regularity of the space on the left-hand side nor by decreasing the regularities of any of the spaces on the right-hand side of (2.14).

3. Proof of the local result

This section is devoted to providing the proof to the local well-posedness result, Theorem 1.1. As mentioned in the introduction, we divide the proof into two different cases.

3.1. Local result for $\alpha = -1$

Proof. We use the contraction mapping principle. Let $(\phi,\psi) \in H^s \times H^{s+1}$, $s \geqslant -\frac{1}{2}$, with $\|(\phi,\psi)\|_{H^s\times H^{s+1}} = r$. Let us define

$$\mathcal{M}_r := \big\{(u,v) \in Y_s \times Y_{s+1}\colon \big\|(u,v)\big\|_{Y_s\times Y_{s+1}} \leqslant 2c_0 r\big\},$$

where c_0 here is the constant in (2.3), which depends only on the fixed cut-off function φ. Then $\mathcal{M}_r$ is a Banach space with norm

$$\big|\big|\big|(u,v)\big|\big|\big| := \|u\|_{Y_s} + \|v\|_{Y_{s+1}}.$$

For $(u,v) \in \mathcal{M}_r$, let us define the maps

$$\begin{cases} \Phi_\phi[u,v] = U(t)\phi - \int_0^t U(t-t')(6u\partial_x u - 2\beta v\partial_x v)(t')\,dt' \\ \Psi_\psi[u,v] = U(t)\psi - \int_0^t U(t-t')(3u\partial_x v)(t')\,dt'. \end{cases} \tag{3.1}$$

These, of course, result from the substitution of the nonlinear terms of (1.4) into (1.13). As we are only interested, at the moment, in getting a local in time solution, we can replace (3.1) by

$$\begin{cases} \Phi_\phi[u,v] = \varphi(t)U(t)\phi - \varphi(t)\int_0^t U(t-t')\varphi(t')(6u\partial_x u - 2\beta v\partial_x v)(t')\,dt' \\ \Psi_\psi[u,v] = \varphi(t)U(t)\psi - \varphi(t)\int_0^t U(t-t')\varphi(t')(3u\partial_x v)(t')\,dt'. \end{cases} \tag{3.2}$$

We will show that $\Phi \times \Psi$ maps $\mathcal{M}_r$ into $\mathcal{M}_r$ and is a contraction.

Using (2.3), (2.7) and (2.8), as well as the fact that $u\partial_x u = \frac{1}{2}\partial_x(u^2)$, we get

$$\begin{aligned} \|\Phi\|_{Y_{-\frac12}} &\leqslant \|\varphi(t)U(t)\phi\|_{Y_{-\frac12}} + \left\|\varphi(t)\int_0^t U(t-t')\varphi(t')(6u\partial_x u - 2\beta v\partial_x v)(t')\,dt'\right\|_{Y_{-\frac12}} \\ &\leqslant c_0\|\phi\|_{H^{-\frac12}} + c_1\|\varphi(t')u\partial_x u(t')\|_{Z_{-\frac12}} + c_1\|\varphi(t')v\partial_x v(t')\|_{Z_{-\frac12}} \\ &\leqslant c_0\|\phi\|_{H^{-\frac12}} + c_2\lambda^{0+}\|u\|^2_{X_{-\frac12,\frac12}} + c_2\lambda^{0+}\|v\|^2_{X_{-\frac12,\frac12}} \\ &\leqslant c_0\|\phi\|_{H^{-\frac12}} + c_2\lambda^{0+}\big(\|u\|^2_{Y_{-\frac12}} + \|v\|^2_{Y_{\frac12}}\big). \end{aligned} \tag{3.3}$$

Similarly, the use of (2.3), (2.7) and (2.9) yields

$$\begin{aligned} \|\Psi\|_{Y_{\frac12}} &\leqslant \|\varphi(t)U(t)\psi\|_{Y_{\frac12}} + \left\|\varphi(t)\int_0^t U(t-t')\varphi(t')(3u\partial_x v)(t')\,dt'\right\|_{Y_{\frac12}} \\ &\leqslant c_0\|\psi\|_{H^{\frac12}} + c_1\|\varphi(t')u\partial_x v(t')\|_{Z_{\frac12}} \\ &\leqslant c_0\|\psi\|_{H^{\frac12}} + c_1\lambda^{0+}\|u\|_{X_{-\frac12,\frac12}}\|v\|_{X_{\frac12,\frac12}} \\ &\leqslant c_0\|\psi\|_{H^{\frac12}} + c_1\lambda^{0+}\|u\|_{Y_{-\frac12}}\|v\|_{Y_{\frac12}} \\ &\leqslant c_0\|\psi\|_{H^{\frac12}} + \tilde{c}_2\lambda^{0+}\big(\|u\|^2_{Y_{-\frac12}} + \|v\|^2_{Y_{\frac12}}\big). \end{aligned} \tag{3.4}$$

Therefore, for $C = \max\{c_2, \tilde{c}_2\}$ and choosing initial data for which r is sufficiently small, we can make $8\lambda^{0+}Cc_0r < 1$, to get

$$\begin{aligned} \|(\Phi,\Psi)\|_{Y_{-\frac12}\times Y_{\frac12}} &\leqslant c_0\|(\phi,\psi)\|_{H^{-\frac12}\times H^{\frac12}} + 2C\lambda^{0+}\|(u,v)\|^2_{Y_{-\frac12}\times Y_{\frac12}} \\ &\leqslant c_0r + 2C\lambda^{0+}(2c_0r)^2 \leqslant 2c_0r. \end{aligned} \tag{3.5}$$

Hence $\Phi \times \Psi$ maps $\mathcal{M}_r$ into $\mathcal{M}_r$. Now we move to show that $\Phi \times \Psi$ is a contraction.

Using the same argument used in (3.3) and (3.4) we can obtain

$$\left\|\Phi[u,v]-\Phi[u_1,v_1]\right\|_{Y_{-\frac{1}{2}}} \leqslant C\lambda^{0+}2c_0 r\left(\|u-u_1\|_{Y_{-\frac{1}{2}}}+\|v-v_1\|_{Y_{\frac{1}{2}}}\right) \tag{3.6}$$

and

$$\left\|\Psi[u,v]-\Psi[u_1,v_1]\right\|_{Y_{\frac{1}{2}}} \leqslant C\lambda^{0+}2c_0 r\left(\|u-u_1\|_{Y_{-\frac{1}{2}}}+\|v-v_1\|_{Y_{\frac{1}{2}}}\right). \tag{3.7}$$

Hence, for $8\lambda^{0+}Cc_0 r < 1$, we see that $\Phi \times \Psi$ is a contraction. Now with a standard argument, the proof of the theorem follows for initial data with small $H^s \times H^{s+1}$ norm.

Our next task is to remove the smallness assumption on the initial data. For this, we use a scaling argument. Let $\lambda = \lambda_0$ be fixed, $\eta > 1$ and $(\phi, \psi) \in H^s(0,\lambda_0) \times H^{s+1}(0,\lambda_0)$. Then the IVP (1.4) is well-posed on a small interval of time $[0, \eta^{-3}]$ if and only if the IVP for the η-rescaled functions $u^\eta(x,t) = \eta^{-2}u(\frac{x}{\eta}, \frac{t}{\eta^3})$, and similarly for v^η, in $(x,t) \in \mathbb{R}/\eta\lambda_0\mathbb{Z} \times \mathbb{R}$,

$$\begin{cases} \partial_t u^\eta + \partial_x^3 u^\eta + 6u^\eta \partial_x u^\eta = 2\beta v^\eta \partial_x v^\eta, \\ \partial_t v^\eta + \partial_x^3 v^\eta + 3u^\eta \partial_x v^\eta = 0, \\ u^\eta(x,0) = \eta^{-2}\phi\left(\dfrac{x}{\eta}\right), \quad v^\eta(x,0) = \eta^{-2}\psi\left(\dfrac{x}{\eta}\right), \end{cases} \tag{3.8}$$

is well-posed on [0, 1].

After rescaling, we have that

$$\left\|\left(\phi^\eta, \psi^\eta\right)\right\|_{H^s(0,\eta\lambda_0)\times H^{s+1}(0,\eta\lambda_0)} \leqslant c\eta^{-\frac{3}{2}-s}\left\|(\phi,\psi)\right\|_{H^s(0,\lambda_0)\times H^{s+1}(0,\lambda_0)}. \tag{3.9}$$

So, we can make

$$8(\eta\lambda_0)^{0+}Cc_0\left\|\left(\phi^\eta,\psi^\eta\right)\right\|_{H^s(0,\eta\lambda_0)\times H^{s+1}(0,\eta\lambda_0)} \leqslant 8(\eta\lambda_0)^{0+}Cc_0\eta^{-3/2-s}r < 1$$

by choosing $\eta = \eta(\lambda_0, \|(\phi,\psi)\|_{H^s(0,\lambda_0)\times H^{s+1}(0,\lambda_0)})$ sufficiently large. This proves that the IVP (3.8) is well-posed on the time interval [0, 1]. Therefore, the original IVP is locally well-posed on the time interval $[0, \eta^{-3}]$ for any initial data. □

3.2. *Local result for $\alpha \neq -1, 0$*

Since the periods of w and v in the IVP (1.5) are different, it is necessary to make some modifications in the definitions of the spaces that we used in the previous case.

For $s, b \in \mathbb{R}$, we now define $X_{s,b,\gamma}$ as the closure of the Schwartz space $\mathcal{S}(\mathbb{R}/\gamma\lambda\mathbb{Z} \times \mathbb{R})$ in the norm

$$\|f\|_{X_{s,b,\gamma}} := \left\|\langle k\rangle^s\left\langle\tau - 4\pi^2k^3\right\rangle^b \hat{f}(k,\tau)\right\|_{L^2((dk)_{\gamma\lambda}d\tau)}. \tag{3.10}$$

Note that for $\gamma = 1$, the $X_{s,b,\gamma}$ becomes simply the space $X_{s,b}$ defined in (1.17). In a similar manner, we also modify the spaces Y_s and Z_s to $Y_{s,\gamma}$ and $Z_{s,\gamma}$ with the following norms

$$\|f\|_{Y_{s,\gamma}} := \|f\|_{X_{s,\frac{1}{2},\gamma}} + \left\|\langle k\rangle^s \hat{f}(k,\tau)\right\|_{L^2((dk)_{\gamma\lambda})L^1(d\tau)}, \tag{3.11}$$

and

$$\|f\|_{Z_{s,\gamma}} := \|f\|_{X_{s,-\frac{1}{2},\gamma}} + \left\| \frac{\langle k\rangle^s \hat{f}(k,\tau)}{\langle \tau - 4\pi^2 k^3\rangle} \right\|_{L^2((dk)_{\gamma\lambda})L^1(d\tau)}, \tag{3.12}$$

respectively.

As before, multiplication by $\varphi(t)$ is a bounded operator on the spaces $X_{s,b,\gamma}$, $Y_{s,\gamma}$ and $Z_{s,\gamma}$. Also note that the estimates (2.3), (2.7)–(2.9) are valid on these new spaces too.

We are now in position to provide the proof of Theorem 1.1 in this case.

Proof. We will also use the contraction mapping principle.

Define $\phi_\alpha = \phi(\theta x)$. Let us consider $(\phi_\alpha, \psi) \in H^s(0, \frac{\lambda}{\theta}) \times H^{s+1}(0,\lambda)$, $s \geqslant -\frac{1}{2}$, with $\|(\phi_\alpha,\psi)\|_{H^s(0,\frac{\lambda}{\theta})\times H^{s+1}(0,\lambda)} = r$. Let us define

$$\mathcal{M}_r := \left\{(u,v) \in Y_{s,\frac{1}{\theta}} \times Y_{s+1,1}\colon \left\|(u,v)\right\|_{Y_{s,\frac{1}{\theta}}\times Y_{s+1,1}} \leqslant 2c_0 r\right\}.$$

Then $\mathcal{M}_r$ is a Banach space with norm

$$\left|\!\left|\!\left|(u,v)\right|\!\right|\!\right| := \|u\|_{Y_{s,\frac{1}{\theta}}} + \|v\|_{Y_{s+1,1}}.$$

For $(u,v) \in \mathcal{M}_r$, let us define the maps

$$\begin{cases} \Phi_{\phi_\alpha}[w,v] = U(t)\phi_\alpha - \displaystyle\int_0^t U(t-t')\left(6\alpha^{\frac{2}{3}} w\partial_x w + 2\beta\alpha^{-\frac{1}{3}} p_\alpha \partial_x p_\alpha\right)(t')\,dt', \\ \Psi_\psi[w,v] = U(t)\psi - \displaystyle\int_0^t U(t-t')(3q_\alpha\partial_x v)(t')\,dt'. \end{cases} \tag{3.13}$$

As we are interested to get local in time solution, we can replace (3.13) by

$$\begin{cases} \Phi_{\phi_\alpha}[w,v] = \varphi(t)U(t)\phi_\alpha - \varphi(t)\displaystyle\int_0^t U(t-t')\varphi(t')\left(6\alpha^{\frac{2}{3}} w\partial_x w + 2\beta\alpha^{-\frac{1}{3}} p_\alpha \partial_x p_\alpha\right)(t')\,dt', \\ \Psi_\psi[w,v] = \varphi(t)U(t)\psi - \varphi(t)\displaystyle\int_0^t U(t-t')\varphi(t')(3q_\alpha\partial_x v)(t')\,dt'. \end{cases} \tag{3.14}$$

We will show that $\Phi \times \Psi$ maps $\mathcal{M}_r$ into $\mathcal{M}_r$ and is a contraction.

Now using (2.3), (2.7) and (2.8), we get

$$\begin{aligned} \|\Phi\|_{Y_{-\frac{1}{2},\frac{1}{\theta}}} &\leqslant \left\|\varphi(t)U(t)\phi_\alpha\right\|_{Y_{-\frac{1}{2},\frac{1}{\theta}}} \\ &\quad + \left\|\varphi(t)\int_0^t U(t-t')\varphi(t')(6w\partial_x w + 2\beta p_\alpha\partial_x p_\alpha)(t')\,dt'\right\|_{Y_{-\frac{1}{2},\frac{1}{\theta}}} \\ &\leqslant c_0\|\phi_\alpha\|_{H^{-\frac{1}{2}}} + c_1\left\|\varphi(t')w\partial_x w\right\|_{Z_{-\frac{1}{2},\frac{1}{\theta}}} + c_1\left\|\varphi(t')p_\alpha\partial_x p_\alpha(t')\right\|_{Z_{-\frac{1}{2},\frac{1}{\theta}}} \\ &\leqslant c_0\|\phi_\alpha\|_{H^{-\frac{1}{2}}} + c_2\lambda^{0+}\|w\|^2_{X_{-\frac{1}{2},\frac{1}{2},\frac{1}{\theta}}} + c_2\lambda^{0+}\|p_\alpha\|^2_{X_{-\frac{1}{2},\frac{1}{2},\frac{1}{\theta}}} \\ &\leqslant c_0\|\phi_\alpha\|_{H^{-\frac{1}{2}}} + c_2\lambda^{0+}\left(\|w\|^2_{Y_{-\frac{1}{2},\frac{1}{\theta}}} + \|v\|^2_{Y_{\frac{1}{2},1}}\right). \end{aligned} \tag{3.15}$$

Similarly, the use of (2.3), (2.7) and (2.9) yields

$$\begin{aligned}
\|\Psi\|_{Y_{\frac{1}{2},1}} &\leqslant \big\|\varphi(t)U(t)\psi\big\|_{Y_{\frac{1}{2}}} + \left\|\varphi(t)\int_0^t U(t-t')\varphi(t')(3q_\alpha\partial_x v)(t')\,dt'\right\|_{Y_{\frac{1}{2},1}} \\
&\leqslant c_0\|\psi\|_{H^{\frac{1}{2}}} + c_1\big\|\varphi(t')q_\alpha\partial_x v\big\|_{Z_{\frac{1}{2},1}} \\
&\leqslant c_0\|\psi\|_{H^{\frac{1}{2}}} + c_1\lambda^{0+}\|q_\alpha\|_{X_{-\frac{1}{2},\frac{1}{2},1}}\|v\|_{X_{\frac{1}{2},\frac{1}{2},1}} \\
&\leqslant c_0\|\psi\|_{H^{\frac{1}{2}}} + c_1\lambda^{0+}\|w\|_{X_{-\frac{1}{2},\frac{1}{2},\frac{1}{\theta}}}\|v\|_{X_{\frac{1}{2},\frac{1}{2},1}} \\
&\leqslant c_0\|\psi\|_{H^{\frac{1}{2}}} + \tilde{c}_2\lambda^{0+}\big(\|w\|^2_{Y_{-\frac{1}{2},\frac{1}{\theta}}} + \|v\|^2_{Y_{\frac{1}{2},1}}\big).
\end{aligned} \tag{3.16}$$

Therefore, for $C = \max\{c_2, \tilde{c}_2\}$ and, like before, choosing r sufficiently small, we can make $8\lambda^{0+}Cc_0r < 1$, to obtain

$$\begin{aligned}
\big\|(\Phi,\Psi)\big\|_{Y_{-\frac{1}{2},\frac{1}{\theta}}\times Y_{\frac{1}{2},1}} &\leqslant c_0\big\|(\phi_\alpha,\psi)\big\|_{H^{-\frac{1}{2}}\times H^{\frac{1}{2}}} + 2C\lambda^{0+}\big\|(w,v)\big\|^2_{Y_{-\frac{1}{2},\frac{1}{\theta}}\times Y_{\frac{1}{2},1}} \\
&\leqslant c_0r + 2C\lambda^{0+}(2c_0r)^2 \\
&\leqslant 2c_0r.
\end{aligned} \tag{3.17}$$

Hence $\Phi\times\Psi$ maps $\mathcal{M}_r$ into $\mathcal{M}_r$. Now we move to show that $\Phi\times\Psi$ is a contraction.

Using the same argument used in (3.15) and (3.16) we can obtain

$$\big\|\Phi[w,v]-\Phi[u_1,v_1]\big\|_{Y_{-\frac{1}{2}}} \leqslant C\lambda^{0+}2c_0r\big(\|w-w_1\|_{Y_{-\frac{1}{2},\frac{1}{\theta}}} + \|v-v_1\|_{Y_{\frac{1}{2},1}}\big) \tag{3.18}$$

and

$$\big\|\Psi[w,v]-\Psi[w_1,v_1]\big\|_{Y_{\frac{1}{2}}} \leqslant C\lambda^{0+}2c_0r\big(\|w-w_1\|_{Y_{-\frac{1}{2},\frac{1}{\theta}}} + \|v-v_1\|_{Y_{\frac{1}{2},1}}\big). \tag{3.19}$$

Hence, for $8\lambda^{0+}Cc_0r < 1$, we see that $\Phi\times\Psi$ is a contraction. As in the previous case, this completes the proof for small data. The proof for any data follows from the scaling argument as we did in the previous case, so we omit the details. □

4. Proof of the global result

As mentioned in the introduction, a global solution to the IVP (1.4) (i.e., IVP (1.1) for $\alpha = -1$) is known to exist for given data $(\phi,\psi)\in H^s\times H^{s+1}$, $s\geqslant 0$. Therefore, our interest in this work is to extend the local solution obtained in the previous section to the global one, for data $(\phi,\psi)\in H^s\times H^{s+1}$, $-\frac{1}{2}\leqslant s<0$.

Consider fixed $0<\epsilon\ll 1$, $\lambda\gg 1$, $N\gg 1$, (λ, N may depend on ϵ and $\|(\phi,\psi)\|_{H^s\times H^{s+1}}$ but they will only be properly adjusted to suit our needs at the end of the proof, once we have all the final inequalities in place).

Let $m(\xi)$ be a nonnegative smooth symbol on $\mathbb{R}$ given by

$$m(\xi) = \begin{cases} 1, & |\xi|\leqslant 1, \\ |\xi|^{-1}, & |\xi|\geqslant 2. \end{cases} \tag{4.1}$$

For $\sigma \in \mathbb{R}$, let I_N^σ be a Fourier multiplier operator defined by

$$\widehat{I_N^\sigma f}(\xi) = m\left(\frac{\xi}{N}\right)^\sigma \hat{f}(\xi).$$

Let $I := I_N^{-s}$, then I has the following property.

Lemma 4.1. *The Fourier multiplier operator I maps $H^s \times H^{s+1}$ to $L^2 \times H^1$ with*

$$\big\|(Iu, Iv)\big\|_{L^2\times H^1} \leqslant cN^{-s}\big\|(u, v)\big\|_{H^s\times H^{s+1}}. \tag{4.2}$$

Let $X_{s,b}^\delta$ denote the Bourgain space defined on the time interval $[0,\delta]$ with norm $\|f\|_{X_{s,b}^\delta} := \inf\{\|F\|_{X_{s,b}}\colon F = f,\ \forall t \in [0,\delta]\}$, and similarly for Y_s^δ and Z_s^δ. Now, we state a variant of the local well-posedness result after introducing the Fourier multiplier operator I. The proof of this result follows from the argument used to obtain the local result in Section 3.

Theorem 4.2. *Let $s \geqslant -1/2$, then for any (ϕ, ψ) such that $(I\phi, I\psi) \in L^2 \times H^1$, there exist $\delta = \delta(\|(I\phi, I\psi)\|_{L^2\times H^1})$ (with $\delta(\rho) \to \infty$ as $\rho \to 0$) and a unique solution to the IVP* (1.4) *in the time interval* $[0,\delta]$. *Moreover, the solution satisfies the estimate*

$$\big\|(Iu, Iv)\big\|_{Y_0^\delta\times Y_1^\delta} \lesssim \big\|(I\phi, I\psi)\big\|_{L^2\times H^1}. \tag{4.3}$$

To prove the global well-posedness result, we use the scaling argument introduced earlier. To be more precise, if (u, v) is a solution to the IVP (1.4) with initial data (ϕ, ψ) then so is (u^λ, v^λ) with initial data $(\phi^\lambda, \psi^\lambda)$, where

$$u^\lambda(x,t) = \lambda^{-2}u\left(\frac{x}{\lambda}, \frac{t}{\lambda^3}\right), \qquad \phi^\lambda(x) = \lambda^{-2}u\left(\frac{x}{\lambda}\right)$$

and similarly for v^λ and ψ^λ.

Observe that,

$$\big\|\big(\phi^\lambda, \psi^\lambda\big)\big\|_{H^s\times H^{s+1}} \leqslant c\lambda^{-\frac{3}{2}-s}\big\|(\phi, \psi)\big\|_{H^s\times H^{s+1}}. \tag{4.4}$$

With this observation and (4.2) we get

$$\big\|\big(I\phi^\lambda, I\psi^\lambda\big)\big\|_{L^2\times H^1} \leqslant cN^{-s}\lambda^{-\frac{3}{2}-s}\big\|(\phi, \psi)\big\|_{H^s\times H^{s+1}}. \tag{4.5}$$

Now, if we choose $\lambda = \lambda(N)$ suitable, we can make the norm $\|(I\phi^\lambda, I\psi^\lambda)\|_{L^2\times H^1}$ as small as we please. In fact, by choosing

$$\lambda \sim N^{\frac{-2s}{3+2s}} \tag{4.6}$$

we can have

$$\big\|\big(I\phi^\lambda, I\psi^\lambda\big)\big\|_{L^2\times H^1} \leqslant \epsilon. \tag{4.7}$$

In view of (4.7), we can guarantee that the rescaled solution (Iu^λ, Iv^λ) exists in the time interval $[0, 1]$.

Note. The period of the rescaled solution changes and this fact needs to be taken into consideration to define spaces depending on the corresponding interval of periodicity. From here onwards we will work on the rescaled solution.

To extend the local solution to the global one, we need to obtain appropriate estimates for the Hamiltonian and the L^2-conserved quantity. Recall that

$$H\big(u^\lambda, v^\lambda\big) := \beta \int \big(v_x^{\lambda 2} - u^\lambda v^{\lambda 2}\big)\,dx \tag{4.8}$$

and

$$G\big(u^\lambda, v^\lambda\big) := \frac{1}{2}\int \left\{u^{\lambda 2} + \frac{2}{3}\beta v^{\lambda 2}\right\} dx, \tag{4.9}$$

are the Hamiltonian and L^2-conserved quantity respectively, associated to the IVP (1.4).

Now we prove some estimates that are crucial to extend the local solution to the global one.

First, observe that, using Lemma 4.1, the Gagliardo–Nirenberg inequality and the choice of λ in (4.6), we have

$$H\big(I\phi^\lambda, I\psi^\lambda\big) \leqslant \epsilon^2 \tag{4.10}$$

and

$$G\big(I\phi^\lambda, I\psi^\lambda\big) \leqslant \epsilon^2. \tag{4.11}$$

The essential estimates, that yield the almost conserved character of these quantities are contained in the next theorem.

Theorem 4.3. *The following hold*:

$$\big|H\big(Iu^\lambda, Iv^\lambda\big)(1) - H\big(Iu^\lambda, Iv^\lambda\big)(0)\big| \leqslant c\lambda^{0+}N^{-\frac{1}{2}} \tag{4.12}$$

and

$$\big|G\big(Iu^\lambda, Iv^\lambda\big)(1) - G\big(Iu^\lambda, Iv^\lambda\big)(0)\big| \leqslant c\lambda^{0+}N^{-\frac{1}{2}}. \tag{4.13}$$

Proof. For simplicity of notation, let us replace Iu^λ, Iv^λ by Iu, Iv, respectively.

First, observe that we can write $\partial_t H(Iu, Iv)$ and $\partial_t G(Iu, Iv)$ as

$$\begin{aligned}\partial_t H(Iu, Iv) = -2\beta \int \big\{&3(Iv_{xx} + IuIv)\big[IuIv_x - I(uv_x)\big] + 3(Iv)^2\big[IuIu_x - I(uu_x)\big] \\ &+ \beta(Iv)^2\big[I(vv_x) - IvIv_x\big]\big\}\,dx,\end{aligned} \tag{4.14}$$

and

$$\begin{aligned}\partial_t G(Iu, Iv) = \int \big\{&6Iu\big[IuIu_x - I(uu_x)\big] + 2\beta Iv\big[IuIv_x - I(uv_x)\big] \\ &+ 2\beta Iu\big[I(vv_x) - IvIv_x\big]\big\}\,dx.\end{aligned} \tag{4.15}$$

Therefore, to prove (4.12) and (4.13), it is enough to show that

$$\begin{aligned}&\bigg|2\beta\iint\chi_{[0,1]}(t)\big\{3(Iv_{xx}+IuIv)\big[IuIv_x-I(uv_x)\big]+3(Iv)^2\big[IuIu_x-I(uu_x)\big]\\&\quad+\beta(Iv)^2\big[I(vv_x)-IvIv_x\big]\big\}\,dx\,dt\bigg|\leqslant c\lambda^{0+}N^{-\frac{1}{2}},\end{aligned}\tag{4.16}$$

and

$$\begin{aligned}&\bigg|\iint\chi_{[0,1]}(t)\big\{6Iu\big[IuIu_x-I(uu_x)\big]+2\beta Iv\big[IuIv_x-I(uv_x)\big]\\&\qquad+2\beta Iu\big[I(vv_x)-IvIv_x\big]\big\}\,dx\,dt\bigg|\leqslant c\lambda^{0+}N^{-\frac{1}{2}},\end{aligned}\tag{4.17}$$

respectively.

To prove the estimates (4.16) and (4.17) we use Lemma 2.1 for an appropriate s. First let us proceed to prove (4.16). Using the triangle inequality we have

$$\begin{aligned}&\bigg|2\beta\iint\chi_{[0,1]}(t)\big\{3(Iv_{xx}+IuIv)\big[IuIv_x-I(uv_x)\big]+3(Iv)^2\big[IuIu_x-I(uu_x)\big]\\&\qquad+\beta(Iv)^2\big[I(vv_x)-IvIv_x\big]\big\}\,dx\,dt\bigg|\\&\quad\lesssim\bigg|\iint\chi_{[0,1]}(t)(Iv_{xx}+IuIv)\big[IuIv_x-I(uv_x)\big]\,dx\,dt\bigg|\\&\qquad+\bigg|\iint\chi_{[0,1]}(t)(Iv)^2\big[IuIu_x-I(uu_x)\big]\,dx\,dt\bigg|\\&\qquad+\bigg|\iint\chi_{[0,1]}(t)(Iv)^2\big[I(vv_x)-IvIv_x\big]\,dx\,dt\bigg|\\&\quad:=J_1+J_2+J_3.\end{aligned}\tag{4.18}$$

Estimate for J_1. Using Lemma 2.1 with $s=-1$, we get

$$J_1\lesssim\|Iv_{xx}+IuIv\|_{Y_{-1}}\big\|IuIv_x-I(uv_x)\big\|_{Z_1}.\tag{4.19}$$

Now,

$$\|Iv_{xx}+IuIv\|_{Y_{-1}}\lesssim\|Iv\|_{Y_1}+\|IuIv\|_{Y_{-1}}.\tag{4.20}$$

From (2.14), we have

$$\|IuIv\|_{Y_{-1}}\lesssim\lambda^{0+}\|Iu\|_{Y_0}\|Iv\|_{Y_1}.\tag{4.21}$$

Therefore, using (4.20) and (4.21) along with (4.3) and (4.7) in (4.19), we obtain

$$J_1\lesssim\lambda^{0+}\big\|IuIv_x-I(uv_x)\big\|_{Z_1}.\tag{4.22}$$

Estimate for J_2. Using Lemma 2.1 with $s=0$, we get

$$J_2\lesssim\big\|(Iv)^2\big\|_{Y_0}\big\|IuIu_x-I(uu_x)\big\|_{Z_0}.\tag{4.23}$$

Also, using Theorem 3 in [8], it is easy to show that

$$\big\|(Iv)^2\big\|_{Y_0}\lesssim\lambda^{0+}\|Iv\|_{Y_1}^2.\tag{4.24}$$

Therefore, using (4.24) along with (4.3) and (4.7) in (4.23), we get

$$J_2 \lesssim \lambda^{0+} \left\| IuIu_x - I(uu_x) \right\|_{Z_0}. \tag{4.25}$$

Estimate for J_3. Using Lemma 2.1 with $s = -1$, we get

$$J_3 \lesssim \left\| (Iv)^2 \right\|_{Y_{-1}} \left\| IvIv_x - I(vv_x) \right\|_{Z_1}. \tag{4.26}$$

As in (4.24), we have

$$\left\| (Iv)^2 \right\|_{Y_{-1}} \lesssim \lambda^{0+} \|Iv\|_{Y_1}^2. \tag{4.27}$$

Therefore, using (4.27) along with (4.3) and (4.7) in (4.26), we obtain

$$J_3 \lesssim \lambda^{0+} \left\| IvIv_x - I(vv_x) \right\|_{Z_1}. \tag{4.28}$$

Hence the proof of the estimate (4.16) reduces to getting the estimates:

$$\left\| IuIv_x - I(uv_x) \right\|_{Z_1} \lesssim \lambda^{0+} N^{-\frac{1}{2}}, \tag{4.29}$$

$$\left\| IuIu_x - I(uu_x) \right\|_{Z_0} \lesssim \lambda^{0+} N^{-\frac{1}{2}}, \tag{4.30}$$

and

$$\left\| IvIv_x - I(vv_x) \right\|_{Z_1} \lesssim \lambda^{0+} N^{-\frac{1}{2}}. \tag{4.31}$$

Proof of (4.29). We need to consider three different cases, taking into account the low–low, high–high and low–high frequency interactions. In low frequencies I is just an identity operator, so the estimate (4.29) follows trivially. Therefore, we only need to consider the other two cases.

(a) High–high interactions.

Using (4.3) and (4.7) we have $\|Iu\|_{Y_0} \leqslant \epsilon$ and $\|Iv\|_{Y_1} \leqslant \epsilon$. So the estimate (4.29), in this case will follow, if we show

$$\left\| IuIv_x - I(uv_x) \right\|_{Z_1} \lesssim \lambda^{0+} N^{-\frac{1}{2}} \|Iu\|_{Y_0} \|Iv\|_{Y_1}. \tag{4.32}$$

Recall that $I := I_N^{-s}$. Now observe that,

$$m^{-s}\left(\frac{\xi_1}{N}\right) m^{-s}\left(\frac{\xi_2}{N}\right) \lesssim m^{-s}\left(\frac{\xi_1 + \xi_2}{N}\right). \tag{4.33}$$

In view of this observation, we obtain

$$\left\| IuIv_x - I(uv_x) \right\|_{Z_1} \leqslant \|IuIv_x\|_{Z_1} + \left\| I(uv_x) \right\|_{Z_1} \lesssim \left\| I(uv_x) \right\|_{Z_1}. \tag{4.34}$$

Therefore, to get (4.32), it is enough to prove

$$\left\| I(uv_x) \right\|_{Z_1} \lesssim \lambda^{0+} N^{-\frac{1}{2}} \|Iu\|_{Y_0} \|Iv\|_{Y_1}. \tag{4.35}$$

For this, we use

$$\|Iv\|_{Z_1} \lesssim N^{-s} \left\| \langle \nabla \rangle^s v \right\|_{Z_1} \sim N^{-s} \left\| \langle \nabla \rangle^{s+\frac{1}{2}} v \right\|_{Z_{\frac{1}{2}}},$$

where $\langle \nabla \rangle$ is a Fourier multiplier with symbol $\langle \xi \rangle$.

Now, the LHS of (4.35) can be estimated as

$$\left\|I(uv_x)\right\|_{Z_1} \lesssim N^{-s}\left\|\langle\nabla\rangle^{s+\frac{1}{2}}(uv_x)\right\|_{Z_{\frac{1}{2}}}. \tag{4.36}$$

Using the fractional Leibnitz rule we may apply $\langle\nabla\rangle^{s+1/2}$ to one of the functions, say v_x, i.e., we can write (4.36) as

$$\left\|I(uv_x)\right\|_{Z_1} \lesssim N^{-s}\left\|u\big(\langle\nabla\rangle^{s+\frac{1}{2}}v\big)_x\right\|_{Z_{\frac{1}{2}}}. \tag{4.37}$$

Now, using (2.13) for $s=\frac{1}{2}$, we get

$$\left\|I(uv_x)\right\|_{Z_1} \lesssim N^{-s}\lambda^{0+}\|u\|_{Y_{-\frac{1}{2}}}\left\|\langle\nabla\rangle^{s+\frac{1}{2}}v\right\|_{Y_{\frac{1}{2}}}. \tag{4.38}$$

Note that

$$\left\|\langle\nabla\rangle^{s+\frac{1}{2}}v\right\|_{Y_{\frac{1}{2}}} \lesssim N^{s}\|Iv\|_{Y_1}. \tag{4.39}$$

Also, as u has Fourier support in the region $\xi \geqslant N/100$, we have

$$\|u\|_{Y_{-\frac{1}{2}}} \lesssim N^{-\frac{1}{2}}\|Iu\|_{Y_0}. \tag{4.40}$$

Inserting (4.39) and (4.40) in (4.38), we get the desired estimate (4.35).

(b) Low–high interactions.

Let v_x be supported in the region $\xi \geqslant N/5$ and u in $\xi \leqslant N/100$. With this assumption, we need to show

$$\left\|uIv_x - I(uv_x)\right\|_{Z_1} \lesssim \lambda^{0+}N^{-\frac{1}{2}}\|Iu\|_{Y_0}\|Iv\|_{Y_1}. \tag{4.41}$$

From the mean value theorem, we have

$$\left|m^{-s}(\xi'+\xi) - m^{-s}(\xi)\right| \lesssim \frac{|\xi'|}{|\xi|}m^{-s}(\xi), \quad \text{if } |\xi| \geqslant 1/5,\ |\xi'| \leqslant 1/100. \tag{4.42}$$

Thus, one can obtain

$$\left|m^{-s}\left(\frac{\xi_1+\xi_2}{N}\right) - m^{-s}\left(\frac{\xi_2}{N}\right)\right| \lesssim \frac{|\xi_1|}{|\xi_2|}m^{-s}\left(\frac{\xi_2}{N}\right) \lesssim m^{1-s}\left(\frac{\xi_2}{N}\right),$$
$$\text{if } |\xi| \geqslant N/5,\ |\xi'| \leqslant N/100. \tag{4.43}$$

Therefore, we may estimate the LHS of (4.41) as

$$\left\|uIv_x - I(uv_x)\right\|_{Z_1} \lesssim \left\|u\big(I_N^{1-s}v\big)_x\right\|_{Z_1} \sim \left\|\langle\nabla\rangle^{\frac{1}{2}}\big(u\big(I_N^{1-s}v\big)_x\big)\right\|_{Z_{\frac{1}{2}}}. \tag{4.44}$$

Again, we can apply the derivative $\langle\nabla\rangle^{1/2}$ on v. Now, using (2.13) we get

$$\left\|u\langle\nabla\rangle^{\frac{1}{2}}\big(I_N^{1-s}v\big)_x\right\|_{Z_{\frac{1}{2}}} \lesssim \lambda^{0+}\|u\|_{Y_{-\frac{1}{2}}}\left\|\langle\nabla\rangle^{\frac{1}{2}}\big(I_N^{1-s}v\big)\right\|_{Y_{\frac{1}{2}}}. \tag{4.45}$$

From the frequency support of v we can see that

$$\left\|\langle\nabla\rangle^{\frac{1}{2}}\big(I_N^{1-s}v\big)\right\|_{Y_{\frac{1}{2}}} \lesssim N^{-\frac{1}{2}}\|Iv\|_{Y_1}. \tag{4.46}$$

As u is supported on $|\xi| \leqslant N/100$, we have

$$\|u\|_{Y_{-\frac{1}{2}}} \lesssim \|Iu\|_{Y_0}. \tag{4.47}$$

Now, combining (4.46), (4.47) and (4.45), we obtain the required estimate (4.41).

Proof of (4.30). As in the proof of (4.29) we need to consider three different cases taking into account the low–low, high–high and low–high frequency interactions. In low frequencies I is just an identity operator so the estimate (4.30) follows trivially. Therefore, we only need to consider the other two cases.

(a) High–high interactions.

Using (4.3) and (4.7) we have $\|Iu\|_{Y_0} \leqslant \epsilon$. So the estimate (4.30), in this case will follow, if we show

$$\big\|IuIu_x - I(uu_x)\big\|_{Z_0} \lesssim \lambda^{0+}N^{-\frac{1}{2}}\|Iu\|_{Y_0}^2. \tag{4.48}$$

Using the observation in (4.33), to get (4.48), it is enough to prove

$$\big\|I(uu_x)\big\|_{Z_0} \lesssim \lambda^{0+}N^{-\frac{1}{2}}\|Iu\|_{Y_0}^2. \tag{4.49}$$

As earlier, we use

$$\|Iv\|_{Z_0} \lesssim N^{-s}\big\|\langle\nabla\rangle^s v\big\|_{Z_0} \sim N^{-s}\big\|\langle\nabla\rangle^{s+\frac{1}{2}} v\big\|_{Z_{-\frac{1}{2}}}.$$

Now the LHS of (4.49) can be estimated as

$$\big\|I(uu_x)\big\|_{Z_0} \lesssim N^{-s}\big\|\langle\nabla\rangle^{s+\frac{1}{2}}(uu_x)\big\|_{Z_{-\frac{1}{2}}} \lesssim N^{-s}\big\|\partial_x\big(\langle\nabla\rangle^{s+\frac{1}{2}}u^2\big)\big\|_{Z_{-\frac{1}{2}}}. \tag{4.50}$$

Therefore, using (2.8), we get

$$\big\|I(uu_x)\big\|_{Z_0} \lesssim N^{-s}\lambda^{0+}\|u\|_{Y_{-\frac{1}{2}}}\big\|\langle\nabla\rangle^{s+\frac{1}{2}}u\big\|_{Y_{-\frac{1}{2}}}. \tag{4.51}$$

Note that

$$\big\|\langle\nabla\rangle^{s+\frac{1}{2}}u\big\|_{Y_{-\frac{1}{2}}} \lesssim N^s\|Iu\|_{Y_0}. \tag{4.52}$$

Also, as u has Fourier support in the region $\xi \geqslant N/100$, we have

$$\|u\|_{Y_{-\frac{1}{2}}} \lesssim N^{-\frac{1}{2}}\|Iu\|_{Y_0}. \tag{4.53}$$

Inserting (4.52) and (4.53) in (4.51), we get the desired estimate (4.49).

(b) Low–high interactions.

Let u_x has Fourier support in the region $\xi \geqslant N/5$ and u in $\xi \leqslant N/100$. With this assumption, we need to show

$$\big\|uIu_x - I(uu_x)\big\|_{Z_0} \lesssim \lambda^{0+}N^{-\frac{1}{2}}\|Iu\|_{Y_0}^2. \tag{4.54}$$

From the mean value theorem, as earlier, we may estimate LHS of (4.54) as

$$\big\|uIu_x - I(uu_x)\big\|_{Z_0} \lesssim \big\|u\big(I_N^{1-s}u\big)_x\big\|_{Z_0} \sim \big\|u\langle\nabla\rangle^{\frac{1}{2}}\big(I_N^{1-s}u\big)_x\big\|_{Z_{-\frac{1}{2}}}. \tag{4.55}$$

With the same procedure used in the proof of estimate (4.29), we obtain from (4.55)

$$\left\| uIu_x - I(uu_x)\right\|_{Z_0} \lesssim \lambda^{0+}\|u\|_{Y_{-\frac{1}{2}}}\left\|\langle\nabla\rangle^{\frac{1}{2}}\left(I_N^{1-s}u\right)\right\|_{Y_{-\frac{1}{2}}}. \tag{4.56}$$

From frequency support of u we can see that

$$\left\|\langle\nabla\rangle^{\frac{1}{2}}\left(I_N^{1-s}u\right)\right\|_{Y_{-\frac{1}{2}}} \lesssim N^{-\frac{1}{2}}\|Iu\|_{Y_0}, \tag{4.57}$$

and

$$\|u\|_{Y_{-\frac{1}{2}}} \lesssim \|Iu\|_{Y_0}. \tag{4.58}$$

Now, combining (4.57), (4.58) and (4.56), we obtain the required estimate (4.54).

Proof of (4.31). The proof of (4.31) is similar to that of (4.29) with trivial modifications.

Now, we move to give proof of the estimate (4.17).

Using the triangle inequality we have

$$\begin{aligned}
&\left|\iint \chi_{[0,1]}(t)\left\{6Iu\left[IuIu_x - I(uu_x)\right] + 2\beta Iv\left[IuIv_x - I(uv_x)\right]\right.\right.\\
&\qquad \left.\left. + \beta Iu\left[I(vv_x) - IvIv_x\right]\right\}dx\,dt\right| \\
&\lesssim \left|\iint \chi_{[0,1]}(t)Iu\left[IuIu_x - I(uu_x)\right]dx\,dt\right| + \left|\iint \chi_{[0,1]}(t)Iv\left[IuIv_x - I(uv_x)\right]dx\,dt\right| \\
&\qquad + \left|\iint \chi_{[0,1]}(t)Iu\left[I(vv_x) - IvIv_x\right]dx\,dt\right| \\
&:= J_1' + J_2' + J_3'.
\end{aligned} \tag{4.59}$$

Estimate for J_1'. Using Lemma 2.1 with $s = 0$ and $\|Iu\|_{Y_0} \leqslant \epsilon$, we get

$$J_1' \lesssim \|Iu\|_{Y_0}\left\|IuIu_x - I(uu_x)\right\|_{Z_0} \lesssim \left\|IuIu_x - I(uu_x)\right\|_{Z_0}. \tag{4.60}$$

Estimate for J_2'. Using Lemma 2.1 with $s = -1$ and $\|Iv\|_{Y_1} \leqslant \epsilon$,

$$J_2' \lesssim \|Iv\|_{Y_{-1}}\left\|IuIv_x - I(uv_x)\right\|_{Z_1} \lesssim \left\|IuIv_x - I(uv_x)\right\|_{Z_1}. \tag{4.61}$$

Estimate for J_3'. Also, using Lemma 2.1 with $s = -1$ and $\|Iu\|_{Y_0} \leqslant \epsilon$,

$$J_3' \lesssim \|Iu\|_{Y_{-1}}\left\|I(vv_x) - IvIv_x\right\|_{Z_1} \lesssim \left\|I(vv_x) - IvIv_x\right\|_{Z_1}. \tag{4.62}$$

Hence, the proof of the estimate (4.17) reduces to getting estimates for (4.60)–(4.62). But these estimates can be proved exactly in a similar way to that of (4.29)–(4.31), respectively.

Combining all these we conclude the proof of the theorem. $\square$

Using the estimates (4.10), (4.11) and Theorem 4.3, we have the following almost conservation laws.

Corollary 4.4. *The following hold*:

$$\left|H\left(Iu^\lambda, Iv^\lambda\right)(1)\right| \leqslant \epsilon^2 + c\lambda^{0+}N^{-\frac{1}{2}} \tag{4.63}$$

and

$$\left|G\left(Iu^\lambda, Iv^\lambda\right)(1)\right| \leqslant \epsilon^2 + c\lambda^{0+}N^{-\frac{1}{2}}. \tag{4.64}$$

Now we are in position to supply the proof of the global well-posedness result.

Proof of Theorem 1.2. Our aim is to extend the local solution to the IVP (1.1) with $\alpha = -1$ and $\beta > 0$, obtained in Theorem 1.1, to the time interval $[0, T]$ for any $T > 0$. For this purpose we use the scaling argument introduced earlier. Observe that the solution (u, v) exists in $[0, T]$ if and only if (u^λ, v^λ) exists in $[0, \lambda^3 T]$. Therefore, we need to extend the rescaled solution to $[0, \lambda^3 T]$. For this, we develop an iteration process.

Recall that, with our choice of λ in (4.6), we have an existence of a $\lambda(N)$-periodic solution in the interval $[0, 1]$. Now, using the almost conserved quantities (4.63) and (4.64) obtained in Corollary 4.4 we can get $\|(Iu^\lambda, Iv^\lambda)(1)\|_{L^2\times H^1} \leqslant \epsilon^2$ for $\beta > 0$. So we can iterate this process $c^{-1}\lambda^{0-}N^{1/2}$ times to extend the local solution to the interval $[0, c^{-1}\lambda^{0-}N^{1/2}]$ before doubling $|H(Iu^\lambda, Iv^\lambda)(1)|$ and $|G(Iu^\lambda, Iv^\lambda)(1)|$. As we are interested in extending the solution to the time interval $[0, \lambda^3 T]$, let us choose $N = N(T)$ such that $c^{-1}\lambda^{0-}N^{1/2} > \lambda^3 T$. Therefore, with our choice of λ in (4.6) we need to have

$$N^{\frac{1}{2}} \geqslant c\left(\lambda(N)\right)^{3+}T \sim TN^{\frac{-6s}{3+2s}+\epsilon}, \tag{4.65}$$

i.e.,

$$TN^{-\frac{1}{2}-\frac{6s}{3+2s}+\epsilon} < c.$$

This is possible if $s > -3/14$. Hence the IVP (1.1) has global solution in the case $\alpha = -1$ and $\beta > 0$, whenever $s > -3/14$. □

Remark 4.5. It would be interesting to obtain a global result for the case $\alpha \neq -1, 0$. But the presence of the parameters in the Hamiltonian prevents us from having obvious cancellation, thereby suggesting the possible difficulties to obtain almost conserved quantities in this case. Work in this direction is currently being pursued by the authors.

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