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The velocity and shear stress functions of the Falkner–Skan equation arising in boundary layer theory

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Abstract

New compactness results on the velocity functions and shear stress functions of the well-known Falkner–Skan equation are obtained. The methodology is to utilize the equivalence between the Falkner–Skan equation and a singular integral equation established recently by Lan and Yang.

Keywords: Falkner–Skan equation; Compactness; Velocity function; Shear stress function; Singular integral equation; Boundary layer problem

1. Introduction

The following well-known Falkner–Skan equation

$$f'''(\eta) + f(\eta)f''(\eta) + \lambda\big[1 - (f')^2(\eta)\big] = 0 \quad \text{on } \eta \in (0, \infty), \tag{1.1}$$

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subject to the boundary condition

$$f(0) = f'(0) = 0, \qquad f'(\infty) = 1, \tag{1.2}$$

and the side condition

$$0 < f'(\eta) < 1 \quad \text{for } \eta \in (0, \infty), \tag{1.3}$$

is one of the most important equations in laminar boundary layer theory, where η is the similarity boundary-layer ordinate, $f(\eta)$ is the similarity stream function and $f'(\eta)$ and $f''(\eta)$ are the velocity and the shear stress, respectively.

(1.1)–(1.3) has been extensively studied analytically and numerically. We refer to [1,4–10,16–21] and references therein for the analytical treatments and to [3,11,13] for the numerical results. The physical significance of (1.1)–(1.3) can be found, for example in [2,12,14].

It is well known that there exists $\lambda^* < 0$ such that (1.1)–(1.3) has multiple solutions for each $\lambda \in (\lambda^*, 0)$, has a unique solution for $\lambda = \lambda^*$ or $\lambda \geqslant 0$ and has no solutions for $\lambda < \lambda^*$ (see [6, Theorems 6.1, 7.1 and 8.1], [7, Proposition 1.1 and Theorem 1.1], [8, Theorem]). Moreover, every solution f of (1.1)–(1.3) satisfies the following condition:

$$f''(\eta) > 0 \quad \text{for } \eta \in (0, \infty) \tag{1.4}$$

(see [6, Theorems 6.1, 7.1 and 8.1]).

Recently, we have estimated the number λ^* analytically and shown $\lambda^* \in [-0.4, -0.12]$ in [10]. We refer to [3,11,13] for the numerical results, where $\lambda^* = -0.1988$. Moreover, some useful properties of solutions f of (1.1), (1.2), (1.4) are given in [10]. It is clear that f is increasing and concave up on $(0, \infty)$ and it is shown in [10] that f satisfies $f(\eta) < \eta$ for $\eta \in (0, \infty)$ and $\lim_{\eta \to \infty} f(\eta)/\eta = 1$.

In this paper, we study compactness of the set of velocity functions f' and of the set of shear stress functions f''. We shall prove that the two sets are compact when $\lambda \in [\lambda^*, 0]$ in $BC(\mathbb{R}_+)$. The main technique is to utilize the equivalence between (1.1), (1.2), (1.4) and a singular integral equation of the form

$$z(t) = \int_t^1 \frac{(1-s)(\lambda + \lambda s + s)}{z(s)}\, ds + (1-t) \int_0^t \frac{s}{z(s)}\, ds \quad \text{for } t \in (0, 1), \tag{1.5}$$

which was established in [10]. In Section 2, we prove some new properties of positive solutions of (1.5). These properties, together with the Helly selection principle for an infinite sequence of functions of bounded variation will be used to prove that the set of positive solutions of (1.5) is compact in $C[0, 1]$ when $\lambda \in [\lambda^*, 0]$. In Section 3 we apply the compactness result, together with the equivalence of (1.5) and (1.1), (1.2), (1.4), to prove that both the set of velocity functions and the set of shear stress functions are compact in $BC(\mathbb{R}_+)$ when $\lambda \in [\lambda^*, 0]$.

2. Compactness of the set of positive solutions of (1.5)

Let $z \in C(0, 1)$ with $z(t) > 0$ for $t \in (0, 1)$. We define

$$Az(t) = \int_t^1 f_z(s)\, ds \quad \text{for } t \in [0, 1] \quad \text{and} \quad Bz(t) = \int_0^t \frac{s}{z(s)}\, ds \quad \text{for } t \in [0, 1),$$

where $f_z(s) := \frac{(1-s)(\lambda+\lambda s+s)}{z(s)}$ for $s \in (0,1)$. Let $\delta := \delta(\lambda) = \frac{-\lambda}{1+\lambda}$. Then $\delta \in [0,1)$ if and only if $\lambda \in (-1/2, 0]$. It is shown in [10] that if $\delta \in (0,1)$, then

$$f_z(s) \leqslant 0 \quad \text{for } s \in (0,\delta) \quad \text{and} \quad f_z(s) \geqslant 0 \quad \text{for } s \in [\delta,1), \tag{2.1}$$

and

$$Az \quad \text{is increasing on } (0,\delta) \text{ and decreasing on } [\delta,1). \tag{2.2}$$

Let

$$Q = \{z \in C[0,1]\colon\ z(t) > 0 \text{ for } t \in (0,1)\}.$$

Then, if $z \in Q$, then the improper integral $Az(t)$ is a Lebesgue integral for $t \in [0,1)$ and

$$Az(t) \geqslant 0 \quad \text{for } t \in [0,1], \tag{2.3}$$

and if $z \in Q$ is a solution of (1.5), then

$$Bz(1) = \lim_{t\to 1^-} Bz(t) = \infty \quad \text{and} \quad \lim_{t\to 1^-}(1-t)Bz(t) = 0.$$

If a function $z : [0,1] \to \mathbb{R}_+$ satisfies (1.5), then $z \in C(0,1)$.

The following result shows that the limits of z at the end-points exist under suitable conditions on z and $Az(0)$ (see [10, Proposition 2.2]).

Lemma 2.1. *Let $\lambda > -1/2$ and let $z : (0,1) \to \mathbb{R}_+$ be bounded. Assume that (λ, z) satisfies (1.5) and $Az(0) \in \mathbb{R}_+$. Then $\lim_{t\to 0^+} z(t) = Az(0)$ and $\lim_{t\to 1^-} z(t) = 0$.*

It is showed in [10] that (1.5) is equivalent to some differential equations with suitable boundary conditions which we give below and will use later.

Theorem 2.1.

(1) *Let $(\lambda, z) \in (-1/2, \infty) \times Q$. Then (λ, z) satisfies (1.5) if and only if $z(1) = 0$ and*

$$z'(t) = \frac{-\lambda(1-t^2)}{z(t)} - Bz(t) \quad \textit{for } t \in (0,1). \tag{2.4}$$

(2) *Let $(\lambda, z) \in \mathbb{R}_+ \times Q$. Then (λ, z) satisfies (1.5) if and only if (λ, z) is a solution of the following second order differential equation of the form*

$$z''(t) = -\lambda\left(\frac{1-t^2}{z(t)}\right)' - \frac{t}{z(t)} \quad \textit{for } t \in (0,1), \tag{2.5}$$

subject to the boundary condition:

$$z(0) > 0, \qquad z(1) = 0 \quad \textit{and} \quad z'(0) = -\lambda/z(0). \tag{2.6}$$

It is known that there exists $\lambda^* < 0$ such that (1.1)–(1.3) has at least one solution in Q for each $\lambda \in (\lambda^*, 0)$ (in this case, the solutions are not unique), has a unique solution for either $\lambda = \lambda^*$ or $\lambda \geqslant 0$ and has no solutions for $\lambda < \lambda^*$ (see [6,10]). Recently, Lan and Yang [10] have proved $\lambda^* \in [-0.4, -0.12]$. Since (1.1)–(1.3) is equivalent to (1.5) (see Lemma 3.1 below or Theorem 3.2 in [10]), these results on (1.1)–(1.3) just mentioned above hold for (1.5). Hence, (1.5) has multiple positive solutions when $\lambda \in (\lambda^*, 0)$. We write

$$\Sigma := \{(\lambda, z) \in [\lambda^*, 0] \times Q\colon\ (\lambda, z) \text{ satisfies (1.5)}\}. \tag{2.7}$$

Our main purpose of this section is to prove that the set Σ is compact in $\mathbb{R} \times C[0,1]$. We need the following result obtained in [10] which gives the upper and lower bounds of positive solutions of (1.5).

Lemma 2.2. *Assume that* $(\lambda, z) \in \Sigma$. *Then the following assertions hold*:

(H_1) $2/27 \leqslant \|z\| \leqslant 1$ *and*
(H_2) $z(t) \geqslant \frac{1}{2}(1-t)t^2$ *for* $t \in [0,1]$.

We denote by $BV[0,1]$ the space of functions of bounded variation on $[0,1]$ and by V_z the (total) variation of z over $[0,1]$.

The following result provides some new properties of positive solutions of (1.5).

Proposition 2.1. *Assume that* $(\lambda, z) \in \Sigma$ *and* $\|z\| = z(t_0)$ *for some* $t_0 \in (0,1)$. *Then the following assertions hold*:

(P_1) $t_0 < \sqrt{2}/2$.
(P_2) z *is strictly increasing on* $[0, t_0]$ *and strictly decreasing on* $[t_0, 1]$.
(P_3) $V_z = 2\|z\| - z(0)$.

Proof. (P_1) Let $t_0 \in (0,1)$ be such that $\|z\| = z(t_0)$. Then it follows from Fermat's theorem that $z'(t_0) = 0$ and by (2.4), we have

$$z'(t_0) \leqslant \frac{-\lambda(1-t_0^2)}{z(t_0)} - \frac{1}{z(t_0)} \int_0^{t_0} s\,ds = \frac{1}{z(t_0)}\left[-\lambda + \left(\lambda - \frac{1}{2}\right)t_0^2\right].$$

This implies $-\lambda + (\lambda - \frac{1}{2})t_0^2 \geqslant 0$ and $t_0 < \sqrt{2}/2$.

(P_2) By (2.5), we see that if $z'(t) = 0$ for some $t \in (0,1)$, then $z''(t) < 0$. This implies that z satisfies the following property:

(P) z is not a constant on any interval $[a,b] \subset (0,1)$ with $a < b$.

Let $t_1, t_2 \in [0, t_0]$ with $t_1 < t_2$. Let $\xi \in [t_1, t_0]$ be such that $z(\xi) = \min\{z(t)\colon t \in [t_1, t_0]\}$. Then it follows from (P) that $\xi < t_0$. If $\xi \in (t_1, t_0)$, it follows from Fermat's theorem that $z'(\xi) = 0$ and $z''(\xi) < 0$. It follows that $z(\xi)$ is a local maximum. Hence, there exists $[a,b] \subset (t_1, t_0)$ such that $z(t) \leqslant z(\xi)$ for $t \in [a,b]$. This implies $z(t) = z(\xi)$ for $t \in [a,b]$, which contradicts (P). Hence, $\xi = t_1$ and $z(t_1) < z(t_2)$. This shows that z is strictly increasing on $[0, t_0]$. A similar proof shows that z is strictly decreasing on $[t_0, 1]$.

(P_3) Let $0 < t_1 < t_2 < \cdots < t_m = 1$ and $\sigma = \sum_{i=1}^m |z(t_i) - z(t_{i-1})|$. Then there exists $0 < i < m$ such that $t_0 \in [t_i, t_{i+1})$ and we have

$$\begin{aligned}
\sigma &= \sum_{j=1}^{i} \bigl|z(t_j) - z(t_{j-1})\bigr| + \bigl|z(t_{i+1}) - z(t_i)\bigr| + \sum_{j=i+2}^{m} \bigl|z(t_j) - z(t_{j-1})\bigr| \\
&\leqslant \sum_{j=1}^{i-1} \bigl(z(t_j) - z(t_{j-1})\bigr) + 2z(t_0) - z(t_{i+1}) - z(t_i) - \sum_{j=i+2}^{m} \bigl(z(t_j) - z(t_{j-1})\bigr)
\end{aligned}$$

$$= 2z(t_0) - z(0) - z(1) = 2z(t_0) - z(0).$$

This implies $V_z = 2\|z\| - z(0)$. $\square$

The following new results provide some properties of a sequence of positive solutions of (1.5).

Lemma 2.3. *Let $\{(\lambda_n, z_n)\} \subset \Sigma$ with $\lambda_n \neq 0$ be such that $\lambda_n \to \lambda \in [\lambda^*, 0]$. Assume that there exists a real-valued function z defined on $[0, 1]$ such that*

$$z_n(t) \to z(t) \quad \textit{for each } t \in [0, 1].$$

Then the following assertions hold:

(h_1) *$\{Bz_n(t)\}$ is bounded for each $t \in (0, 1)$ and*

$$Bz_n(t) \leqslant \sqrt{-2\big(t + \ln(1-t)\big)}. \tag{2.8}$$

(h_2) *If $z(t) > 0$ for $t \in (0, 1)$, then $\lim_{n\to\infty} \int_t^1 F_n(s)\,ds = Az(t)$ for each $t \in (0, 1)$, where $F_n(s) = \frac{(1-s)(\lambda_n+\lambda_n s+s)}{z_n(s)}$ for $s \in (0, 1)$.*
(h_3) *$\{z_n\}$ is equicontinuous on $[a, b]$ for each $[a, b] \subset (0, 1)$.*
(h_4) *If $z(0) > 0$, then $\{z_n\}$ is equicontinuous on $[0, b]$ for each $b \in (0, 1)$.*
(h_5) *If $z(0) = 0$, then $\lambda < 0$ and if $Az(0) = 0$, then $\lim_{t_n\to 0^+} z_n(t_n) = 0$.*
(h_6) *$\lim_{t_n\to 1^-} z_n(t_n) = 0$.*

Proof. Since $(\lambda_n, z_n) \in \Sigma$, it follows that

$$z_n(t) = \int_t^1 F_n(s)\,ds + (1-t)Bz_n(t) \quad \text{for } t \in (0, 1). \tag{2.9}$$

(h_1) By Lemma 2.2(H_2) and (2.2), we have $z_n(t) \geqslant \frac{(1-t)t^2}{2}$ for $t \in [0, 1]$ and $\int_t^1 F_n(s)\,ds \geqslant 0$ for $t \in [0, 1]$ and $n \in \mathbb{N}$. This implies $(1-t)Bz_n(t) \leqslant z_n(t)$ for $t \in (0, 1)$ and $n \in \mathbb{N}$. Since $(Bz_n)'(t) = \frac{t}{z_n(t)}$ for $t \in (0, 1)$, we have for $t \in (0, 1)$,

$$(Bz_n)'(t)Bz_n(t) = \frac{tBz_n(t)}{z_n(t)} = \frac{t}{1-t}.$$

Integrating the above inequality from 0 to t implies

$$\frac{1}{2}(Bz_n)^2(t) \leqslant \int_0^t \frac{s}{1-s}\,ds = -t - \ln(1-t) \quad \text{for } t \in (0, 1).$$

This implies $\{Bz_n(t)\}$ is bounded for each $t \in (0, 1)$ and (2.8) holds.

(h_2) Let $\delta_n = \frac{-\lambda_n}{1+\lambda_n}$. By Lemma 2.2$(H_2)$, we have for $s \in [\delta_n, 1)$,

$$0 \leqslant F_n(s) \leqslant 2(\lambda_n + \lambda_n s + s)/s^2 \leqslant (1 + 3s)/s^2 \tag{2.10}$$

and $2(\lambda_n + \lambda_n s + s)/s^2 \leqslant F_n(s) \leqslant 0$ for $s \in (0, \delta_n]$. This implies

$$\big|F_n(s)\big| \leqslant \big|2(\lambda_n + \lambda_n s + s)/s^2\big| \leqslant (1 + 3s)/s^2 \quad \text{for } s \in (0, 1).$$

Let $t \in (0,1)$. Since $\int_t^1 (1+3s)/s^2\,ds < \infty$ and $F_n(s) \to f_z(s)$ for each $s \in (t,1)$, it follows from the Lebesgue dominated convergence theorem that (h_2) holds.

(h_3) Let $[a,b] \subset (0,1)$. By (2.4), we have

$$z_n'(t) = \frac{-\lambda_n(1-t^2)}{z_n(t)} - Bz_n(t) \quad \text{for } t \in (0,1). \tag{2.11}$$

Using Lemma 2.2(H_2), we have for $t \in [a,b]$,

$$\begin{aligned} \left|z_n'(t)\right| &\leqslant \frac{-\lambda_n(1-t^2)}{z_n(t)} + Bz_n(t) \leqslant -\frac{2\lambda_n(1-t^2)}{(1-t)t^2} + Bz_n(b) \\ &\leqslant \frac{1+a}{a^2} + Bz_n(b) < \infty. \end{aligned}$$

This implies that $\{\|z_n'\|_{C[a,b]}\}$ is bounded and $\{z_n\}$ is equicontinuous on $[a,b]$.

(h_4) If $z(0) > 0$, then there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that $z_n(0) \geqslant \varepsilon$ for $n \geqslant n_0$. By (2.8), there exists $a \in (0,1/2)$ such that $Bz_n(t) \leqslant \varepsilon$ for $t \in [0,a]$ and $n \geqslant n_0$. This and (2.11) imply $z_n'(t) \geqslant -Bz_n(t) \geqslant -\varepsilon$ for $t \in (0,a]$ and $n \geqslant n_0$. Let $t \in (0,a]$. Integrating this inequality from 0 to t implies

$$z_n(t) \geqslant -\varepsilon t + z_n(0) \geqslant \varepsilon/2 \quad \text{for } t \in (0,a] \text{ and } n \geqslant n_0.$$

Hence, we obtain for $t \in (0,a]$,

$$\left|z_n'(t)\right| \leqslant \frac{-\lambda_n(1-t^2)}{z_n(t)} + Bz_n(t) \leqslant -\frac{2\lambda_n}{\varepsilon} + Bz_n(a) < \infty.$$

This implies that $\{\|z_n'\|_{C[0,a]}\}$ is bounded. By the proof of (h_3), $\{\|z_n'\|_{C[a,b]}\}$ is bounded and $\{z_n\}$ is equicontinuous on $[0,b]$.

(h_5) If $z(0) = 0$, then $z_n(0) \to 0$. Let $s_n \in (0,1)$ such that $z_n(s_n) = \|z_n\|$. Differentiating (2.9), we have $z_n(t)z_n'(t) \leqslant -\lambda_n(1-t^2)$ for $t \in (0,1)$. Integrating the inequality from 0 to s_n, we obtain

$$\frac{1}{2}\left(z_n^2(s_n) - z_n^2(0)\right) \leqslant (-\lambda_n)\int_0^{s_n}\left(1-t^2\right)dt \leqslant (2/3)(-\lambda_n)$$

and $\frac{1}{2}(\|z_n\|^2 - z_n^2(0)) \leqslant (2/3)(-\lambda_n)$. It follows from Lemma 2.2(H_1) that $\frac{1}{2}(2/27)^2 \leqslant (-\lambda)(2/3)$. This implies $\lambda < 0$ and $\delta = \lambda/(1+\lambda) > 0$. Let $t_n \to 0^+$. Since $\lambda_n \to \lambda$, there exists $n_0 \in \mathbb{N}$ such that $\delta_n \geqslant \delta/2$ and $t_n < \delta/2$ for $n \geqslant n_0$. Let $t \in (0,\delta/2)$. Then there exists $n_1 \geqslant n_0$ such that $t_n < t$ and $\lambda_n + \lambda_n s + s \leqslant 0$ for $s \in [t_n, t]$ and $n \geqslant n_1$. This implies $\int_{t_n}^t F_n(s)\,ds \leqslant 0$ and

$$z_n(t_n) = \int_{t_n}^1 F_n(s)\,ds + (1-t_n)Bz_n(t_n) \leqslant \int_t^1 F_n(s)\,ds + (1-t_n)Bz_n(t_n).$$

It follows from (h_2) that $0 \leqslant \limsup_{n\to\infty} z_n(t_n) \leqslant Az(t)$ for each $t \in (0,\delta/2)$. This implies $0 \leqslant \limsup_{n\to\infty} z_n(t_n) \leqslant \lim_{t\to 0^+} Az(t)$. By Lemma 2.1 and $Az(0) = 0$, we obtain $\lim_{t\to 0^+} Az(t) = Az(0)$ and $\lim_{n\to\infty} z_n(t_n) = 0$.

(h_6) Since $\lambda \geqslant \lambda^* > -1/2$, we have $(1+\delta(\lambda))/2 < 1$. Let $\{t_n\} \subset (0,1)$ with $t_n \to 1^-$. Then there exists $n_0 \in \mathbb{N}$ such that $\delta_n < (1+\delta(\lambda))/2 < t_n < 1$ for $n \geqslant n_0$. It follows from (2.10) that

$$0 \leqslant \int_{t_n}^{1} F_n(s)\,ds \leqslant \int_{t_n}^{1} \frac{1+3s}{s^2}\,ds \quad \text{for } n \geqslant n_0.$$

This implies $\lim_{t_n \to 1} \int_{t_n}^1 F_n(s)\,ds = 0$. Let $\gamma \in (0,1)$. Then there exists $n_1 \geqslant n_0$ such that $\gamma \leqslant t_n$ for $n \geqslant n_1$. By Lemma 2.2(H_2), we have

$$\begin{aligned} Bz_n(t_n) &= Bz_n(\gamma) + \int_{\gamma}^{t_n} \frac{s}{z_n(s)}\,ds \leqslant Bz_n(\gamma) + 2\int_{\gamma}^{t_n} \frac{1}{s(1-s)}\,ds \\ &= Bz_n(\gamma) + 2\ln t_n - 2\ln(1-t_n) - 2\ln\frac{\gamma}{1-\gamma}. \end{aligned}$$

Noting that $\{Bz_n(\gamma)\}$ is bounded and $\lim_{t_n \to 1^-}(1-t_n)\ln(1-t_n) = 0$, we have

$$\lim_{t_n \to 1^-} (1-t_n)Bz_n(t_n) = 0.$$

This implies

$$\lim_{t_n \to 1^-} z_n(t_n) = \lim_{t_n \to 1^-} \int_{t_n}^{1} F_n(s)\,ds + \lim_{t_n \to 1^-} (1-t_n)Bz_n(t_n) = 0$$

and (h_6) holds. □

In order to prove compactness of Σ, we need the following Helly selection principle (see [15, Corollary 3.2]).

Lemma 2.4. *Let $\{z_n(t)\} \subset BV[0,1]$ be an infinite sequence. Assume that $\{V_{z_n}\}$ is bounded and there exists $K > 0$ such that $|z_n(t)| \leqslant K$ for $t \in [0,1]$ and $n \in \mathbb{N}$. Then there exist a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ and $z \in BV[0,1]$ such that $z_{n_k}(t) \to z(t)$ for each $t \in [0,1]$.*

We are now in a position to prove our main result of this section.

Theorem 2.2. *The set Σ defined in* (2.7) *is compact in $\mathbb{R} \times C[0,1]$.*

Proof. Let $(\lambda_m, z_m) \in \Sigma$ be such that $\lambda_m < 0$ and $\lambda_m \to \lambda \in [\lambda^*, 0]$. Then

$$z_m(t) = \int_{t}^{1} F_m(s)\,ds + (1-t)Bz_m(t) \quad \text{for } t \in (0,1). \tag{2.12}$$

It suffices to show that there exist a subsequence $\{z_n\}$ of $\{z_m\}$ and $z \in Q$ such that $z_n \to z$ in $C[0,1]$ and $(\lambda, z) \in \Sigma$. In fact, by Lemma 2.2(H_1) and Lemma 2.3(h_2), we have $\|z_m\| \leqslant 1$ and $V_{z_m} = 2\|z_m\| - z_m(0) \leqslant 2$. By Lemma 2.4, there exist a subsequence $\{z_n\}$ of $\{z_m\}$ and $z \in BV[0,1]$ such that $z_n(t) \to z(t)$ for each $t \in [0,1]$. Let $t \in (0,1)$ and $\gamma \in (0,t)$. Then

$$z_n(t) = \int_{t}^{1} F_n(s)\,ds + (1-t)\int_{\gamma}^{t} \frac{s}{z_n(s)}\,ds + (1-t)Bz_n(\gamma). \tag{2.13}$$

By Lemma 2.2(H_2), we have $z_n(t) \geqslant \frac{1}{2}(1-t)t^2$ for $t \in (0,1)$. This implies $z(t) \geqslant \frac{1}{2}(1-t)t^2 > 0$ for $t \in (0,1)$. By Lemma 2.3(h_2), we have

$$\lim_{n\to\infty} \int_t^1 F_n(s)\,ds = Az(t) \quad \text{for } t \in (0,1),$$

and by Lemma 2.3(h_1), $\{Bz_n(\gamma)\}$ is bounded. We may assume that $Bz_n(\gamma) \to \eta(\gamma)$. By (2.8), we have $0 \leqslant \eta(\gamma) \leqslant \sqrt{-2(\gamma + \ln(1-\gamma))}$ and $\lim_{\gamma\to 0}\eta(\gamma) = 0$. Since $\lim_{n\to\infty}\int_\gamma^t \frac{s}{z_n(s)}\,ds = \int_\gamma^t \frac{s}{z(s)}\,ds$, it follows from (2.13) that

$$z(t) = Az(t) + (1-t)\int_\gamma^t \frac{s}{z(s)}\,ds + (1-t)\eta(\gamma) \quad \text{for } \gamma \in (0,t).$$

Taking limit as $\gamma \to 0$ implies that (λ, z) satisfies (1.5) and $z \in C(0,1)$. We prove

$$0 \leqslant \lim_{t\to 0^+} Az(t) < \infty. \tag{2.14}$$

Indeed, if $\lambda < 0$, then there exists $n_0 \in \mathbb{N}$ such that $\delta/2 < \delta_n$ for $n \geqslant n_0$ and

$$0 \leqslant z_n(0) = Az_n(0) \leqslant Az_n(t) \quad \text{for } t \in (0,\delta/2) \subset (0,\delta_n] \text{ and } n \geqslant n_0.$$

This implies

$$0 \leqslant z(0) \leqslant Az(t) \leqslant Az(t) + Bz(t) = z(t) \leqslant V_z \quad \text{for } t \in (0,\delta/2). \tag{2.15}$$

Since Az is increasing on $(0,\delta/2)$ by (2.2), it follows from (2.15) that $\lim_{t\to 0} Az(t)$ exists and (2.14) holds. If $\lambda = 0$, then $(Az)(t) \geqslant 0$ for $t \in [0,1]$ and Az is decreasing on $(0,1]$. Since $0 \leqslant Az(t) \leqslant Az(t) + Bz(t) = z(t) \leqslant Vz$ for $t \in (0,1)$, it follows that $\lim_{t\to 0} Az(t)$ exists and (2.14) holds. By Lemma 2.1, we have $\lim_{t\to 0^+} z(t) = Az(0)$ and $\lim_{t\to 1^-} z(t) = 0$. Since $z_n(1) = 0$ and $z_n(1) \to z(1)$, we have $z(1) = 0$ and $\lim_{t\to 1^-} z(t) = z(1)$. Hence, z is continuous from the left at 1. Now, we prove that z is continuous from the right at 0. Since $\lim_{t\to 0^+} z(t) = Az(0)$ and $z_n(0) \to z(0)$, it suffices to show $z_n(0) = \int_0^1 F_n(s)\,ds \to Az(0)$. We consider two cases:

(i) If $z(0) > 0$, then there exists $n_0 > 0$ such that $z_n(0) \geqslant z(0)/2 > 0$ for $n \geqslant n_0$. By Lemma 2.3(h_4), there exists $\gamma_0 \in (0,1)$ such that $|z_n(t) - z_n(0)| < z(0)/4$ for $t \in [0,\gamma_0]$. This implies $z_n(t) \geqslant z(0)/4$ for $t \in [0,\gamma_0]$ and $n \geqslant n_0$. Hence, we have

$$\lim_{n\to\infty} \int_0^{\gamma_0} F_n(s)\,ds = \int_0^{\gamma_0} f_z(s)\,ds.$$

It follows from Lemma 2.3(h_2) that

$$\begin{aligned} z(0) &= \lim_{n\to\infty} z_n(0) = \lim_{n\to\infty} \int_0^1 F_n(s)\,ds = \lim_{n\to\infty}\left(\int_0^{\gamma_0} F_n(s)\,ds + \int_{\gamma_0}^1 F_n(s)\,ds\right) \\ &= \int_0^1 f_z(s)\,ds = Az(0). \end{aligned}$$

(ii) If $z(0) = 0$, then $z_n(0) \to 0$. By Lemma 2.3(h_5), $\lambda < 0$ and $\tau = \inf\{\frac{-\lambda_n}{1+\lambda_n}:\ n \in \mathbb{N}\} > 0$. Hence, $\lambda_n + \lambda_n s + s \leqslant 0$ for $s \in [0, \tau]$ and $n \in \mathbb{N}$. Since $z_n(t) \geqslant \int_t^1 F_n(s)\,ds$ for $t \in (0, 1)$ and $F_n(t) \leqslant 0$ for $t \in (0, \tau)$, we obtain

$$-(\lambda_n + \lambda_n t + t) \geqslant G_n(t)G_n'(t) \quad \text{for } t \in (0, \tau), \tag{2.16}$$

where $G_n(t) = \int_t^1 F_n(s)\,ds$. Since G_n is increasing on $(0, \tau)$, we have for $\gamma \in (0, \tau)$,

$$\left(\int_\gamma^1 F_n(s)\,ds\right)^2 - \left(\int_0^1 F_n(s)\,ds\right)^2 \geqslant 0.$$

Integrating (2.16) from 0 to γ, we have

$$\int_0^\gamma -(\lambda_n + \lambda_n t + t)\,dt \geqslant \frac{1}{2}\left[G_n^2(\gamma) - G_n^2(0)\right] \geqslant 0.$$

This implies $\int_0^\gamma -(\lambda + \lambda t + t)\,dt \geqslant \frac{1}{2}[(Az(\gamma))^2 - G_n^2(0)] \geqslant 0$ for $\gamma \in (0, \tau)$. Taking limit as $\gamma \to 0^+$ implies $(Az(0))^2 = \lim_{n\to\infty}(G_n(0))^2$ and

$$z(0) = \lim_{n\to\infty} z_n(0) = \lim_{n\to\infty} \int_0^1 F_n(s)\,ds = Az(0).$$

It follows that $\lim_{t\to 0^+} z(t) = Az(0) = z(0)$, so z is continuous from the right at 0. Hence, z is continuous on $[0, 1]$. It follows that (λ, z) is a solution of (1.5). Next, we prove that $\{z_n\}$ converges to z in $C[0, 1]$, that is, $\lim_{n\to\infty} \|z_n - z\| = 0$. In fact, if not, then there exist $\varepsilon_0 > 0$, a subsequence $\{z_j\}$ of $\{z_n\}$ and $\{t_j\} \subset [0, 1]$ with $t_j \to t_0 \in [0, 1]$ such that

$$\|z_j - z\| = \left|z_j(t_j) - z(t_j)\right| \geqslant \varepsilon_0. \tag{2.17}$$

By Lemma 2.3(h_3) and (h_4), we have either $t_0 = 0$ and $z(0) = 0$ or $t_0 = 1$. If $t_0 = 0$ and $z(0) = 0$, then $Az(0) = z(0) = 0$ and it follows from Lemma 2.3(h_5) that $z_j(t_j) \to 0$ and by Lemma 2.1, we have $\lim_{t_j\to 0^+} z(t_j) = z(0) = 0$. Hence, we have $|z_j(t_j) - z(t_j)| \to 0$, which contradicts (2.17). If $t_0 = 1$, then it follows from Lemma 2.3(h_6) that $z_j(t_j) \to 0$. Since $z(t_j) \to z(1) = 0$, we have $|z_j(t_j) - z(t_j)| \to 0$, which contradicts (2.17). Hence, $z_n \to z$ in $C[0, 1]$ and Σ is compact in $\mathbb{R} \times C[0, 1]$. □

3. Compactness on velocity and shear stress functions

In this section we apply Theorem 2.2 and the following known equivalence result obtained in [10] to prove results on compactness of velocity and shear stress functions.

Lemma 3.1.

(1) *If* $(\lambda, f) \in \mathbb{R} \times C^2(\mathbb{R}_+)$ *satisfies* (1.1), (1.2), (1.4), *then* (λ, z) *satisfies* (1.5), *where* $z : [0, 1] \to \mathbb{R}_+$ *is defined by*

$$z(t) = \begin{cases} f''((f')^{-1}(t)) & \text{if } t \in [0, 1), \\ 0 & \text{if } t = 1. \end{cases} \tag{3.1}$$

(2) *If* $(\lambda, z) \in (-1/2, \infty) \times Q$ *satisfies* (1.5), *then* $(\lambda, f) \in \mathbb{R} \times C^2(\mathbb{R}_+)$ *and satisfies* (1.1), (1.2), (1.4), *where* $f : \mathbb{R}_+ \to \mathbb{R}_+$ *is defined by*

$$f(\eta) = \int_0^{g^{-1}(\eta)} \frac{s}{z(s)}\, ds \tag{3.2}$$

and $g : [0, 1) \to \mathbb{R}_+$ *is defined by*

$$g(t) = \int_0^t \frac{1}{z(s)}\, ds. \tag{3.3}$$

It is shown in [10] that there exists $\lambda^* \in [-0.4, -0.12]$ such that (1.1), (1.2), (1.4) has at least one solution for $\lambda \in (\lambda^*, 0)$ and has a unique solution for $\lambda \in \{\lambda^*, 0\}$. We denote by Γ the set of solutions of (1.1), (1.2), (1.4), that is,

$$\Gamma := \bigl\{(\lambda, f) \in \bigl[\lambda^*, 0\bigr] \times C^2(\mathbb{R}_+)\text{: } (\lambda, f) \text{ satisfies (1.1), (1.2), (1.4)}\bigr\}.$$

Let $\Gamma' = \{(\lambda, f')\text{: } (\lambda, f) \in \Gamma\}$ and $\Gamma'' = \{(\lambda, f'')\text{: } (\lambda, f) \in \Gamma\}$. We denote by $BC(\mathbb{R}_+)$ the Banach space of continuous bounded functions defined on $\mathbb{R}_+$ with the norm $\|f\| = \sup\{|f(x)|\text{: } x \in \mathbb{R}_+\}$.

Using Theorem 2.2 and Lemma 3.1, we prove the following compactness result.

Theorem 3.1. *Γ' and Γ'' are compact in $R \times BC(\mathbb{R}_+)$.*

Proof. Let $(\lambda_n, f_n) \in \Gamma$. By Lemma 3.1, there exists $(\lambda_n, z_n) \in \Sigma$ such that (2.9) holds and

$$f_n(\eta) = \int_0^{g_n^{-1}(\eta)} \frac{s}{z_n(s)}\, ds, \qquad g_n(t) = \int_0^t \frac{1}{z_n(s)}\, ds \quad \text{and} \quad f_n'(\eta) = g_n^{-1}(\eta).$$

By Theorem 2.2, one may assume that $(\lambda_n, z_n) \to (\lambda, z) \in \Sigma$ and there exists $(\lambda, f) \in \Gamma$ such that (3.2) and (3.3) hold and $f'(\eta) = g^{-1}(\eta)$. We prove

$$\lim_{t \to 0^+} \limsup_{n \to \infty} g_n(t) = 0. \tag{3.4}$$

In fact, if $z(0) > 0$, it follows from Lemma 2.3(h_4) that there exist $\delta > 0$, $b > 0$ and $n_0 \in \mathbb{N}$ such that $z_n(s) \geqslant \sigma$ for $s \in [0, b]$ and $n \geqslant n_0$. This implies $g_n(t) \leqslant t/\sigma$ for $t \in [0, b]$ and (3.4) holds. If $z(0) = 0$, then by Lemma 2.3(h_5) we have $\lambda < 0$ and there exists $n_1 > 0$ such that $\lambda_n \leqslant \lambda/2$ for $n \geqslant n_1$. Since $(1 - t)Bz_n(t) \leqslant z_n(t)$ for $t \in (0, 1)$, it follows from Lemma 2.2(H_1) that

$$Bz_n(t) \leqslant \frac{z_n(t)}{1 - t} \leqslant \frac{1}{1 - t} \quad \text{for } t \in (0, 1).$$

This, together with (2.11) implies

$$\frac{-\lambda(1 - t^2)}{2z_n(s)} \leqslant \frac{-\lambda_n(1 - s^2)}{z_n(s)} \leqslant z_n'(s) + \frac{1}{1 - s} \quad \text{for } t \in (0, 1) \text{ and } s \in (0, t).$$

Integrating the above inequality from 0 to t, we have

$$g_n(t) \leqslant \frac{2}{-\lambda(1 - t^2)}\bigl[z_n(t) - z_n(0) - \ln(1 - t)\bigr].$$

This implies $\limsup_{n\to\infty} g_n(t) \leqslant \frac{2}{-\lambda(1-t^2)}[z(t) - \ln(1-t)]$ for $t \in (0,1)$ and (3.4) holds. Let $\eta \in \mathbb{R}_+$ and let $t_n = f_n'(\eta)$ and $t = f'(\eta)$. Then $t_n, t \in (0,1)$ and $\eta = g_n(t_n) = g(t)$. We prove $f_n'(\eta) \to f'(\eta)$. It is obviously true when $\eta = 0$, so we assume that $\eta > 0$ and $t_n \to t_0 \in [0,1]$. We prove $t_0 > 0$. In fact, if $t_0 = 0$, then for each $\gamma \in (0,1)$ there exists $n_\gamma \in \mathbb{N}$ such that $t_n < \gamma$ for $n \geqslant n_\gamma$. It follows that $\eta = g_n(t_n) \leqslant g_n(\gamma)$ and by (3.4) we have $\eta = 0$, a contradiction. By Lemma 2.2(H_1), we have $z_n(s) \geqslant s^2(1-s)/2$ for $s \in (0,1)$. This implies $\lim_{n\to\infty} \int_{t_0}^{t_n} \frac{1}{z_n(s)}\,ds = 0$. Since $g(t) = g_n(t_n) = g_n(t_0) + \int_{t_0}^{t_n} \frac{1}{z_n(s)}\,ds$, it follows that $\lim_{n\to\infty} g_n(t_0) = g(t)$. On the other hand, we have for $\gamma \in (0, t_0)$,

$$\lim_{n\to\infty} g_n(t_0) = \lim_{n\to\infty} \int_\gamma^{t_0} \frac{1}{z_n(s)}\,ds + \lim_{n\to\infty} \int_0^\gamma \frac{1}{z_n(s)}\,ds$$
$$= \int_\gamma^{t_0} \frac{1}{z(s)}\,ds + \lim_{n\to\infty} \int_0^\gamma \frac{1}{z_n(s)}\,ds.$$

Taking limit as $\gamma \to 0^+$ implies $\lim_{n\to\infty} g_n(t_0) = g(t)$. Hence, we obtain $g(t_0) = g(t)$. Since g is strictly increasing on $(0,1)$, $t = t_0$ and $f_n'(\eta) \to f'(\eta)$.

In order to prove that Γ' and Γ'' are compact, it is sufficient to prove that $\lim_{n\to\infty} \|f_n' - f'\| = 0$ and $\lim_{n\to\infty} \|f_n'' - f''\| = 0$. We prove

$$\lim_{n\to\infty} \left\| f_n' - f' \right\| = 0. \tag{3.5}$$

In fact, if (3.5) is false, then there exist a subsequence $\{\eta_{n_k}\}$ with $\eta_{n_k} \to \eta_0 \in [0,\infty]$ and $\epsilon > 0$ such that

$$\left| f_{n_k}'(\eta_{n_k}) - f'(\eta_{n_k}) \right| \geqslant \epsilon. \tag{3.6}$$

If $\eta_0 < \infty$, then noting that $f_n''(\eta) = z_n(f'(\eta)) \leqslant 1$ for $\eta \in \mathbb{R}_+$, we have $|f_{n_k}'(\eta_{n_k}) - f_{n_k}'(\eta_0)| \to 0$. Since

$$\begin{aligned} &\left| f_{n_k}'(\eta_{n_k}) - f'(\eta_{n_k}) \right| \\ &\quad \leqslant \left| f_{n_k}'(\eta_{n_k}) - f_{n_k}'(\eta_0) \right| + \left| f_{n_k}'(\eta_0) - f'(\eta_0) \right| + \left| f'(\eta_0) - f'(\eta_{n_k}) \right| \end{aligned}$$

and $|f_{n_k}'(\eta_0) - f'(\eta_0)| \to 0$ and $|f'(\eta_0) - f'(\eta_{n_k})| \to 0$, we have $|f_{n_k}'(\eta_{n_k}) - f'(\eta_{n_k})| \to 0$, which contradicts (3.6). If $\eta_0 = \infty$, then for each $\eta \in \mathbb{R}_+$, there exists $n_0 \in \mathbb{N}$ such that $\eta \leqslant \eta_{n_k}$ for $n_k \geqslant n_0$. Since f_{n_k}' is increasing on $(0,\infty)$, we have $f_{n_k}'(\eta) \leqslant f_{n_k}'(\eta_{n_k}) < 1$. Taking limit implies

$$f'(\eta) \leqslant \limsup_{n_k\to\infty} f_{n_k}'(\eta_{n_k}) \leqslant 1 \quad \text{for } \eta \in (0,\infty).$$

This, together with $f'(\infty) = 1$ implies $\lim_{n_k\to\infty} f_{n_k}'(\eta_{n_k}) = 1$. Since $f'(\eta_{n_k}) \to 1$, we have $|f_{n_k}'(\eta_{n_k}) - f'(\eta_{n_k})| \to 0$, which contradicts (3.6). This implies $f_n' \to f'$.

Now, we prove $\lim_{n\to\infty} \|f_n'' - f''\| = 0$. For $\varepsilon > 0$, there exists $n_1 > 0$ such that $\|z_n - z\| < \varepsilon/2$ for $n \geqslant n_1$. Since z is uniformly continuous on $[0,1]$, there exists $\delta > 0$ such that whenever $|t_2 - t_1| < \delta$, $|z(t_2) - z(t_1)| < \varepsilon/2$. Since $f_n' \to f'$, there exists $n_2 \geqslant n_1$ such that $\|f_n' - f\| < \delta$ for $n \geqslant n_2$. Hence we have for $n \geqslant n_2$ and $\eta \in \mathbb{R}_+$,

$$\begin{aligned} \left| f_n''(\eta) - f''(\eta) \right| &= \left| z_n\big(f_n'(\eta)\big) - z\big(f'(\eta)\big) \right| \\ &\leqslant \left| z_n\big(f_n'(\eta)\big) - z\big(f_n'(\eta)\big) \right| + \left| z\big(f_n'(\eta)\big) - z\big(f'(\eta)\big) \right| \end{aligned}$$

$$\leqslant \|z_n - z\| + \left|z\left(f_n'(\eta)\right) - z\left(f'(\eta)\right)\right|$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This implies $\lim_{n\to\infty} \|f_n'' - f''\| = 0$ and Γ'' is compact in $BC(\mathbb{R}_+)$. $\square$

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