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A drop theorem without vector topology

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Abstract

Daneš' drop theorem is extended to bornological vector spaces. An immediate application is to establish Ekeland-type variational principle and its equivalence, Caristi fixed point theorem, in bornological vector spaces. Meanwhile, since every locally convex space becomes a convex bornological vector space when equipped with the canonical von Neumann bornology, Qiu's generalization of Daneš' work to locally convex spaces is recovered.

Keywords: Drop theorem; Ekeland variational principle; Caristi fixed point theorem; Bornological vector space

1. Introduction

Let E be a vector space. If $a \in E$ and $B \subseteq E$, then the *drop associated with a and B*, which is denoted by $D(a, B)$, is the convex hull of $\{a\} \cup B$. If B is convex as well, then

$$D(a, B) = \big\{a + t(b - a):\ b \in B,\ 0 \leqslant t \leqslant 1\big\}. \tag{1}$$

In 1972, Daneš [5] proved his renowned drop theorem.

Theorem 1.1. *Let A be a complete subset of a normed linear space $(E, \|\cdot\|)$, let $x_0 \in A$, $b \in E \setminus A$ and $B(b, r)$ the closed ball centered at b with radius $r < \inf\{\|b - x\|:\ x \in A\}$. Then there exists $a \in A \cap D(x_0, B(b, r))$ such that*

$$D\big(a, B(b, r)\big) \cap A = \{a\}.$$

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In the two decades after its birth, generalizations of Theorem 1.1 have been made but the authors mainly working within the Banach setting [7,11,15,18,19]. In 1996, Daneš' result was pushed to more abstract spaces. Cheng et al. [4] proved a drop theorem in sequentially-complete locally convex spaces (see also [8,10]), which was later extended by Zheng [21] to sequentially-complete topological vector spaces, without assuming any local convexity. However, as pointed out by Qiu [16], Zheng's result was reduced to the one in [4] when the underlying space was locally convex. The contribution of Qiu [16] is the following theorem.

Theorem 1.2. *Let A be a locally closed subset of a locally convex space E and B be a locally closed, bounded, convex subset of E. Moreover, assume that there exists a locally convex topology $\mathcal{T}$ on E such that $0 \notin \mathrm{cl}_{\mathcal{T}}(A - B)$ (the closure of $A - B = \{a - b\colon a \in A,\ b \in B\}$ in $(E, \mathcal{T})$). Then for any $x_0 \in A$, there exists $a \in D(x_0, B) \cap A$ such that $D(a, B) \cap A = \{a\}$ provided that either of the following conditions is satisfied*:

(Q1) *the local closure of $B \cap L(A)$ is locally complete, where $L(A)$ denotes the linear manifold generated by A*;
(Q2) *A is locally complete.*

Recall that a sequence $\{x_n\}$ in a Hausdorff locally convex vector space is locally convergent if there is a bounded disk B in E such that $\{x_n\}$ is convergent in the normed space (E_B, p_B) where

$$E_B = \bigcup_{\lambda > 0} \lambda \cdot B$$

and p_B is the Minkowski gauge of B: for all $x \in E_B$,

$$p_B(x) = \inf\{r > 0\colon x \in r \cdot B\}.$$

In the following, we do not distinguish p_B from its extension to the whole space E defined by

$$p_B(x) = \begin{cases} \inf\{r > 0\colon x \in r \cdot B\} & \text{if } x \in E_B, \\ +\infty & \text{if } x \in E \setminus E_B. \end{cases}$$

Similarly, $\{x_n\}$ is locally Cauchy if there is a bounded disk B in E such that $\{x_n\}$ is Cauchy in (E_B, p_B). In addition, a subset A of E is locally closed if the limit of every locally convergent sequence in A remains in A while A is locally complete if every locally Cauchy sequence in A is locally convergent to some element in A.

Since local-completeness is strictly weaker than sequential-completeness, Qiu's result generalizes Daneš' drop theorem to an extremely wide class of locally convex spaces (see the hierarchy in [3]).

Instead of working in the category of topological vector spaces, we shall proceed in another direction. We extend Theorem 1.1 to a class of bornological vector spaces. Roughly speaking, our attention is shifted from the collection of 'open subsets' of the underlying space to the collection of 'bounded subsets.' One motivation is the recent work of Michor who argued that bounded subsets, rather than open subsets, should play the fundamental role in infinite-dimensional calculus [13].

This paper is organized as following. After recalling some preliminary results in bornological vector spaces in Section 2, of which our main reference is [9], our main theorem is stated and proved in Section 3. We then apply the main theorem in different contexts in Section 4. On one hand, counterparts of well-known results in topological vector spaces are transplanted to bornological vector spaces. On the other hand, results in sequentially complete locally convex spaces are strengthened to locally complete locally convex spaces. To be precise, bornological counterparts of a couple of well-known results in locally convex spaces are proved in Section 4.1, namely, an Ekeland-type variational principle and a Caristi-type fixed point theorem in bornological vector space are obtained in Sections 4.1.1 and 4.1.2, respectively. The equivalence of these two theorems is also established. In Section 4.2, Qiu's drop theorem (Theorem 1.2) is recovered.

2. Fundamentals of bornological vector spaces

Throughout this paper, E is a vector space over $\mathbb{R}$ and we shall denote the zero vector in E by 0_E.

A *vector bornology* on E is a collection $\mathcal{B}$ of subsets of E that satisfies the following conditions:

(B1) $x \in E$ implies that $\{x\} \in \mathcal{B}$.
(B2) $B_1 \subseteq B_2$ and $B_2 \in \mathcal{B}$ implies that $B_1 \in \mathcal{B}$.
(B3) $B_1, B_2 \in \mathcal{B}$ implies that $B_1 \cup B_2 \in \mathcal{B}$.
(B4) $B_1, B_2 \in \mathcal{B}$ implies that

$$B_1 + B_2 = \{x_1 + x_2\colon\ x_1 \in \mathcal{B}_1,\ x_2 \in \mathcal{B}_2\} \in \mathcal{B}.$$

(B5) For any bounded interval $I \subseteq \mathbb{R}$, $B \in \mathcal{B}$ implies that

$$I \cdot B = \{\alpha x\colon\ \alpha \in I,\ x \in B\} \in \mathcal{B}.$$

In view of (B5), if $B \in \mathcal{B}$, so is its balanced hull B_{b} which is defined by $B_{\mathrm{b}} = [-1, 1] \cdot B$.

The ordered pair $(E, \mathcal{B})$ is called a *bornological vector space* (BVS) and every candidate of $\mathcal{B}$ is called a *bounded subset* (with respect to $\mathcal{B}$). We shall simply denote $(E, \mathcal{B})$ by E if there is no ambiguity.

Example 2.1. Let $E = \mathbb{R}^n$ and $\|\cdot\|$ be the usual Euclidean norm on $\mathbb{R}^n$. The topology determined by $\|\cdot\|$ is denoted by $\mathcal{T}_{\mathrm{norm}}$. We define $\mathcal{B}_{\mathcal{T}_{\mathrm{norm}}}$ to be the collection of all subsets $B \subseteq R^n$ satisfying $\sup\{\|b\|\colon\ b \in B\} < \infty$. It is easy to see that $(\mathbb{R}^n, \mathcal{B}_{\mathcal{T}_{\mathrm{norm}}})$ is a BVS.

Lemma 2.2. *If $x \in E$ and B is a convex subset of E, then*

(1) *$D(x, B) \in \mathcal{B}$ whenever $B \in \mathcal{B}$,*
(2) *$D(y, B) \subseteq D(x, B)$ whenever $y \in D(x, B)$.*

A sequence $\{x_n\}$ in E is said to be *Mackey-convergent* (*or M-convergent*) *to a point x*, denoted by $\lim^{\flat}{}_{n\to\infty} x_n = x$, if there is a balanced $B \in \mathcal{B}$ and a sequence of positive real numbers $\{\lambda_n\}$ such that $\lim_{n\to\infty} \lambda_n = 0$ and

$$x_n - x \in \lambda_n \cdot B \quad \text{for all } n \in \mathbb{N}.$$

Also, we say that x is a bornological limit of $\{x_n\}$.

Similarly, a sequence $\{x_n\}$ in E is said to be *Mackey–Cauchy* (*or M-Cauchy*) if there is a balanced $B \in \mathcal{B}$ and a double sequence of positive real numbers $\{\lambda_{mn}\}$ such that $\lim_{m,n\to\infty} \lambda_{mn} = 0$ and

$$x_m - x_n \in \lambda_{mn} \cdot B \quad \text{for all } m, n \in \mathbb{N}.$$

Remark 2.3. It is very easy to check that bornological limits enjoy the following standard algebraic properties [9]: if $\{\lambda_n\}$ is a convergent sequence of reals, $\{x_n\}$ and $\{y_n\}$ are M-convergent sequences in E, then both ${\lim^\flat}_{n\to\infty}(x_n + y_n)$ and ${\lim^\flat}_{n\to\infty}(\lambda_n x_n)$ exist; in addition,

$$\lim_{n\to\infty}{}^{\flat}(x_n + y_n) = \lim_{n\to\infty}{}^{\flat} x_n + \lim_{n\to\infty}{}^{\flat} y_n \tag{2a}$$

and

$$\lim_{n\to\infty}{}^{\flat}(\lambda_n x_n) = \Big(\lim_{n\to\infty} \lambda_n\Big) \cdot \Big(\lim_{n\to\infty}{}^{\flat} x_n\Big). \tag{2b}$$

If $A \subseteq E$, then the set of all bornological limits of sequences in A is denoted by $A^{(1)}$. Clearly, $A \subseteq A^{(1)}$. A is said to be *Mackey-closed* (*or M-closed*) if $A = A^{(1)}$.

Remark 2.4. Though the collection of all (complements of) M-closed subsets of E defines a topology on E, it is rarely a vector topology with respect to the algebraic structure of E (cf. [9]). In other words, those known results in topological vector spaces do not cover ours.

It is known that, in a general topological space, limit of a convergent sequence need not to be unique without assuming Hausdorffness. Similarly, a BVS $(E, \mathcal{B})$ is said to be *separated* if every M-convergent sequence is M-convergent to exactly one bornological limit.

Remark 2.5. It could be proved [9] that a BVS is separated if and only if there is no non-trivial bounded subspace in E.

From now on, *we will consider only separated BVS.*

The following theorem yields the M-closedness of a drop relative to some M-closed set.

Theorem 2.6. *If $B \in \mathcal{B}$ is convex and M-closed, then the drop $D(x, B)$ is M-closed for every $x \in E$.*

Proof. Note that for each $b \in B$, $D(x - b, B - \{b\}) = D(x, B) - \{b\}$; and it is easy to see that $D(x, B)$ is M-closed if and only if $D(x, B) - \{b\}$ is. We may thus assume that $0_E \in B$. Suppose that z is the bornological limit of a sequence in $D(x, B)$. Then there exists a sequence $\{(b_n, t_n)\}$ in $B \times [0, 1]$ such that ${\lim^\flat}_{n\to\infty}(1 - t_n)x + t_n b_n = z$.

As every subsequence of $\{(1 - t_n)x + t_n b_n\}$ will be M-convergent to z as well, by passing to a subsequence if necessary, we may further assume that $\lim_{n\to\infty} t_n = t_0$, which is in $[0, 1]$. As

$$(1 - t_n)x - (1 - t_0)x = (t_0 - t_n)x \in (t_0 - t_n) \cdot \{x\} \subseteq |t_0 - t_n| \cdot \{x\}_{\mathrm{b}}$$

for all $n \in \mathbb{N}$, ${\lim^\flat}_{n\to\infty}(1 - t_n)x = (1 - t_0)x$, whereby $\{t_n b_n\}$ is M-convergent (cf. (2a)). We claim that ${\lim^\flat}_{n\to\infty} t_n b_n \in t_0 \cdot B$. Consequently,

$$z = (1 - t_0)x + \lim_{n\to\infty}{}^{\flat} t_n b_n \in (1 - t_0)x + t_0 \cdot B \subseteq D(x, B)$$

and we are done.

Since we shall need the above claim later, it is worth treating it separately in the coming Lemma 2.7. □

Lemma 2.7. *Let $B \in \mathcal{B}$ be M-closed. If $\{t_n\}$ is a sequence of positive real numbers convergent to t and $\{b_n\}$ is a sequence in B such that $\{t_n b_n\}$ is M-convergent to $b \in B$, then $b \in t \cdot B$.*

Proof. Note that

$$t_n b_n \in t_n \cdot B \subseteq t_n \cdot B_{\mathrm{b}} \quad \text{for all } n \in \mathbb{N}.$$

If $\lim_{n\to\infty} t_n = 0$, then $\lim^{\flat}{}_{n\to\infty} t_n b_n = 0$. Since E is separated, $b = 0 \in 0 \cdot B$ as claimed.

Now we turn to the case $t > 0$. By considering a subsequence of $\{t_n\}$ if necessary, we may assume that $t_n > 0$ for all $n \in \mathbb{N}$. Observe that $\lim_{n\to\infty} 1/t_n = 1/t$ and hence, with the aid of (2b),

$$\lim_{n\to\infty}^{\flat} b_n = \lim_{n\to\infty}^{\flat}\left(\frac{1}{t_n} \cdot (t_n b_n)\right) = \left(\lim_{n\to\infty} \frac{1}{t_n}\right) \cdot \left(\lim_{n\to\infty}^{\flat} t_n b_n\right) = \frac{1}{t} b.$$

Since B is M-closed, $b \in t \cdot B$. $\square$

A common feature of all those known drop theorems in a topological vector space $(E, \mathcal{T})$ is the 'separateness' between A and B (e.g., Theorems 1.1 and 1.2), which could be encoded in the form $0 \notin \mathrm{cl}_{\mathcal{T}}(A - B)$. To describe its counterpart in BVS, we need the notion of bornivorous subsets.

Definition 2.8. Let $(E, \mathcal{B})$ be a BVS. A subset $A \subseteq E$ is bornivorous if it absorbs every bounded subset of E. To be precise, for all $B \in \mathcal{B}$, there is $\lambda_0 > 0$ such that

$$[0, \lambda_0] \cdot B \subseteq A.$$

Remark 2.9. It is well known [9] that a set A is bornivorous if and only if

$$0 \notin (E \setminus A)^{(1)}.$$

To end this section, we introduce a kind of completeness in a BVS, which we shall employ in later discussion.

Definition 2.10. Let $(E, \mathcal{B})$ be a BVS. A subset A of E is Mackey-complete (or M-complete) if every M-Cauchy sequence in A will be M-convergent to some element in A.

Lemma 2.11. *Let $(E, \mathcal{B})$ be a BVS and suppose that $B \in \mathcal{B}$ is convex and M-complete. Then, for all $\lambda \in \mathbb{R}$ and $a \in E$, both $\lambda \cdot B$ and the drop $D(a, B)$ are M-complete.*

Proof. Let $\{b_n\}$ be an M-Cauchy sequence in $\lambda \cdot B$. We need only consider the case $\lambda \neq 0$. By definition, $b_n = \lambda b'_n$ for some $b'_n \in B$. Also, there are a balanced $B_1 \in \mathcal{B}$ and a double sequence of real numbers $\{\lambda_{n,m}\}$ such that $\lim_{n,m\to\infty} \lambda_{n,m} = 0$ and

$$b_n - b_m = \lambda \cdot \left(b'_n - b'_m\right) \in \lambda_{n,m} \cdot B_1.$$

Clearly, $b'_n - b'_m \in \lambda_{n,m}/\lambda \cdot B_1$ and $\lim_{n,m\to\infty} \lambda_{n,m}/\lambda = 0$. It follows that $\{b'_n\}$ is M-Cauchy in B and hence is M-convergent to some $b'_0 \in B$. It is easy to show that $\{b_n\}$ is M-convergent to $\lambda b'_0 \in \lambda B$. So $\lambda \cdot B$ is M-complete.

We proceed to prove the M-completeness of $D(a, B)$. As explained in the proof of Theorem 2.6, we may assume that $0_E \in B$.

Let $\{a + t_n(b_n - a)\}$ be a M-Cauchy sequence in $D(a, B)$. In view of the next Lemma 2.12, it suffices to prove that $\{a + t_n(b_n - a)\}$ has a M-convergent subsequence.

Suppose that $\{t_{n_k}\}$ is a convergent subsequence of $\{t_n\}$ and $\lim_{k\to\infty} t_{n_k} = t$. Then

$$\left[(1 - t_{n_k})a\right] - \left[(1 - t)a\right] = (t - t_{n_k})a \in |t - t_{n_k}| \cdot \{a\}_{\mathrm{b}}.$$

As a result, $\{(1-t_{n_k})a\}$ is M-convergent to $(1-t)a$. Note also that $0_E \in B$ and hence $(1-t)a \in D(a, B)$.

Since M-convergent sequences are M-Cauchy and the sum of two M-Cauchy sequences is M-Cauchy as well, we may thus conclude that $\{t_{n_k} b_{n_k}\}$ is M-Cauchy. Since B is convex, contains 0, and $t_n \in [0, 1]$ for all n, we conclude that $\{t_{n_k} b_{n_k}\}$ is a M-Cauchy sequence in B. By the M-completeness of B and Lemma 2.7, $\{t_{n_k} b_{n_k}\}$ is M-convergent to some element tb for some $b \in B$.

In conclusion, $\lim^{\mathrm{b}}{}_{k\to\infty}[a+t_{n_k}(b_{n_k}-a)] = a+t(b-a) \in D(a, B)$ and thus $\{a+t_{n_k}(b_{n_k}-a)\}$ is a convergent subsequence of $\{a + t_n(b_n - a)\}$. □

Lemma 2.12. *If $\{x_n\}$ is a M-Cauchy sequence and $\{x_{n_k}\}$ is a subsequence M-convegent to x, then $\{x_n\}$ is also M-convergent to x.*

Proof. Since $\{x_n\}$ is M-Cauchy, there is a balanced $B_1 \in \mathcal{B}$ and a double sequence $\{\lambda_{m,n}\}$ such that

$$\lim_{n,m\to\infty} \lambda_{m,n} = 0 \quad \text{and} \quad x_n - x_m \in \lambda_{n,m} \cdot B_1.$$

On the other hand, $\{x_{n_k}\}$ is M-convergent to x implies that there is a balanced $B_2 \in \mathcal{B}$ and a sequence of real numbers $\{\lambda_{n_k}\}$ such that

$$\lim_{k\to\infty} \lambda_{n_k} = 0 \quad \text{and} \quad x_{n_k} - x \in \lambda_{n_k} \cdot B_2.$$

Then, we define

$$B = \big(B_1 \cup B_2 \cup \{x_n - x\colon\ n = 1, 2, \dots, n_1 - 1\}\big)_{\mathrm{b}}$$

and

$$\mu_n = \begin{cases} 1 & \text{if } 1 \leqslant n < n_1, \\ \max\{\lambda_{n_k}, \lambda_{n,n_k}\} & \text{if } n_k \leqslant n < n_{k+1}. \end{cases}$$

Note that $B \in \mathcal{B}$ by (B3) and (B5), and in addition, $\lim_{n\to\infty} \mu_n = 0$. Clearly, $x_n - x \in \mu_n \cdot B$ if $1 \leqslant n < n_1$. If $n \geqslant n_1$ satisfying $n_k \leqslant n < n_{k+1}$, then

$$\{x_n - x\} = \big\{(x_n - x_{n_k}) + (x_{n_k} - x)\big\} \subseteq \lambda_{n,n_k} \cdot B_1 + \lambda_{n_k} \cdot B_2 \subseteq \mu_n \cdot B$$

as well. Consequently, $\{x_n\}$ is M-convergent to x as claimed. □

Finally, the following lemma demonstrates a relation between M-closedness and M-completeness.

Lemma 2.13. *Let $(E, \mathcal{B})$ be a BVS and $A \subseteq E$ be an M-complete subset. Then A is M-closed. On the other hand, if $(E, \mathcal{B})$ is M-complete and $A \subseteq E$ is M-closed, then A is M-complete.*

3. Drop theorem in bornological vector spaces

Let A and B be subsets of a BVS $(E, \mathcal{B})$. For each $x_0 \in A$, we set

$$C_{x_0} = D(x_0, B) - B.$$

We first present a key lemma.

Lemma 3.1. *If $B \in \mathcal{B}$, B is convex, and $E \setminus (A - B)$ is bornivorous, then C_{x_0} is convex and bounded. In addition, for each $x \in A \cap D(x_0, B)$ and $\varepsilon > 0$, there exists $a(x, \varepsilon) \in D(x, B) \cap A$ such that*

$$\sup_{y \in A \cap D(a(x,\varepsilon), B)} p_{C_{x_0}}\big(a(x, \varepsilon) - y\big) \leqslant \varepsilon.$$

Proof. As $(E, \mathcal{B})$ is a BVS and $B \in \mathcal{B}$, by Lemma 2.2(1), $C_{x_0} \in \mathcal{B}$. Moreover, since $D(x_0, B)$ and $-B$ is convex, C_{x_0} is convex.

We then claim that for each $x \in A \cap D(x_0, B)$,

$$\alpha_x := \inf p_{C_{x_0}}\big(\big(A \cap D(x, B)\big) - B\big) \in (0, 1]. \tag{3}$$

Instead of proving this claim, we assume it and proceed. Fix $x \in A \cap D(x_0, B)$ and $\varepsilon > 0$. Then there exist $b(x, \varepsilon) \in B$ and $a(x, \varepsilon) \in A \cap D(x, B)$ such that

$$p_{C_{x_0}}\big(a(x, \varepsilon) - b(x, \varepsilon)\big) \leqslant (1 + \varepsilon)\alpha_x. \tag{4}$$

We are going to establish that

$$\sup_{y \in A \cap D(a(x,\varepsilon), B)} p_{C_{x_0}}\big(a(x, \varepsilon) - y\big) \leqslant \varepsilon.$$

Let $y \in A \cap D(a(x, \varepsilon), B)$ be arbitrary. By definition, $y = (1 - t)a(x, \varepsilon) + tb$ for some $t \in [0, 1]$ and $b \in B$. On one hand,

$$p_{C_{x_0}}\big(a(x, \varepsilon) - y\big) = p_{C_{x_0}}\big(t\big(a(x, \varepsilon) - b\big)\big) = t p_{C_{x_0}}\big(a(x, \varepsilon) - b\big) \leqslant t, \tag{5}$$

where the second equality follows from the positive homogeneity of $p_{C_{x_0}}$ and the last equality follows from

$$\big\{a(x, \varepsilon) - b\big\} \subseteq D(x, B) - B \subseteq D(x_0, B) - B = C_{x_0}$$

and Lemma 2.2(2).

On the other hand, $y - (tb + (1 - t)b(x, \varepsilon)) \in (A \cap D(x, B)) - B$ and hence, with the aid of (3) and (4),

$$\begin{aligned}
\alpha_x &\leqslant p_{C_{x_0}}\big(y - \big(tb + (1 - t)b(x, \varepsilon)\big)\big) \\
&= p_{C_{x_0}}\big((1 - t)\big(a(x, \varepsilon) - b(x, \varepsilon)\big)\big) = (1 - t)p_{c_{x_0}}\big(a(x, \varepsilon) - b(x, \varepsilon)\big) \\
&\leqslant (1 - t)\alpha_x(1 + \varepsilon).
\end{aligned}$$

As $\alpha_x > 0$, we have $t \leqslant (1 - t)\varepsilon \leqslant \varepsilon$. Combining this with (5), we are done.

It remains to establish (3) for any $x \in A \cap D(x_0, B)$. On the one hand, for each $x \in D(x_0, B)$, $D(x, B) \subset D(x_0, B)$ by Lemma 2.2. It follows that

$$\big(A \cap D(x, B)\big) - B \subset C_{x_0}$$

and hence $\alpha_x \leqslant 1$. On the other hand, since $E \setminus (A - B)$ is bornivorous and C_{x_0} is bounded, there is a positive real number $\lambda_0 > 0$ such that $[0, \lambda_0] \cdot C_{x_0} \subseteq E \setminus (A - B)$, whereby

$$\big([0, \lambda_0] \cdot C_{x_0}\big) \cap (A - B) = \emptyset.$$

As a result, for each $x \in A \cap D(x_0, B)$,

$$p_{C_{x_0}}\big(\big(A \cap D(x, B)\big) - B\big) \subseteq p_{C_{x_0}}(A - B) \subseteq [\lambda_0, \infty].$$

The proof of Lemma 3.1 is completed. $\square$

We need one more notion before we state our main Theorem 3.3.

Definition 3.2. Let $(E, \mathcal{B})$ be a BVS. Suppose that $A, B \subseteq E$. We say that A is drop-Mackey-complete relative to B if every M-Cauchy sequence $\{x_n\}$ in A satisfying

$$x_{n+1} \in D(x_n, B) \quad \text{for all } n \in \mathbb{N},$$

is M-convergent in E.

Theorem 3.3. *Let $(E, \mathcal{B})$ be a BVS. If $A \subseteq E$ is M-closed, $B \in \mathcal{B}$ is M-closed and convex, $E \setminus (A - B)$ is bornivorous, and A is drop-Mackey-complete relative to B, then for all $x_0 \in A$, there exists $a \in D(x_0, B) \cap A$ such that $D(a, B) \cap A = \{a\}$.*

Proof. Using Lemma 3.1, we can construct a sequence $\{a_n\}$ in A recursively by setting $a_1 = x_0$ and $a_{n+1} = a(a_n, 1/n)$ so that $a_1 \in D(x_0, B) \cap A$, $a_{n+1} \in D(a_n, B) \cap A$, and

$$\sup_{y \in D(a_n, B) \cap A} p_{C_{x_0}}(a_n - y) \leqslant \frac{1}{n} \tag{6}$$

for all $n \in \mathbb{N}$.

Firstly, we claim that $\{a_n\}$ chosen in this way is M-Cauchy. Since $a_{n+1} \subseteq D(a_n, B)$, in view of Lemma 2.2(2), for all $k \in \mathbb{N}$,

$$D(a_{n+k}, B) \subseteq D(a_{n+k-1}, B) \subseteq \cdots \subseteq D(a_{n+1}, B) \subseteq D(a_n, B). \tag{7}$$

By (6), for all $k \in \mathbb{N}$, $p_{C_{x_0}}(a_n - a_{n+k}) \leqslant 1/n$ and thus

$$a_{n+k} - a_n \in \frac{1}{n} \cdot [C_{x_0}]_{\mathrm{b}}.$$

Therefore, $\{a_n\}$ is M-Cauchy as claimed. Since A is drop-Mackey-complete relative to B and A is M-closed, there exists $a \in A$ such that $\lim^{\flat}{}_{n\to\infty} a_n = a$.

Secondly, we shall prove that

$$D(a, B) \subseteq \bigcap_{n=1}^{\infty} D(a_n, B). \tag{8}$$

In fact, for each $n \in \mathbb{N}$, the hierarchy in (7) implies that the truncated sequence $\{a_{n+1}, a_{n+2}, \ldots\}$ is a M-convergent sequence in $D(a_n, B)$, which is M-closed by Theorem 2.6. As a result, $a \in D(a_n, B)$ for every $n \in \mathbb{N}$. Again, in view of Lemma 2.2(2), we have $D(a, B) \subseteq D(a_n, B)$ for all $n \in \mathbb{N}$. We have thus established (8).

Thirdly, we shall show that $D(a, B) \cap A = \{a\}$. If $x \in D(a, B) \cap A$, then $x \in \bigcap_{n=1}^{\infty} D(a_n, B)$ by (8). Applying (6), we have

$$p_{C_{x_0}}(a_n - x) \leqslant \frac{1}{n}$$

for all $n \in \mathbb{N}$, which simply means that $\lim^{\flat}{}_{n\to\infty} a_n = x$. As E is separated, x must be a.

It remains to prove that $a \in D(x_0, B)$. By (8), $a \in D(a_1, B)$. In addition, by our choice of a_1 and Lemma 2.2(2), $D(a_1, B) \subseteq D(x_0, B)$. The result follows. □

An essential assumption in Theorem 3.3 is that A is drop-Mackey-complete relative to B. Clearly, if A is M-complete, then A is drop-Mackey-complete relative to B. The same conclusion is also valid when B is M-complete.

Lemma 3.4. *If either A or B is M-complete, then A is drop-Mackey-complete relative to B.*

The following corollary is immediate.

Corollary 3.5. *Let $(E, \mathcal{B})$ be a BVS. If $A \subseteq E$ is a M-closed, $B \in \mathcal{B}$ is M-closed and convex, $E \setminus (A - B)$ is bornivorous, and moreover, either A or B is M-complete, then for all $x_0 \in A$, there is $a \in D(x_0, B) \cap A$ such that $D(a, B) \cap A = \{a\}$.*

4. Applications

In this section, several applications of Theorem 3.3, to be more precise, Corollary 3.5, are illustrated. We divide them into two categories: in separated bornological vector spaces and in Hausdorff locally convex spaces.

4.1. In bornological vector spaces

It is well known that the following three theorems are equivalent: Daneš' drop theorem, Ekeland's variational principle, and Caristi fixed point theorem (see [6] or [15] when E is Banach; and see [21] for general topological vector spaces). We shall transplant some of these results to bornological vector spaces.

4.1.1. Ekeland-type variational principle

Let $(E, \mathcal{B})$ be a BVS and $\mathcal{B}_{\mathcal{T}_{\mathrm{norm}}}$ be as defined in Example 2.1. Then $(E \times \mathbb{R}, \mathcal{B} \times \mathcal{B}_{\mathcal{T}_{\mathrm{norm}}})$ is also a BVS. In addition, $B_1 \times B_2 \in \mathcal{B} \times \mathcal{B}_{\mathcal{T}_{\mathrm{norm}}}$ is balanced if and only if both B_1 and B_2 are balanced. The following lemma is another simple observation.

Lemma 4.1. *If $\{(x_n, t_n)\}$ is a M-Cauchy sequence, then $\{t_n\}$ is convergent and $\{x_n\}$ is M-Cauchy. In case that $\{(x_n, t_n)\}$ is M-convergent, $\{x_n\}$ is M-convergent too. To be more precise, if $\lim^{\flat}{}_{n\to\infty}(x_n, t_n) = (x, t)$, then $\lim^{\flat}{}_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} t_n = t$.*

We state first our Ekeland-type Variational Principle in BVS.

Theorem 4.2. *Let $(E, \mathcal{B})$ be a BVS equipped with a subadditive, positively homogeneous funtion $P : E \to \mathbb{R} \cup \{+\infty\}$. If $f : E \to \mathbb{R} \cup \{+\infty\}$ is bounded from below, i.e., $\inf_{x\in E} f(x) > -\infty$, and there exists $x_0 \in E$ such that $f(x_0) < +\infty$ and satisfies the following conditions*:

(EVP1) *the intersection of the epigraph of f and $E \times (-\infty, f(x_0)]$,*

$$A := \{(x, t) \in E \times \mathbb{R}:\ f(x) \leqslant t \leqslant f(x_0)\},$$

is M-closed in $(E \times \mathbb{R}, \mathcal{B} \times \mathcal{B}_{\mathcal{T}_{\mathrm{norm}}})$;

(EVP2) *the set*

$$C = \{x \in E:\ P(x) \leqslant 1\}$$

is M-closed and bounded;

(EVP3) *either $\{x \in E:\ f(x) \leqslant f(x_0)\}$ or C is M-complete.*

Then, for all $\varepsilon > 0$, there exists $x_\varepsilon \in E$ such that

$$f(x_\varepsilon) + \varepsilon P(x_\varepsilon - x_0) \leqslant f(x_0), \tag{9a}$$

$$f(x) + \varepsilon P(x - x_\varepsilon) > f(x_\varepsilon) \quad \text{for all } x \neq x_\varepsilon. \tag{9b}$$

Before we present its proof, we state without proof two simple lemmas. In the sequel, we shall employ the same notations as in Theorem 4.2.

Lemma 4.3. *If $\{x \in E\colon\ f(x) \leqslant f(x_0)\}$ is M-complete and* (EVP1) *holds, then A is M-complete.*

Lemma 4.4. *If $P : E \to \mathbb{R} \cup \{+\infty\}$ is a subadditive and positively homogeneous function, then C is convex. If, in addition, C is bounded, then $x \neq 0$ implies that $P(x) > 0$.*

We need also the following technical lemma, of which the proof is deferred to the end of this section.

Lemma 4.5. *Suppose that f is bounded from below* (*cf. Theorem* 4.2). *Let $m = \inf\{t\colon\ (x,t) \in A\}$ and $P : E \to \mathbb{R} \cup \{\infty\}$ be a subadditive, positively homogeneous function. We set for $\varepsilon > 0$,*

$$B = \left\{(x, m-1)\colon\ x \in \frac{1-m}{\varepsilon} \cdot C\right\}.$$

Then

(i) *B is convex,*
(ii) *$(E \times \mathbb{R}) \setminus (A - B)$ is bornivorous,*
(iii) *B is M-closed and bounded provided that* (EVP2) *holds,*
(iv) *B is M-complete whenever C is.*

Assuming these lemmas, we present the proof of Theorem 4.2.

Proof of Theorem 4.2. Without loss of generality, we may assume that $f(x_0) = 0$ and $x_0 = 0_E$. Thus, $m = \inf\{t\colon\ (x,t) \in A\} \leqslant 0$.

First of all, under the assumptions (EVP1)–(EVP3), and according to Lemmas 4.3 and 4.5, the hypotheses of Corollary 3.5, applied to A from (EVP1) and B from Lemma 4.5, are satisfied. As a result, we get a pair $(x_\varepsilon, \bar{t}) \in D((0_E, 0), B) \cap A$ such that

$$D\big((x_\varepsilon, \bar{t}), B\big) \cap A = \big\{(x_\varepsilon, \bar{t})\big\}.$$

We claim that

$$P(x_\varepsilon) \leqslant \frac{-\bar{t}}{\varepsilon} \quad \text{and} \quad f(x_\varepsilon) = \bar{t}. \tag{10}$$

Once (10) is established, we are ready to prove (9a) and (9b). In fact, (9a) follows immediately from (10):

$$f(x_\varepsilon) + \varepsilon P(x_\varepsilon) \leqslant \bar{t} - \bar{t} = 0 = f(x_0).$$

Next, we establish (9b). Let $x \in E \setminus \{x_\varepsilon\}$. By Lemma 4.4,

$$P(x - x_\varepsilon) > 0.$$

We investigate the two mutually exclusive cases $(x, f(x)) \notin A$ and $(x, f(x)) \in A$ separately.

Case 1: $(x, f(x)) \notin A$. It follows that $f(x) > 0$. Hence

$$f(x) + \varepsilon P(x - x_\varepsilon) > 0 \geqslant f(x_\varepsilon)$$

and we are done.

Case 2: $(x, f(x)) \in A$. Assume to the contrary that

$$f(x) + \varepsilon P(x - x_\varepsilon) \leqslant f(x_\varepsilon). \tag{11}$$

We shall prove that $(x, f(x)) \in D((x_\varepsilon, \bar{t}), B)$. However, if it were the case, then $(x, f(x)) \in \{(x_\varepsilon, \bar{t})\}$ and hence $x = x_\varepsilon$. It contradicts our choice of x. Therefore, (11) must be violated and therefore (9b) holds.

Now we proceed to prove that $(x, f(x)) \in D((x_\varepsilon, \bar{t}), B)$ provided that (11) is valid. In view of (10), it amounts to establish

$$\bigl(x, f(x)\bigr) \in D\bigl(\bigl(x_\varepsilon, f(x_\varepsilon)\bigr), B\bigr). \tag{12}$$

By the definition of m and the positivity of $P(x - x_\varepsilon)$, we have

$$m - 1 < f(x) < f(x_\varepsilon).$$

It follows that

$$\frac{f(x_\varepsilon) - m + 1}{f(x_\varepsilon) - f(x)} > 1.$$

Further, a direct computation yields that

$$\begin{aligned}\bigl(x, f(x)\bigr) &= \frac{f(x) - m + 1}{f(x_\varepsilon) - m + 1} \cdot \bigl(x_\varepsilon, f(x_\varepsilon)\bigr) \\ &\quad + \left(1 - \frac{f(x) - m + 1}{f(x_\varepsilon) - m + 1}\right) \cdot \left(x_\varepsilon + \frac{f(x_\varepsilon) - m + 1}{f(x_\varepsilon) - f(x)}(x - x_\varepsilon), m - 1\right).\end{aligned}$$

Thus, (12) is proved provided that

$$\left(x_\varepsilon + \frac{f(x_\varepsilon) - m + 1}{f(x_\varepsilon) - f(x)}(x - x_\varepsilon), m - 1\right) \in B,$$

i.e.,

$$P\left(x_\varepsilon + \frac{f(x_\varepsilon) - m + 1}{f(x_\varepsilon) - f(x)}(x - x_\varepsilon)\right) \leqslant \frac{1 - m}{\varepsilon}.$$

Indeed, by (10), (11) and the sublinearity of P, we have

$$\begin{aligned}P\left(x_\varepsilon + \frac{f(x_\varepsilon) - m + 1}{f(x_\varepsilon) - f(x)}(x - x_\varepsilon)\right) &\leqslant P(x_\varepsilon) + \frac{f(x_\varepsilon) - m + 1}{f(x_\varepsilon) - f(x)} P(x - x_\varepsilon) \\ &\leqslant P(x_\varepsilon) + \frac{f(x_\varepsilon) - m + 1}{f(x_\varepsilon) - f(x)} \cdot \frac{f(x_\varepsilon) - f(x)}{\varepsilon} \\ &\leqslant \frac{-\bar{t}}{\varepsilon} + \frac{\bar{t} - m + 1}{\varepsilon} = \frac{1 - m}{\varepsilon}.\end{aligned}$$

Therefore,

$$\left(x_\varepsilon + \frac{f(x_\varepsilon) - m + 1}{f(x_\varepsilon) - f(x)}(x - x_\varepsilon), m - 1\right) \in B$$

as claimed.

It remains to justify our *claim* (10).

Since $(x_\varepsilon, \bar{t}) \in D((0_E, 0), B)$, there exist $\lambda \in [0, 1]$ and $(b, m-1) \in B$ such that

$$(x_\varepsilon, \bar{t}) = \big(\lambda b, \lambda(m-1)\big).$$

It follows that $\varepsilon/(1-m) \cdot b \in C$ and $-\bar{t} = \lambda(1-m)$, whereby

$$P(x_\varepsilon) = \frac{\lambda(1-m)}{\varepsilon} \cdot P\left(\frac{\varepsilon}{1-m} \cdot b\right) \leqslant \frac{\lambda(1-m)}{\varepsilon} = \frac{-\bar{t}}{\varepsilon}.$$

The first assertion in (10) is thus established. We proceed to prove that $\bar{t} = f(x_\varepsilon)$.

Note that $(x_\varepsilon, \bar{t}) \in A$ implies that $\bar{t} \geqslant m > m-1$ and hence

$$P(x_\varepsilon) < \frac{1-m}{\varepsilon}.$$

Therefore, $x_\varepsilon \in (1-m)/\varepsilon \cdot C$, i.e., $(x_\varepsilon, m-1) \in B$.

Recall that $\lambda(m-1) = \bar{t} \geqslant f(x_\varepsilon) > m-1$. So, there is $\mu \in [\lambda, 1)$ such that

$$f(x_\varepsilon) = \mu(m-1).$$

Using this μ, we see that

$$\big(x_\varepsilon, f(x_\varepsilon)\big) = \frac{1-\mu}{1-\lambda}(x_\varepsilon, \bar{t}) + \left(1 - \frac{1-\mu}{1-\lambda}\right)(x_\varepsilon, m-1)$$

and therefore $(x_\varepsilon, f(x_\varepsilon)) \in D((x_\varepsilon, \bar{t}), B)$.

Clearly, $f(x_\varepsilon) \leqslant 0$, whereby $(x_\varepsilon, f(x_\varepsilon)) \in D((x_\varepsilon, \bar{t}), B) \cap A = \{(x_\varepsilon, \bar{t})\}$. Hence, $\bar{t}$ must be $f(x_\varepsilon)$. This finishes our proof of (10) and thus the proof of Theorem 4.2. □

Proof of Lemma 4.5. (iv) is trivial. Meanwhile, (i) follows from the convexity of C, which is asserted in Lemma 4.4.

We proceed to prove (ii). Suppose the contrary. In view of Remark 2.9, we can find sequences $(x_n, t_n) \in A$ and $(x'_n, m-1) \in B$ such that

$$\lim_{n\to\infty}{}^{\flat} \big(x_n - x'_n, t_n - m + 1\big) = (0_E, 0).$$

By Lemma 4.1, $\lim_{n\to\infty} t_n = m-1$. But this is impossible as $t_n \geqslant m > m-1$ for all $n \in \mathbb{N}$.

Finally, we establish (iii). By (EVP2), $C \in \mathcal{B}$ and therefore

$$B = \left[\frac{1-m}{\varepsilon} \cdot C\right] \times \{m-1\} \in \mathcal{B} \times \mathcal{B}_{\mathbb{R}}.$$

It remains to show that B is M-closed. Suppose that $\{(x_n, m-1)\}_n$ is an M-convergent sequence in B with $\lim^{\flat}{}_{n\to\infty}(x_n, m-1) = (x, r)$, where $x \in E$ and $r \in \mathbb{R}$. We shall show that $(x, r) \in B$.

It follows immediately from Lemma 4.1 that $\lim^{\flat}{}_{n\to\infty} x_n = x$ and $r = m-1$. It is easy to see that $(1-m)/\varepsilon \cdot C$ is M-closed whenever C is M-closed. Thus, $x \in (1-m)/\varepsilon \cdot C$. We are done. □

4.1.2. Caristi-type fixed point theorem

In this section, an extension of the well-known Caristi fixed point theorem [1] to BVS is presented. Indeed, we may better call it Browder–Caristi fixed point theorem: according to Caristi [2], it is a strengthened form suggested by Browder. Though the original theorem handled only single-valued functions, the proof is valid for set-valued functions with *ad hoc* assumptions. Moreover, we would like to remark that Caristi-type fixed point theorem has already been established in different topological spaces in the past decades; for instance, it has been extended to a partially ordered complete metric space by Mizoguchi [14] and to a topological vector space by Qiu [17]. We denote the collection of all nonempty subsets of E by 2^E.

Theorem 4.6. *Let* $(E, \mathcal{B})$ *be a separated convex BVS and* $T : E \to 2^E$ *be a set-valued map. Suppose that there is a subadditive, positively homogeneous function* $P : E \to \mathbb{R} \cup \{+\infty\}$ *with*

$$C = \left\{x \in E:\ P(x) \leqslant 1\right\}$$

being M*-closed and bounded. Suppose also that there are a function* $f : E \to \mathbb{R} \cup \{+\infty\}$ *which is bounded from below and a point* $x_0 \in E$ *such that* $f(x_0) < +\infty$ *and the set*

$$\left\{(x, t) \in E \times \mathbb{R}:\ f(x) \leqslant t \leqslant f(x_0)\right\}$$

is M*-closed. If*

(1) *either* $\{x \in E:\ f(x) \leqslant f(x_0)\}$ *or* C *is* M*-complete, and*
(2) *for all* $x \in E$,

$$P(y - x) \leqslant f(x) - f(y) \quad \text{for some } y \in T(x), \tag{13}$$

then T *has a fixed point, i.e., there is* $x \in E$ *with* $x \in T(x)$.

In addition, if (13) *is strengthened to*

$$P(y - x) \leqslant f(x) - f(y) \quad \text{for all } y \in T(x), \tag{14}$$

then there exists $x \in E$ *such that* $T(x) = \{x\}$.

Proof. Under the given hypothesis, A, C, and P satisfy (EVP1)–(EVP3). Let $\varepsilon \in (0, 1)$ be fixed. In view of Theorem 4.2, we could find x_ε such that (9b) holds.

Suppose that (13) is true. Then take $y \in T(x_\varepsilon)$ so that the inequality in (13) holds. If T has no fixed point, then $y \neq x_\varepsilon$. By (9b),

$$f(y) + \varepsilon P(y - x_\varepsilon) > f(x_\varepsilon).$$

It follows that

$$\varepsilon P(y - x_\varepsilon) > f(x_\varepsilon) - f(y) \geqslant P(y - x_\varepsilon).$$

But it is impossible. Therefore, T must have a fixed point.

Now suppose that (14) is true. If there exists $y \in T(x_\varepsilon) \setminus \{x_\varepsilon\}$, then we may proceed as above and arrive at a contradiction. In other words, $T(x_\varepsilon) \subseteq \{x_\varepsilon\}$. But $T(x_\varepsilon) \neq \emptyset$ implies that $T(x_\varepsilon) = \{x_\varepsilon\}$. $\square$

To end this section, we shall show that Theorems 4.2 and 4.6 are indeed equivalent, as their topological vector space counterpart.

Theorem 4.7. *Ekeland's variational principle* (*Theorem* 4.2) *is equivalent to Caristi fixed point theorem* (*Theorem* 4.6).

Proof. We have seen that Caristi fixed point theorem is a consequence of Ekeland's variational principle. It remains to establish the converse.

Suppose that Ekeland's principle is not true. Then there is $\varepsilon_0 > 0$ such that for all $x \in E$, either

$$f(x) + \varepsilon_0 P(x - x_0) > f(x_0)$$

or there is $y_x \in E \setminus \{x\}$ satisfying

$$f(y_x) \leqslant f(x) - \varepsilon_0 P(y_x - x).$$

Firstly, we set $\tilde{P} = \varepsilon_0 P$. Then $\tilde{P}$ is again a subadditive, positively homogeneous function from E to $\mathbb{R} \cup \{\infty\}$, and the set

$$\tilde{C} = \{x \in E\colon\ \tilde{P}(x) \leqslant 1\} = \frac{1}{\varepsilon_0} \cdot C$$

is M-closed and bounded as well.

Secondly, let

$$S = \{x \in E\colon\ f(x) + \tilde{P}(x - x_0) \leqslant f(x_0)\}.$$

Clearly, $x_0 \in S$. Define $\tilde{f} : E \to \mathbb{R} \cup \{+\infty\}$ by

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in S, \\ +\infty & \text{if } x \in E \setminus S, \end{cases}$$

and $T : E \to 2^E$ by

$$T(x) = \begin{cases} \{y \in E \setminus \{x\}\colon\ f(y) \leqslant f(x) - \tilde{P}(y - x)\} & \text{if } x \in S, \\ \{x_0\} & \text{if } x \in E \setminus S. \end{cases}$$

Since $y_x \in T(x)$ whenever $x \in S$, we have $T(x) \neq \emptyset$ for all $x \in E$. In addition, as $x_0 \in S$, we may conclude that T has no fixed point.

Note that T satisfies (14) with P and f being replaced by $\tilde{P}$ and $\tilde{f}$, respectively. To see this, we divide into two cases: $x \in E \setminus S$ and $x \in S$. When $x \in E \setminus S$, observe that $\tilde{f}(x) = +\infty$ and $T(x) = \{x_0\}$; thus (14) follows immediately. When $x \in S$, observe that $\tilde{f}(x) = f(x)$ is finite, by the definition of S, which in turn implies that $f(y)$ is finite for any $y \in T(x)$ as well. Consequently, (14) holds as claimed.

Further, if

$$\tilde{A} := \{(x,t)\colon\ \tilde{f}(x) \leqslant t \leqslant \tilde{f}(x_0)\} = \{(x,t)\colon\ x \in S,\ f(x) \leqslant t \leqslant f(x_0)\} \tag{15}$$

is M-closed, then T has a fixed point in view of Theorem 4.6. A contradiction arises and we are done.

It remains to prove the M-closedness of $\tilde{A}$ and we shall present its proof in the following lemma. □

Lemma 4.8. *Suppose that all hypotheses of Theorem* 4.2 *hold. Then $\tilde{A}$, defined in* (15)*, is M-closed.*

Proof. We shall continue using the notations of Theorem 4.7 and its proof. Let $(z,s) \in [\tilde{A}]^{(1)}$ and $\{(x_n, t_n)\}$ be a sequence in $\tilde{A}$ such that

$$\lim_{n\to\infty}{}^{\flat}\,(x_n, t_n) = (z, s).$$

As $\{(x_n, t_n)\}$ is also a sequence in A and A is M-closed, we have $(z, s) \in A$. It remains to prove that $z \in S$.

Note that $\{f(x_n)\}$ is bounded above by $f(x_0)$, and since f is bounded below, the sequence $\{f(x_n)\}$ has a convergent subsequence $\{f(x_{n_k})\}$. We put $\lim_{k\to\infty} f(x_{n_k}) = \alpha$. In addition, by definition,

$$f(x_{n_k}) + \tilde{P}(x_{n_k} - x_0) \leqslant f(x_0) \quad \text{for all } k \in \mathbb{N}. \tag{16}$$

Firstly, if there are infinitely many x_{n_k}'s being equal to x_0, then $\lim^\flat{}_{n\to\infty} x_n = x_0$ by Lemma 2.12. Since E is separated, $x_0 = z$ and we are done.

From now on, by resorting to a subsequence if necessary, we shall assume that $x_{n_k} \neq x_0$ for all $k \in \mathbb{N}$. Moreover, we claim that $f(z) \leqslant \alpha$. Suppose not. Then there is $K \in \mathbb{N}$ such that

$$f(x_{n_k}) < \frac{\alpha + f(z)}{2} < f(z) \quad \text{for all } k \geqslant K,$$

whereby $(x_{n_k}, (\alpha + f(z))/2)$ is a sequence in A. But $\lim^\flat{}_{k\to\infty} x_{n_k} = z$ implies that $\lim^\flat{}_{k\to\infty}(x_{n_k}, (\alpha + f(z))/2) = (z, (\alpha + f(z))/2)$. But, $(z, (\alpha + f(z))/2) \notin A$! Contradicting the fact that A is M-closed. It establishes our claim that $f(z) \leqslant \alpha$. We shall now digress to two cases: $\alpha = f(x_0)$ and $\alpha < f(x_0)$.

We treat the case $\alpha = f(x_0)$ first. As $x_{n_k} \neq x_0$, we have $\tilde{P}(x_{n_k} - x_0) > 0$ for all $k \in \mathbb{N}$ (cf. Lemma 4.4); in addition, according to (16), we have $\lim_{k\to\infty} \tilde{P}(x_{n_k} - x_0) = 0$. As a result,

$$x_{n_k} - x_0 = \tilde{P}(x_{n_k} - x_0) \cdot \frac{x_{n_k} - x_0}{\tilde{P}(x_{n_k} - x_0)} \in \tilde{P}(x_{n_k} - x_0) \cdot [\tilde{C}]_{\mathrm{b}},$$

which is balanced and bounded, and hence $\lim^\flat{}_{k\to\infty} x_{n_k} = x_0$. But E is separated implies that $z = x_0 \in S$.

Finally, we treat the case that $\alpha < f(x_0)$. Without loss of generality, we may assume that $f(x_{n_k}) < f(x_0)$ for all $k \in \mathbb{N}$. It follows from (16) that

$$\tilde{P}\left(\frac{x_{n_k} - x_0}{f(x_0) - f(x_{n_k})}\right) \leqslant 1,$$

i.e.,

$$\frac{x_{n_k} - x_0}{f(x_0) - f(x_{n_k})} \in \tilde{C}.$$

However, as

$$\lim_{k\to\infty}{}^\flat \frac{x_{n_k} - x_0}{f(x_0) - f(x_{n_k})} = \lim_{k\to\infty} \frac{1}{f(x_0) - f(x_{n_k})} \cdot \lim_{k\to\infty}{}^\flat (x_{n_k} - x_0) = \frac{z - x_0}{f(x_0) - \alpha}$$

and $\tilde{C}$ is M-closed, we have

$$\tilde{P}\left(\frac{z - x_0}{f(x_0) - \alpha}\right) \leqslant 1.$$

In particular,

$$f(z) + \tilde{P}(z - x_0) \leqslant \alpha + \tilde{P}(z - x_0) \leqslant f(x_0).$$

That is, $z \in S$ as claimed. $\square$

4.2. In locally convex spaces

In this section, we show that how Theorem 3.3 improves some known results in locally convex space (LCS). In the sequel, *all LCS's are Hausdorff.*

Let $(E, \mathcal{T})$ be a LCS and $\{p_i\}_{i \in I}$ be a family of continuous semi-norms generating the topology $\mathcal{T}$. We set

$$\mathcal{B}_T = \{B \subseteq E\colon\ B \text{ is bounded in } (E, \mathcal{T})\}.$$

It could be checked that $\mathcal{B}_T$ is a bornology. We called it the von Neumann bornology associated with $\mathcal{T}$. Recall that $B \subseteq E$ is bounded if for any neighbourhood V of 0 in E, there is $\lambda_0 > 0$ such that $[0, \lambda_0] \cdot B \subseteq V$. Since there is a neighbourhood base at 0 consisting of convex open sets, the convex hull of every $B \in \mathcal{B}_T$, denoted by B_c, remains in $\mathcal{B}_T$ (in sophisticated terms, a bornology having this property is called a convex bornology). As a result, if $B \in \mathcal{B}_T$ is balanced, then B_c is a bounded disk.

The following lemma demonstrates that the class of locally convergent/locally Cauchy sequences in $(E, \mathcal{T})$ and the class of all M-convergent/M-Cauchy sequence in $(E, \mathcal{B}_T)$ are the same.

Lemma 4.9. *Let $(E, \mathcal{T})$ be a LCS and $\mathcal{B}_T$ be the associated von Neumann bornology. A sequence $\{x_n\}$ in $(E, \mathcal{T})$ is locally convergent if and only if it is M-convergent in $(E, \mathcal{B}_T)$; and it is locally Cauchy if and only if it is M-Cauchy in $(E, \mathcal{B}_T)$.*

Proof. Suppose that $\{x_n\}$ is locally convergent to x_0. By definition, there is a bounded disk B such that $\lim_{n\to\infty} p_B(x_n - x_0) = 0$. Note also that, for every $n \in \mathbb{N}$, $x_n - x_0 \in [(1 + 1/n) \times p_B(x_n - x_0)] \cdot B$. Since $\lim_{n\to\infty}[(1+1/n)p_B(x_n - x_0)] = 0$, we conclude that $\lim^{\flat}{}_{n\to\infty} x_n = x_0$. Conversely, suppose that $\lim^{\flat}{}_{n\to\infty} x_n = x_0$. So we could find a balanced bounded set B and a sequence of positive real numbers $\{\lambda_n\}$ such that $\lim_{n\to\infty} \lambda_n = 0$ and $x_n - x_0 \in \lambda_n \cdot B \subseteq \lambda_n \cdot B_c$. Recall that B_c is a bounded disk. Clearly,

$$0 \leqslant p_{B_c}(x_n - x_0) \leqslant \lambda_n, \quad \text{for all } n \in \mathbb{N}.$$

By the Sandwich principle, $\lim_{n\to\infty} p_{B_c}(x_n - x_0) = 0$. Therefore, $\{x_n\}$ is locally convergent to x_0. The equivalence between locally Cauchy and M-Cauchy sequences is analogous and is therefore omitted. □

We are now able to state a general drop theorem in locally convex spaces, which is a direct consequence of our main theorem (Theorem 3.3).

Theorem 4.10. *Let $(E, \mathcal{T})$ be a LCS and A be a locally closed subset of E. If $B \in \mathcal{B}$ is locally-closed and convex, $0 \notin (A - B)^{(1)}$, and A is drop-Mackey-complete relative to B, then for all $x_0 \in A$, there is $a \in D(x_0, B) \cap A$ such that $D(a, B) \cap A = \{a\}$.*

Proof. In view of Remark 2.9 and Lemma 4.9, it is simply a restatement of Theorem 3.3. □

Since there are both a topological structure and a bornological structure on E, a sequence could be convergent topologically or bornologically. The following lemma tells us how these convergence concepts are related. Note that it works for arbitrary topological vector spaces.

Lemma 4.11. *Let $(E,\mathcal{T})$ be a topological vector space. Then, every M-Cauchy sequence in $(E,\mathcal{B}_T)$ is Cauchy and every M-convergent sequence is convergent. As a result, if $A \subseteq E$ is closed, then it is M-closed.*

Proof. Let $\{x_n\}$ be an M-Cauchy sequence. Then there is a balanced $B \in \mathcal{B}_T$ and a double sequence of real numbers $\{\lambda_{n,m}\}$ such that $\lim_{n,m\to\infty} \lambda_{n,m} = 0$ and

$$x_n - x_m \in \lambda_{n,m} \cdot B.$$

Suppose that V is a neighbourhood of 0_E. Then, as B is bounded, there is $\alpha > 0$ such that

$$[0,\alpha] \cdot B \subseteq V.$$

For this $\alpha > 0$, there are $N, M \in \mathbb{N}$ such that $|\lambda_{n,m}| < \alpha$ whenever $n \geqslant N$ and $m \geqslant M$. In other words, $x_n - x_m \in V$ whenever $n \geqslant N$ and $m \geqslant M$. As a result, $\{x_n\}$ is Cauchy. In case that $\{x_n\}$ is M-convergent, the proof is similar and is omitted.

Finally, let $\{x_n\}$ be a M-convergent sequence in A. By the above discussion, $\{x_n\}$ will be convergent. Since A is closed, we are done. □

Before we show that Theorem 1.2 by Qiu is a particular case of Theorem 4.10, however, we would like to add one remark on an assumption therein, namely, $0 \notin \mathrm{cl}_\tau(A - B)$ for some locally convex topology τ on E. One should be careful that τ is not completely arbitrary but with which A is kept to be locally closed while B is kept to be locally closed and locally bounded in (E,τ) as well.

In fact, it is the content of Mackey's theorem [12] which gives a precise description of all such locally convex topology.

Lemma 4.12. *Let $(E,\mathcal{T})$ be a LCS and E^* be its topological dual. Now suppose that τ is a topology of the pair (E,E^*), i.e., (E,τ) is locally convex and $(E,\tau)^* = E^*$; then*

(1) *B is convex implies that* $\mathrm{cl}_{\mathcal{T}} B = \mathrm{cl}_\tau B$ *and*
(2) *$B \in \mathcal{B}_T$ if and only if $B \in \mathcal{B}_\tau$.*

In addition, a locally convex topology τ on E is a topology of the pair of (E,E^*) if and only if

$$\sigma(E,E^*) \subseteq \tau \subseteq \tau(E,E^*),$$

where $\sigma(E,E^*)$ and $\tau(E,E^*)$ denote the weak topology and Mackey topology on E, respectively.

Now, we are ready to propose a version of Theorem 1.2.

Theorem 4.13. *Let A be a locally closed subset of a LCS $(E,\mathcal{T})$ and B be a locally closed, bounded, convex subset of E. Moreover, assume that $0 \notin \mathrm{cl}_\tau(A - B)$ for some locally convex topology τ lying between $\sigma(E,E^*)$ and $\tau(E,E^*)$. Then for any $x_0 \in A$, there exists $a \in D(x_0,B) \cap A$ such that $D(x_0,B) \cap A = \{a\}$ provided that either* (Q1) *or* (Q2) *is satisfied.*

Proof. By definition, $B \in \mathcal{B}_T$. Moreover, A, B are M-closed by Lemma 4.9 while $E \setminus (A - B)$ is bornivorous according to Remark 2.9.

We claim that either one of (Q1) and (Q2) will imply that A is drop-Mackey-complete relative to B. Then our assertion will follow from Theorem 4.10 immediately.

We proceed to prove that A is drop-Mackey-complete relative to B. Let $\{a_n\}$ be a M-Cauchy sequence in A with $a_{n+1} \in D(a_n, B)$ for all $n \in \mathbb{N}$. By Lemma 4.9, it is locally Cauchy.

Clearly, if (Q2) holds, it must be locally convergent and hence is M-convergent according to Lemma 4.9.

Now suppose that (Q1) holds. Note that $a_{n+1} \in D(a_n, B)$ implies that there are $t_n \in [0, 1]$ and $b_n \in B$ such that

$$a_{n+1} = (1 - t_n)a_n + t_n b_n$$

for all $n \in \mathbb{N}$.

If there is a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that

$$a_{n_1} = a_{n_2} = \cdots = a_{n_k} = \cdots,$$

then we are done by Lemma 2.12. So, by considering a subsequence if necessary, we may assume that $a_{n+1} \neq a_n$ for all $n \in \mathbb{N}$. It follows that $t_n \neq 0$ for all $n \in \mathbb{N}$. As a result,

$$b_n = \frac{1}{t_n}a_{n+1} - \frac{1 - t_n}{t_n}a_n \in B \cap L(A)$$

and hence, by Lemma 2.2(2),

$$a_{n+1} \in D\big(a_n, B \cap L(A)\big) \subseteq \cdots \subseteq D\big(a_1, B \cap L(A)\big) \subseteq D\big(a_1, \big[B \cap L(A)\big]^{(1)}\big)$$

for all $n \in \mathbb{N}$. In other words, $\{a_n\}$ is a M-Cauchy sequence in $D(a_1, [B \cap L(A)]^{(1)})$.

As $B \cap L(A) \subseteq B$, we have $[B \cap L(A)]^{(1)} \subseteq B^{(1)} = B$ as B is M-closed. Therefore $[B \cap L(A)]^{(1)}$ is bounded by (B2). We claim that $[B \cap L(A)]^{(1)}$ is convex. To prove this claim, we let $z_1, z_2 \in [B \cap L(A)]^{(1)}$, $\alpha \in (0, 1)$; and we shall show that $(1 - \alpha)z_1 + \alpha z_2 \in [B \cap L(A)]^{(1)}$. Let $\{u_n\}$ and $\{v_n\}$ be sequences in $B \cap L(A)$ with $z_1 = \lim^\flat{}_{n\to\infty} u_n$ and $z_2 = \lim^\flat{}_{n\to\infty} v_n$. Since $B \cap L(A)$ is convex, $\{(1 - \alpha)u_n + \alpha v_n\}$ is a sequence in $B \cap L(A)$ and

$$\lim_{n\to\infty}{}^\flat \big[(1 - \alpha)u_n + \alpha v_n\big] = (1 - \alpha)\lim_{n\to\infty}{}^\flat u_n + \alpha \lim_{n\to\infty}{}^\flat v_n = (1 - \alpha)z_1 + \alpha z_2.$$

In conclusion, $[B \cap L(A)]^{(1)}$ is bounded and convex, which is also M-complete by (Q1). We can then apply Lemma 2.11 and conclude that $D(a_1, [B \cap L(A)]^{(1)})$ is M-complete. Consequently, $\{a_n\}$ is M-convergent. Finally, $\lim^\flat{}_{n\to\infty} a_n \in A$ as A is M-closed. Therefore, A is drop-Mackey-complete relative to B as claimed. □

We end this section with a variant of drop theorem in LCS's, which asserts that the drop property will be valid on a whole spectrum of LCS's whenever it holds somewhere between the weak topology and the Mackey topology.

Theorem 4.14. *Let $(E, \mathcal{T})$ be a LCS and A be an M-closed subset of $(E, \mathcal{B}_T)$. If $B \in \mathcal{B}_T$ is $\mathcal{T}'$-closed, convex and $0 \notin \mathrm{cl}_{\mathcal{T}'}(A - B)$ and A is drop-Mackey-complete relative to B, where $\mathcal{T}'$ denotes a locally convex topology on E with $\mathcal{T} \subseteq \mathcal{T}' \subseteq \tau(E, E^*)$. Then for any $x_0 \in A$, there is $a \in D(x_0, B) \cap A$ such that $D(a, B) \cap A = \{a\}$.*

Proof. Firstly, according to Theorem 4.12, $B \in \mathcal{B}_{T'}$ and A is M-closed as well as drop-Mackey-complete relative to B in $(E, \mathcal{T}')$. Secondly, $\mathcal{T} \subseteq \mathcal{T}'$ implied that $0 \notin \mathrm{cl}_{\mathcal{T}'}(A - B)$. Lastly, B is M-closed by Lemma 4.11. By Theorem 4.13, we are done. □

Remark 4.15. In view of Lemma 4.9, one could easily deduce the topological counterparts of Theorems 4.2, 4.6 and 4.7 in a locally complete LCS (see also a recent paper by Qiu [17]).

5. Conclusion

In this work, Daneš' drop theorem is extended to bornological vector spaces. On the one hand, an Ekeland-type variational principle, and hence the equivalent Caristi-type fixed point theorem, is proved to be valid in general bornological vector spaces under some mild completeness conditions. On the other hand, this suggests the possibilities to extend several well-known results in analysis beyond sequentially complete spaces.

One motivation to this project is the infinite-dimensional calculus developed by Kriegl and Michor [13], which depends only on the von Neumann bornology of the underlying locally convex space. In other words, we have developed tools available in a very general class of locally convex spaces, namely, convenient spaces (see [13]). Our work may shed new lights on analysis (for example, optimization problems) in infinite-dimensional spaces beyond Fréchet spaces (see [20]).

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