

A regularity classification of boundary points for p -harmonic functions and quasiminimizers

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Abstract

In this paper it is shown that irregular boundary points for p -harmonic functions as well as for quasiminimizers can be divided into semiregular and strongly irregular points with vastly different boundary behaviour. This division is emphasized by a large number of characterizations of semiregular points. The results hold in complete metric spaces equipped with a doubling measure supporting a Poincaré inequality. They also apply to Cheeger p -harmonic functions and in the Euclidean setting to $\mathcal{A}$ -harmonic functions, with the usual assumptions on $\mathcal{A}$.

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1. Introduction

Let Ω be a nonempty bounded open subset of $\mathbf{R}^n$, $n \geq 2$, and $1 < p < \infty$. We follow Heinonen–Kilpeläinen–Martio [11] in the study of p -harmonic functions. For $f \in C(\partial\Omega)$, the Perron method provides a unique solution Pf (denoted H_f in [11]) of the Dirichlet (boundary value) problem, i.e. Pf is p -harmonic in Ω and takes the boundary values f in a weak sense. A point $x_0 \in \partial\Omega$ is said to be *regular* if

$$\lim_{\Omega \ni y \rightarrow x_0} Pf(y) = f(x_0)$$

for every $f \in C(\partial\Omega)$, and *irregular* otherwise.

We can rephrase this in the following way: A point $x_0 \in \partial\Omega$ is regular if the following two conditions hold:

(a) for all $f \in C(\partial\Omega)$ the limit

$$\lim_{\Omega \ni y \rightarrow x_0} Pf(y) \text{ exists;}$$

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(b) for all $f \in C(\partial\Omega)$ there is a sequence $\{y_j\}_{j=1}^\infty$ such that

$$\Omega \ni y_j \rightarrow x_0 \quad \text{and} \quad Pf(y_j) \rightarrow f(x_0), \quad \text{as } j \rightarrow \infty.$$

It turns out that for irregular points *exactly one* of these two properties fails; a priori one would assume that it is possible that both fail but this can never happen. We will say that x_0 is *semiregular* if (a) holds but not (b); and *strongly irregular* if (b) holds but not (a).

There has been much written on the dichotomy between regular and irregular points, in particular in the linear case. In this paper we promote the *trichotomy* between regular, semiregular and strongly irregular points.

The importance of the distinction between semiregular and strongly irregular points is perhaps best illustrated by the equivalent characterizations of semiregular points in Theorem 3.3. Similar characterizations of regular points have been provided in Björn–Björn [4]. Characterizations of strongly irregular points are easily deduced from the characterizations of regular and semiregular points.

The distinction between semiregular and strongly irregular points has been used in some papers in the literature. In the linear case the trichotomy was developed in detail in Lukeš–Malý [15]. The author is not aware of any paper in which this distinction has been fully developed in the nonlinear case, not even in $\mathbf{R}^2$.

In this paper we show that the trichotomy can be deduced from two basic results, the Kellogg property and a removability result. In [15] they deduce the trichotomy (and more) using considerably more theory.

After the preprint version of this paper was written Martio used and referred to this classification in [16] where he primarily studied the boundary behaviour at strongly irregular points in the borderline case $p = n$.

The first example of an irregular point was given by Zaremba [17] in (1911), in which he showed that the centre of a punctured disk is irregular. This is an example of a semiregular point. Shortly afterwards, Lebesgue [14] gave his famous example of the Lebesgue spine; an example of a strongly irregular point.

In Section 2 we give a simple proof of the trichotomy. In Section 3 we give various characterizations of semiregular points and of sets consisting entirely of semiregular points, this is done in terms of Perron solutions, removability, semibarriers and the obstacle problem. It is also shown that semiregularity is a local property of the boundary. In Section 4 we show that the sets of semiregular and of strongly irregular boundary points are similar in size. In Sections 5 and 6 we look at the corresponding results in weighted $\mathbf{R}^n$ and metric spaces as well as for quasiminimizers.

2. The trichotomy

To help the understanding of the underlying principles, in this section we concentrate on showing the trichotomy. Theorem 2.1 is a special case of Theorem 3.3, and the proof of Theorem 3.3 gives an independent proof of Theorem 2.1.

Theorem 2.1. *Let $1 < p < \infty$, let Ω be a nonempty bounded open subset of (unweighted) $\mathbf{R}^n$, $n \geq 2$, and let $x_0 \in \partial\Omega$. Then x_0 is either regular, semiregular or strongly irregular.*

Proof. We distinguish two cases.

Case 1. There is $r > 0$ such that $C_p(B \cap \partial\Omega) = 0$, where $B = B(x_0, r) := \{x \in \mathbf{R}^n: |x - x_0| < r\}$ and C_p is the Sobolev capacity, see Section 2.35 in Heinonen–Kilpeläinen–Martio [11].

Since sets of zero capacity cannot separate sets, which follows, e.g., from the proof of Lemma 2.46 in [11] (or Lemma 8.6 in Björn–Björn–Shanmugalingam [7]), we must have $B \subset \overline{\Omega}$ and thus $C_p(B \setminus \Omega) = C_p(B \cap \partial\Omega) = 0$. Let $f \in C(\partial\Omega)$. By Theorem 7.36 in [11] (or Theorem 6.2 in Björn [1]), the Perron solution Pf has a p -harmonic extension U to $\Omega \cup B$. Since U is continuous we have

$$\lim_{\Omega \ni y \rightarrow x_0} Pf(y) = U(x_0),$$

i.e. (a) holds and x_0 is either regular or semiregular.

Case 2. The capacity $C_p(B(x_0, r) \cap \partial\Omega) > 0$ for all $r > 0$. (Note that this is complementary to Case 1.)

For every $j = 1, 2, \dots$, we thus have $C_p(B(x_0, 1/j) \cap \partial\Omega) > 0$, and by the Kellogg property (Theorem 9.11 in [11] or Theorem 3.9 in Björn–Björn–Shanmugalingam [6]) there is a regular boundary point $x_j \in B(x_0, 1/j) \cap \partial\Omega$. (We do not require the x_j to be distinct.) Let $f \in C(\partial\Omega)$. Since x_j is regular we can find $y_j \in B(x_j, 1/j) \cap \Omega$ so that

$|Pf(y_j) - f(x_j)| < 1/j$. It follows directly that $y_j \rightarrow x_0$ and $Pf(y_j) \rightarrow f(x_0)$, i.e. (b) holds, and thus x_0 is either regular or strongly irregular. $\square$

With a little extra work one can actually say more. In Case 1 one can deduce that x_0 is always semiregular: This is most easily obtained using Corollary 6.2 in Björn–Björn–Shanmugalingam [7], see the proof of (d') $\Rightarrow$ (a') in Theorem 3.1.

We observe that semiregular points are only obtained in the first case, and thus the relatively open set

$$S = \{x \in \partial\Omega : \text{there is } r > 0 \text{ such that } C_p(B(x, r) \cap \partial\Omega) = 0\} \quad (2.1)$$

consists exactly of all semiregular boundary points. On the other hand the closed set $\partial\Omega \setminus S$ consists of all points which are either regular or strongly irregular, and is moreover the closure of the set of all regular boundary points.

Also in Case 2 it is possible to improve upon the result above. Namely one can show that the sequence $\{y_j\}_{j=1}^\infty$ can be chosen independently of f , see the proof of $\neg(e) \Rightarrow \neg(c)$ in Theorem 3.3.

3. Characterizations of semiregular points

Assume in this section that Ω is a nonempty bounded open subset of (unweighted) $\mathbf{R}^n$, $n \geq 2$, and that $1 < p \leq n$. Let also μ denote the Lebesgue measure on $\mathbf{R}^n$.

Theorem 3.1. *Let $V \subset \partial\Omega$ be relatively open. Then the following are equivalent:*

- (a') *The set V consists entirely of semiregular points.*
- (b') *The set V does not contain any regular point.*
- (c') *It is true that $C_p(V) = 0$.*
- (d') *The set $\Omega \cup V$ is open and every bounded p -harmonic function on Ω has a p -harmonic extension to $\Omega \cup V$.*
- (e') *The set $\Omega \cup V$ is open, $\mu(V) = 0$, and every bounded p -superharmonic function on Ω has a p -superharmonic extension to $\Omega \cup V$.*
- (f') *For $f \in C(\partial\Omega)$, the Perron solution Pf depends only on $f|_{\partial\Omega \setminus V}$ (i.e. if $f, h \in C(\partial\Omega)$ and $f = h$ on $\partial\Omega \setminus V$, then $Pf \equiv Ph$).*

The equivalence of (b') and (c') as well as of the vanishing of the p -harmonic measure of V was proved in Björn–Björn–Shanmugalingam [6, Proposition 9.1].

Together with the implication (a) $\Rightarrow$ (f) in Theorem 3.3 this theorem shows that the set S of all semiregular boundary points is a relatively open set which can be characterized as the largest relatively open subset of $\partial\Omega$ having any of the properties above, or, e.g., by (2.1).

By (d') we also see that S is contained in the interior of $\overline{\Omega}$, i.e., $S \subset \partial\Omega \setminus \partial\overline{\Omega}$. Note however that it can happen that $S \neq \partial\Omega \setminus \partial\overline{\Omega}$: for $p = 2$ and $\Omega = B(0, 2) \setminus ([-1, 0] \cup \{1\})$ (using complex notation in $\mathbf{R}^2 = \mathbf{C}$) we have $S = \{1\}$ but $\partial\Omega \setminus \partial\overline{\Omega} = [-1, 0] \cup \{1\}$.

Proof. (a') $\Rightarrow$ (b') This is trivial.

(b') $\Rightarrow$ (c') This follows directly from the Kellogg property (Theorem 9.11 in Heinonen–Kilpeläinen–Martio [11] or Theorem 3.9 in Björn–Björn–Shanmugalingam [6]).

(c') $\Rightarrow$ (e') Sets of capacity zero cannot separate space, see the proof of Lemma 2.46 in [11] (or Lemma 8.6 in Björn–Björn–Shanmugalingam [7]), and hence $\Omega \cup V$ must be open. That $\mu(V) = 0$ follows directly from the fact that $C_p(V) = 0$. The extension is now provided by Theorem 7.35 in [11] (or by Theorem 6.3 in Björn [1]).

(e') $\Rightarrow$ (d') The first part is clear. Let u be a bounded p -harmonic function on Ω . By assumption u has a p -superharmonic extension U to $\Omega \cup V$. Also $-u$ has a p -superharmonic extension W to $\Omega \cup V$. Thus $-W$ is a p -subharmonic extension of u to $\Omega \cup V$. By Proposition 6.5 in Björn [1], $U = -W$ is p -harmonic.

(d') $\Rightarrow$ (a') Let $x_0 \in V$. Let $f \in C(\partial\Omega)$. Then Pf has a p -harmonic extension U to $\Omega \cup V$. It follows that

$$\lim_{\Omega \ni y \rightarrow x_0} Pf(y) = U(x_0),$$

and thus the limit in the left-hand side always exists.

It remains to show that x_0 is irregular. Let $h(x) = \text{dist}(x, \partial\Omega \setminus V)$ and let U be a p -harmonic extension of Ph to $\Omega \cup V$. By continuity and the Kellogg property we have

$$\lim_{\Omega \cup V \ni y \rightarrow x} U(y) = \lim_{\Omega \ni y \rightarrow x} Ph(y) = 0 \quad \text{for q.e. } x \in \partial(\Omega \cup V).$$

Corollary 6.2 in Björn–Björn–Shanmugalingam [7], shows that $U = P_{\Omega \cup V} 0 \equiv 0$. As $h(x_0) > 0$ we conclude that x_0 is irregular.

(c') $\Rightarrow$ (f') This follows from Theorem 6.1 in [7].

(f') $\Rightarrow$ (b') Let $f \equiv 0$ and $h(x) = \text{dist}(x, \partial\Omega \setminus V)$. By assumption $Ph = Pf \equiv 0$, but as $h(x) \neq 0$ for $x \in V$, we see that there is no regular point in V . $\square$

In Björn–Björn [4, Theorem 4.2], it was shown that the existence of a barrier is equivalent to regularity of a boundary point. Also weak barriers were discussed in the end of Section 6 in [4] and in Björn [3]. Here we introduce semibarriers and weak semibarriers (the latter are called weak barriers in Lukeš–Malý [15, Corollary 9]). Below we show that the existence of a semibarrier or of a weak semibarrier for a boundary point is equivalent to the fact that the point is *not* semiregular.

Definition 3.2. A function u is a *semibarrier* (with respect to Ω) at $x_0 \in \partial\Omega$ if

- (i) u is p -superharmonic in Ω ;
- (ii) $\liminf_{\Omega \ni y \rightarrow x} u(y) > 0$ for every $x \in \partial\Omega \setminus \{x_0\}$;
- (iii) $\liminf_{\Omega \ni y \rightarrow x_0} u(y) = 0$.

A function u is a *weak semibarrier* (with respect to Ω) at $x_0 \in \partial\Omega$ if it is a positive p -superharmonic function in Ω such that (iii) holds.

Barriers (of all types) are often defined in a local way, but since we obtain the localness of regularity in other ways, we prefer global definitions; the equivalence of the local definitions considered elsewhere follows directly using that (semi)regularity is a local property.

In $\mathbf{R}^n$ it is clear that every semibarrier is a weak semibarrier.

Theorem 3.3. Let $x_0 \in \partial\Omega$, $\delta > 0$ and $d(y) = |y - x_0|$. Then the following are equivalent:

- (a) The point x_0 is semiregular.
- (b) The point x_0 is semiregular with respect to $G := \Omega \cap B(x_0, \delta)$.
- (c) There is no sequence $\{y_j\}_{j=1}^\infty$ such that $\Omega \ni y_j \rightarrow x_0$, as $j \rightarrow \infty$, and

$$\lim_{j \rightarrow \infty} Pf(y_j) = f(x_0) \quad \text{for all } f \in C(\partial\Omega).$$

- (d) The point x_0 is not regular nor strongly irregular.
- (e) It is true that $x_0 \notin \overline{\{x \in \partial\Omega : x \text{ is regular}\}}$.
- (f) There is a neighbourhood V of x_0 such that $C_p(V \cap \partial\Omega) = 0$.
- (g) There is a neighbourhood V of x_0 such that $C_p(V \setminus \Omega) = 0$.
- (h) There is a neighbourhood V of x_0 such that every bounded p -harmonic function on Ω has a p -harmonic extension to $\Omega \cup V$.
- (i) There is a neighbourhood V of x_0 such that every bounded p -superharmonic function on Ω has a p -superharmonic extension to $\Omega \cup V$, and $\mu(V \setminus \Omega) = 0$.
- (j) There is a neighbourhood V of x_0 such that for $f \in C(\partial\Omega)$, the Perron solution Pf depends only on $f|_{\partial\Omega \setminus V}$.
- (k) It is true that

$$\liminf_{\Omega \ni y \rightarrow x_0} Pd(y) > 0.$$

- (l) There is no weak semibarrier at x_0 .
- (m) The continuous solution u of the $\mathcal{K}_{d,d}$ -obstacle problem is not a semibarrier at x_0 .

For the obstacle problem, we refer the reader to Section 3.19 in Heinonen–Kilpeläinen–Martio [11] (or Section 3 in Kinnunen–Martio [13]).

Proof. (e) $\Leftrightarrow$ (f) $\Leftrightarrow$ (j) $\Rightarrow$ (a) This follows directly from Theorem 3.1, with V in Theorem 3.1 corresponding to $V \cap \partial\Omega$ here.

(a) $\Rightarrow$ (k) The limit

$$c := \lim_{\Omega \ni y \rightarrow x_0} Pd(y)$$

exists. If c were 0, then x_0 would be regular, by Theorem 4.2 in Björn–Björn [4], a contradiction. Thus $c > 0$.

(k) $\Rightarrow$ (d) $\Rightarrow$ (c) This is trivial.

$\neg$ (e) $\Rightarrow$ $\neg$ (c) For each $j \geq 1$, $B(x_0, 1/j) \cap \partial\Omega$ contains a regular boundary point x_j . Let $f_j(x) = \max\{\min\{2 - jd(x), 1\}, 0\}$. Then we can find $y_j \in B(x_j, 1/j) \cap \Omega$ so that

$$\frac{1}{j} > |f_j(x_j) - Pf_j(y_j)| = |1 - Pf_j(y_j)|.$$

Then $y_j \rightarrow x_0$ and $Pf_j(y_j) \rightarrow 1$, as $j \rightarrow \infty$.

Let now $f \in C(\partial\Omega)$. Without loss of generality we may assume that $0 \leq f \leq 2$ and that $f(x_0) = 1$. Let $\varepsilon > 0$. Then we can find k such that

$$f \geq 1 - \varepsilon \quad \text{on } B(x_0, 2/k) \cap \partial\Omega.$$

It follows that $f \geq f_j - \varepsilon$ for $j \geq k$, and thus

$$\liminf_{j \rightarrow \infty} Pf(y_j) \geq \liminf_{j \rightarrow \infty} Pf_j(y_j) - \varepsilon = 1 - \varepsilon.$$

Letting $\varepsilon \rightarrow 0$ gives $\liminf_{j \rightarrow \infty} Pf(y_j) \geq 1$. Applying this to $\tilde{f} := 2 - f$ instead gives $\limsup_{j \rightarrow \infty} Pf(y_j) \leq 1$, and the implication is proved.

(f) $\Leftrightarrow$ (b) Note first that (f) is equivalent to the existence of a neighbourhood W of x_0 with $C_p(W \cap \partial G) = 0$. But this is equivalent to (b), by the already proved (f) $\Leftrightarrow$ (a) applied to G instead of Ω .

(f) $\Rightarrow$ (g) By Theorem 3.1, (c') $\Rightarrow$ (e'), the set $\Omega \cup (V \cap \partial\Omega)$ is open, and we can use $V \cap \partial\Omega$ as our set V in (g).

(g) $\Rightarrow$ (f) This is trivial.

(g) $\Rightarrow$ (i) $\Rightarrow$ (h) Both in (g) and (i) it follows directly that $V \subset \overline{\Omega}$. Thus the implications follow directly from Theorem 3.1, with V in Theorem 3.1 corresponding to $V \cap \partial\Omega$ here.

(h) $\Rightarrow$ (g) We may assume that V is connected (replace otherwise V by the component of V containing x_0). Proposition 7.5 in Björn [1] now shows that $C_p(V \setminus \Omega) = 0$.

(i) $\Rightarrow$ (l) Let u be a positive p -superharmonic function on Ω . Then $\min\{u, 1\}$ has a p -superharmonic extension U to $\Omega \cup V$. Since U is lower semicontinuously regularized, see Theorem 7.22 in Heinonen–Kilpeläinen–Martio [11] (or Theorem 7.14 in Kinnunen–Martio [13]) and $\mu(V \setminus \Omega) = 0$, it follows that $U \geq 0$ in $\Omega \cup V$. If $U(x_0)$ were 0, then it would follow from the minimum principle that $U \equiv 0$ in the component of $\Omega \cup V$ containing x_0 , but this would contradict the fact that u is positive in Ω . Thus

$$0 < U(x_0) \leq \liminf_{\Omega \ni y \rightarrow x_0} u(x_0),$$

and hence there is no weak semibarrier.

(l) $\Rightarrow$ (m) This is trivial.

$\neg$ (e) $\Rightarrow$ $\neg$ (m) It is clear that u satisfies (i) and (ii) in Definition 3.2.

Let $\{x_j\}_{j=1}^\infty$ be a sequence of regular boundary points such that $d(x_j) < 1/j$. By Corollary 5.2 in Björn–Björn [4], $\lim_{\Omega \ni y \rightarrow x_j} u(y) = d(x_j)$. We can therefore find $y_j \in B(x_j, 1/j) \cap \Omega$ such that $u(y_j) < 2/j$. Thus

$$0 \leq \liminf_{\Omega \ni y \rightarrow x_0} u(y) \leq \liminf_{j \rightarrow \infty} u(y_j) = 0. \quad \square$$

4. The sets of semiregular and of strongly irregular points

Assume in this section that Ω is a nonempty bounded open subset of (unweighted) $\mathbf{R}^n$ and that $1 < p < \infty$.

Let us consider the partition of $\partial\Omega$ into

$$R = \{x \in \partial\Omega: x \text{ is regular}\},$$

$$S = \{x \in \partial\Omega: x \text{ is semiregular}\},$$

$$I = \{x \in \partial\Omega: x \text{ is strongly irregular}\}.$$

By the Kellogg property, $C_p(S) = C_p(I) = 0$, and thus R is the significantly largest of these three sets. It is natural to ask if one can compare the sizes of S and I . For any given compact K with $C_p(K) = 0$, if we let B be a ball containing K and let $\Omega = B \setminus K$, then $S = K$ and $I = \emptyset$. On the other hand the Lebesgue spine, see Lebesgue [14], shows that we can have S empty and I being a point. As strong irregularity is a local property it is also possible to have S empty and I countable. It is less obvious that I can equal any prescribed compact set with zero capacity. But, indeed this is possible as we show below, and thus one can really say that the sizes of S and I are similar.

Theorem 4.1. *Let S and I be two compact disjoint sets with zero capacity. Then there is an open set Ω with $S \cup I \subset \partial\Omega$ and such that*

$$S = \{x \in \partial\Omega: x \text{ is semiregular}\},$$

$$I = \{x \in \partial\Omega: x \text{ is strongly irregular}\}.$$

To prove this we will need the following result, where cap_p is the variational capacity introduced on p. 28 in Heinonen–Kilpeläinen–Martio [11], and μ is the Lebesgue measure on $\mathbf{R}^n$.

Theorem 4.2. *Let $x_0 \in \partial\Omega$. If*

$$\int_0^1 \left(\frac{\text{cap}_p(B(x_0, r) \setminus \Omega, B(x_0, 2r))}{r^{-p} \mu(B(x_0, r))} \right)^{1/p} \frac{dr}{r} < \infty,$$

then x_0 is irregular.

Theorem 4.2 follows directly from the Wiener criterion obtained by Kilpeläinen–Malý [12] (or results by J. Björn [9,10]).

Proof of Theorem 4.1. We have already observed that this result is true when I is empty. Also, if $p > n$, then only the empty set has capacity zero and there is nothing to prove. We therefore assume that $I \neq \emptyset$ and that $1 < p \leq n$ below. In particular singleton sets have capacity zero.

Let B be a ball containing $S \cup I$. Assume without loss of generality that $\text{dist}(I, S \cup \partial B) > 2$. Observe that, as balls have positive capacity, $I \subset \overline{B \setminus I}$. We can therefore find a sequence of points $\{x_j\}_{j=1}^\infty$ in $B \setminus I$, such that $\delta_j := \text{dist}(x_j, I) < \frac{1}{2} \delta_{j-1}$, $j = 1, 2, \dots$, where $\delta_0 := 1$, and such that $I = \bigcap_{k=1}^\infty \overline{\{x_j: j \geq k\}}$.

Since $\text{cap}_p(\{x_j\}, B(x_j, \frac{1}{2} \delta_j)) = 0$, we can find $r_j < \frac{1}{4} \delta_j$, $j = 1, 2, \dots$, such that

$$\text{cap}_p \left(B(x_j, r_j), B \left(x_j, \frac{1}{2} \delta_j \right) \right) \leq \delta_j^n. \quad (4.1)$$

Next we use Corollary 6.32 in Heinonen–Kilpeläinen–Martio [11] (or Theorem 1.1 in Björn–Björn [5]) to find bounded regular sets Ω_j such that $B \setminus B(x_j, r_j) \subset \Omega_j \subset \mathbf{R}^n \setminus \{x_j\}$. Let $F_j = B(x_j, r_j) \setminus \Omega_j$ and

$$\Omega = B \setminus \left(S \cup I \cup \bigcup_{j=1}^\infty F_j \right).$$

We shall show that Ω has the required properties.

The boundary $\partial\Omega$ consists of the (pairwise disjoint) pieces ∂B , S , I , ∂F_1 , $\partial F_2, \dots$. These sets are moreover separated from each other, except for that $I \subset \overline{\bigcup_{j=1}^{\infty} \partial F_j}$. As regularity is a local property we see directly that ∂B , ∂F_1 , $\partial F_2, \dots$ consist entirely of regular boundary points (with respect to Ω). Using Theorem 3.1 we also see that S consists entirely of semiregular points.

Let $x_0 \in I \subset \overline{\bigcup_{j=1}^{\infty} \partial F_j}$. As x_0 is in the closure of the set of regular boundary points, Theorem 3.3 shows that x_0 is either regular or strongly irregular, and it is enough to show that x_0 is irregular. We first make the following estimate for $r < 1$,

$$\begin{aligned} \text{cap}_p(B(x_0, r) \setminus \Omega, B(x_0, 2r)) &\leq \text{cap}_p\left(B(x_0, r) \cap \bigcup_{|x_j - x_0| \leq 4r/3} B(x_j, r_j), B(x_0, 2r)\right) \\ &\leq \sum_{|x_j - x_0| \leq 4r/3} \text{cap}_p(B(x_0, r) \cap B(x_j, r_j), B(x_0, 2r)) \\ &\leq \sum_{\delta_j \leq 4r/3} \text{cap}_p\left(B(x_j, r_j), B\left(x_j, \frac{1}{2}\delta_j\right)\right) \\ &\leq \sum_{\delta_j \leq 4r/3} \delta_j^n \end{aligned} \quad (4.2)$$

$$\leq 2\left(\frac{4r}{3}\right)^n. \quad (4.3)$$

It thus follows that

$$\int_0^1 \left(\frac{\text{cap}_p(B(x_0, r) \setminus \Omega, B(x_0, 2r))}{r^{-p}\mu(B(x_0, r))} \right)^{1/p} \frac{dr}{r} \leq C \int_0^1 \left(\frac{r^n}{r^{-p}r^n} \right)^{1/p} \frac{dr}{r} = C < \infty. \quad (4.4)$$

By Theorem 4.2, x_0 is irregular and the proof is complete. $\square$

5. Generalizations to weighted $\mathbf{R}^n$ and metric spaces

With the exception of Proposition 7.5 in Björn [1] all the quoted results in Sections 2 and 3 hold for $\mathcal{A}$ -(super)harmonic functions on weighted $\mathbf{R}^n$ for $1 < p < \infty$. Here, as is usual, we assume that $\mathcal{A}$ satisfies the degenerate ellipticity conditions (3.3)–(3.7) on p. 56 of Heinonen–Kilpeläinen–Martio [11] and that the weight is p -admissible, see Chapters 1 and 20 in [11]. The results also hold for p -(super)harmonic functions in metric spaces, where we assume that X is a complete metric space equipped with a doubling measure μ and supporting a weak $(1, p)$ -Poincaré inequality, and that $\Omega \subset X$ is a nonempty bounded open set satisfying $C_p(X \setminus \Omega) > 0$ (which is immediate if X is unbounded). See either [1], Björn–Björn [4] or Björn–Björn–Shanmugalingam [7] for the necessary definitions. (We have provided references in brackets to metric space results whenever necessary.)

Proposition 7.5 in [1] holds under the assumption that singleton sets have capacity zero, and under this assumption no changes are needed, nor under any other condition implying that removable sets for bounded p -harmonic functions have capacity zero. In weighted $\mathbf{R}^n$ it is not clear if all removable sets have capacity zero, and in metric spaces (actually on $\mathbf{R}$) there are even examples of removable sets with positive capacity, see Section 9 in [1]. Because of this one needs to add either that “ x_0 is irregular” or that “ $V \subset \overline{\Omega}$ ” to (h) in Theorem 3.3. (The modification of the proof is straightforward.)

When generalizing Theorem 4.1 to weighted $\mathbf{R}^n$ and metric spaces we restrict ourselves, for simplicity, to spaces in which singleton sets either all have positive capacity (in which case the theorem is trivial) or all have zero capacity. As balls may have nonregular boundary points (see Example 3.1 in Björn–Björn [5]) we let B be a bounded regular set containing $S \cup I$ in the beginning of the proof (using Theorem 1.1 in [5]). By compactness $\inf_{x \in I} \mu(B(x, 1)) > 0$. It therefore follows from the doubling property that there are constants $C > 0$ and $\kappa \geq 1$ such that $\mu(B(x, r)) \geq Cr^\kappa$ for all $x \in I$ and $0 < r \leq 1$. We need to replace n by κ in (4.1)–(4.4). To obtain (4.1) we also need to know that cap_p is an outer capacity, which was shown in Björn–Björn–Shanmugalingam [8].

6. Generalizations to quasiminimizers

We refer the reader to Björn [1,2] for the definition of quasiminimizers and for the definition of the Newtonian (Sobolev) spaces $N^{1,p}(\overline{\Omega})$ and $N_0^{1,p}(\Omega)$. A *quasiharmonic* function is a continuous quasiminimizer.

Assume that Ω is a subset of a metric space X as in Section 5.

Definition 6.1. Let $x_0 \in \partial\Omega$ and consider the following two properties:

(a) For all $f \in C(\partial\Omega) \cap N^{1,p}(\overline{\Omega})$ and all quasiharmonic u on Ω with $u - f \in N_0^{1,p}(\Omega)$, the limit

$$\lim_{\Omega \ni y \rightarrow x_0} u(y) \text{ exists.}$$

(b) For all $f \in C(\partial\Omega) \cap N^{1,p}(\overline{\Omega})$ and all quasiharmonic u on Ω with $u - f \in N_0^{1,p}(\Omega)$, there is a sequence $\{y_j\}_{j=1}^\infty$ such that

$$\Omega \ni y_j \rightarrow x_0 \quad \text{and} \quad u(y_j) \rightarrow f(x_0), \quad \text{as } j \rightarrow \infty.$$

Then x_0 is *regular for quasiharmonic functions* if both (a) and (b) hold; x_0 is *semiregular for quasiharmonic functions* if (a) holds but not (b); and x_0 is *strongly irregular for quasiharmonic functions* if (b) holds but not (a).

To obtain the trichotomy in this case, all that is needed is a removability result, see Theorem 6.2 in [1], and the Kellogg property. It is not known if the Kellogg property holds for quasiharmonic functions, but in fact the weak Kellogg property obtained in Theorem 4.1 in [2] is enough. The proof remains essentially the same as the proof of Theorem 2.1.

Using the removability result and the weak Kellogg property one can also modify the proofs in Section 3 to see that quasiversions of (a'), (d') and (e') are equivalent to the statements in Theorem 3.1, and quasiversions of (a), (b), (d), (h) (suitably modified, see Section 5) and (i) are equivalent to the statements in Theorem 3.3. It is not known if quasiversions of (b'), (c) and (e) are equivalent to the other statements.

Theorem 4.1 generalizes directly to the corresponding result for quasiminimizers. As the Kellogg property is open for quasiminimizers it is however not clear if the set of all strongly irregular boundary points for quasiminimizers has capacity zero.

Let us also mention that for fixed $Q > 1$, we obtain similar results if we replace “quasi” by “ Q -quasi” in this section.

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