J. Math. Anal. Appl. 355 (2009) 821–829

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Local analytic solutions of the generalized Dhombres functional equation II ☆

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ARTICLE INFO

Article history:
Received 17 December 2007
Available online 14 February 2009
Submitted by M. Passare

Keywords:
Local analytic function
Holomorphic function
Iterative functional equation

ABSTRACT

We study local analytic solutions f of the generalized Dhombres functional equation $f(zf(z)) = \varphi(f(z))$, where φ is holomorphic at $w_0 \neq 0$, f is holomorphic in some open neighborhood of 0, depending on f, and $f(0) = w_0$. After deriving necessary conditions on φ for the existence of nonconstant solutions f with $f(0) = w_0$ we describe, assuming these conditions, the structure of the set of all formal solutions, provided that w_0 is not a root of 1. If $|w_0| \neq 1$ or if w_0 is a Siegel number we show that all formal solutions yield local analytic ones. For w_0 with $0 < |w_0| < 1$ we give representations of these solutions involving infinite products.

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1. Introduction

The iterative functional equation of Dhombres, the first time appearing in [2], and some of its generalizations were studied in the real case with respect to continuous solutions in [5,6] and, in particular, in [8] and [9]. The investigation of generalized Dhombres equations in the complex domain with respect to holomorphic solutions begins with [10] and [11]. For a survey of results and problems concerning iterative functional equations we refer to [7] (cf. also [1] and [12] for related problems).

In the present paper we study at $z = 0$ analytic nonconstant solutions f with $f(0) = w_0$, $w_0 \neq 0$, of the generalized Dhombres functional equation

$$f\big(zf(z)\big) = \varphi\big(f(z)\big) \tag{1}$$

for z in some neighborhood of 0, where φ is given and supposed to be holomorphic at w_0. In [10] we considered the set of local analytic solutions of (1) with $f(0) = 0$. It is easy to see that the point $z = 0$ plays a particular role for solutions of (1). The methods used in [10] and in the present paper are similar, but there are also specific differences arising from the fact that now we assume $f(0) \neq 0$. Indeed, the role of Böttcher functions in [10] is replaced here by the application of Schröder functions, i.e., solutions of certain Schröder equations (see Section 2). The results in the present paper, in particular the main result, Theorem 1, are proved only if $|w_0| \neq 1$ or w_0 is a Siegel number. The representation of local analytic solutions of (1) involving certain infinite products (Theorem 2) is only proved for the case $|w_0| < 1$, and these infinite products do not make sense in the formal case using the order topology, while analogous infinite products used in [10] are also defined for formal solutions.

☆ The research was supported, in part, by projects 201/06/0318 from the Czech Science Foundation, MSM4781305904 from the Czech Ministry of Education. Support of these institutions is gratefully acknowledged.

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doi:10.1016/j.jmaa.2009.02.020

It turns out that the development of our ideas becomes clearer if, before turning to proofs of analyticity, we study somewhat more systematically formal solutions of (1) in Section 2. Even in this simplified situation we get only a complete description of the general solution if we assume that w_0 is not a root of 1 (Proposition 2). However, if we only assume that $w_0 \neq 0$, then there are necessary conditions on φ in order that (1) has a nonconstant solution, namely, there has to exist a $k \in \mathbb{N}$ such that

$$\varphi(w) = w_0 + w_0^k(w - w_0) + d_2(w - w_0)^2 + \cdots. \tag{2}$$

If we write the nonconstant solutions f of (1) as $f(z) = w_0 + g(z)$, and if w_0 is not a root of 1, then necessarily $g(z) = c_k z^k + \cdots$, with $c_k \neq 0$. On the other hand, if g has the form mentioned above then to each $c_k \neq 0$ there exists exactly one (formal) solution f with $f(z) = w_0 + c_k z^k + \cdots$ (Proposition 2b).

If now $|w_0| \neq 1$ or w_0 is a Siegel number and φ is supposed to converge, then we will see that all formal solutions obtained in Proposition 2 are convergent, so describing all local analytic solutions f with $f(0) = w_0$ under our assumptions. At this stage we are also able to solve the so-called converse problem for Eq. (1) (see [9] and [10]) under the hypothesis made in the present work (Theorem 2). This converse problem is, roughly speaking, the question whether to a given f there exists φ such that (1) holds.

The last part of this introduction is devoted to deducing the necessary form of φ in (1) in order to obtain nonconstant solutions f with $f(0) = w_0$ for a given $w_0 \in \mathbb{C}^*(= \mathbb{C} \setminus \{0\})$, and to transform (1) into an equivalent form (1′) which is useful for what follows. By a solution we mean a local analytic f, if φ is holomorphic at $f(0) = w_0$, or a formal series if φ is merely a formal power series in $w - w_0$.

Proposition 1. *Let $w_0 \in \mathbb{C}^*$ and let φ be analytic in a neighborhood of w_0 (or a formal power series $\varphi(w)$ in $w - w_0$). Then the following holds true.*

(a) *If (1) has a nonconstant solution f with $f(0) = w_0$ then there exists $k \in \mathbb{N}$ such that*

$$f(z) = w_0 + g(z) \tag{3}$$

with $\operatorname{ord}_{z=0} g = k$, and

$$\varphi(w) = w_0 + w_0^k(w - w_0) + d_2\big(w - w_0^2\big) + \cdots. \tag{4}$$

(b) *If w_0 is not a root of 1, then the integer $k \in \mathbb{N}$ from (a) such that (3) and (4) hold is uniquely determined.*

(c) *If we introduce g by (3) then, putting $\varphi(w) = w_0 + \psi(w - w_0)$, (1) is equivalent with the functional equation*

$$g\big(w_0 z + z g(z)\big) = \psi\big(g(z)\big) \tag{1′}$$

in a neighborhood of $z = 0$ or for formal series g, where $\psi(y) = w_0^k y + d_2 y^2 + \cdots$.

Proof. Cases (a) and (c). If f fulfills $f(0) = w_0$, $f \neq \text{const}$, then $f(z) = w_0 + g(z)$, with $\operatorname{ord}_{z=0} g = k \in \mathbb{N}$, $g(0) = 0$. If we substitute this into (1) and put $z = 0$, then we find $f(0 \cdot f(0)) = \varphi(f(0))$, hence

$$w_0 = \varphi(w_0),$$

and

$$\varphi(w) = w_0 + d_1(w - w_0) + d_2(w - w_0)^2 + \cdots = w_0 + \psi(w_0).$$

If we again substitute the last expression into (1) we find

$$g\big(w_0 z + z g(z)\big) = \psi\big(g(z)\big) \tag{1′}$$

and hence, since $g(z) = c_k z^k + \cdots$, with $k \geqslant 1$, $c_k \neq 0$, we get from (1′) comparing coefficients of z^k that

$$w_0^k c_k = d_1 c_k,$$

hence $d_1 = w_0^k$ which means (4).

The case (b) is obvious. □

2. Formal solutions

Now we study formal solutions $g(z) = c_k z^k + \cdots$, $k \geqslant 1$, $c_k \neq 0$ of the functional equation

$$g\big(w_0 z + z g(z)\big) = \psi\big(g(z)\big) \tag{1′}$$

where $\psi(y) = w_0^k y + d_2 y^2 + \cdots$, $w_0 \neq 0$. In this section we consider ψ and g as elements in the ring of formal power series $\mathbb{C}[[z]]$. For basic notations and operations in $\mathbb{C}[[z]]$ we refer the reader to [4] or [3]. For $F, G \in \mathbb{C}[[z]]$ we will use the notation

$$F(z) \equiv G(z) \pmod{z^l} \quad \text{if } \operatorname{ord}(F - G) \geqslant l \text{ (for } l \in \mathbb{N}).$$

$(\Gamma, \circ)$ is the group of invertible formal series in $\mathbb{C}[\![z]\!]$, with respect to the substitution, hence $\Gamma = \{F;\ F \in \mathbb{C}[\![z]\!],\ \operatorname{ord} F = 1\}$. Γ_1 denotes the subgroup of Γ consisting of all F with $F(z) \equiv z \pmod{z^2}$. Furthermore, let $\widetilde{\mathbb{C}[\![z]\!]} = \{F;\ F \in \mathbb{C}[\![z]\!],\ \operatorname{ord} F \geqslant 1\}$ which is a vector subspace of $\mathbb{C}[\![z]\!]$, and let $\mathcal{U}_{\mathbb{C}[\![z]\!]} = \{F;\ F \in \mathbb{C}[\![z]\!],\ F(z) \equiv 1 \pmod{z}\}$ which is a subgroup with respect to multiplication $\cdot$ and a subgroup of the group $\{F;\ F \in \mathbb{C}[\![z]\!],\ \operatorname{ord} F = 0\}$ of all multiplicative units in $\mathbb{C}[\![z]\!]$. For $k \in \mathbb{N}$ we introduce $\mathbb{C}[\![z^k]\!] = \{F;\ F \in \mathbb{C}[\![z]\!],\ F(z) = \sum_{\nu=0}^{\infty} c_\nu z^\nu$ with $c_\nu = 0$ if $\nu \not\equiv 0 \pmod{k}\}$. This is a subring of $\mathbb{C}[\![z]\!]$, closed under substitution $\circ$. Then we put $\mathbb{C}^{(k)}[\![z]\!] = \{F;\ F \in \mathbb{C}[\![z]\!],\ F(z) = \sum_{\nu=0}^{\infty} c_\nu z^\nu$ with $c_\nu = 0$ if $\nu \not\equiv 1 \pmod{k}\}$, $\Gamma^{(k)} = \Gamma \cap \mathbb{C}^{(k)}[\![z]\!]$, $\Gamma_1^{(k)} = \Gamma_1 \cap \mathbb{C}^{(k)}[\![z]\!]$. We already used $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. For $\alpha \in \mathbb{C}$ we write $L_\alpha(z) = \alpha z \in \mathbb{C}[\![z]\!]$ hence, for $\alpha \in \mathbb{C}^*$, $L_\alpha^{-1} = L_{\alpha^{-1}}$, $F(\alpha z) = (F \circ L_\alpha)z$, $\alpha F(z) = (L_\alpha \circ F)(z)$. By E we denote the set of all complex roots of 1 which is a subgroup of $(\mathbb{C}^*, \cdot)$.

We begin by collecting some simple, partly well-known facts about formal series which will be used throughout this section.

Lemma 1. *Let $\eta \in E$, a primitive root of 1 of order $k \geqslant 1$, and let $F, G, \ldots$ denote elements of $\mathbb{C}[\![z]\!]$. Then the following holds true.*

(a) *$F \in \mathbb{C}[\![z^k]\!]$ if and only if $F \circ L_\eta = F$.*

(b) *$\mathbb{C}[\![z^k]\!]$ is a subring of $\mathbb{C}[\![z]\!]$ containing $\mathbb{C}$, closed under substitution and complete in the order topology in $\mathbb{C}[\![z]\!]$.*
The substitution τ_k of z^k (hence with $\tau_k(F)(z) = F(z^k)$ for $F \in \mathbb{C}[\![z]\!]$) is an injective ring homomorphism of $\mathbb{C}[\![z]\!]$ onto $\mathbb{C}[\![z^k]\!]$, continuous in the order topology.
Let $F \in \mathbb{C}[\![z]\!]$, $G = \mathbb{C}[\![z^k]\!]$, $\operatorname{ord} G \geqslant 1$. Then $F \circ G \in \mathbb{C}[\![z^k]\!]$.

(c) *$F \in \mathbb{C}^{(k)}[\![z]\!]$ if and only if $F \circ L_\eta = L_\eta \circ F$. Moreover, $\mathbb{C}^{(k)}[\![z]\!] = z\mathbb{C}[\![z^k]\!]$, $(\Gamma^{(k)}, \circ)$ is a subgroup of Γ, and $(\Gamma_1^{(k)}, \circ)$ a subgroup of $(\Gamma_1, \circ)$.*

(d) *Let $\exp z = \sum_{\nu=0}^{\infty} \frac{z^\nu}{\nu!}$, $l(z) = \ln(1+z) = \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu-1}}{\nu} z^\nu$. Then the mapping*

$$\operatorname{Exp} : \big(\widetilde{\mathbb{C}[\![z]\!]}, +\big) \longrightarrow (\mathcal{U}_{\mathbb{C}[\![z]\!]}, \cdot)$$

defined by $\operatorname{Exp}(F)(z) = (\exp \circ F)(z)$ is a group isomorphism, continuous in the order topology, and $(\operatorname{Exp})^{-1} = \operatorname{Ln}$ is given by

$$\operatorname{Ln}(\phi) = l \circ (\phi - 1) \quad (\phi \in \mathcal{U}_{\mathbb{C}[\![z]\!]}).$$

Proof. (a) Let $F(z) = \sum_{\nu=0}^{\infty} c_\nu z^\nu$. Then $F \circ L_\eta = F$ if and only if $(\eta^\nu - 1)c_\nu = 0$, $\nu \geqslant 0$ if and only if $c_\nu = 0$ if $\nu \not\equiv 0 \pmod{k}$ if and only if $F \in \mathbb{C}[\![z^k]\!]$.

(b) The first assertions are well known. To show the last one we observe by (a) for $G \in \mathbb{C}[\![z^k]\!]$, $\operatorname{ord} G \geqslant 1$, that $(F \circ G)(\eta z) = F(G(\eta z)) = F(G(z))$ hence, by (a), $F \circ G \in \mathbb{C}[\![z^k]\!]$.

(c) Let $F(z) = \sum_{\nu=0}^{\infty} c_\nu z^\nu$. Then $F \circ L_\eta = L_\eta \circ F$ if and only if $\sum_{\nu=0}^{\infty} c_\nu \eta^\nu z^\nu = \sum_{\nu=0}^{\infty} \eta c_\nu z^\nu$ if and only if $(\eta^\nu - 1)c_\nu = 0$, $\nu \geqslant 0$ if and only if $c_\nu = 0$ if $\nu \not\equiv 1 \pmod{k}$ if and only if $F \in \mathbb{C}^{(k)}[\![z]\!]$. We show that $(\Gamma^{(k)}, \circ)$ is a group. Clearly, $z \in \Gamma^{(k)}$. Let $F, G \in \Gamma^{(k)}$. Then $F \circ G \in \Gamma$, and $((F \circ G) \circ L_\eta)(z) = F(G(\eta z)) = F(\eta G(z)) = \eta F(G(z)) = (L_\eta \circ (F \circ G))(z)$, hence $F \circ G = \mathbb{C}^{(k)}[\![z]\!] \cap \Gamma = \Gamma^{(k)}$, by the characterization of $\mathbb{C}[\![z^k]\!]$ just proved. Eventually let $F \in \Gamma^{(k)}$. Since F^{-1} is characterized by the relation $F(F^{-1}(z)) = z$, we have $\eta z = \eta F(F^{-1}(z)) = F(\eta F^{-1}(z))$. Substituting $\eta^{-1}z$ for z we get $z = F(\eta F^{-1}(\eta^{-1}z))$ hence $F^{-1}(z) = \eta F^{-1}(\eta^{-1}z)$ or $\eta^{-1}F^{-1}(z) \cdot F^{-1}(\eta^{-1}z)$. Since $\eta^{-1} \in E$ and is primitive of order k we find $F^{-1}(z) \in \mathbb{C}^{(k)}[\![z]\!] \cap \Gamma = \Gamma^{(k)}$.

(d) This is well known, see, e.g., [4]. □

As in [10] we will need some facts about multiplicative roots of formal power series.

Lemma 2.

(a) *Let $k \in \mathbb{N}$, $l \in \mathbb{N}_0$. Let $\phi(z) \in \mathbb{C}[\![z]\!]$ with $\operatorname{ord}\phi = kl$. Then there exist exactly k different solutions ψ_j, $j = 0, \ldots, k-1$ in $\mathbb{C}[\![z]\!]$ of the equation*

$$\big(\psi(z)\big)^k = \phi(z). \tag{5}$$

We have $\operatorname{ord}\psi_j = l$ for $j = 0, \ldots, k-1$, and if η denotes a primitive root of 1 of order k, then the solutions of (5) are given by

$$\psi_j(z) = \eta^j \psi_0(z), \quad j = 0, \ldots, k-1.$$

Hence ψ_j is uniquely determined as a solution of (5) by the coefficient of z^l.

(b) *Let, in particular, $\phi(z) = \widetilde{\phi}(z^k)$ for a $\widetilde{\phi} \in \Gamma$ (hence $\phi \in \mathbb{C}[\![z^k]\!]$, $\operatorname{ord}\phi = k$). Then each solution ψ of*

$$\psi(z)^k = \widetilde{\phi}\big(z^k\big) \tag{5′}$$

belongs to $\Gamma^{(k)}$.

Proof. (a) These assertions are well known and we omit the proof (cf. [10]).

(b) Let ψ be a solution of (5′) as given by (a). Substitute ηz for z in (5′) to obtain

$$\psi(\eta z)^k = \widetilde{\phi}\big(z^k\big). \tag{5′}$$

We have $\operatorname{ord}\psi = 1$ and $\psi(\eta z) \equiv \eta\psi(z) \pmod{z^2}$, hence by the uniqueness part of (a) and the description of the solutions of (5′) (here $l = 1$), we get $\psi(\eta z) = \eta\psi(z)$ hence, by Lemma 1c, $\psi \in \mathbb{C}[\![z^k]\!] \cap \Gamma = \Gamma^{(k)}$. □

Another basic tool in our study of (1) (and (1′)) is the Schröder equation. Let $\phi \in \Gamma$, $\phi(z) \equiv \lambda z \pmod{z^2}$. Schröder's equation is the functional equation

$$\big(S^{-1} \circ \phi \circ S\big)(z) = \lambda z \tag{6}$$

for the unknown $S \in \Gamma$.

Lemma 3. *Assume that $\phi(z) \equiv \lambda z \pmod{z^2}$ such that $\lambda \in \mathbb{C}^* \setminus E$ (hence, in particular, $\phi \in \Gamma$).*

(a) *Then there exists exactly one solution S_ϕ of (6) in Γ_1. The general solution of (6) in Γ is given by*

$$\{S = S_\phi \circ L_{s_1};\ s_1 \in \mathbb{C}^*\},$$

and a special solution S is uniquely determined by s_1. We call S_ϕ the Schröder function of ϕ.

(b) *Assume, in addition to the above assumptions, that $\phi \in \Gamma^{(k)}$. Then each solution of (6) belongs to $\Gamma^{(k)}$.*

Proof. (a) It is well known, cf., e.g., [13] or [7].

(b) Write (6) in the form $\phi \circ S_\phi = S_\phi \circ L_\eta$. By the characterization of $\mathbb{C}[\![z^k]\!]$ in Lemma 1b we get from (6)

$$\phi \circ (L_\eta \circ S_\phi) = (\phi \circ L_\eta) \circ S_\phi = (L_\eta \circ \phi) \circ S_\phi = (L_\eta \circ S_\phi) \circ L_\lambda.$$

Hence $L_\eta \circ S_\phi$ is a solution of (6) and $(L_\eta \circ S_\phi)(z) \equiv \eta z \pmod{z^2}$. By Lemma 3a, $S_\phi \circ L_\eta$ is a solution of (6) too, and $(S_\phi \circ L_\eta)(z) \equiv \eta z \pmod{z^2}$. By the uniqueness assertion of (a) we get $L_\eta \circ S_\phi = S_\phi \circ L_\eta$ hence, by Lemma 1b, $S_\phi \in \Gamma^{(k)}$, and again by Lemma 3a, $S \in \Gamma^{(k)}$ for each solution S of (6) for $\phi \in \Gamma^{(k)}$. □

Now we are ready to describe the general nonconstant solution of (1′) in $\mathbb{C}[\![z]\!]$. By Proposition 1, for the existence of nonconstant solutions g of (1′) it is necessary that there exists a $k \in \mathbb{N}$ such that

$$\psi(y) \equiv w_0^k y \pmod{y^2}$$

and

$$\operatorname{ord} g(z) = k.$$

If $w_0 \in \mathbb{C}^* \setminus E$ then such a k is unique.

Proposition 2. *Assume that*

$$w_0 = \mathbb{C}^* \setminus E, \tag{H_1}$$

$$\psi(y) \equiv w_0^k y \pmod{y^2}, \quad \text{for some } k \in \mathbb{N}. \tag{H_2}$$

(a) *Then there exists a unique $\widetilde{g}_0 \in \Gamma_1$ such that the set of nonconstant solutions g of*

$$g\big(w_0 z + z g(z)\big) = \psi\big(g(z)\big) \tag{1′}$$

in $\mathbb{C}[\![z]\!]$ is given by

$$\mathcal{L}_\psi = \big\{g;\ g(z) = \widetilde{g}_0\big(c_k z^k\big),\ c_k \in \mathbb{C}^*\big\}.$$

Hence, $\mathcal{L}_\psi \subset \mathbb{C}[\![z^k]\!]$. Moreover, (1′) has exactly one solution g with $g(z) \equiv c_k z^k \pmod{z^{k+1}}$, for each $c_k \in \mathbb{C}^*$.

(b) *For $\widetilde{g}_0$ from (a), with $\widetilde{g}_0 = \widetilde{h}_0^{-1}$, $\widetilde{h}_0$ is the unique solution of the linear functional equation*

$$(w_0 + z)^k \widetilde{h}_0(z) = \widetilde{h}_0\big(\psi(z)\big).$$

Proof of part (a). We use some ideas already applied in [10] to transform (1′) into an equivalent linear functional equation. The remaining parts of the proof are somewhat different. We divide our procedure into several steps, including Lemma 4.

Step 1. Firstly we observe that a possible solution g of (1′) with $g(z) \equiv c_k z^k \pmod{z^{k+1}}$, $c_k \neq 0$ can, according to Lemma 2b, always be written as

$$g(z) = T(z)^k, \quad \operatorname{ord} T = 1. \tag{6}$$

We take an arbitrary solution T of (6). Inserting (6) into (1′) yields

$$\big(T\big(w_0 z + zT(z)^k\big)\big)^k = \psi\big(T(z)^k\big). \tag{7}$$

Substituting $T^{-1}(z)$ for z in (7) gives

$$\big(T\big(w_0 T^{-1}(z) + T^{-1}(z) z^k\big)\big)^k = \psi\big(z^k\big). \tag{8}$$

We have, by our hypothesis (H_2),

$$\psi\big(z^k\big) \equiv w_0^k z^k \pmod{z^{k+1}}.$$

By Lemma 2, $\psi(z^k)$ has exactly one multiplicative root $\widetilde{\psi}$ of order k such that $\widetilde{\psi}(z) \equiv w_0 z \pmod{z^2}$. Since $\operatorname{ord} z^k T^{-1}(z) > 1$, we have

$$T\big(w_0 T^{-1}(z) + z^k T^{-1}(z)\big) \equiv w_0 z \pmod{z^2},$$

and hence, by Lemma 2a,

$$\widetilde{\psi}(z) = T\big(w_0 T^{-1}(z) + z^k T^{-1}(z)\big). \tag{9}$$

By Lemma 2b, since $\psi(z^k) \in \mathbb{C}[\![z^k]\!]$, we obtain $\widetilde{\psi}(z) \in \Gamma^{(k)}$. Applying T^{-1} on both sides of (9) we get

$$\big(w_0 + z^k\big) T^{-1}(z) = T^{-1}\big(\widetilde{\psi}(z)\big),$$

and putting

$$U = T^{-1} \in \Gamma \tag{10}$$

we find

$$\big(w_0 + z^k\big) U(z) = U\big(\widetilde{\psi}(z)\big) \tag{11}$$

with $U \in \Gamma$, $\widetilde{\psi} \in \Gamma^{(k)}$, $\widetilde{\psi}(z) \equiv w_0 z \pmod{z^2}$. Thus, (11) is equivalent to (1′) via the transformations (6) and (10).

Step 2. Now we find the general solution of (11). Here the application of the Schröder function $S_{\widetilde{\psi}}$ of $\widetilde{\psi}$ (see Lemma 3) is useful. Putting

$$V = U \circ S_{\widetilde{\psi}}^{-1}, \tag{12}$$

one can easily see that (11) is equivalent to

$$\big(w_0 + \big(S_{\widetilde{\psi}}^{-1}(z)^k\big) V(z) = V(w_0 z)\big), \quad V \in \Gamma. \tag{13}$$

Since (see (11)) $\widetilde{\psi} \in \Gamma^{(k)}$ we deduce from Lemma 3b that

$$S_{\widetilde{\psi}} \in \Gamma_1^{(k)}, \tag{14a}$$

and, by Lemma 1c,

$$S_{\widetilde{\psi}}^{-1} \in \Gamma_1^{(k)}, \tag{14b}$$

and by writing $S_{\widetilde{\psi}}^{-1}(z) = z \cdot M(z)$ with $M(z) \in \mathbb{C}[\![z^k]\!]$ (Lemma 1c) we see, by Lemma 1b, that

$$\operatorname{ord}\big(\big(S_{\widetilde{\psi}}^{-1}\big)^k\big) = k, \quad \big(S_{\widetilde{\psi}}^{-1}(z)\big)^k \in \mathbb{C}\big[\big[z^k\big]\big]. \tag{14c}$$

Step 3. We proceed with the proof of Proposition 2a by the next lemma.

Lemma 4. *Assume (H_1) and (H_2). Then the general solution of* (13) *in Γ is given by*

$$\big\{V(z);\ V(z) = v_1 V^{(0)}(z),\ v_1 \in \mathbb{C}^*\big\}$$

where $V^{(0)}$ is the unique solution of (13) *in Γ_1. Furthermore, $V^{(0)} \in \Gamma^{(k)}$. A solution V of* (13) *is uniquely determined by $V(z) \equiv v_1 z \pmod{z^2}$.*

Proof. Each possible solution $V \in \Gamma$ of (13) has a representation $V(z) = v_1 V^{(0)(z)}$ with $V^0 \in \Gamma_1$, a solution of (13). We write

$$V^{(0)}(z) = zV^*(z), \quad \text{with } V^*(z) \in \mathcal{U}_{\mathbb{C}[\![z]\!]}. \tag{15}$$

If we put (15) in (13), we find

$$\big(1 + w_0^{-1}\big(S_{\widetilde{\psi}}^{-1}(z)\big)^k\big)V^*(z) = V^*(w_0 z). \tag{15}$$

Since $1 + w_0^{-1}(S_{\widetilde{\psi}}^{-1}(z))^k$, $V^*(z) \in \mathcal{U}_{\mathbb{C}[\![z]\!]}$ we may consider the formal logarithms of these series (see Lemma 1d), namely

$$A(z) = \operatorname{Ln}\big(1 + w_0^{-1}\big(S_{\widetilde{\psi}}^{-1}(z)\big)^k\big), \qquad Y(z) = \operatorname{Ln} V^*(z), \tag{16}$$

with $\operatorname{ord} A \geqslant 1$, $\operatorname{ord} Y \geqslant 1$, and get from (15)

$$A(z) = Y(w_0 z) - Y(z). \tag{17}$$

By Lemma 1b, 1d and by (15c) we get

$$A(z) \in \mathbb{C}\big[\!\big[z^k\big]\!\big]. \tag{18}$$

If we write $A(z) = \sum_{\nu=1}^{\infty} \alpha_\nu z^\nu$, and $Y(z) = \sum_{\nu=1}^{\infty} \gamma_\nu z^\nu$, (17) is the same as

$$\big(w_0^\nu - 1\big)\gamma_\nu = \alpha_\nu, \quad \nu \geqslant 1. \tag{19}$$

Since $w_0 \in \mathbb{C}^* \setminus E$, Y exists and is uniquely determined. From (18) we get $\alpha_\nu = 0$ if $\nu \not\equiv 0 \pmod k$ and so, by (19), $\gamma_\nu = (w_0^\nu - 1)^{-1}\alpha_\nu$ for $\nu \not\equiv 0 \pmod k$, i.e., $Y(z) \in \mathbb{C}[\![z^k]\!]$ and hence, again by the last sentence of Lemma 1b and 1d,

$$V^*(z) = \exp\big(Y(z)\big) \in \mathbb{C}\big[\!\big[z^k\big]\!\big], \qquad V^{(0)}(z) = zV^*(z) \in \mathbb{C}^{(k)}[\![z]\!] \cap \Gamma_1 = \Gamma_1^{(k)},$$

whence

$$V(z) = v_1 V^{(0)}(z) \in \Gamma^{(k)}. \quad \square \tag{20}$$

Step 4. We now go back to Eq. (11). By (12) we have $U = V \circ S_{\widetilde{\psi}}$ and, because of (20), (15a) and Lemma 1c, also $U \in \Gamma^{(k)}$. From (20) we find $U(z) = v_1(V^{(0)} \circ S_{\widetilde{\psi}})(z) = v_1 U^{(0)}(z)$ with $U^{(0)} \in \Gamma_1^{(k)}$. Using (10) we have $T = U^{-1}$, $T(z) = ((U^{(0)})^{-1} \circ L_{v_1^{-1}})(z) = (U^{(0)})^{-1}(t_1 z)$, with $t_1 = v_1^{-1}$, where, by Lemma 1c, $(U^{(0)})^{-1} \in \Gamma_1^{(k)}$. By (6) we find $g(z) = T(z)^k = ((U^{(0)})^{-1}(t_1 z))^k$ and since $U^{(0)} \in \Gamma_1^{(k)}$, by Lemma 1b and 1c, $g(z) \in \mathbb{C}[\![z^k]\!]$. The last inclusion can be worked out in more detail: We write $(U^{(0)})^{-1}(z) = z(1 + \sum_{\nu \geqslant 1} \beta_\nu (z^k)^\nu)$, $(U^{(0)})^{-1}(t_1 z) = t_1 z(1 + \sum_{\nu \geqslant 1} \beta_\nu (t_1^k z^k)^\nu)$, and putting $c_k = t_1^k$, $g(z) = T(z)^k = ((U^{(0)})^{-1}(t_1 z))^k = c_k z^k (1 + \sum_{\nu \geqslant 1} \beta_\nu (c_k z^k)^\nu)^k = c_k z^k (1 + \sum_{\nu \geqslant 1} \delta_\nu (c_k z^k)^\nu)$. Introducing

$$\widetilde{g}_0(y) = y\left(1 + \sum_{\nu \geqslant 1} \delta_\nu y^\nu\right) \in \Gamma_1 \subset \mathbb{C}[\![y]\!], \tag{21}$$

we have

$$g(z) = \widetilde{g}_0\big(c_k z^k\big), \qquad g(z) \equiv c_k z^k \pmod{z^{k+1}}, \tag{22}$$

for each solution of (1′).

We have then to show that for each $c_k \in \mathbb{C}^*$, $g(z)$ defined by (22) is indeed a solution of (1′). This follows from the fact that the steps in the proof can be all reversed. It is also easy to see, using again (22) and comparing coefficients of z^k, that there is a unique solution g of (1′) such that $g(z) \equiv c_k z^k \pmod{z^{k+1}}$ for a given c_k. (Note that $c_k = 0$ yields the trivial solution $g = 0$ of (1′).) This completes the proof of part (a) of Proposition 2. $\square$

Proof of part (b) of Proposition 2. Assume that $\widetilde{\widetilde{g}}_0 \in \Gamma_1$, and that for each $c_k \in \mathbb{C}^*$, $\widetilde{\widetilde{g}}_0(c_k z^k)$ is a solution of (1′). We want to show that $\widetilde{\widetilde{g}}_0 = \widetilde{g}_0$. In fact, the substitution of $c_k z^k$, $c_k \neq 0$, for z is an injective ring homomorphism (see Lemma 1b). Since $\widetilde{g}_0(c_k z^k)$ and $\widetilde{\widetilde{g}}_0(c_k z^k)$ are both solutions of (1′) with $\widetilde{g}_0(c_k z^k) \equiv \widetilde{\widetilde{g}}_0(c_k z^k) \equiv c_k z^k \pmod{z^{k+1}}$ we find by the uniqueness part of Proposition 1a that $\widetilde{g}_0(c_k z^k) = \widetilde{\widetilde{g}}_0(c_k z^k)$, hence $\widetilde{g}_0 = \widetilde{\widetilde{g}}_0$.

Since $\widetilde{g}_0(c_k z^k)$ satisfies (1′), for each $c_k \in \mathbb{C}^*$ we get

$$\widetilde{g}_0\big(c_k z^k\big(w_0 + \widetilde{g}_0\big(c_k z^k\big)\big)\big) = \psi\big(\widetilde{g}_0\big(c_k z^k\big)\big)$$

and, again by Lemma 1b,

$$\widetilde{g}_0\big(y\big(w_0 + \widetilde{g}_0(y)\big)\big) = \psi\big(\widetilde{g}_0(y)\big) \quad \text{in } \mathbb{C}[\![y]\!].$$

Introducing $\widetilde{h}_0 = \widetilde{g}_0^{-1}$, this yields

$$(w_0 + y)\widetilde{h}_0(y) = \psi\big(\widetilde{h}_0(y)\big). \tag{23}$$

This is a linear functional equation which can be solved by a similar procedure as used for (11). Hence $\widetilde{h}_0$ is the unique solution of (23) with $\widetilde{h}_0(y) \equiv y \pmod{y^2}$. This finishes the proof of Proposition 2. □

We now turn to the so-called converse problem (see [9,10]). We ask under which conditions f with $f(z) = w_0 + g(z)$, $\operatorname{ord} g \geqslant k$, $w_0 \notin E$, is a solution of an appropriate equation (1), i.e., g is a solution of Eq. (1′). Using the approach to Proposition 1 we obtain

Proposition 3.

(a) *Let $w_0 \in \mathbb{C}$, $g \in \mathbb{C}[\![z]\!]$ with $\operatorname{ord} g = k \in \mathbb{N}$. If there exists $g \in \Gamma$ such that $g(z) = \widetilde{g}(z^k)$, then there exists a unique ψ such that (1′) holds for these g and ψ.*

(b) *Let $w_0 \in \mathbb{C}^* \setminus E$, $g \in \mathbb{C}[\![z]\!]$ with $\operatorname{ord} g = k \in \mathbb{N}$. Then there exists $\psi \in \mathbb{C}[\![w]\!]$ such that g and ψ satisfy (1′) with these g and ψ if and only if there exists a $\widetilde{g} \in \Gamma$ such that $g(z) = \widetilde{g}(z^k)$.*

Proof. (a) By assumption we have $g(z(w_0 + g(z))) = \widetilde{g}(z^k(w_0 + \widetilde{g}(z^k)^k))$. Now, if there exists $\psi \in \mathbb{C}[\![w]\!]$ such that g and ψ satisfy (1′), we obtain

$$\widetilde{g}\big(z^k\big(w_0 + \widetilde{g}\big(z^k\big)\big)^k\big) = \psi\big(\widetilde{g}\big(z^k\big)\big)$$

necessarily and hence, by Lemma 1b,

$$\psi(y) = \widetilde{g}\big(\widetilde{g}^{-1}(y)(w_0 + y)^k\big) \tag{24}$$

in $\mathbb{C}[\![y]\!]$. Defining ψ by (24), a direct calculation gives that g and ψ satisfy (1′).

(b) If we assume $w_0 \in \mathbb{C}^* \setminus E$, then Proposition 1a shows that for a solution g of (1) necessarily $g(z) = \widetilde{g}(z^k)$, for some $\widetilde{g}$. □

We conclude this section by three remarks.

Remark 1. In the same way as in [10] we can also under the hypotheses of the present paper establish connections between (1′) (and hence, between (1)) and certain Briot–Bouquet differential equations and with analytic iteration groups of type I. We will not give details here.

Remark 2. The two essential steps in the proof of Proposition 2a, namely (A) to write a solution g of (1′) with $\operatorname{ord} g = k$ as $g(z) = T(z)^k$ and to study a linear functional equation for T^{-1}, then (B) to use the Schröder function $S_{\widetilde{\psi}}$ of $\widetilde{\psi}$, with $\widetilde{\psi}(z)^k = \psi(z^k)$, may be interchanged. One may firstly transform (1′) by introducing the Schröder function S_ψ of ψ, and then in the transformed functional equation for $h := S_\psi \circ g$ use the representation $h(z) = R(z)^k$. We will not give the details here.

Remark 3. A third approach to (1′) starts with determining a $\widetilde{g}_0 \in \Gamma_1$ such that for each $c_k \in \mathbb{C}^*$, $g(z) := \widetilde{g}_0(c_k z^k)$ is a solution of (1′). If we substitute this into (1′) we deduce the linear functional equation (2) for $\widetilde{h}_0$ which has a unique solution $\widetilde{h}_0 \in \Gamma_1$. Hence we find by $\widetilde{g}_0(c_k z^k)$ a solution of (1′) with $\widetilde{g}_0(c_k z^k) \equiv c_k z^k \pmod{z^{k+1}}$. We have now to prove that such a solution is unique. To prove this, assume that g_1 and g_2 are distinct solutions of (1′) such that $g_j(z) \equiv c_k z^k \pmod{z^{k+1}}$, for $j = 1, 2$. Then $g_j(z) = c_k z^k + \cdots + c_{l-1} z^{l-1} + c_l^{(j)} z^l + \cdots$, $j = 1, 2$, with $l > k$ and $c_l^{(1)} \neq c^{(2)}$. We substitute these solutions in (1′), which yields

$$g_j\big(w_0 z + z g_j(z)\big) = w_0^k g_j(z) + d_2 g_j(z)^2 + \cdots, \quad j = 1, 2.$$

Subtracting one equation from the other, side by side, we obtain an equation where the right-hand side starts with $w_0^k(c_l^{(1)} - c_l^{(2)})z^l$ while the left-hand side starts with $w_0^l(c_l^{(1)} - c_l^{(2)})z^l$. Since w_0 is not a root of 1, $l > k$, and $c_l^{(1)} \neq c_l^{(2)}$, we get a contradiction. Thus, $g_1 = g_2$.

3. Local analytic solutions

In order to prove that the formal solutions of (1′), described in Proposition 2a, are all convergent we make an additional assumption on w_0 and we already suppose that ψ is holomorphic in a neighborhood of $w = 0$.

For w_0 we assume that either

$$|w_0| \neq 1 \tag{$H_1, 1$}$$

or

$$w_0 \text{ is a Siegel number.} \tag{$H_1, 2$}$$

Below we recall the definition of a Siegel number and some consequences of this definition.

If ψ is given by a convergent power series, then $\widetilde{\psi}$ constructed in the proof of Proposition 2a is convergent. If $|w_0| \neq 1$ or if w_0 is a Siegel number, then the Schröder function $S_{\widetilde{\psi}}$ (see Lemma 3) is a convergent power series, and $w_0 + S_{\widetilde{\psi}}^{-1}(z)^k$ on the left-hand side of (13) as well as $A(z)$ on the left-hand side of (17) are convergent power series. It is then sufficient to prove that under the hypotheses $(H_1, 1)$ or $(H_1, 2)$ the unique solution Y of (17) is a convergent power series since then, successively, V^* in (15), $V^{(0)}$, V, U (see (12)), $T = U^{-1}$, and g (see (6)) and $\widetilde{g}_0$ are convergent power series. We show now that assuming $(H_1, 1)$ or $(H_1, 2)$ we get a convergent power series Y from (17).

Assume firstly $(H_1, 1)$. For the coefficients γ_ν of $Y(z) = \sum_{\nu \geqslant 1} \gamma_\nu z^\nu$ we have, by (19),

$$\gamma_\nu = \left(w_0^\nu - 1\right)^{-1} \alpha_\nu, \quad \nu \geqslant 1. \tag{25}$$

Since $|w_0| \neq 1$ there exists a $\theta_0 > 0$ such that $|w_0^\nu - 1|^{-1} \leqslant \theta_0$ for all $\nu \geqslant 1$, and hence,

$$|\gamma_\nu| \leqslant \theta_0 |\alpha_\nu|, \quad \nu \in \mathbb{N}.$$

Since $A(z) = \sum_{\nu \geqslant 1} \alpha_\nu z^\nu$ is a convergent power series, so is $Y(z)$, and we have proved that all formal solutions of $(1')$ are convergent power series.

Assume secondly that $(H_1, 2)$ holds, i.e., that w_0 is a Siegel number. By a Siegel number (cf. [13]) we mean a complex number $w_0 = \mathrm{e}^{2\pi i \alpha}$ with $\alpha \in [0, 1)$ such that the condition (22) on p. 166 of [13] is satisfied. A consequence of this property of α is that there exists $\mu > 0$ such that

$$\left|w_0^\nu - 1\right|^{-1} < (2\nu)^\mu, \quad \nu \in \mathbb{N} \tag{26}$$

(cf. (2), par. 24, p. 166 of [13]), where the notation has been changed according to our previous notations. Property (26) is essential in the proof of Siegel's theorem on the convergence of the Schröder function, here $S_{\widetilde{\psi}}$.

From (25) and (26) we deduce

$$|\gamma_\nu| \leqslant (2\nu)^\mu |\alpha_\nu|, \quad \nu \in \mathbb{N}. \tag{27}$$

If $\delta > 1$ then there exists $n_0(\delta, \mu) =: n_0 \in \mathbb{N}$ such that $(2\nu)^\mu < \delta^\nu$ if $\nu > n_0$. Therefore (27) yields

$$|\gamma_\nu| < |\alpha_\nu| \delta^\nu, \quad \text{for } \nu > n_0. \tag{28}$$

Since $A(z) = \sum_{\nu \geqslant 1} \alpha_\nu z^\nu$ is a convergent power series there exists $\beta > 0$ such that

$$|\alpha_\nu| < \beta^\nu, \quad \nu > n_0, \tag{29}$$

and from (28) and (29) we get

$$|\alpha_\nu| < (\beta\delta)^\nu, \quad \nu > n_0$$

which means that $Y(z)$ is a convergent power series.

We summarize these results for the generalized Dhombres equation (1) in the following theorem.

Theorem 1. *Let $k \in \mathbb{N}$, $w_0 \in \mathbb{C}$ such that either $|w_0| \neq 1$ or w_0 is a Siegel number. Assume that φ with $\varphi(w) = w_0 + w_0^k(w - w_0) + \cdots$ is holomorphic in a neighborhood of w_0. Then the following holds true*:

(a) *There exists a unique convergent power series $\widetilde{g}_0$, $\widetilde{g}_0(y) \equiv y \pmod{y^2}$ such that the set of all local analytic solutions f of*

$$f\big(zf(z)\big) = \varphi\big(f(z)\big) \tag{1}$$

with $f(0) = w_0$, $f \neq \text{const}$, is given by

$$\big\{f;\ f(z) = w_0 + \widetilde{g}_0\big(c_k z^k\big),\ c_k \in \mathbb{C}^*\big\}.$$

(b) *A function f with $f(z) = w_0 + g(z)$, $\operatorname{ord} g = k \in \mathbb{N}$, holomorphic in a neighborhood of 0, is the solution of a generalized Dhombres equation (1) if and only if there exists a (convergent) power series $\widetilde{g}$ with $\operatorname{ord} \widetilde{g} = 1$ such that*

$$f(z) = w_0 + \widetilde{g}\big(z^k\big)$$

in a neighborhood of 0. For a given f of this form the function φ in (1) is uniquely determined.

Proof. Part (a) has already been proved in detail, part (b) follows from Proposition 3. □

If $0 < |w_0| < 1$, then as in [10], the local analytic solutions of (1) have representations involving infinite products. It should be observed that these representations are not valid, if the solutions are considered as formal power series, when $\mathbb{C}[\![x]\!]$ is equipped with the order topology like in Section 2. However, a detailed analysis would show that these representations make sense if the infinite products are considered as limits in the so-called weak (coefficientwise) topology of $\mathbb{C}[\![x]\!]$, also if φ (or ψ) are not necessarily convergent power series. We have

Theorem 2. *Assume* $0 < |w_0| < 1$, *and let* $\varphi(w) = w_0 + w_0^k(w - w_0) + d_2(w - w_0)^2 + \cdots$ *be holomorphic at* w_0. *Then the local analytic solutions* f *of* (1) *with* $f \neq w_0$ *are given by*

$$f(z) = w_0 + \left(\left[S_\psi(\zeta) \cdot \prod_{\nu \geqslant 1} \frac{1}{1 + w_0^{-1}(\widetilde{\psi}^\nu(\zeta))^k} \right]_{\zeta = t_1 z}^{[-1]} \right)^k \tag{30}$$

with $t_1 \in \mathbb{C}^*$ *and also by*

$$f(z) = w_0 + \left(\left[\zeta \cdot \prod_{\nu \geqslant 0} \frac{\widetilde{\psi}^{\nu+1}(\zeta)}{\widetilde{\psi}^\nu(\zeta)(w_0 + (\widetilde{\psi}^\nu(\zeta))^k)} \right]_{\zeta = t_1 z}^{[-1]} \right)^k \tag{31}$$

with $t_1 \in \mathbb{C}^*$, *where the infinite products converge uniformly in each compact subset of certain neighborhood of* 0, *depending on* f, $^{[-1]}$ *denotes the inverse with respect to substitution, and* $\zeta = t_1 z$ *denotes the substitution of* $t_1 z$ *for* ζ.

Proof. Since it is based on standard calculations involving results obtained in the proof of Proposition 2, we give here only a sketch. The classical iteration procedure for solving linear functional equations can be applied to (15) or to (17). Since $0 < |w_0| < 1$, if the number of iterations tends to ∞ we obtain (30) and (31), and the uniform convergence on compact subsets, using well-known standard estimates for the iterates $\widetilde{\psi}^\nu$, $\nu \geqslant 0$. The iteration process is particularly simple for (17) where it yields $Y(z) = -\sum_{\nu \geqslant 0} A(w_0^\nu z)$. A representation similar to (30) or (31) can also be proved for the function $\widetilde{g}_0$ of Theorem 1 which generates all local analytic solutions. □

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