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Spectral transverse instability of solitary waves in Korteweg fluids

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ABSTRACT

The motion of Korteweg fluids is governed by the Euler–Korteweg model, which admits planar solitary waves for nonmonotone pressure laws such as the van der Waals law below critical temperature. In an earlier work with Danchin, Descombes and Jamet, it was shown by variational arguments and numerical computations that some of these solitary waves are stable in one space dimension. The purpose here is to study their stability with respect to transverse perturbations in several space dimensions. By Evans functions techniques and Rouché's theorem, it is shown that transverse perturbations of *large* wave length always destabilize solitary waves in the Euler–Korteweg model, whereas energy estimates show that perturbations of *short* wave length tend to stabilize them.

1. Introduction

A Korteweg fluid is by definition endowed with internal capillarity, and sensitive to surface tension effects. Capillary/surface tension effects are known to be important in thin tubes, but also at liquid-vapor interfaces, which is the main application we have in mind. As shown for instance in [14,20], the motion of a Korteweg fluid is governed by the Euler–Korteweg model, made of the standard Euler equations for compressible fluids supplemented with the so-called Korteweg tensor, which encodes capillary/surface tension effects. This tensor is obtained by allowing the free energy of the fluid to depend not only on its density ρ but also on $\nabla\rho$, the density gradient, in the following way

$$F(\rho,\nabla\rho) = F_0(\rho) + \frac{1}{2}K(\rho)|\nabla\rho|^2,$$

where $K(\rho)$ is a capillarity coefficient that can depend on ρ. Then, if dissipation phenomena are neglected, the classical principles of mechanics lead to the following conservation laws for isothermal flows

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho\mathbf{u}) = 0, \\ \partial_t(\rho\mathbf{u}) + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla p = \nabla\big(\rho\operatorname{div}(K\nabla\rho)\big) - \operatorname{div}(K\nabla\rho\otimes\nabla\rho), \end{cases} \tag{1.1}$$

where $\mathbf{u}$ is the velocity of the fluid, and $p := \rho\frac{\partial F}{\partial\rho} - F$ is a generalized pressure depending on both ρ and $\nabla\rho$. A model of this kind also arises in quantum hydrodynamics, with $\rho K \equiv$ constant, see for instance [16]. When $p_0 := \rho\frac{\mathrm{d}F_0}{\mathrm{d}\rho} - F_0$ is nonmonotone, which is typically the case with van der Waals fluids below critical temperature (*i.e.* when $p_0 = \frac{RT\rho}{1-b\rho} - a\rho^2$ with $T < (8a)/(27bR)$), the Euler–Korteweg model (1.1) is known to admit two classes of planar traveling waves:

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- heteroclinic waves, representing diffuse phase boundaries [3] (a 'phase' being defined as a region where p_0 is monotonically increasing with ρ, and the term 'diffuse' meaning the interface is not sharp, its width even becoming infinite when approaching the critical temperature in van der Waals fluids),
- homoclinic, or *solitary* waves [6], of which the physical significance remains unclear.

The neutral spectral stability of diffuse phase boundaries was shown by 'direct' energy estimates in [3], and an orbital stability result was proved in [6] in one space dimension, by introducing a suitable Hamiltonian framework and using variational arguments analogous to those of Grillakis, Shatah and Strauss [12].

Concerning solitary waves, several kinds of them were identified in [6], depending on the location of their endstates. Their one-dimensional orbital stability was investigated by (slightly) adapting the method of Grillakis, Shatah and Strauss to solitary waves with nonzero endpoints. The starting point was their interpretation as critical points under constraint of the total energy, the constraint being linked to spatial translational invariance. This led to a sufficient condition for their stability. Numerical evidence was given that this condition is satisfied by some dynamic solitary waves, whereas it fails for solitary waves closer to thermodynamic equilibrium. To be more precise, for a solitary wave of speed σ, the stability condition is $m''(\sigma) > 0$, where m is the constrained energy of the wave, which can be evaluated in the phase plane (*i.e.* without integrating the ODE governing the wave profile, see Section 2 below for more details). Various plottings were displayed in [6], showing that for some solitary waves m is indeed strictly convex ($m''(\sigma) > 0$), and for others it can be strictly concave ($m''(\sigma) < 0$). This paper will mainly concern solitary waves for which $m''(\sigma) > 0$.

Another approach to (spectral) stability is by *Evans functions* techniques, as for instance in [18], where Pego and Weinstein show that $m''(\sigma)$ is linked to the low frequency behavior of the Evans function associated with the linearized equations about solitary waves for three Hamiltonian PDEs (namely, generalized Korteweg–de Vries, Benjamin–Bona–Mahoney, and Boussinesq equations). A similar link has been pointed out by Zumbrun [22] (and independently by Bridges and Derks [9]) for a special case (with a particular capillarity) of (1.1) in Lagrangian coordinates. In particular, Zumbrun proved that $m''(\sigma) \geqslant 0$ is necessary for linearized stability in that context. This result will be revisited here in Section 3, with general capillarities in Eulerian coordinates. This will serve as a milestone for the *multi-dimensional* stability analysis, which is the main purpose of the present paper.

Variational tools are not appropriate for the stability analysis of planar solitary waves in several space dimensions, because we lose their interpretation as critical points (under constraint) of the total energy (which is not even properly defined) in the whole space $\mathbb{R}^d$, $d \geqslant 2$. Nevertheless, the Evans function technique does extend to arbitrary space dimensions, and its low frequency behavior can be computed explicitly, as far as the Euler–Korteweg system is concerned. This will be done in Section 4, by computations resembling the one-dimensional ones but with more equations (to take into account transverse velocities) and including a wave vector η corresponding to perturbations in transverse directions. Our main result (Theorem 2) follows from the low frequency behavior of the Evans function and an argument pointed out by Zumbrun and Serre [23] in the framework of viscous shocks. It says that one-d stable planar solitary wave solutions of the Euler–Korteweg model are spectrally unstable with respect to transverse perturbations of large wave length. By contrast, energy estimates as performed in [3] show that they are neutrally stable with respect to transverse perturbations of short wave length.

2. Main assumptions and definitions

The capillarity $K(\rho)$ will be assumed to depend smoothly of ρ, and to be positive for all positive values of ρ. Recall that by definition,

$$p(\rho, \nabla\rho) = p_0(\rho) + \frac{1}{2}\big(\rho K'(\rho) - K(\rho)\big)|\nabla\rho|^2, \quad p_0 := \rho\frac{\mathrm{d}F_0}{\mathrm{d}\rho} - F_0.$$

Then for smooth solutions, (1.1) is easily seen to be equivalent to

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho\mathbf{u}) = 0, \\ \partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla g_0 = \nabla\left(K\Delta\rho + \dfrac{1}{2}K'_\rho|\nabla\rho|^2\right), \end{cases} \tag{2.2}$$

where $K'_\rho := \frac{\mathrm{d}K}{\mathrm{d}\rho}$, and $g_0 := \frac{\mathrm{d}F_0}{\mathrm{d}\rho}$ is the standard chemical potential of the fluid, which satisfies

$$\frac{\mathrm{d}g_0}{\mathrm{d}\rho} = \frac{1}{\rho}\frac{\mathrm{d}p_0}{\mathrm{d}\rho}.$$

In one space dimension, (2.2) reduces to

$$\begin{cases} \partial_t\rho + \partial_x(\rho u) = 0, \\ \partial_t u + u\partial_x u + \partial_x(g_0) = \partial_x\left(K\partial_x^2\rho + \dfrac{1}{2}K'_\rho(\partial_x\rho)^2\right), \end{cases} \tag{2.3}$$

which admits the formal Hamiltonian formulation

$$\partial_t\mathbf{U} = \mathcal{J}\delta\mathcal{H}[\mathbf{U}] \tag{2.4}$$

where

$$\mathbf{U} := \begin{pmatrix} \rho \\ u \end{pmatrix}, \qquad \mathcal{J} := \begin{pmatrix} 0 & -\partial_x \\ -\partial_x & 0 \end{pmatrix},$$

$$\mathcal{H}[\mathbf{U}] := \int H(\mathbf{U}, \partial_x \mathbf{U})\,\mathrm{d}x, \quad H(\mathbf{U}, \partial_x \mathbf{U}) = \frac{1}{2}\rho u^2 + F_0(\rho) + \frac{1}{2}K(\rho)(\partial_x \rho)^2,$$

and

$$\delta\mathcal{H}[\mathbf{U}] = \begin{pmatrix} \frac{1}{2}u^2 + g_0(\rho) - K(\rho)\partial_x^2\rho - \frac{1}{2}\frac{\mathrm{d}K}{\mathrm{d}\rho}(\rho)(\partial_x\rho)^2 \\ \rho u \end{pmatrix}.$$

To make this formulation correct we may prescribe the behavior of $\mathbf{U}$ at infinity, and change the integral of H accordingly, in order to turn it into a convergent one. As far as perturbations of solitary waves are concerned, we may assume that $\mathbf{U}$ converges (exponentially fast) to some limit $\mathbf{U}_\infty$ at $\pm\infty$. Then

$$\widetilde{\mathcal{H}}[\mathbf{U}; \mathbf{U}_\infty] := \int \big(H(\mathbf{U}, \partial_x \mathbf{U}) - H(\mathbf{U}_\infty, 0) - \delta\mathcal{H}[\mathbf{U}_\infty]\cdot(\mathbf{U} - \mathbf{U}_\infty)\big)\,\mathrm{d}x$$

is well defined for $\mathbf{U} \in \mathbf{U}_\infty + (H^1 \times L^2)$, and for such $\mathbf{U}$, (2.3) equivalently reads

$$\partial_t \mathbf{U} = \mathcal{J}\delta\widetilde{\mathcal{H}}[\mathbf{U}; \mathbf{U}_\infty]. \tag{2.5}$$

Here above, the notation δ stands for the variational gradient with respect to $\mathbf{U}$, the endstate $\mathbf{U}_\infty$ being kept fixed. A solitary wave is by definition a homoclinic traveling wave solution, that is, a solution that propagates a same profile, say $\underline{\mathbf{U}}$, at constant speed, say σ, with a same endstate $\mathbf{U}_\infty$ at $+\infty$ and $-\infty$. For a nonmonotone pressure law $p_0 = p_0(\rho)$, or equivalently, for a nonconvex free energy $F_0 = F_0(\rho)$, (2.3) is known to admit solitary waves, that is, global smooth solutions of the form

$$\mathbf{U}(x,t) = \underline{\mathbf{U}}(x - \sigma t), \qquad \lim_{\xi\to\pm\infty} \underline{\mathbf{U}}(\xi) = \mathbf{U}_\infty.$$

The existence of solitary waves follows from a simple phase portrait analysis of the governing ODEs, which appear to be Hamiltonian too (a general fact, see [2, pp. 11–12]), see [6] for more details. Solitary waves – unlike heteroclinic connections – persist under perturbation of the speed σ. Moreover, solitary waves can be viewed, in one space dimension, as critical points of the Hamiltonian $\widetilde{\mathcal{H}}$ under the constraint

$$\widetilde{\mathcal{Q}}[\mathbf{U}; \mathbf{U}_\infty] := \int \big((\rho - \rho_\infty)(u - u_\infty)\big)\,\mathrm{d}x.$$

Indeed, working in the abstract Hamiltonian setting described above, we may write the traveling wave ODEs as

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\big(-\sigma\underline{\mathbf{U}} + \mathbf{J}\delta\widetilde{\mathcal{H}}[\underline{\mathbf{U}}; \mathbf{U}_\infty]\big) = 0, \quad \mathbf{J} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \xi = x - \sigma t,$$

hence, multiplying the ODE by $\mathbf{J}$ and using that $\mathbf{J}^2 = \mathbf{I}$,

$$\delta\widetilde{\mathcal{H}}[\underline{\mathbf{U}}; \mathbf{U}_\infty] - \sigma\mathbf{J}\underline{\mathbf{U}} \equiv \text{constant}.$$

Evaluating at $\pm\infty$, we see that the constant must be $-\sigma\mathbf{J}\mathbf{U}_\infty$, and since $\mathbf{J}(\underline{\mathbf{U}} - \mathbf{U}_\infty) = \delta\widetilde{\mathcal{Q}}[\underline{\mathbf{U}}; \mathbf{U}_\infty]$, we obtain

$$\delta(\widetilde{\mathcal{H}} - \sigma\widetilde{\mathcal{Q}})[\underline{\mathbf{U}}; \mathbf{U}_\infty] \equiv \mathbf{0}. \tag{2.6}$$

As claimed above, this means that $\underline{\mathbf{U}}$ is a critical point of $\widetilde{\mathcal{H}}$ under the constraint $\widetilde{\mathcal{Q}}$, with associated Lagrange multiplier σ (the speed of the wave). The fact that $\widetilde{\mathcal{Q}}$ is a conserved quantity along solutions of (2.3) (in $\mathbf{U}_\infty + \mathcal{C}^1(\mathbb{R}; H^1 \times L^2)$) is linked to translational invariance. Indeed, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\mathcal{Q}}[\mathbf{U}; \mathbf{U}_\infty] &= \int \big(\delta\widetilde{\mathcal{Q}}[\mathbf{U}; \mathbf{U}_\infty]\cdot\partial_t\mathbf{U}\big)\,\mathrm{d}x = -\int \big(\mathbf{J}(\mathbf{U} - \mathbf{U}_\infty)\cdot\mathbf{J}\partial_x\delta\widetilde{\mathcal{H}}[\mathbf{U}; \mathbf{U}_\infty]\big)\,\mathrm{d}x \\ &= \int \big(\delta\widetilde{\mathcal{H}}[\mathbf{U}; \mathbf{U}_\infty]\cdot\partial_x\mathbf{U}\big)\,\mathrm{d}x \end{aligned}$$

after integration by parts (and using that $\mathbf{J}^t\mathbf{J} = \mathbf{I}$), and the nullity of the last integral follows from the equality

$$\frac{\mathrm{d}}{\mathrm{d}s}\widetilde{\mathcal{H}}[\mathbf{U}_s; \mathbf{U}_\infty] = 0$$

for $\mathbf{U}_s(x,t) := \mathbf{U}(x+s,t)$. This very same translational invariance also implies that solitary waves of given speed σ and endstate $\mathbf{U}_\infty$, form a one-parameter family $(\underline{\mathbf{U}}_s)_{s\in\mathbb{R}}$, with $\underline{\mathbf{U}}(\xi) = \underline{\mathbf{U}}(\xi + s)$. In addition, we see on (2.6) that

$$\widetilde{\mathcal{H}}[\underline{\mathbf{U}}_s; \mathbf{U}_\infty] - \sigma\widetilde{\mathcal{Q}}[\underline{\mathbf{U}}_s; \mathbf{U}_\infty]$$

does not depend on s. So there is no ambiguity in defining

$$m(\sigma;\mathbf{U}_\infty) := \widetilde{\mathcal{H}}[\underline{\mathbf{U}};\mathbf{U}_\infty] - \sigma\,\widetilde{\mathcal{Q}}[\underline{\mathbf{U}};\mathbf{U}_\infty].$$

This constrained energy plays a crucial role in the one-dimensional stability analysis of the wave $\underline{\mathbf{U}}$. As observed in [6], the actual computation of $m(\sigma;\mathbf{U}_\infty)$ does not require the resolution of the traveling wave ODEs, and can be done in the phase plane. Indeed, $\widetilde{\mathcal{H}}[\mathbf{U};\mathbf{U}_\infty]$ has the special form of

$$\widetilde{\mathcal{H}}[\mathbf{U};\mathbf{U}_\infty] = \int\left(\widetilde{H}_0(\mathbf{U};\mathbf{U}_\infty) + \frac{1}{2}K(\rho)(\partial_x\rho)^2\right)\mathrm{d}x,$$

where

$$\widetilde{H}_0(\mathbf{U};\mathbf{U}_\infty) := H_0(\mathbf{U}) - H_0(\mathbf{U}_\infty) - \mathrm{d}H_0(\mathbf{U}_\infty)\cdot(\mathbf{U}-\mathbf{U}_\infty), \quad H_0(\mathbf{U}) := \frac{1}{2}\rho u^2 + F_0(\rho).$$

(The actual expression of $\widetilde{H}_0(\mathbf{U};\mathbf{U}_\infty)$ is rather complicated and is not important for what follows.) This implies that

$$\delta\widetilde{\mathcal{H}}[\mathbf{U};\mathbf{U}_\infty]\cdot\partial_x\mathbf{U} = \partial_x\left(\widetilde{H}_0(\mathbf{U};\mathbf{U}_\infty) - \frac{1}{2}K(\rho)(\partial_x\rho)^2\right),$$

so that $\mathrm{d}\underline{\mathbf{U}}/\mathrm{d}\xi$ is an integrating factor of (2.6). The integrated equation reads

$$\widetilde{H}_0(\underline{\mathbf{U}};\mathbf{U}_\infty) - \sigma(\underline{\rho}-\rho_\infty)(\underline{u}-u_\infty) - \frac{1}{2}K(\underline{\rho})\left(\frac{\mathrm{d}\underline{\rho}}{\mathrm{d}\xi}\right)^2 \equiv 0,$$

hence

$$m(\sigma;\mathbf{U}_\infty) = \int K(\underline{\rho})\left(\frac{\mathrm{d}\underline{\rho}}{\mathrm{d}\xi}\right)^2\mathrm{d}\xi = 2\int_{\xi_0}^{+\infty} K(\underline{\rho})\left(\frac{\mathrm{d}\underline{\rho}}{\mathrm{d}\xi}\right)^2\mathrm{d}\xi,$$

where ξ_0 is the center of symmetry of the soliton. To compute $m(\sigma;\mathbf{U}_\infty)$ in the phase plane it suffices to make the change of variables $r = \underline{\rho}(\xi)$ for $\xi\in(\xi_0,+\infty)$ and use the formula

$$\frac{\mathrm{d}\underline{\rho}}{\mathrm{d}\xi} = \pm\left(\frac{2}{K(\underline{\rho})}\big(\widetilde{H}_0(\underline{\mathbf{U}};\mathbf{U}_\infty) - \sigma(\underline{\rho}-\rho_\infty)(\underline{u}-u_\infty)\big)\right)^{1/2}.$$

3. One-dimensional stability criterion

In what follows we omit the tilda on $\mathcal{H}$ and $\mathcal{Q}$ for simplicity, and we emphasize with a superscript the dependence on σ of solitary waves.

Theorem 1. *We fix an endstate $\mathbf{U}_\infty$, and assume that, for all σ in an open interval there exists a solitary wave solution of* (2.3), $\underline{\mathbf{U}}^\sigma$, *of speed σ and endstate $\mathbf{U}_\infty$. We consider the function m defined by*

$$m(\sigma;\mathbf{U}_\infty) := \mathcal{H}\big[\underline{\mathbf{U}}^\sigma;\mathbf{U}_\infty\big] - \sigma\,\mathcal{Q}\big[\underline{\mathbf{U}}^\sigma;\mathbf{U}_\infty\big],$$

the functionals $\mathcal{H}$ and $\mathcal{Q}$ being defined by

$$\mathcal{H}[\mathbf{U};\mathbf{U}_\infty] := \int\left(H_0(\mathbf{U}) - H_0(\mathbf{U}_\infty) + \frac{1}{2}K(\rho)(\partial_x\rho)^2 - \partial_\rho H_0(\mathbf{U}_\infty)(\rho-\rho_\infty) - \partial_u H_0(\mathbf{U}_\infty)(u-u_\infty)\right)\mathrm{d}x$$

with

$$H_0(\rho,u) := \frac{1}{2}\rho u^2 + F_0(\rho),$$

and

$$\mathcal{Q}[\mathbf{U};\mathbf{U}_\infty] := \int\big((\rho-\rho_\infty)(u-u_\infty)\big)\,\mathrm{d}x.$$

- *The solitary wave $\underline{\mathbf{U}}^\sigma$ is* orbitally stable *if*

$$\frac{\partial^2 m}{\partial\sigma^2}(\sigma;\mathbf{U}_\infty) > 0.$$

- *It is* linearly unstable *if*

$$\frac{\partial^2 m}{\partial\sigma^2}(\sigma;\mathbf{U}_\infty) < 0.$$

Remark 1. As mentioned before, solitary waves can be found by phase portrait analysis. For double-well free energy, typical of van der Waals fluids, this matter is investigated in details in [6], with a classification of solitary waves according to their endstate (liquid or vapor) and their amplitude.

Proof of Theorem 1. The sufficient condition $m''(\sigma) > 0$ for orbital stability can be deduced from the abstract result of Grillakis, Shatah and Strauss [12]: this was already pointed out by Bona and Sachs in [8] for the 'good' Boussinesq equation, a special case of (2.3) rewritten in Lagrangian coordinates; for the general system (2.3), see [6]. That $m''(\sigma) < 0$ implies instability cannot be deduced from the Grillakis–Shatah–Strauss result – which is an if and only if result for orbital stability – basically because the operator $\mathcal{J}$ is not onto. However, an Evans function calculation does yield a necessary condition for stability, as was shown by Zumbrun [22] in a Lagrangian framework (also see [9]) with a constant capillarity coefficient κ, related to the Eulerian capillarity coefficient by $\kappa = K\rho^5$. We are going to perform this calculation in the Eulerian framework with an arbitrary capillarity coefficient K. We first make standard observations on the profile equation

$$(\delta\mathcal{H} - \sigma\delta\mathcal{Q})\big[\underline{\mathbf{U}}^\sigma; \mathbf{U}_\infty\big] \equiv 0 \tag{3.7}$$

(which is just (2.6) with slightly different notations). The variational form of (3.7) has two crucial consequences regarding the second-order differential operator

$$\mathcal{L}^\sigma := (\operatorname{Hess}\mathcal{H} - \sigma \operatorname{Hess}\mathcal{Q})\big[\underline{\mathbf{U}}^\sigma; \mathbf{U}_\infty\big].$$

The first consequence is linked to translational invariance. Indeed, all translated profiles $\underline{\mathbf{U}}^\sigma_s : \xi \mapsto \underline{\mathbf{U}}^\sigma(\xi + s)$ satisfy the same equation (3.7). Therefore, differentiating

$$(\delta\mathcal{H} - \sigma\delta\mathcal{Q})\big[\underline{\mathbf{U}}^\sigma_s; \mathbf{U}_\infty\big] \equiv 0$$

with respect to s and evaluating at $s = 0$ we find that $\partial_\xi\underline{\mathbf{U}}^\sigma$ is in the kernel of $\mathcal{L}^\sigma$. The second consequence is obtained by differentiating (3.7) with respect to σ. This yields

$$\mathcal{L}^\sigma \cdot \partial_\sigma\underline{\mathbf{U}}^\sigma = \delta\mathcal{Q}\big[\underline{\mathbf{U}}^\sigma; \mathbf{U}_\infty\big]. \tag{3.8}$$

To address the linearized stability of $\underline{\mathbf{U}}^\sigma$, the first, usual step consists in making a change of Galilean frame $(x, t) \mapsto (\xi := x - \sigma t, t)$, so as to make the wave stationary. This clearly changes the abstract form of (2.3),

$$\partial_t\mathbf{U} = -\partial_x\mathbf{J}\delta\mathcal{H}[\mathbf{U}; \mathbf{U}_\infty],$$

into

$$\partial_t\mathbf{U} - \sigma\,\partial_\xi\mathbf{U} = -\partial_\xi\mathbf{J}\delta\mathcal{H}[\mathbf{U}; \mathbf{U}_\infty].$$

Linearizing about $\underline{\mathbf{U}}^\sigma$ we are led to

$$\partial_t\dot{\mathbf{U}} - \sigma\,\partial_\xi\dot{\mathbf{U}} = -\partial_\xi\mathbf{J}(\operatorname{Hess}\mathcal{H})\big[\underline{\mathbf{U}}^\sigma; \mathbf{U}_\infty\big] \cdot \dot{\mathbf{U}},$$

or equivalently, observing that $\dot{\mathbf{U}} = \mathbf{J}^2\dot{\mathbf{U}} = \mathbf{J}(\operatorname{Hess}\mathcal{Q})[\underline{\mathbf{U}}^\sigma; \mathbf{U}_\infty] \cdot \dot{\mathbf{U}}$,

$$\partial_t\dot{\mathbf{U}} = -\partial_\xi\mathbf{J}\mathcal{L}^\sigma \cdot \dot{\mathbf{U}}.$$

Introducing the third-order differential operator $L^\sigma := -\partial_\xi\mathbf{J}\mathcal{L}^\sigma$, we infer from (3.8) that $L^\sigma \cdot \partial_\sigma\underline{\mathbf{U}}^\sigma = -\partial_\xi\mathbf{J}\delta\mathcal{Q}[\underline{\mathbf{U}}^\sigma; \mathbf{U}_\infty]$, that is,

$$L^\sigma \cdot \partial_\sigma\underline{\mathbf{U}}^\sigma = -\partial_\xi\underline{\mathbf{U}}^\sigma. \tag{3.9}$$

Since

$$L^\sigma \cdot \partial_\xi\underline{\mathbf{U}}^\sigma = -\partial_\xi\mathbf{J}\mathcal{L}^\sigma \cdot \partial_\xi\underline{\mathbf{U}}^\sigma = 0,$$

this means that 0 is an eigenvalue of L^σ of algebraic multiplicity greater or equal to 2. It will turn out that, if

$$\frac{\partial^2 m}{\partial\sigma^2}(\sigma; \mathbf{U}_\infty) \neq 0,$$

the eigenvalue 0 is exactly of multiplicity 2, or equivalently, the Evans function associated to L^σ has a zero of multiplicity two at zero. This will follow from Lemma 1 below and the more explicit formula

$$\frac{\partial^2 m}{\partial\sigma^2}(\sigma; \mathbf{U}_\infty) = -\int \big((\underline{\rho}^\sigma - \rho_\infty)\partial_\sigma\underline{u}^\sigma + (\underline{u}^\sigma - u_\infty)\partial_\sigma\underline{\rho}^\sigma\big)\,\mathrm{d}\xi. \tag{3.10}$$

The latter comes from the definition of m, which implies

$$\frac{\partial m}{\partial\sigma}(\sigma; \mathbf{U}_\infty) = \int (\delta\mathcal{H} - \sigma\delta\mathcal{Q})\big[\underline{\mathbf{U}}^\sigma; \mathbf{U}_\infty\big] \cdot \partial_\sigma\underline{\mathbf{U}}^\sigma\,\mathrm{d}\xi - \mathcal{Q}\big[\underline{\mathbf{U}}^\sigma; \mathbf{U}_\infty\big] = -\mathcal{Q}\big[\underline{\mathbf{U}}^\sigma; \mathbf{U}_\infty\big]$$

because of (3.7), hence

$$\frac{\partial^2 m}{\partial\sigma^2}(\sigma;\mathbf{U}_\infty) = -\int \delta\mathcal{Q}\big[\underline{\mathbf{U}}^\sigma;\mathbf{U}_\infty\big]\cdot\partial_\sigma\underline{\mathbf{U}}^\sigma\,\mathrm{d}\xi. \tag{3.11}$$

Lemma 1 below shows that $\partial^2 m/\partial\sigma^2$ is proportional to the second-order derivative of the Evans function at $\lambda = 0$. More precisely, if $\partial^2 m/\partial\sigma^2$ is negative, then the Evans function changes sign in between 0 and $+\infty$, so that by the mean value theorem it must vanish at some positive λ, which is therefore an unstable eigenvalue of the linear operator L^σ. □

Remark 2. The profile $\underline{\mathbf{U}}^\sigma$ is a critical point of the constrained functional $\mathcal{H}-\sigma\mathcal{Q}$, and the Hessian at $\underline{\mathbf{U}}^\sigma$ of that functional is precisely

$$\mathcal{L}^\sigma = \begin{pmatrix}\mathcal{M}_0 & \underline{u}^\sigma-\sigma\\ \underline{u}^\sigma-\sigma & \underline{\rho}^\sigma\end{pmatrix},\quad \mathcal{M}_0 := -\partial_\xi\underline{K}^\sigma\partial_\xi + \underline{\alpha}^\sigma \quad \text{with } \underline{K}^\sigma := K\big(\underline{\rho}^\sigma\big),\quad \text{and}$$

$$\underline{\alpha}^\sigma := \frac{\mathrm{d}g_0}{\mathrm{d}\rho}\big(\underline{\rho}^\sigma\big) - \frac{\mathrm{d}K}{\mathrm{d}\rho}\big(\underline{\rho}^\sigma\big)\partial_\xi^2\underline{\rho}^\sigma - \frac{1}{2}\frac{\mathrm{d}^2K}{\mathrm{d}\rho^2}\big(\underline{\rho}^\sigma\big)\big(\partial_\xi\underline{\rho}^\sigma\big)^2.$$

The operator $\mathcal{L}^\sigma$ is *not* monotone if $\underline{\mathbf{U}}^\sigma$ is homoclinic. It would be monotone if the Sturm–Liouville operator

$$\mathcal{M} := \mathcal{M}_0 - \frac{1}{\underline{\rho}^\sigma}\big(\underline{u}^\sigma-\sigma\big)^2$$

were so. But, $\mathcal{L}^\sigma\cdot\partial_\xi\underline{\mathbf{U}}^\sigma = 0$ implies that $\partial_\xi\underline{\rho}^\sigma$ is in the kernel of $\mathcal{M}$, and since $\partial_\xi\underline{\rho}^\sigma$ vanishes (once), 0 is the *second* eigenvalue of $\mathcal{M}$. In fact, this implies that 0 is also the second eigenvalue of $\mathcal{L}^\sigma$ (see Appendix B in [6] for details). Note in addition that by (3.8) and (3.11),

$$\frac{\partial^2 m}{\partial\sigma^2}(\sigma;\mathbf{U}_\infty) = -\big\langle\mathcal{L}^\sigma\cdot\partial_\sigma\underline{\mathbf{U}}^\sigma,\partial_\sigma\underline{\mathbf{U}}^\sigma\big\rangle_{L^2}.$$

Hence the stable case $\partial^2 m/\partial\sigma^2 > 0$ corresponds to when

$$\big\langle\mathcal{L}^\sigma\cdot\partial_\sigma\underline{\mathbf{U}}^\sigma,\partial_\sigma\underline{\mathbf{U}}^\sigma\big\rangle_{L^2} < 0.$$

The main result in [12] shows that this 'bad' direction $\partial_\sigma\underline{\mathbf{U}}^\sigma$ can then be factored out, in that

$$\big\langle\mathcal{L}^\sigma\cdot\mathbf{Y},\mathbf{Y}\big\rangle_{L^2} \geqslant 0 \quad \text{for all } \mathbf{Y} \quad \text{such that} \quad \big\langle\delta\mathcal{Q}\big[\underline{\mathbf{U}}^\sigma;\mathbf{U}_\infty\big],\mathbf{Y}\big\rangle_{L^2} = 0.$$

Lemma 1. *If* (3.7) *admits a homoclinic solution then the endstate is necessarily subsonic, that is,*

$$\frac{\mathrm{d}p_0}{\mathrm{d}\rho}(\rho_\infty) > (u_\infty-\sigma)^2, \tag{3.12}$$

and the essential spectrum of the linear operator

$$L^\sigma = -\partial_\xi\mathbf{J}(\mathrm{Hess}\,\mathcal{H}-\sigma\,\mathrm{Hess}\,\mathcal{Q})\big[\underline{\mathbf{U}}^\sigma;\mathbf{U}_\infty\big]$$

consists of the imaginary axis. Furthermore, L^σ can be associated with a smooth Evans function $D^\sigma:\lambda\in[0,+\infty)\to\mathbb{R}$, such that

$$\forall\lambda>0,\quad \big(D^\sigma(\lambda)=0 \iff \mathrm{Ker}\big(L^\sigma-\lambda\big)\neq\{0\}\big),$$

and $D^\sigma(0)=0$, $(D^\sigma)'(0)=0$, $D^\sigma(\lambda)>0$ for $\lambda\gg 1$,

$$\mathrm{sgn}\big(D^\sigma\big)''(0) = -\,\mathrm{sgn}\int\big(\big(\underline{\rho}^\sigma-\rho_\infty\big)\partial_\sigma\underline{u}^\sigma + \big(\underline{u}^\sigma-u_\infty\big)\partial_\sigma\underline{\rho}^\sigma\big)\,\mathrm{d}\xi.$$

Proof. The profile equation (3.7) can be rewritten more explicitly as

$$\begin{cases}\underline{\rho}^\sigma\big(\underline{u}^\sigma-\sigma\big) \equiv \rho_\infty(u_\infty-\sigma),\\ K\big(\underline{\rho}^\sigma\big)\partial_\xi^2\underline{\rho}^\sigma + \frac{1}{2}\partial_\xi K\big(\underline{\rho}^\sigma\big)\partial_\xi\underline{\rho}^\sigma - g_0\big(\underline{\rho}^\sigma\big) + g_0(\rho_\infty) - \frac{1}{2}\big(\underline{u}^\sigma-\sigma\big)^2 + \frac{1}{2}(u_\infty-\sigma)^2 = 0.\end{cases} \tag{3.13}$$

• *Subsonicity of the enstate.* We may eliminate the velocity $\underline{u}^\sigma$ from (3.13) and rewrite the second equation (of second-order) as the planar system

$$\begin{cases}\phi' = \dfrac{1}{\sqrt{K(\phi)}}\psi,\\ \psi' = \dfrac{1}{\sqrt{K(\phi)}}\left(g_0(\phi) + \dfrac{1}{2}\dfrac{j^2}{\phi^2} - \mu\right),\end{cases} \tag{3.14}$$

with the simplifying notations $\phi := \underline{\rho}^\sigma$, $j := \rho_\infty(u_\infty - \sigma)$, and $\mu := g_0(\rho_\infty) + \frac{1}{2}\frac{j^2}{\rho_\infty^2}$. (Note that $\psi = \sqrt{K(\phi)}\phi'$ implies $\psi' = (K\phi'' + \frac{1}{2}K(\phi)'\phi')/\sqrt{K(\phi)}$.) The matrix of the linearized system at $(\rho_\infty, 0)$ is

$$\frac{1}{\sqrt{K(\phi)}}\begin{pmatrix} 0 & 1 \\ \frac{\mathrm{d}g_0}{\mathrm{d}\rho}(\rho_\infty) - \frac{j^2}{\rho_\infty^3} & 0 \end{pmatrix},$$

which is hyperbolic if and only if

$$\frac{1}{\rho_\infty}\frac{\mathrm{d}p_0}{\mathrm{d}\rho}(\rho_\infty) = \frac{\mathrm{d}g_0}{\mathrm{d}\rho}(\rho_\infty) > \frac{j^2}{\rho_\infty^3} = \frac{(u_\infty - \sigma)^2}{\rho_\infty}.$$

In other words, the fixed point $(\rho_\infty, 0)$ of (3.14) is a saddle-point if (3.12) holds true, and a center if $\frac{\mathrm{d}p_0}{\mathrm{d}\rho}(\rho_\infty) < (u_\infty - \sigma)^2$. For a homoclinic connection to exist, $(\rho_\infty, 0)$ must be a saddle-point, hence the necessary condition (3.12). Note that (3.12) implies in particular

$$\frac{\mathrm{d}p_0}{\mathrm{d}\rho}(\rho_\infty) > 0,$$

which means that the density ρ_∞ corresponds to a thermodynamically stable state, where we have a real sound speed

$$c_\infty := \sqrt{\frac{\mathrm{d}p_0}{\mathrm{d}\rho}(\rho_\infty)}.$$

(Recall that the existence and classification of solitary waves has been discussed in [6,7].)

• *Essential spectrum of the linearized operator.* Regarding the essential spectrum of L^σ, we have to concentrate on the asymptotic operator L^σ_∞, obtained by freezing the coefficients at $\pm\infty$,

$$L^\sigma_\infty \cdot \dot{\mathbf{U}} := \begin{pmatrix} -(u_\infty - \sigma)\partial_\xi\dot\rho - \rho_\infty\partial_\xi\dot u \\ -(u_\infty - \sigma)\partial_\xi\dot u - \frac{\mathrm{d}g_0}{\mathrm{d}\rho}(\rho_\infty)\partial_\xi\dot\rho + K(\rho_\infty)\partial_\xi^3\dot\rho \end{pmatrix}.$$

By Fourier transform, we find that $\lambda \in \mathbb{C}$ belongs to the spectrum of L^σ_∞ if and only if there exists $\zeta \in \mathbb{R}$ such that

$$\big(\lambda + i(u_\infty - \sigma)\zeta\big)^2 + \rho_\infty\left(\frac{\mathrm{d}g_0}{\mathrm{d}\rho}(\rho_\infty) + K(\rho_\infty)\zeta^2\right)\zeta^2 = 0. \tag{3.15}$$

Since by assumption $K(\rho_\infty) > 0$, and as we have seen above, $\frac{\mathrm{d}g_0}{\mathrm{d}\rho}(\rho_\infty) > 0$ (a necessary condition for the homoclinic wave to exist), (3.15) has no solution $\zeta \in \mathbb{R}$ for $\lambda \notin i\mathbb{R}$. By standard (Coppel–Palmer [10,17], or Henry [13]) arguments, this implies that the essential spectrum of the variable-coefficients operator L^σ is contained in $i\mathbb{R}$ (and in fact equal to $i\mathbb{R}$ because all elements of $i\mathbb{R}$ are 'approximate eigenvalues' of L^σ).

• *Construction of the Evans function.* In order to construct an Evans function [1,18], we first rewrite the eigenvalue equations $(L^\sigma - \lambda)\cdot\dot{\mathbf{U}} = 0$ as a first-order system of ODEs, where ξ is viewed as a 'time'-variable. By definition,

$$L^\sigma \cdot \dot{\mathbf{U}} = \begin{pmatrix} -\partial_\xi((\underline{u}^\sigma - \sigma)\dot\rho + \underline{\rho}^\sigma\dot u) \\ \partial_\xi(-(\underline{u}^\sigma - \sigma)\dot u - \underline{\alpha}^\sigma\dot\rho + \underline{K}^\sigma\partial_\xi^2\dot\rho + \partial_\xi\underline{K}^\sigma\partial_\xi\dot\rho) \end{pmatrix}.$$

We recall that

$$\underline{K}^\sigma := K\big(\underline{\rho}^\sigma\big) \quad \text{and} \quad \underline{\alpha}^\sigma := \frac{\mathrm{d}g_0}{\mathrm{d}\rho}\big(\underline{\rho}^\sigma\big) - \frac{\mathrm{d}K}{\mathrm{d}\rho}\big(\underline{\rho}^\sigma\big)\partial_\xi^2\underline{\rho}^\sigma - \frac{1}{2}\frac{\mathrm{d}^2K}{\mathrm{d}\rho^2}\big(\underline{\rho}^\sigma\big)\big(\partial_\xi\underline{\rho}^\sigma\big)^2.$$

So $(L^\sigma - \lambda)\cdot\dot{\mathbf{U}} = 0$ is equivalent to

$$\big(B^\sigma\Phi\big)' = A(\lambda)\Phi, \tag{3.16}$$

where the prime ($'$) stands for $\mathrm{d}/\mathrm{d}\xi$, and

$$\Phi := \begin{pmatrix} \dot\rho \\ \dot\rho' \\ \dot\rho'' \\ \dot u \end{pmatrix}, \qquad B^\sigma := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\underline{\alpha}^\sigma & (\underline{K}^\sigma)' & \underline{K}^\sigma & -(\underline{u}^\sigma - \sigma) \\ (\underline{u}^\sigma - \sigma) & 0 & 0 & \underline{\rho}^\sigma \end{pmatrix}, \qquad A(\lambda) := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \lambda \\ -\lambda & 0 & 0 & 0 \end{pmatrix}.$$

The eigenvalues of the asymptotic system $(B^\sigma_\infty\Phi)' = A(\lambda)\Phi$, with

$$B^\sigma_\infty := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -c_\infty^2/\rho_\infty & 0 & K_\infty & -(u_\infty - \sigma) \\ (u_\infty - \sigma) & 0 & 0 & \rho_\infty \end{pmatrix}, \quad K_\infty := K(\rho_\infty),$$

are the roots ω of the dispersion relation

$$\big(\lambda+(u_\infty-\sigma)\omega\big)^2-\big(c_\infty^2-\rho_\infty K_\infty\omega^2\big)\omega^2=0. \tag{3.17}$$

(Alternatively, (3.17) can be derived from (3.15) by substituting ω for $i\zeta$.) We easily see that, for $\operatorname{Re}\lambda>0$, (3.17) has no purely imaginary root ω, and by studying the case $\lambda\in\mathbb{R}$, $\lambda\gg 1$, we find that (3.17) has exactly two roots of negative real parts, say $\omega_1(\lambda)$ and $\omega_2(\lambda)$ (either both real or complex conjugate), and two roots of positive real parts, say $\omega_3(\lambda)$ and $\omega_4(\lambda)$ (either both real or complex conjugate). When λ goes to zero, the four roots are real, and two of them go to zero. We choose their numbering so that

$$\omega_2\sim\frac{-\lambda}{c_\infty+u_\infty-\sigma},\qquad \omega_3\sim\frac{\lambda}{c_\infty-u_\infty+\sigma},$$
$$\omega_1\to-\sqrt{\big(c_\infty^2-(u_\infty-\sigma)^2\big)/(\rho_\infty K_\infty)},\qquad \omega_4\to+\sqrt{\big(c_\infty^2-(u_\infty-\sigma)^2\big)/(\rho_\infty K_\infty)}$$

when λ goes to 0. In addition, at points λ where ω_1 and ω_2 are distinct, respectively where ω_3 and ω_4 are distinct, which is the case for large real λ and for λ close to zero, the corresponding eigenvectors, $\mathbf{W}_1^\sigma(\lambda)$, $\mathbf{W}_2^\sigma(\lambda)$, and respectively $\mathbf{W}_3^\sigma(\lambda)$, $\mathbf{W}_4^\sigma(\lambda)$, span the stable, and respectively the unstable, subspace (in $\mathbb{C}^4$) of the matrix $(B_\infty^\sigma)^{-1}A(\lambda)$. They can be chosen of the form

$$\mathbf{W}_j^\sigma(\lambda):=\begin{pmatrix}\rho_\infty\\ \rho_\infty\omega_j^\sigma(\lambda)\\ \rho_\infty\omega_j^\sigma(\lambda)^2\\ -\frac{\lambda}{\omega_j^\sigma(\lambda)}-(u_\infty-\sigma)\end{pmatrix}. \tag{3.18}$$

Then their limits at $\lambda=0$ are easily found to be

$$\mathbf{W}_{1,4}^\sigma(0)=\begin{pmatrix}\rho_\infty\\ \rho_\infty\omega_{1,4}^\sigma(0)\\ \rho_\infty\omega_{1,4}^\sigma(0)^2\\ -(u_\infty-\sigma)\end{pmatrix},\qquad \mathbf{W}_2^\sigma(0)=\begin{pmatrix}\rho_\infty\\0\\0\\c_\infty\end{pmatrix},\qquad \mathbf{W}_3^\sigma(0)=\begin{pmatrix}\rho_\infty\\0\\0\\-c_\infty\end{pmatrix}. \tag{3.19}$$

We can construct a so-called Evans function D^σ, which is analytic and real-valued for $\lambda\in[0,+\infty)$, such that

$$D^\sigma(\lambda)=0,\quad \lambda>0\quad\Longleftrightarrow\quad \operatorname{Ker}\big(L^\sigma-\lambda\big)\neq\{0\}.$$

(See [1,18] for $\lambda>0$, and [11,15] for the extension to $\lambda=0$.) More precisely, D^σ can be taken of the form

$$D^\sigma(\lambda)=\det\big(\widetilde{\Phi}_1^\sigma(\lambda),\widetilde{\Phi}_2^\sigma(\lambda),\widetilde{\Phi}_3^\sigma(\lambda),\widetilde{\Phi}_4^\sigma(\lambda)\big)\big|_{\xi=0},$$

where $(\widetilde{\Phi}_1^\sigma(\lambda),\widetilde{\Phi}_2^\sigma(\lambda))$ (respectively $(\widetilde{\Phi}_3^\sigma(\lambda),\widetilde{\Phi}_4^\sigma(\lambda))$), span the real stable (respectively unstable) manifold of (3.16). These real-valued $\widetilde{\Phi}_j^\sigma$ can be constructed in a simple way from the complex-valued solutions Φ_j^σ of (3.16) characterized, away from collision points, by

$$\Phi_{1,2}^\sigma(\lambda)\overset{\xi\to+\infty}{\sim}\mathrm{e}^{\omega_{1,2}^\sigma(\lambda)\xi}\mathbf{W}_{1,2}^\sigma(\lambda),\qquad \Phi_{3,4}^\sigma(\lambda)\overset{\xi\to-\infty}{\sim}\mathrm{e}^{\omega_{3,4}^\sigma(\lambda)\xi}\mathbf{W}_{3,4}^\sigma(\lambda). \tag{3.20}$$

It suffices to define

$$\widetilde{\Phi}_1^\sigma:=\Phi_1^\sigma+\Phi_2^\sigma,\qquad \widetilde{\Phi}_2^\sigma:=\frac{\Phi_1^\sigma-\Phi_2^\sigma}{\omega_1-\omega_2},$$
$$\widetilde{\Phi}_3^\sigma:=\Phi_3^\sigma+\Phi_4^\sigma,\qquad \widetilde{\Phi}_4^\sigma:=\frac{\Phi_3^\sigma-\Phi_4^\sigma}{\omega_3-\omega_4}.$$

These $\widetilde{\Phi}_j^\sigma$s, as the Φ_j^σs, depend analytically on λ away from collision points. Furthermore, they are obviously real-valued when the Φ_j^σs are so. Otherwise, when (ω_1,ω_2) is a conjugate pair, so is $(\Phi_1^\sigma,\Phi_2^\sigma)$ and therefore the $\widetilde{\Phi}_{1,2}^\sigma$ are still real-valued. Of course the same observation holds true with the indices $(3,4)$ instead of $(1,2)$. Note also that the $\widetilde{\Phi}_j^\sigma$s do not depend on the numbering of stable and unstable modes. As usual, it is trickier to define the Evans function at collision points, that is, where either ω_1 and ω_2, or ω_3 and ω_4, collide (which does happen, as a closer examination of the algebraic equation (3.17) shows). Indeed, even though the eigenvectors $\mathbf{W}_j^\sigma$ are such that

$$\widetilde{\mathbf{W}}_2^\sigma:=\frac{\mathbf{W}_1^\sigma-\mathbf{W}_2^\sigma}{\omega_1-\omega_2}\quad\text{and}\quad \widetilde{\mathbf{W}}_4^\sigma:=\frac{\mathbf{W}_3^\sigma-\mathbf{W}_4^\sigma}{\omega_3-\omega_4}$$

do have limits at collision points that are independent of $\mathbf{W}_2^\sigma$ and $\mathbf{W}_4^\sigma$ respectively (as is easily found from (3.18)) – which means that $(B_\infty^\sigma)^{-1}A(\lambda)$ has a 2×2 Jordan block at those points – the behavior of the individual $\widetilde{\Phi}_{2,4}^\sigma$ is unclear. However, working with wedge products [1] we can make sure that the Evans function crosses collision points in a continuous (and even analytic) manner.

• *Low frequency expansion of the Evans function.* Observing that by definition

$$D^\sigma(\lambda) = \frac{\det(\Phi_1^\sigma(\lambda), \Phi_2^\sigma(\lambda), \Phi_3^\sigma(\lambda), \Phi_4^\sigma(\lambda))|_{\xi=0}}{(\omega_2(\lambda)-\omega_1(\lambda))(\omega_4(\lambda)-\omega_3(\lambda))},$$

where the denominator in the neighborhood of $\lambda = 0$ is

$$\big(\omega_2(\lambda)-\omega_1(\lambda)\big)\big(\omega_4(\lambda)-\omega_3(\lambda)\big) \sim \frac{c_\infty^2 - (u_\infty - \sigma)^2}{\rho_\infty K_\infty} > 0,$$

we see that $D^\sigma(\lambda)$ has the same sign as

$$\Delta^\sigma(\lambda) := \det\big(\Phi_1^\sigma(\lambda), \Phi_2^\sigma(\lambda), \Phi_3^\sigma(\lambda), \Phi_4^\sigma(\lambda)\big)\big|_{\xi=0} \tag{3.21}$$

for λ close to 0.

Since $L^\sigma \cdot (\underline{\mathbf{U}}^\sigma)' = 0$ and $(\underline{\mathbf{U}}^\sigma)'$ goes exponentially fast to zero at $\pm\infty$, the one-dimensional stable/unstable manifold of (3.16) with $\lambda = 0$ is spanned by $(\underline{\mathbf{U}}^\sigma)'$. This means that both $\Phi_1^\sigma(0)$ and $\Phi_4^\sigma(0)$ must be proportional to $(\underline{\mathbf{U}}^\sigma)'$. Now we have to be careful to comply with (3.18) and (3.20), which imply in particular that the first component of $\Phi_1^\sigma(0)$, respectively $\Phi_4^\sigma(0)$, must be positive when ξ goes to $+\infty$, respectively $-\infty$. Since $(\underline{\rho}^\sigma)'$ has different signs at $+\infty$ and $-\infty$, this means there exists a nonzero real number r such that

$$\Phi_1^\sigma(0) = -r\begin{pmatrix} (\underline{\rho}^\sigma)' \\ (\underline{\rho}^\sigma)'' \\ (\underline{\rho}^\sigma)''' \\ (\underline{u}^\sigma)' \end{pmatrix}, \qquad \Phi_4^\sigma(0) = r\begin{pmatrix} (\underline{\rho}^\sigma)' \\ (\underline{\rho}^\sigma)'' \\ (\underline{\rho}^\sigma)''' \\ (\underline{u}^\sigma)' \end{pmatrix}. \tag{3.22}$$

The actual value of r can be deduced from the phase portrait of the profile equation (which is symmetric with respect to the horizontal axis), its sign depending on the type of soliton considered. It is of no importance though. We only need to know that the sign of $D^\sigma(\lambda)$ (for small λ) is opposite to the sign of

$$\check{\Delta}^\sigma(\lambda) := \det\big(\check{\Phi}_1^\sigma(\lambda), \Phi_2^\sigma(\lambda), \Phi_3^\sigma(\lambda), \check{\Phi}_4^\sigma(\lambda)\big)\big|_{\xi=0}, \quad \check{\Phi}_1 := -(1/r)\Phi_1,\ \check{\Phi}_4 := (1/r)\Phi_4.$$

Taking (3.22) into account in (3.21) we readily find that $\check{\Delta}^\sigma(0) = 0$. Furthermore, $(\check{\Delta}^\sigma)'(0) = 0$. This can be seen as follows. Denoting by $\phi_j^\sigma(\lambda)$ and $\mu_j^\sigma(\lambda)$ the first and fourth components of $\Phi_j^\sigma(\lambda)$ (or $\check{\Phi}_j^\sigma(\lambda)$ for $j = 1$ or 4) respectively, we find by differentiation of $(B^\sigma \Phi_j^\sigma(\lambda))' = A(\lambda)\Phi_j^\sigma(\lambda)$ with respect to λ that, thanks to (3.22) and (3.9),

$$L^\sigma \cdot \begin{pmatrix} \partial_\lambda \phi_{1,4}^\sigma(0) \\ \partial_\lambda \mu_{1,4}^\sigma(0) \end{pmatrix} = \begin{pmatrix} (\underline{\rho}^\sigma)' \\ (\underline{u}^\sigma)' \end{pmatrix} = -L^\sigma \cdot \begin{pmatrix} \partial_\sigma \underline{\rho}^\sigma \\ \partial_\sigma \underline{u}^\sigma \end{pmatrix},$$

which implies

$$\begin{pmatrix} \partial_\lambda \phi_{1,4}^\sigma(0) + \partial_\sigma \underline{\rho}^\sigma \\ \partial_\lambda \mu_{1,4}^\sigma(0) + \partial_\sigma \underline{u}^\sigma \end{pmatrix} \parallel \begin{pmatrix} (\underline{\rho}^\sigma)' \\ (\underline{u}^\sigma)' \end{pmatrix}, \text{ a generator of the one-dimensional kernel of } L^\sigma.$$

Therefore, using (3.22) again and up to adding a constant times $\lambda\Phi_{1,4}^\sigma(\lambda)$ to $\Phi_{1,4}^\sigma(\lambda)$, we may assume without loss of generality that

$$\partial_\lambda \check{\Phi}_1^\sigma(0) = \partial_\lambda \check{\Phi}_4^\sigma(0) = -\begin{pmatrix} \partial_\sigma(\underline{\rho}^\sigma) \\ \partial_\sigma(\underline{\rho}^\sigma)' \\ \partial_\sigma(\underline{\rho}^\sigma)'' \\ \partial_\sigma(\underline{u}^\sigma) \end{pmatrix}. \tag{3.23}$$

Together with (3.22), this obviously implies that $(\check{\Delta}^\sigma)'(0) = 0$. Differentiating once more, we find that

$$\big(\check{\Delta}^\sigma\big)''(0) = \det\big(\check{\Phi}_1^\sigma(0), \Phi_2^\sigma(0), \Phi_3^\sigma(0), \partial^2_{\lambda\lambda}\big(\check{\Phi}_4^\sigma - \check{\Phi}_1^\sigma\big)(0)\big)\big|_{\xi=0}.$$

To evaluate this determinant, we first observe that $\det B^\sigma|_{\xi=0} = \underline{\rho}^\sigma(0)\underline{K}^\sigma(0) \neq 0$, so that

$$\begin{aligned} &\det\big(\check{\Phi}_1^\sigma(0), \Phi_2^\sigma(0), \Phi_3^\sigma(0), \partial^2_{\lambda\lambda}\big(\check{\Phi}_4^\sigma - \check{\Phi}_1^\sigma\big)(0)\big)\big|_{\xi=0} \\ &\quad = \frac{1}{\underline{\rho}^\sigma(0)\underline{K}^\sigma(0)} \det\big(B^\sigma\check{\Phi}_1^\sigma(0), B^\sigma\Phi_2^\sigma(0), B^\sigma\Phi_3^\sigma(0), \partial^2_{\lambda\lambda}B^\sigma\big(\check{\Phi}_4^\sigma - \check{\Phi}_1^\sigma\big)(0)\big)\big|_{\xi=0}. \end{aligned}$$

For simplicity, in what follows, we just denote by Φ_j the function $\Phi_j^\sigma(0)$, and by ϕ_j and μ_j its first and last components, and $\Theta_j = \partial^2_{\lambda\lambda}\check{\Phi}_j^\sigma(0)$, with θ_j and χ_j its first and last components. By construction of Φ_j, since the last two rows of $A(0)$ are zeroes, we have

$$B^\sigma \Phi_j = \begin{pmatrix} \phi_j \\ \phi_j' \\ R_j \end{pmatrix},$$

where R_j is a *constant vector* in $\mathbb{R}^2$. More specifically, R_1 is the null vector, while

$$\lim_{\xi\to+\infty}\phi_2(\xi)=\rho_\infty,\qquad \lim_{\xi\to+\infty}\mu_2(\xi)=c_\infty,\qquad \lim_{\xi\to-\infty}\phi_3(\xi)=\rho_\infty,\qquad \lim_{\xi\to-\infty}\mu_3(\xi)=-c_\infty$$

(which come from (3.19) and (3.20)), imply that

$$R_2=\begin{pmatrix}-c_\infty(u_\infty-\sigma+c_\infty)\\ \rho_\infty(u_\infty-\sigma+c_\infty)\end{pmatrix},\qquad R_3=\begin{pmatrix}c_\infty(u_\infty-\sigma-c_\infty)\\ \rho_\infty(u_\infty-\sigma-c_\infty)\end{pmatrix}.$$

Furthermore, we claim that

$$B^\sigma\Theta_{1,4}=\begin{pmatrix}\theta_{1,4}\\ \theta'_{1,4}\\ S_{1,4}\end{pmatrix},$$

with $S_{1,4}:\xi\to S_{1,4}(\xi)\in\mathbb{R}^2$ such that

$$S_4-S_1=2\int_{-\infty}^{+\infty}\begin{pmatrix}-\partial_\sigma\underline{u}^\sigma\\ \partial_\sigma\underline{\rho}^\sigma\end{pmatrix}\,\mathrm{d}\xi. \tag{3.24}$$

Indeed, differentiating twice $(B^\sigma\Phi_j^\sigma(\lambda))'=A(\lambda)\Phi_j^\sigma(\lambda)$ with respect to λ at $\lambda=0$, and using (3.23), we find that

$$L^\sigma\cdot\begin{pmatrix}\theta_{1,4}\\ \chi_{1,4}\end{pmatrix}=-2\begin{pmatrix}\partial_\sigma\underline{\rho}^\sigma\\ \partial_\sigma\underline{u}^\sigma\end{pmatrix},$$

hence

$$\begin{cases}(\underline{u}^\sigma-\sigma)\theta_1+\underline{\rho}^\sigma\chi_1=-2\displaystyle\int_\xi^{+\infty}\partial_\sigma\underline{\rho}^\sigma,\\[2ex] \underline{K}^\sigma\theta_1''+\big(\underline{K}^\sigma\big)'\theta_1'-\underline{\alpha}^\sigma\theta_1-\big(\underline{u}^\sigma-\sigma\big)\chi_1=2\displaystyle\int_\xi^{+\infty}\partial_\sigma\underline{u}^\sigma,\end{cases}$$

$$\begin{cases}(\underline{u}^\sigma-\sigma)\theta_4+\underline{\rho}^\sigma\chi_4=2\displaystyle\int_{-\infty}^{\xi}\partial_\sigma\underline{\rho}^\sigma,\\[2ex] \underline{K}^\sigma\theta_4''+\big(\underline{K}^\sigma\big)'\theta_4'-\underline{\alpha}^\sigma\theta_4-\big(\underline{u}^\sigma-\sigma\big)\chi_4=-2\displaystyle\int_{-\infty}^{\xi}\partial_\sigma\underline{u}^\sigma,\end{cases}$$

which imply (3.24) by definition of S_1 and S_4. To complete the computation of $(\check{\Delta}^\sigma)''(0)$, we observe that

$$\det(R_2,R_3)=2\rho_\infty c_\infty\big(c_\infty^2-(u_\infty-\sigma)^2\big)>0$$

by (3.12), and we introduce (the unique) real numbers d_2 and d_3 such that

$$S_4-S_1=d_2R_2-d_3R_3.$$

Therefore,

$$\big(\check{\Delta}^\sigma\big)''(0)=\frac{1}{\underline{\rho}^\sigma(0)\underline{K}^\sigma(0)}\begin{vmatrix}(\underline{\rho}^\sigma)' & \phi_2 & \phi_3 & \widetilde{\theta}_4-\widetilde{\theta}_1\\ (\underline{\rho}^\sigma)'' & \phi_2' & \phi_3' & \widetilde{\theta}_4'-\widetilde{\theta}_1'\\ 0_2 & R_2 & R_3 & 0_2\end{vmatrix}_{|\xi=0}=\frac{\det(R_2,R_3)}{\underline{\rho}^\sigma(0)\underline{K}^\sigma(0)}\begin{vmatrix}(\underline{\rho}^\sigma)' & \widetilde{\theta}_4-\widetilde{\theta}_1\\ (\underline{\rho}^\sigma)'' & \widetilde{\theta}_4'-\widetilde{\theta}_1'\end{vmatrix}_{|\xi=0}$$

with

$$\widetilde{\theta}_4:=\theta_4+d_3\phi_3,\qquad \widetilde{\theta}_1:=\theta_1+d_2\phi_2.$$

It thus only remains to compute $\delta_{|\xi=0}$, with

$$\delta:=\begin{vmatrix}(\underline{\rho}^\sigma)' & \widetilde{\theta}_4-\widetilde{\theta}_1\\ (\underline{\rho}^\sigma)'' & \widetilde{\theta}_4'-\widetilde{\theta}_1'\end{vmatrix},$$

knowing that $(\underline{\rho}^\sigma)'$ and $\widetilde{\theta}_{1,4}$ all satisfy an ODE of the form

$$\underline{K}^\sigma y''+\big(\underline{K}^\sigma\big)'y'-\underline{\alpha}^\sigma y+\frac{1}{\underline{\rho}^\sigma}\big(\underline{u}^\sigma-\sigma\big)^2y=s[y],$$

and more precisely,

$$s\big[\big(\underline{\rho}^\sigma\big)'\big]=0,\qquad s[\widetilde{\theta}_4]=\big(1,\big(\underline{u}^\sigma-\sigma\big)/\underline{\rho}^\sigma\big)(S_4+d_3R_3)=\big(1,\big(\underline{u}^\sigma-\sigma\big)/\underline{\rho}^\sigma\big)(S_1+d_2R_2)=s[\widetilde{\theta}_1].$$

The rest of the computation is based on the Melnikov technique. Decomposing δ as

$$\delta=\delta_4-\delta_1,\quad \delta_{1,4}:=\begin{vmatrix}(\underline{\rho}^\sigma)' & \widetilde{\theta}_{1,4}\\ (\underline{\rho}^\sigma)'' & \widetilde{\theta}'_{1,4}\end{vmatrix},\quad \text{with } \delta_4(-\infty)=0,\ \delta_1(+\infty)=0,$$

and integrating the ODEs

$$\frac{\mathrm{d}\delta_{1,4}}{\mathrm{d}\xi}=-\frac{(\underline{K}^\sigma)'}{\underline{K}^\sigma}\delta_{1,4}+\frac{(\underline{\rho}^\sigma)'}{\underline{K}^\sigma}s[\widetilde{\theta}_{1,4}]$$

on $(0,+\infty)$ and $(-\infty,0)$ respectively, we find that

$$\delta_{|\xi=0}=\frac{1}{\underline{K}^\sigma(0)}\int_{-\infty}^{+\infty}s[\widetilde{\theta}_{1,4}]\big(\underline{\rho}^\sigma\big)'.$$

Now, thanks to the identity

$$\big(\underline{u}^\sigma-\sigma\big)\big(\underline{\rho}^\sigma\big)'=-\underline{\rho}^\sigma\big(\underline{u}^\sigma\big)',$$

we have

$$\int_{-\infty}^{+\infty}s[\widetilde{\theta}_4]\big(\underline{\rho}^\sigma\big)'=\int_{-\infty}^{+\infty}\big(\big(\underline{\rho}^\sigma\big)',-\big(\underline{u}^\sigma\big)'\big)(S_4+d_3R_3).$$

Clearly (since $\underline{\rho}^\sigma$, $\underline{u}^\sigma$ have the same limits at $+\infty$ and $-\infty$) the constant vector R_3 does not contribute to this integral, and recalling that

$$S_4(\xi)=2\int_{-\infty}^{\xi}\begin{pmatrix}-\partial_\sigma\underline{u}^\sigma\\ \partial_\sigma\underline{\rho}^\sigma\end{pmatrix},$$

after integration by parts we finally arrive at

$$\delta_{|\xi=0}=\frac{2}{\underline{K}^\sigma(0)}\int_{-\infty}^{+\infty}\big(\big(\underline{\rho}^\sigma-\rho_\infty\big)\partial_\sigma\underline{u}^\sigma+\big(\underline{u}^\sigma-u_\infty\big)\partial_\sigma\underline{\rho}^\sigma\big).$$

This yields the formula

$$\big(\check{\Delta}^\sigma\big)''(0)=\frac{4\rho_\infty c_\infty(c_\infty^2-(u_\infty-\sigma)^2)}{\underline{\rho}^\sigma(0)(\underline{K}^\sigma(0))^2}\int_{-\infty}^{+\infty}\big(\big(\underline{\rho}^\sigma-\rho_\infty\big)\partial_\sigma\underline{u}^\sigma+\big(\underline{u}^\sigma-u_\infty\big)\partial_\sigma\underline{\rho}^\sigma\big),$$

hence

$$\big(\Delta^\sigma\big)''(0)=-\frac{4r^2\rho_\infty c_\infty(c_\infty^2-(u_\infty-\sigma)^2)}{\underline{\rho}^\sigma(0)(\underline{K}^\sigma(0))^2}\int_{-\infty}^{+\infty}\big(\big(\underline{\rho}^\sigma-\rho_\infty\big)\partial_\sigma\underline{u}^\sigma+\big(\underline{u}^\sigma-u_\infty\big)\partial_\sigma\underline{\rho}^\sigma\big).$$

• *High frequency behavior of the Evans function.* This part of the analysis could be omitted – and is indeed omitted in [22] – in view of the sufficient stability condition provided by the Grillakis–Shatah–Strauss method. It is of interest though, for the method – which can be useful in other frameworks – and as a way to double-check that the stability condition is indeed

$$\int\big(\big(\underline{\rho}^\sigma-\rho_\infty\big)\partial_\sigma\underline{u}^\sigma+\big(\underline{u}^\sigma-u_\infty\big)\partial_\sigma\underline{\rho}^\sigma\big)\,\mathrm{d}\xi<0. \tag{3.25}$$

By means of an energy estimate based on a 'symmetrized' reformulation of the linearized system (see [7, Proposition 3.4]), we can find $\lambda_0>0$ such that L^σ has no eigenvalue $\lambda>\lambda_0$. We may then argue by homotopy. For $\theta\in[0,1]$, consider the operator the operator L_θ^σ defined by

$$L_\theta^\sigma\cdot\dot{\mathbf{U}}=\begin{pmatrix}-\partial_\xi(u_\theta^\sigma\dot\rho+\rho_\theta^\sigma\dot u)\\ \partial_\xi(-u_\theta^\sigma\dot u-\alpha_\theta^\sigma\dot\rho+K_\theta^\sigma\partial_\xi^2\dot\rho+\partial_\xi K_\theta^\sigma\partial_\xi\dot\rho)\end{pmatrix},$$

where

$$u_\theta^\sigma := \theta(\underline{u}^\sigma - \sigma), \qquad \rho_\theta^\sigma := \rho_\infty + \theta(\underline{\rho}^\sigma - \rho_\infty), \qquad K_\theta^\sigma := K(\rho_\theta^\sigma),$$
$$\alpha_\theta^\sigma := \theta \frac{\mathrm{d}g_0}{\mathrm{d}\rho}(\rho_\theta^\sigma) - \frac{\mathrm{d}K}{\mathrm{d}\rho}(\rho_\theta^\sigma)\partial_\xi^2 \rho_\theta^\sigma - \frac{1}{2}\frac{\mathrm{d}^2K}{\mathrm{d}\rho^2}(\rho_\theta^\sigma)(\partial_\xi \rho_\theta^\sigma)^2.$$

At $\theta = 1$ we recover L^σ and at $\theta = 0$ we get the constant-coefficients operator

$$L_0 \cdot \dot{\mathbf{U}} := \begin{pmatrix} -\rho_\infty \partial_\xi \dot{u} \\ K_\infty \partial_\xi^3 \dot{\rho} \end{pmatrix}.$$

The spectrum of L_0 is found to be exactly $i\mathbb{R}$ by Fourier transform. Furthermore, the aforementioned energy estimate can be adapted to deal with L_θ^σ and show that for all $\theta \in [0,1]$, L_θ^σ has no eigenvalue of real part greater than some threshold $\lambda_* \geqslant \lambda_0$. Let us describe how to obtain this estimate, which is not straightforward. Assume that $\dot{\mathbf{U}} = (\dot{\rho}, \dot{u})^{\mathrm{t}}$ is an eigenvector associated with a nonzero eigenvalue λ of L_θ^σ (viewed as an unbounded operator on $H^1 \times L^2$ with domain $H^3 \times H^2$). We look for a λ_* independent of $\dot{\mathbf{U}}$ and θ such that

$$(\operatorname{Re}\lambda - \lambda_*)\|\dot{\mathbf{U}}\|_{H^1\times L^2}^2 \leqslant 0.$$

Since the principal part of L_θ^σ is not dissipative, the elimination of higher order derivatives is not straightforward. It requires a 'symmetrized' reformulation of the eigenvalue equation $(\lambda - L_\theta^\sigma)\dot{\mathbf{U}} = \mathbf{0}$. As observed in earlier work [4,5], a suitable reformulation makes use of the change of variables $\rho \mapsto \zeta := R(\rho)$, where R is a primitive of $\rho \mapsto \sqrt{K(\rho)/\rho}$, which urges us to consider $\dot{\zeta} := R'(\rho_\theta^\sigma)\dot{\rho}$, and derive an estimate for $\|\dot{\zeta}\|_{L^2} + \|\sqrt{\rho_\theta^\sigma}\,\dot{u}\|_{L^2} + \|\sqrt{\rho_\theta^\sigma}\,\dot{w}\|_{L^2}$ with $\dot{w} := \partial_\xi \dot{\zeta}$ instead of the standard norm $\|\dot{\mathbf{U}}\|_{H^1\times L^2}$. We first compute the system satisfied by $(\dot{\zeta}, \dot{u}, \dot{w})$ if $(\lambda - L_\theta^\sigma)\dot{\mathbf{U}} = \mathbf{0}$. Introducing the functions a and h defined by $a(\zeta) := \sqrt{R^{-1}(\zeta)K(R^{-1}(\zeta))}$ and $h(\zeta) := \frac{\mathrm{d}}{\mathrm{d}\zeta} g_0(R^{-1}(\zeta))$, we can write this system as

$$\lambda\dot{\zeta} + u_\theta^\sigma \dot{w} + \dot{u} w_\theta^\sigma + a_\theta^\sigma \partial_\xi \dot{u} + (a_\theta^\sigma)'\dot{\zeta}\partial_\xi u_\theta^\sigma = 0, \tag{3.26}$$
$$\lambda\dot{u} + \partial_\xi\big(u_\theta^\sigma \dot{u} - w_\theta^\sigma \dot{w} - a_\theta^\sigma \partial_\xi \dot{w} - (a_\theta^\sigma)'\dot{\zeta}\partial_\xi w_\theta^\sigma\big) + h_\theta^\sigma \dot{w} + (h_\theta^\sigma)'\dot{\zeta} w_\theta^\sigma = 0, \tag{3.27}$$
$$\lambda\dot{w} + \partial_\xi\big(u_\theta^\sigma \dot{w} + \dot{u} w_\theta^\sigma\big) + \partial_\xi\big(a_\theta^\sigma \partial_\xi \dot{u} + (a_\theta^\sigma)'\dot{\zeta}\partial_\xi u_\theta^\sigma\big) = 0, \tag{3.28}$$

where $\zeta_\theta^\sigma := R(\rho_\theta^\sigma)$, $w_\theta^\sigma := R'(\rho_\theta^\sigma)\partial_\xi \rho_\theta^\sigma$, $a_\theta^\sigma := a(\zeta_\theta^\sigma)$, $(a_\theta^\sigma)' := \frac{\mathrm{d}a}{\mathrm{d}\zeta}(\zeta_\theta^\sigma)$, $h_\theta^\sigma := h(\zeta_\theta^\sigma)$, $(h_\theta^\sigma)' := \frac{\mathrm{d}h}{\mathrm{d}\zeta}(\zeta_\theta^\sigma)$. Interestingly, (3.27) and (3.28) can be written as a single equation for the complex-valued function $\dot{z} := \dot{u} + i\dot{w}$,

$$\lambda\dot{z} + \partial_\xi\big(z_\theta^\sigma \dot{z} + i a_\theta^\sigma \partial_\xi \dot{z} + i(a_\theta^\sigma)'\dot{\zeta}\partial_\xi z_\theta^\sigma\big) + h_\theta^\sigma \dot{w} + (h_\theta^\sigma)'\dot{\zeta} w_\theta^\sigma = 0, \tag{3.29}$$

where $z_\theta^\sigma := u_\theta^\sigma + i w_\theta^\sigma$. Taking the real part of the inner product of (3.29) with $\rho_\theta^\sigma \dot{z}$, integrating by part, and using that $a_\theta^\sigma \partial_\xi \rho_\theta^\sigma = \rho_\theta^\sigma w_\theta^\sigma$, we get

$$\begin{aligned}&\operatorname{Re}\lambda\big\|\sqrt{\rho_\theta^\sigma}\,\dot{z}\big\|_{L^2}^2 + \operatorname{Re}\big\langle(\partial_\xi z_\theta^\sigma)\dot{z}, \rho_\theta^\sigma \dot{z}\big\rangle + \big\langle\partial_\xi(\rho_\theta^\sigma u_\theta^\sigma)\dot{z}, \dot{z}\big\rangle \\ &\quad + \operatorname{Re}\big\langle\big(h_\theta^\sigma + i(a_\theta^\sigma)'\partial_\xi z_\theta^\sigma\big)\dot{w}, \rho_\theta^\sigma \dot{z}\big\rangle + \operatorname{Re}\big\langle\big((h_\theta^\sigma)' + i\partial_\xi\big((a_\theta^\sigma)'\partial_\xi z_\theta^\sigma\big)\big)\dot{\zeta}, \rho_\theta^\sigma \dot{z}\big\rangle = 0.\end{aligned}$$

(Without the weight ρ_θ^σ there would have remained a term with the first-order derivative $\partial_\xi \dot{z}$: this is reminiscent of the symmetrization issue for Euler equations.) On the other hand, taking the real part of the inner product of (3.26) with $\dot{\zeta}$ we obtain

$$\operatorname{Re}\lambda\|\dot{\zeta}\|_{L^2}^2 + \operatorname{Re}\big\langle u_\theta^\sigma \dot{w}, \dot{\zeta}\big\rangle + \operatorname{Re}\big\langle\big(w_\theta^\sigma - (a_\theta^\sigma)'\partial_\xi \zeta_\theta^\sigma\big)\dot{u}, \dot{\zeta}\big\rangle - \operatorname{Re}\big\langle a_\theta^\sigma \dot{u}, \dot{w}\big\rangle + \operatorname{Re}\big\langle(a_\theta^\sigma)'\dot{\zeta}\partial_\xi u_\theta^\sigma, \dot{\zeta}\big\rangle = 0.$$

Summing these two identities we find indeed by Cauchy–Schwarz a λ_* (depending only on the $W^{2,\infty}$ norm of $(\zeta_\theta^\sigma, z_\theta^\sigma)$, which is uniformly bounded for $\theta \in [0,1]$) such that

$$(\operatorname{Re}\lambda - \lambda_*)\big(\|\dot{\zeta}\|_{L^2}^2 + \big\|\sqrt{\rho_\theta^\sigma}\,\dot{z}\big\|_{L^2}^2\big) \leqslant 0,$$

which obviously implies, if $\dot{\mathbf{U}}$, and thus $(\dot{\zeta}, \dot{z})$ is nonzero, that $\operatorname{Re}\lambda \leqslant \lambda_*$.

Now, we can construct an Evans function, D_θ^σ say, depending smoothly on θ, and determine the sign of $D^\sigma = D_1^\sigma$ for $\lambda > \lambda_*$ by computing the sign of D_0^σ, which is constant on $[0,+\infty)$. Denoting by $\omega_j^\sigma(\lambda;\theta)$ the eigenvalues of $(B_{\theta,\infty}^\sigma)^{-1}A(\lambda)$, with

$$B_{\theta,\infty}^\sigma := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\theta c_\infty^2/\rho_\infty & 0 & K_\infty & -\theta(u_\infty - \sigma) \\ \theta(u_\infty - \sigma) & 0 & 0 & \rho_\infty \end{pmatrix},$$

and by $\mathbf{W}_j^\sigma(\lambda;\theta)$ the associated eigenvectors,

$$\mathbf{W}_j^\sigma(\lambda;\theta) := \begin{pmatrix} \rho_\infty \\ \rho_\infty \omega_j^\sigma(\lambda;\theta) \\ \rho_\infty \omega_j^\sigma(\lambda;\theta)^2 \\ -\frac{\lambda}{\omega_j^\sigma(\lambda;\theta)} - \theta(u_\infty - \sigma) \end{pmatrix}, \tag{3.30}$$

we can find $\Phi_j^\sigma(\lambda;\theta)$, solutions of the first-order ODE equivalent to $(L_\theta^\sigma - \lambda)\dot{\mathbf{U}} = 0$, characterized by their asymptotic behavior as in (3.20). In particular, for $\theta = 0$ they are explicitly given by

$$\Phi_j^\sigma(\lambda;0) = \mathrm{e}^{\omega_j^\sigma(\lambda;0)\xi}\mathbf{W}_j^\sigma(\lambda;0),$$

with the $\omega_j^\sigma(\lambda;0)$ occurring in complex conjugate pairs such that

$$\sum_{j=1}^4 \omega_j^\sigma(\lambda;0) = 0.$$

(Indeed, they are roots of $\lambda^2 + \rho_\infty K_\infty \omega^4 = 0$.) Therefore, we have

$$D_0^\sigma(\lambda) = \det\left(\mathbf{V}_1 + \mathbf{V}_2, \frac{\mathbf{V}_2 - \mathbf{V}_1}{\nu_2 - \nu_1}, \mathbf{V}_3 + \mathbf{V}_4, \frac{\mathbf{V}_4 - \mathbf{V}_3}{\nu_4 - \nu_3}\right),$$

where ν_j and $\mathbf{V}_j$ are simplifying notations for $\omega_j^\sigma(\lambda;0)$ and $\mathbf{W}_j^\sigma(\lambda;0)$ respectively. The ν_j are of the form $\pm(1 \pm i)\upsilon$ with

$$\upsilon := \sqrt{\frac{\lambda}{2\sqrt{\rho_\infty K_\infty}}}.$$

Recall that the ordering of ν_1 and ν_2, and of ν_3 and ν_4, does not play any role. To fix the ideas, we can take

$$\nu_1 = -(1+i)\upsilon, \qquad \nu_2 = (-1+i)\upsilon, \qquad \nu_3 = (1-i)\upsilon, \qquad \nu_4 = (1+i)\upsilon.$$

Then

$$D_0^\sigma(\lambda) = 4\rho_\infty^3\lambda \begin{vmatrix} 1 & 0 & 1 & 0 \\ \mathrm{Re}\,\nu_1 & 1 & \mathrm{Re}\,\nu_3 & 1 \\ \mathrm{Re}(\nu_1^2) & \nu_1 + \nu_2 & \mathrm{Re}(\nu_3^2) & \nu_3 + \nu_4 \\ -\mathrm{Re}(\frac{1}{\nu_1}) & \frac{1}{\nu_1\nu_2} & -\mathrm{Re}(\frac{1}{\nu_3}) & \frac{1}{\nu_3\nu_4} \end{vmatrix} = 32\rho_\infty^3\lambda > 0.$$

This ends the proof of Lemma 1. $\square$

4. Multi-dimensional stability criterion

The Grillakis–Shatah–Strauss argument invoked for one-dimensional (orbital) stability breaks down in several space dimensions because planar solitary waves do not have an interpretation in terms of critical points. However, the form of the linearized system makes it possible to extend the Evans function calculation of Lemma 1, and eventually show that one-d stable planar solitary waves are unstable with respect to transverse perturbations.

4.1. The linearized operator

By definition, the profile $(\underline{\rho}^\sigma, \underline{\mathbf{u}}^\sigma)$ of a planar solitary wave solution of (1.1) propagating in direction $\mathbf{n}$ (a unitary vector in $\mathbb{R}^d$) with speed σ and homoclinic to $(\rho_\infty, \mathbf{u}_\infty)$, must satisfy

$$\begin{cases} \underline{\rho}^\sigma(\underline{u}^\sigma - \sigma) \equiv \rho_\infty(u_\infty - \sigma), \\ (\underline{u}^\sigma - \sigma)\partial_\xi \underline{\mathbf{v}}^\sigma = \mathbf{0}, \\ K(\underline{\rho}^\sigma)\partial_\xi^2\underline{\rho}^\sigma + \frac{1}{2}\partial_\xi K(\underline{\rho}^\sigma)\partial_\xi\underline{\rho}^\sigma - g_0(\underline{\rho}^\sigma) + g_0(\rho_\infty) - \frac{1}{2}(\underline{u}^\sigma - \sigma)^2 + \frac{1}{2}(u_\infty - \sigma)^2 = 0, \end{cases} \tag{4.31}$$

where $\underline{u}^\sigma := \underline{\mathbf{u}}^\sigma \cdot \mathbf{n}$ and $\underline{\mathbf{v}}^\sigma := \underline{\mathbf{u}}^\sigma - \underline{u}^\sigma\mathbf{n}$. Therefore, a dynamical solitary wave, for which $\underline{u}^\sigma \neq \sigma$, is such that $\underline{\mathbf{v}}^\sigma$ is constant and $(\underline{\rho}^\sigma, \underline{u}^\sigma)$ satisfy the one-dimensional profile equation (3.13). By change of Galilean frame, we may assume without loss of generality that $\underline{\mathbf{v}}^\sigma$ is zero. Moreover, similarly as in one space dimension, the change of Galilean frame $(\mathbf{x}, t) \mapsto (\mathbf{x} - \sigma t\mathbf{n}, t)$ changes (2.2) into

$$\begin{cases} \partial_t\rho + \mathrm{div}(\rho(\mathbf{u} - \sigma\mathbf{n})) = 0, \\ \partial_t\mathbf{u} + ((\mathbf{u} - \sigma\mathbf{n}) \cdot \nabla)\mathbf{u} + \nabla g_0 = \nabla\left(K\Delta\rho + \frac{1}{2}K'_\rho|\nabla\rho|^2\right), \end{cases} \tag{4.32}$$

of which $(\underline{\rho}^\sigma, \underline{\mathbf{u}}^\sigma)$ is a *stationary* solution. Linearizing (4.32) about $(\underline{\rho}^\sigma, \underline{\mathbf{u}}^\sigma)$ we get

$$\partial_t \dot{\mathbf{U}} = \mathbf{L}^\sigma \cdot \dot{\mathbf{U}}, \quad \text{with } \dot{\mathbf{U}} := \begin{pmatrix} \dot{\rho} \\ \dot{\mathbf{u}} \end{pmatrix},$$
$$\mathbf{L}^\sigma \cdot \dot{\mathbf{U}} := \begin{pmatrix} -\operatorname{div}((\underline{\mathbf{u}}^\sigma - \sigma\mathbf{n})\dot{\rho} + \underline{\rho}^\sigma(\dot{\mathbf{u}} - \sigma\mathbf{n})) \\ -(\underline{u}^\sigma - \sigma)\partial_\xi \dot{\mathbf{u}} - (\dot{u} - \sigma)\partial_\xi \underline{u}^\sigma \mathbf{n} + \nabla(-\underline{\alpha}^\sigma \dot{\rho} + \underline{K}^\sigma \Delta\dot{\rho} + \partial_\xi \underline{K}^\sigma \partial_\xi \dot{\rho}) \end{pmatrix}, \tag{4.33}$$

where $\xi := \mathbf{x}\cdot\mathbf{n} - \sigma t$, and, as in Section 3,

$$\underline{K}^\sigma := K(\underline{\rho}^\sigma), \qquad \underline{\alpha}^\sigma := \frac{\mathrm{d}g_0}{\mathrm{d}\rho}(\underline{\rho}^\sigma) - \frac{\mathrm{d}K}{\mathrm{d}\rho}(\underline{\rho}^\sigma)\partial_\xi^2 \underline{\rho}^\sigma - \frac{\mathrm{d}^2 K}{\mathrm{d}\rho^2}(\underline{\rho}^\sigma)(\partial_\xi \underline{\rho}^\sigma)^2.$$

A necessary condition for the linearized stability of $(\underline{\rho}^\sigma, \underline{\mathbf{u}}^\sigma)$ is that $\mathbf{L}^\sigma$ has no spectrum in the open right half-plane. Equivalently, the operator $L^\sigma(\eta)$, obtained by Fourier transform in the direction normal to $\mathbf{n}$, the corresponding wave vector being denoted by $\eta \in \mathbb{R}^{d-1}$, has no spectrum in the open right half-plane. To obtain the explicit form of $L^\sigma(\eta)$, we may assume without loss of generality – because of rotational invariance of (2.2) – that $\mathbf{n}$ is the last vector e_d of the canonical basis in $\mathbb{R}^d$. Hence we may identify the vector $\dot{\mathbf{v}} \in \mathrm{e}_d^\perp$ with a vector in $\mathbb{R}^{d-1} = \operatorname{span}(\mathrm{e}_1, \dots, \mathrm{e}_{d-1})$, and $\dot{\mathbf{U}}$ and $L^\sigma(\eta)\cdot\dot{\mathbf{U}}$ with

$$\begin{pmatrix} \dot{\rho} \\ \dot{\mathbf{v}} \\ \dot{u} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -\partial_\xi((\underline{u}^\sigma - \sigma)\dot{\rho} + \underline{\rho}^\sigma(\dot{u} - \sigma)) - i\underline{\rho}^\sigma \eta\cdot\dot{\mathbf{v}} \\ -(\underline{u}^\sigma - \sigma)\partial_\xi\dot{\mathbf{v}} + i(-(\underline{\alpha}^\sigma + \underline{K}^\sigma\|\eta\|^2)\dot{\rho} + \underline{K}^\sigma\partial_\xi^2\dot{\rho} + \partial_\xi\underline{K}^\sigma\partial_\xi\dot{\rho})\eta \\ \partial_\xi(-(\underline{u}^\sigma - \sigma)\dot{u} - (\underline{\alpha}^\sigma + \underline{K}^\sigma\|\eta\|^2)\dot{\rho} + \underline{K}^\sigma\partial_\xi^2\dot{\rho} + \partial_\xi\underline{K}^\sigma\partial_\xi\dot{\rho}) \end{pmatrix}$$

respectively. The operator $L^\sigma(\eta)$ is clearly similar to the real-valued operator

$$\widetilde{L}^\sigma(\eta): \begin{pmatrix} \dot{\rho} \\ \dot{\tilde{\mathbf{v}}} \\ \dot{u} \end{pmatrix} \mapsto \begin{pmatrix} -\partial_\xi((\underline{u}^\sigma - \sigma)\dot{\rho} + \underline{\rho}^\sigma(\dot{u} - \sigma)) - \underline{\rho}^\sigma \eta\cdot\dot{\tilde{\mathbf{v}}} \\ -(\underline{u}^\sigma - \sigma)\partial_\xi\dot{\tilde{\mathbf{v}}} - (-(\underline{\alpha}^\sigma + \underline{K}^\sigma\|\eta\|^2)\dot{\rho} + \underline{K}^\sigma\partial_\xi^2\dot{\rho} + \partial_\xi\underline{K}^\sigma\partial_\xi\dot{\rho})\eta \\ \partial_\xi(-(\underline{u}^\sigma - \sigma)\dot{u} - (\underline{\alpha}^\sigma + \underline{K}^\sigma\|\eta\|^2)\dot{\rho} + \underline{K}^\sigma\partial_\xi^2\dot{\rho} + \partial_\xi\underline{K}^\sigma\partial_\xi\dot{\rho}) \end{pmatrix}.$$

Therefore, the spectra of $L^\sigma(\eta)$ and $\widetilde{L}^\sigma(\eta)$ coincide. From now on, we concentrate on $\widetilde{L}^\sigma(\eta)$ and we omit the tildas for simplicity. The asymptotic operator at $\pm\infty$ is

$$L^\sigma_\infty(\eta): \begin{pmatrix} \dot{\rho} \\ \dot{\mathbf{v}} \\ \dot{u} \end{pmatrix} \mapsto \begin{pmatrix} -(u_\infty - \sigma)\partial_\xi\dot{\rho} - \rho_\infty\partial_\xi\dot{u} - \rho_\infty\eta\cdot\dot{\mathbf{v}} \\ -(u_\infty - \sigma)\partial_\xi\dot{\mathbf{v}} - (-(\frac{\mathrm{d}g_0}{\mathrm{d}\rho}(\rho_\infty) + K(\rho_\infty)\|\eta\|^2)\dot{\rho} + K(\rho_\infty)\partial_\xi^2\dot{\rho})\eta \\ -(u_\infty - \sigma)\partial_\xi\dot{u} - (\frac{\mathrm{d}g_0}{\mathrm{d}\rho}(\rho_\infty) + K(\rho_\infty)\|\eta\|^2)\partial_\xi\dot{\rho} + K(\rho_\infty)\partial_\xi^3\dot{\rho} \end{pmatrix}.$$

By Fourier transform in ξ, we find that $\tau \in \mathbb{C}$ belongs to the spectrum of $L^\sigma_\infty(\eta)$ if and only if there exists $\zeta \in \mathbb{R}$ so that, either $\tau = -i(u_\infty - \sigma)\zeta$, or

$$\bigl(\tau + i(u_\infty - \sigma)\zeta\bigr)^2 + \rho_\infty\left(\frac{\mathrm{d}g_0}{\mathrm{d}\rho}(\rho_\infty) + K(\rho_\infty)\bigl(\|\eta\|^2 + \zeta^2\bigr)\right)\bigl(\|\eta\|^2 + \zeta^2\bigr) = 0. \tag{4.34}$$

Therefore, in all cases, τ is purely imaginary. As for the one-dimensional operator L^σ studied in Section 3, this implies the essential spectrum of $L^\sigma(\eta)$ coincides with the imaginary axis. Consequently, the (neutral) linearized stability of $(\underline{\rho}^\sigma, \underline{\mathbf{u}}^\sigma)$ will be determined by the point spectrum of $L^\sigma(\eta)$. As for L^σ, possible unstable eigenvalues τ (with $\operatorname{Re}\tau > 0$) of $L^\sigma(\eta)$ can be characterized as zeroes of an Evans function $\tau \mapsto D(\tau;\eta)$. Viewed as a function of (τ,η), D can be made analytic along rays (as was pointed out by Zumbrun and Serre in [23] for second-order operators associated with viscous shocks; also see [21]). Furthermore, since $L^\sigma(\eta)$ is real-valued, D can be chosen to be real for real τ. Therefore, the comparison of the signs of $D(\lambda\tau;\lambda\eta)$ for λ close to zero and for large λ provides a sufficient condition for instability, by the mean value theorem argument usually valid only in one space dimension. Another way is the one pointed out in [23, Lemma 7.5], which goes as follows in our situation. By nature of the solitary wave there is a function P (which we shall compute explicitly), homogeneous of degree 2, such that $D(\lambda\tau;\lambda\eta) \sim \lambda^2 P(\tau;\eta)$ as λ goes to zero. It will turn out that for a one-d stable solitary wave, P vanishes at points of the form $(\tau_0(\eta),\eta)$. Observing that $p^{(\lambda,\eta)}(\tau) := \lambda^{-2}D(\lambda\tau;\lambda\eta)$ defines a family a holomorphic functions on $\{\operatorname{Re}\tau > 0\}$, depending continuously on $(\lambda,\eta) \in \mathbb{R}^+ \times \mathbb{R}^{d-1}$, Rouché's theorem will then imply the existence of a continuous branch $\tau_*(\lambda,\eta)$ close to $\tau_0(\eta)$ for λ close to 0 such that $p^{(\lambda,\eta)}(\tau_*(\lambda,\eta)) = 0$, hence

$$D\bigl(\tau_\sharp(\eta);\eta\bigr) = 0$$

with $\tau_\sharp(\eta) := \|\eta\|\tau_*(\|\eta\|, \eta/\|\eta\|)$.

4.2. The Evans function computations

We proceed similarly as in Section 3. (The following computation is also close to the one in [3] for heteroclinic planar traveling waves.) We first rewrite the eigenvalue equations $(L^\sigma(\eta) - \tau)\dot{\mathbf{U}} = 0$ as a first-order system of ODEs,

$$\big(B^\sigma(\eta)\Phi\big)' = A^\sigma(\tau;\eta)\Phi,$$

$$\Phi := \begin{pmatrix} \dot\rho \\ \dot\rho' \\ \dot\rho'' \\ \dot{\mathbf{v}} \\ \dot u \end{pmatrix}, \qquad B^\sigma(\eta) := \begin{pmatrix} 1 & 0 & 0 & 0^*_{d-1} & 0 \\ 0 & 1 & 0 & 0^*_{d-1} & 0 \\ -(\underline{\alpha}^\sigma + \underline{K}^\sigma\|\eta\|^2) & (\underline{K}^\sigma)' & \underline{K}^\sigma & 0^*_{d-1} & -(\underline{u}^\sigma - \sigma) \\ 0_{d-1} & 0_{d-1} & 0_{d-1} & (\underline{u}^\sigma - \sigma)\mathbf{I}_{d-1} & 0_{d-1} \\ (\underline{u}^\sigma - \sigma) & 0 & 0 & 0^*_{d-1} & \underline{\rho}^\sigma \end{pmatrix},$$

$$A^\sigma(\tau;\eta) := \begin{pmatrix} 0 & 1 & 0 & 0^*_{d-1} & 0 \\ 0 & 0 & 1 & 0^*_{d-1} & 0 \\ 0 & 0 & 0 & 0^*_{d-1} & \tau \\ (\underline{\alpha}^\sigma + \underline{K}^\sigma\|\eta\|^2)\eta & -(\underline{K}^\sigma)'\eta & -\underline{K}^\sigma\eta & -\tau\mathbf{I}_{d-1} & 0_{d-1} \\ -\tau & 0 & 0 & -\underline{\rho}^\sigma\eta^{\mathrm{t}} & 0 \end{pmatrix}. \tag{4.35}$$

The eigenvalues of the asymptotic system

$$\big(B^\sigma_\infty(\eta)\Phi\big)' = A^\sigma_\infty(\tau;\eta)\Phi \tag{4.36}$$

are $\omega^\sigma_0(\tau) := -\tau/(u_\infty - \sigma)$ and the roots ω of the dispersion relation

$$\big(\tau + (u_\infty - \sigma)\omega\big)^2 + \big(c^2_\infty + \rho_\infty K_\infty\big(\|\eta\|^2 - \omega^2\big)\big)\big(\|\eta\|^2 - \omega^2\big) = 0 \tag{4.37}$$

(obtained from (4.34) by substituting ω for $i\zeta$). We assume from now on that u_∞ is greater than σ, so that $\omega^\sigma_0(\tau)$ is negative for positive τ, and thus contributes to the stable manifold of (4.36). In addition, it is found to be of geometric multiplicity $d-1$, the associated eigenspace of $B^\sigma_\infty(\eta)^{-1}A^\sigma_\infty(\tau;\eta)$ being spanned by the vectors

$$\mathbf{W}^{j,\sigma}_0(\tau;\eta) := \begin{pmatrix} 0 \\ 0 \\ 0 \\ \tau \mathrm{e}_j \\ (u_\infty - \sigma)\eta_j \end{pmatrix}, \quad j \in \{1,\dots,d-1\},$$

for $(\tau,\eta) \neq (0,0)$. Since these vectors $\mathbf{W}^{j,\sigma}_0$ are homogeneous in (τ,η), we may renormalize them and assume that they are homogeneous degree 0, that is, constant along rays $\{(\lambda\tau,\lambda\eta);\ \lambda > 0\}$. Like the simpler equation (3.17), Eq. (4.37) has no purely imaginary root when $\operatorname{Re}\tau$ is positive. Thus the number of roots of negative real parts is independent of (τ,η), within the half-space $\{\operatorname{Re}\tau > 0\}$. As already seen in the case $\eta = 0$ (in which (4.37) degenerates to (3.17)), this number is two. We denote by $\omega^\sigma_1(\tau;\eta)$ and $\omega^\sigma_2(\tau;\eta)$ those roots. In the same way we find that (4.37) has two roots of positive real parts, $\omega^\sigma_3(\tau;\eta)$ and $\omega^\sigma_4(\tau;\eta)$ say. (Observe that $\omega^\sigma_j(\tau;\eta)$ are distinct from $\omega^\sigma_0(\tau)$ for $\tau \neq (u_\infty - \sigma)\|\eta\|$.) We choose their numbering according to their behavior as λ goes to zero along the ray $\{(\lambda\tau,\lambda\eta);\ \lambda > 0\}$. More precisely, we have

$$\omega^\sigma_1(\lambda\tau;\lambda\eta) \to -\sqrt{\big(c^2_\infty - (u_\infty-\sigma)^2\big)/(\rho_\infty K_\infty)}, \qquad \omega^\sigma_4(\lambda\tau;\lambda\eta) \to +\sqrt{\big(c^2_\infty - (u_\infty-\sigma)^2\big)/(\rho_\infty K_\infty)},$$
$$\omega^\sigma_2(\lambda\tau;\lambda\eta) \sim \lambda\underline{\omega}^\sigma_2(\tau;\eta), \qquad \omega^\sigma_3(\lambda\tau;\lambda\eta) \sim \lambda\underline{\omega}^\sigma_3(\tau;\eta),$$

as λ goes to zero, where $\underline{\omega}^\sigma_{2,3}(\tau;\eta)$ are the roots of

$$\big(\tau + (u_\infty - \sigma)\omega\big)^2 + c^2_\infty\big(\|\eta\|^2 - \omega^2\big) = 0. \tag{4.38}$$

By definition, $\operatorname{Re}\underline{\omega}^\sigma_2 < 0$ and $\operatorname{Re}\underline{\omega}^\sigma_3 > 0$. Associated eigenvectors of $B^\sigma_\infty(\eta)^{-1}A^\sigma_\infty(\tau;\eta)$ are

$$\mathbf{W}^\sigma_j(\tau;\eta) := \begin{pmatrix} \rho_\infty \\ \rho_\infty\omega^\sigma_j(\tau;\eta) \\ \rho_\infty\omega^\sigma_j(\tau;\eta)^2 \\ \frac{\tau + (u_\infty-\sigma)\omega^\sigma_j(\tau;\eta)}{\omega^\sigma_j(\tau;\eta)^2 - \|\eta\|^2}\eta \\ -\omega^\sigma_j(\tau;\eta)\frac{\tau + (u_\infty-\sigma)\omega^\sigma_j(\tau;\eta)}{\omega^\sigma_j(\tau;\eta)^2 - \|\eta\|^2} \end{pmatrix}. \tag{4.39}$$

With this choice we have

$$\lim_{\lambda \searrow 0} \mathbf{W}^\sigma_{1,4}(\lambda\tau;\lambda\eta) = \begin{pmatrix} \rho_\infty \\ \rho_\infty\omega^\sigma_{1,4}(0;0) \\ \rho_\infty\omega^\sigma_{1,4}(0;0)^2 \\ 0_{d-1} \\ -(u_\infty - \sigma) \end{pmatrix},$$

$$\lim_{\lambda\searrow 0}\mathbf{W}^{\sigma}_{2,3}(\lambda\tau;\lambda\eta)=\begin{pmatrix}\rho_\infty\\ 0\\ 0\\ \frac{\tau+(u_\infty-\sigma)\underline{\omega}^{\sigma}_{2,3}(\tau;\eta)}{\underline{\omega}^{\sigma}_{2,3}(\tau;\eta)^2-\|\eta\|^2}\eta\\ -\underline{\omega}^{\sigma}_{2,3}(\tau;\eta)\frac{\tau+(u_\infty-\sigma)\underline{\omega}^{\sigma}_{2,3}(\tau;\eta)}{\underline{\omega}^{\sigma}_{2,3}(\tau;\eta)^2-\|\eta\|^2}\end{pmatrix}=\begin{pmatrix}\rho_\infty\\ 0\\ 0\\ \frac{c_\infty^2}{\tau+(u_\infty-\sigma)\underline{\omega}^{\sigma}_{2,3}(\tau;\eta)}\eta\\ -\frac{c_\infty^2\underline{\omega}^{\sigma}_{2,3}(\tau;\eta)}{\tau+(u_\infty-\sigma)\underline{\omega}^{\sigma}_{2,3}(\tau;\eta)}\end{pmatrix}.$$

By the method of Zumbrun et al. [21,23], we can construct an Evans function D^σ, analytic along rays $\{(\lambda\tau,\lambda\eta);\ \lambda>0\}$ and real-valued for $\tau\in[0,+\infty)$, such that

$$D^\sigma(\tau;\eta)=0,\quad \operatorname{Re}\tau>0\quad\Longleftrightarrow\quad \operatorname{Ker}\big(L^\sigma(\eta)-\tau\big)\neq\{0\}.$$

By definition, away from collision points,

$$D^\sigma(\tau;\eta)=\det\big(\Phi_0^{1,\sigma}(\tau;\eta),\dots,\Phi_0^{d-1,\sigma}(\tau;\eta),\Phi_1^\sigma(\tau;\eta),\Phi_2^\sigma(\tau;\eta),\Phi_3^\sigma(\tau;\eta),\Phi_4^\sigma(\tau;\eta)\big)\big|_{\xi=0},$$

where $\Phi_j(\tau;\eta)$ are solutions of (4.35) such that

$$\Phi_0^{j,\sigma}(\tau;\eta)\overset{\xi\to+\infty}{\sim}e^{\omega_0^\sigma(\tau;\eta)\xi}\mathbf{W}_0^{j,\sigma}(\tau;\eta),\qquad \Phi_{0,1,2}(\tau;\eta)\overset{\xi\to+\infty}{\sim}e^{\omega_{0,1,2}(\tau;\eta)\xi}\mathbf{W}^\sigma_{0,1,2}(\tau;\eta),$$
$$\Phi_{3,4}(\tau;\eta)\overset{\xi\to-\infty}{\sim}e^{\omega^\sigma_{3,4}(\tau;\eta)\xi}\mathbf{W}^\sigma_{3,4}(\tau;\eta).\tag{4.40}$$

Since $L^\sigma(0)\cdot(\underline{\mathbf{U}}^\sigma)'=0$ and $(\underline{\mathbf{U}}^\sigma)'$ goes exponentially fast to zero at $\pm\infty$, as in dimension 1 we observe that

$$D^\sigma(\tau;\eta)=-r^2\det\big(\Phi_0^{1,\sigma}(\tau;\eta),\dots,\Phi_0^{d-1,\sigma}(\tau;\eta),\check{\Phi}_1^\sigma(\tau;\eta),\Phi_2^\sigma(\tau;\eta),\Phi_3^\sigma(\tau;\eta),\check{\Phi}_4^\sigma(\tau;\eta)\big)\big|_{\xi=0},$$

where

$$\check{\Phi}_1^\sigma(0;0)=\check{\Phi}_4^\sigma(0;0)=\begin{pmatrix}(\underline{\rho}^\sigma)'\\ (\underline{\rho}^\sigma)''\\ (\underline{\rho}^\sigma)'''\\ 0_{d-1}\\ (\underline{u}^\sigma)'\end{pmatrix}.\tag{4.41}$$

For simplicity, we shall omit the ˇ hats in what follows. Eq. (4.41) obviously implies that $D^\sigma(0;0)=0$. Furthermore, we have

$$\frac{d}{d\lambda}D^\sigma(\lambda\tau;\lambda\eta)\bigg|_{\lambda=0}=0.\tag{4.42}$$

To prove this, we introduce notations for the components of Φ_j^σ and $\Psi_j^\sigma:=\partial_\lambda\Phi_j^\sigma(\lambda\tau;\lambda\eta)$, namely,

$$\Phi_j^\sigma=\begin{pmatrix}\phi_j^\sigma\\ (\phi_j^\sigma)'\\ (\phi_j^\sigma)''\\ \nu_j^\sigma\\ \mu_j^\sigma\end{pmatrix},\qquad \Psi_j^\sigma=\begin{pmatrix}\psi_j^\sigma\\ (\psi_j^\sigma)'\\ (\psi_j^\sigma)''\\ \zeta_j^\sigma\\ \chi_j^\sigma\end{pmatrix}.$$

By differentiation of $(B^\sigma(\lambda\eta)\Phi_j^\sigma(\lambda\tau;\lambda\eta))'=A^\sigma(\lambda\tau;\lambda\eta)\Phi_j^\sigma(\lambda\tau;\lambda\eta)$ with respect to λ, we obtain

$$\big(B^\sigma(0)\Psi_j^\sigma(0;0)\big)'=A^\sigma(0;0)\Psi_j^\sigma(0;0)+A_1^\sigma(\tau;\eta)\Phi_j^\sigma(0;0),$$
$$A_1^\sigma(\tau;\eta):=\frac{d}{d\lambda}A^\sigma(\lambda\tau;\lambda\eta)\bigg|_{\lambda=0}=\begin{pmatrix}0&0&0&0^*_{d-1}&0\\ 0&0&0&0^*_{d-1}&0\\ 0&0&0&0^*_{d-1}&\tau\\ (\underline{\alpha}^\sigma+\underline{K}^\sigma\|\eta\|^2)\eta&-(\underline{K}^\sigma)'\eta&-\underline{K}^\sigma\eta&-\tau\mathbf{I}_{d-1}&0_{d-1}\\ -\tau&0&0&-\underline{\rho}^\sigma\eta^{t}&0\end{pmatrix}.\tag{4.43}$$

By (4.41), we have

$$A_1^\sigma(\tau;\eta)\Phi_{1,4}^\sigma(0;0)=\begin{pmatrix}0\\ 0\\ \tau(\underline{u}^\sigma)'\\ (\underline{\alpha}^\sigma(\underline{\rho}^\sigma)'-(\underline{K}^\sigma)'(\underline{\rho}^\sigma)''-\underline{K}^\sigma(\underline{\rho}^\sigma)''')\eta\\ -\tau(\underline{\rho}^\sigma)'\end{pmatrix}.$$

We thus see that the third row, respectively the last row, in (4.43) for $j = 1,4$ are equivalent to the second (and last) row, respectively the first row, in

$$L^\sigma \cdot \begin{pmatrix} \psi^\sigma_{1,4}(0;0) \\ \chi^\sigma_{1,4}(0;0) \end{pmatrix} = \tau \begin{pmatrix} (\underline{\rho}^\sigma)' \\ (\underline{u}^\sigma)' \end{pmatrix} = -\tau L^\sigma \cdot \begin{pmatrix} \partial_\sigma \underline{\rho}^\sigma \\ \partial_\sigma \underline{u}^\sigma \end{pmatrix},$$

where L^σ is the one-dimensional operator of Section 3. Therefore, up to adding a constant times $\lambda \Phi^\sigma_{1,4}(\lambda\tau;\lambda\eta)$ to $\Phi^\sigma_{1,4}(\lambda\tau;\lambda\eta)$, we may assume that

$$\begin{pmatrix} \psi^\sigma_1(0;0) \\ \chi^\sigma_1(0;0) \end{pmatrix} = \begin{pmatrix} \psi^\sigma_4(0;0) \\ \chi^\sigma_4(0;0) \end{pmatrix} = -\tau \begin{pmatrix} \partial_\sigma(\underline{\rho}^\sigma) \\ \partial_\sigma(\underline{u}^\sigma) \end{pmatrix}. \tag{4.44}$$

Now the intermediate $(d-1)$ rows in (4.43) for $j = 1,4$ read

$$\big((\underline{u}^\sigma - \sigma)\zeta_{1,4}(0;0)\big)' = \big(\underline{\alpha}^\sigma (\underline{\rho}^\sigma)' - (\underline{K}^\sigma)'(\underline{\rho}^\sigma)'' - \underline{K}^\sigma (\underline{\rho}^\sigma)'''\big)\eta = -\left(\frac{1}{2}(\underline{u}^\sigma - \sigma)^2\right)'\eta$$

by the profile equation (3.13). Therefore, by integration,

$$\zeta_1(0;0) = \zeta_4(0;0) = -\frac{1}{2}\left((\underline{u}^\sigma - \sigma) - \frac{(u_\infty - \sigma)^2}{\underline{u}^\sigma - \sigma}\right)\eta.$$

So finally, we have

$$\Psi^\sigma_1(0;0) = \Psi^\sigma_4(0;0), \tag{4.45}$$

which together with (4.41) implies (4.42), and

$$\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2} D^\sigma(\lambda\tau;\lambda\eta)\bigg|_{\lambda=0} = \det\big(\Phi^{1,\sigma}_0, \ldots, \Phi^{d-1,\sigma}_0, \Phi^\sigma_1, \Phi^\sigma_2, \Phi^\sigma_3, \partial^2_{\lambda\lambda}\big(\Phi^\sigma_4 - \Phi^\sigma_1\big)\big)(\lambda\tau;\lambda\eta)\big|_{\lambda=0}\big|_{\xi=0}.$$

For simplicity, in what follows we omit the superscript σ, and we just denote Φ_j for $\Phi^\sigma_j(0;0)$, and Θ_j for $\partial^2_{\lambda\lambda}\Phi^\sigma_j(\lambda\tau;\lambda\eta)|_{\lambda=0}$. The starting point is to evaluate the determinant above is to note that

$$\det B = \underline{\rho}\,\underline{K}(\underline{u} - \sigma)^{d-1} \neq 0,$$

hence

$$\begin{aligned} &\det\big(\Phi^1_0, \ldots, \Phi^{d-1}_0, \Phi_1, \Phi_2, \Phi_3, \partial^2_{\lambda\lambda}\big(\Phi^\sigma_4 - \Phi^\sigma_1\big)\big) \\ &\quad = \frac{1}{\underline{\rho}\,\underline{K}(\underline{u} - \sigma)^{d-1}} \det\big(B\Phi^1_0, \ldots, B\Phi^{d-1}_0, B\Phi_1, B\Phi_2, B\Phi_3, \partial^2_{\lambda\lambda} B(\Phi_4 - \Phi_1)\big). \end{aligned}$$

By construction of Φ_j, since all but the first two rows of $A(0;0)$ are zeroes, we have

$$B(0)\Phi_j = \begin{pmatrix} \phi_j \\ \phi'_j \\ R_j \end{pmatrix},$$

where R_j is a *constant vector* in $\mathbb{R}^{d+1}$ determined by the asymptotic behavior of Φ_j. In particular R_1 is the null vector. We shall compute the other vectors R_j later on. We also need some information on $S_{1,4}: \xi \to S_{1,4}(\xi) \in \mathbb{R}^2$ such that, by definition,

$$B\Theta_{1,4} = \begin{pmatrix} \theta_{1,4} \\ \theta'_{1,4} \\ S_{1,4} \end{pmatrix}.$$

Differentiating twice $(B(\lambda\eta)\Phi_j(\lambda\tau;\lambda\eta))' = A(\lambda\tau;\lambda\eta)\Phi_j(\lambda\tau;\lambda\eta)$ with respect to λ, we obtain

$$\big(B(0)\Theta_j + B_2(\eta)\Phi_j\big)' = A(0;0)\Theta_j + 2A^\sigma_1(\tau;\eta)\Psi_j,$$

$$B_2(\eta) := \frac{\mathrm{d}^2}{\mathrm{d}\lambda^2} B(\lambda\eta)\bigg|_{\lambda=0} = \begin{pmatrix} 0 & 0 & 0 & 0^*_{d-1} & 0 \\ 0 & 0 & 0 & 0^*_{d-1} & 0 \\ -2\underline{K}\|\eta\|^2 & 0 & 0 & 0^*_{d-1} & 0 \\ 0 & 0 & 0 & \mathbf{0}_{d-1} & 0_{d-1} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.46}$$

In particular, we have by (4.41) and (4.44),

$$S'_{1,4} = 2\begin{pmatrix} -\tau^2\partial_\sigma \underline{u} + (\underline{K}\underline{\rho}')'\|\eta\|^2 \\ -\tau(\underline{\alpha}\partial_\sigma\underline{\rho} - \underline{K}'\partial_\sigma\underline{\rho}' - \underline{K}\partial_\sigma\underline{\rho}'')\eta + \frac{1}{2}\tau((\underline{u}-\sigma) - \frac{(u_\infty-\sigma)^2}{\underline{u}-\sigma})\eta \\ \tau^2\partial_\sigma\underline{\rho} + \frac{1}{2}\underline{\rho}((\underline{u}-\sigma) - \frac{(u_\infty-\sigma)^2}{\underline{u}-\sigma})\|\eta\|^2 \end{pmatrix}.$$

Lemma 2. *If Π denotes the projection operator*

$$\Pi : \begin{pmatrix} \phi \\ \phi' \\ R \end{pmatrix} \in \mathbb{C}^{d+3} \mapsto R \in \mathbb{C}^{d+1},$$

then, if $\tau^2 \neq (u_\infty - \sigma)^2\|\eta\|^2$, the vectors $R_0^j := \Pi B(0)\Phi_0^j$, $j = 1,\dots,d-1$, and $R_k := \Pi B(0)\Phi_k$, $k = 2, 3$, are independent.

Remark 3. Points (τ,η) with $\tau^2 = (u_\infty - \sigma)^2\|\eta\|^2$ are collision points, for which ω_2 coincides with ω_0. Our computation below does not imply at all that the second-order derivative of the Evans function vanishes at those points: a different computation should be made to find the actual value of that derivative.

Proof of Lemma 2. We easily compute that

$$R_0^j = (u_\infty - \sigma)\begin{pmatrix} -(u_\infty-\sigma)\eta_j \\ \tau \mathrm{e}_j \\ \rho_\infty\eta_j \end{pmatrix},$$

and for $k = 2, 3$,

$$R_k = \frac{1}{\tau + (u_\infty - \sigma)\underline{\omega}_k}\begin{pmatrix} -c_\infty^2\tau \\ (u_\infty-\sigma)c_\infty^2\eta \\ \rho_\infty((u_\infty-\sigma)(\tau + (u_\infty-\sigma)\underline{\omega}_k) - c_\infty^2\omega_k) \end{pmatrix},$$

hence

$$\det\big(R_0^1,\dots,R_0^{d-1},R_2,R_3\big) = c_\infty^2(\omega_2-\omega_3)\big(c_\infty^2 - (u_\infty-\sigma)^2\big)\big(\tau^2 - (u_\infty-\sigma)^2\|\eta\|^2\big). \qquad \square$$

Thanks to Lemma 2, we may proceed as in Section 3. We introduce (the unique) numbers d_0^j, $j = 1,\dots,d-1$, and $d_{2,3}$ such that

$$S_4 - S_1 = \sum_{j=1}^{d-1} d_0^j R_0^j + d_2R_2 - d_3R_3,$$

and develop the determinant as follows,

$$\begin{aligned}
&\det\big(B(0)\Phi_0^1,\dots,B(0)\Phi_0^{d-1},B\Phi_1,B(0)\Phi_2,B(0)\Phi_3,\partial^2_{\lambda\lambda}B(0)(\Phi_4-\Phi_1)\big) \\
&= \begin{vmatrix} \phi_0^1 & \cdots & \phi_0^{d-1} & \underline{\rho}' & \phi_2 & \phi_3 & \widetilde{\theta}_4 - \widetilde{\theta}_1 \\ (\phi_0^1)' & \cdots & (\phi_0^{d-1})' & \underline{\rho}'' & \phi_2' & \phi_3' & \widetilde{\theta}_4' - \widetilde{\theta}_1' \\ R_0^1 & \cdots & R_0^{d-1} & 0_{d+1} & R_2 & R_3 & 0_{d+1} \end{vmatrix} \\
&= \det\big(R_0^1,\dots,R_0^{d-1},R_2,R_3\big)\begin{vmatrix} \underline{\rho}' & \widetilde{\theta}_4 - \widetilde{\theta}_1 \\ \underline{\rho}'' & \widetilde{\theta}_4' - \widetilde{\theta}_1' \end{vmatrix}
\end{aligned}$$

with

$$\widetilde{\theta}_4 := \theta_4 + d_3\phi_3, \qquad \widetilde{\theta}_1 := \theta_1 + \sum_{j=1}^{d-1} d_0^j\phi_0^j + d_2\phi_2.$$

By the same technique as in Section 3 we find that

$$\begin{vmatrix} \underline{\rho}' & \widetilde{\theta}_4 - \widetilde{\theta}_1 \\ \underline{\rho}'' & \widetilde{\theta}_4' - \widetilde{\theta}_1' \end{vmatrix}_{|\xi=0} = \frac{1}{\underline{K}(0)}\int_{-\infty}^{+\infty} s[\widetilde{\theta}_{1,4}]\underline{\rho}',$$

with

$$s[\widetilde{\theta}_4] := \underline{K}\widetilde{\theta}_4'' + (\underline{K})'\widetilde{\theta}_4' - \underline{\alpha}\widetilde{\theta}_4 + \frac{1}{\underline{\rho}}(\underline{u}-\sigma)^2\widetilde{\theta}_4 = \big(1, 0_{d-1}^*, (\underline{u}-\sigma)/\underline{\rho}\big)(S_4 + d_3 R_3)$$
$$= \big(1, 0_{d-1}^*, (\underline{u}-\sigma)/\underline{\rho}\big)\left(S_1 + \sum_{j=1}^{d-1} d_0^j R_0^j + d_2 R_2\right) =: s[\widetilde{\theta}_1].$$

Since

$$(\underline{u}-\sigma)\underline{\rho}' = -\underline{\rho}\,\underline{u}',$$

we have

$$\int_{-\infty}^{+\infty} s[\widetilde{\theta}_{1,4}]\underline{\rho}' = \int_{-\infty}^{+\infty} \big(\underline{\rho}', 0_{d-1}^*, -\underline{u}'\big)S_4 = 2\int_{-\infty}^{+\infty} \tau^2\big((\underline{\rho}-\rho_\infty)\partial_\sigma \underline{u} + (\underline{u}-u_\infty)\partial_\sigma \underline{\rho}\big)$$
$$+ \int_{-\infty}^{+\infty} \|\eta\|^2 \left(2\underline{K}(\underline{\rho}')^2 + \underline{\rho}(\underline{u}-u_\infty)\left((\underline{u}-\sigma) - \frac{(u_\infty-\sigma)^2}{\underline{u}-\sigma}\right)\right)$$

after integration by part. In factor of τ^2 we recognize $-m''(\sigma)$ (see (3.10)), and the factor of $\|\eta\|^2$ is obviously positive, since

$$2\underline{K}(\underline{\rho}')^2 + \underline{\rho}(\underline{u}-u_\infty)\left((\underline{u}-\sigma) - \frac{(u_\infty-\sigma)^2}{\underline{u}-\sigma}\right) \geqslant \frac{\underline{\rho}}{\underline{u}-\sigma}\big(\underline{u}-\sigma-(u_\infty-\sigma)\big)^2(\underline{u}-\sigma+u_\infty-\sigma) > 0.$$

(Recall that as $j = \underline{\rho}(\underline{u}-\sigma) = \rho_\infty(u_\infty-\sigma)$ has been assumed positive.) In conclusion, if $(\tau;\eta)$ is not a collision point, for λ close to 0, we have $D(\lambda\tau, \lambda\eta) \sim \lambda^2 P(\tau;\eta)$ with

$$P(\tau;\eta) = -r^2\big(-m''(\sigma)\tau^2 + s^2\|\eta\|^2\big),$$

where r and s are nonzero real numbers. If $m''(\sigma) < 0$, which implies that the solitary wave is one-d unstable by Theorem 1, perturbations transverse to the wave makes the local behavior of the Evans function even 'worse'. If $m''(\sigma) > 0$, which implies that the solitary wave is orbitally stable in one space dimension, we find as announced above a continuous branch $\eta \mapsto \tau_\sharp(\eta)$ along which D vanishes. We have thus proved the following.

Theorem 2. *Planar solitary waves satisfying the one-dimensional stability condition* (3.25) (*p.* 348) *are linearly unstable in several space dimensions, in the sense that the linearized equations* (4.33) (*p.* 351) *admit growing mode solutions* $\dot{\mathbf{U}}(\xi)\mathrm{e}^{\tau t + i\eta\cdot\mathbf{y}}$ *with* $\mathrm{Re}\,\tau > 0$.

In view of the method developed recently by Rousset and Tzvetkov [19], we expect that this *linear* transverse instability implies *nonlinear instability*. This will be the purpose of a separate paper.

Remark 4. There are no growing modes $\dot{\mathbf{U}}(\xi)\mathrm{e}^{\tau t + i\eta\cdot\mathbf{y}}$ with $\tau \notin i\mathbb{R}$ if $\|\eta\|$ is large enough. Indeed, if

$$\underline{K}^\sigma\|\eta\|^2 + \underline{\alpha}^\sigma - \big(\underline{u}^\sigma - \sigma\big)^2/\underline{\rho}^\sigma \geqslant 0,$$

the Sturm–Liouville operator

$$\mathcal{M} + \underline{K}^\sigma\|\eta\|^2 = -\partial_\xi \underline{K}^\sigma \partial_\xi + \underline{K}^\sigma\|\eta\|^2 + \underline{\alpha}^\sigma - \big(\underline{u}^\sigma - \sigma\big)^2/\underline{\rho}^\sigma$$

is monotone, and by the energy estimates performed in [3, Theorem 2, pp. 245–247], this is enough to preclude the existence of non-neutral modes.

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