



Note

Slow manifolds for dissipative dynamical systems

A.G. Ramm

Mathematics Department, Kansas State University, Manhattan, KS 66506-2602, USA

ARTICLE INFO

Article history:

Received 2 February 2009

Available online 26 September 2009

Submitted by B. Straughan

Keywords:

Dissipative systems

Dynamical systems

Attractors

Invariant manifolds

Nonlinear evolution

ABSTRACT

A class of infinite-dimensional dissipative dynamical systems is defined for which the slow invariant manifolds can be calculated. Large-time behavior of the evolution of such systems is studied.

© 2009 Elsevier Inc. All rights reserved.

1. Introduction and statement of the results

A dynamical system is described by the evolution problem:

$$\dot{u} = -F(u), \quad u(0) = u_0; \quad \dot{u} := \frac{du}{dt}, \quad (1)$$

where $u_0 \in D(F)$ is arbitrary, $D(F)$ is the domain of F . The system is called dissipative if $F : H \rightarrow H$ is a monotone, closed and hemicontinuous operator in a Hilbert space H : $(F(u) - F(v), u - v) \geq 0$, $u, v \in D(F)$. Here (u, v) is the inner product in H , $D(F)$ is assumed to be a linear set, dense in H , and F is maximal monotone, $R(I + F) = H$, where $R(F)$ is the range of F . Under these assumptions problem (1) has a unique solution $u(t) := S(t)u_0$, defined for all $t \geq 0$, and the operator family $S(t)$ is a semigroup. A set \mathcal{A} is called a global attractor for problem (1) if for any $u_0 \in H$ $\lim_{t \rightarrow \infty} d(u(t), \mathcal{A}) = 0$, where $d(u, v)$ is the distance between u and v , and $u(t)$ solves (1). A set M is called an invariant set for problem (1) if $u_0 \in M$ implies $u(t) \in M$ for all $t > 0$. If an invariant set M is a manifold, it is called an invariant manifold for problem (1). Attractors and invariant manifolds for dissipative dynamical systems are studied in [1,4,5]. In [2, Chapter 3], and in [3] a class of dissipative nonlinear systems is studied. This class consists of passive nonlinear networks.

Assume that $A = A^* \geq m > 0$ is a self-adjoint operator in H , denote by $\sigma(A)$ its spectrum, and by E_δ its resolution of the identity. If $\Delta_\delta = [m, m + \delta]$, the $E(\Delta_\delta) = E_{m+\delta} - E_{m-0}$ is an orthogonal projection operator in H and $E(\Delta_\delta)H$ is an invariant subspace of A . By definition, $E_{s+0} = E_s$. If λ is an eigenvalue of A , then $[E_\lambda - E_{\lambda-0}]H$ is the corresponding eigenspace. By $\sigma(A)$ we denote the spectrum of A , $m = \inf \sigma(A)$.

One is often interested in finding “slow” invariant manifolds for problem (1). If $F = A$ is a linear operator, then its invariant manifold is called “slow”, if it corresponds to the smallest (lowest) eigenvalue of A . The corresponding eigenspace of A is a linear invariant manifold for problem (1). A method for finding “slow” invariant manifolds for problem (1) is

E-mail address: ramm@math.ksu.edu.

proposed in [1]. It consists of solving the problem

$$\dot{u} = -Au + b(u)u, \quad u(0) = u_0; \quad b(t) := b(u(t)) := \frac{(Au, u)}{(u, u)}, \quad (2)$$

and studying the limit $\lim_{t \rightarrow \infty} u(t) := v$. The existence of this limit will be established in this paper under suitable assumptions. One hopes that this limit, if it exists, is an eigenvector of A , corresponding to the smallest eigenvalue Λ of A . Our goal is to find sufficient conditions for the validity of such a conclusion.

In [1] no rigorous results have been established for the global existence of the solution to Eq. (2), for the existence of the limit $\lim_{t \rightarrow \infty} u(t)$, and for finding slow manifolds in an infinite-dimensional Hilbert space. Our aim is to establish such results in this paper.

Let us formulate our results. Their proofs are outlined in Section 2.

Theorem 1. *Problem (2) has a unique global solution $u(t)$ which is given by the formula*

$$u(t) = \frac{e^{-tA}u_0}{(1 - 2 \int_0^t h(\tau) d\tau)^{1/2}}, \quad h(t) := (Ae^{-2tA}u_0, u_0), \quad (3)$$

and $\|u(t)\| = \|u_0\|$ for all $t > 0$.

Remark 1. The last statement of Theorem 1 allows one to assume without loss of generality that $\|u_0\| = 1$. Everywhere below we make this assumption. The closed form solution of the nonlinear evolution problem (2) is quite useful: among other things, it yields existence and uniqueness of the global solution to problem (2) in an infinite-dimensional Hilbert space.

Theorem 2. *Assume that A has a discrete spectrum. Let $\Lambda = m = \inf \sigma(A)$ be the smallest eigenvalue of A . Assume that m is an isolated point of spectrum, P_m is the orthoprojector in H onto the corresponding eigenspace, and $P_mu_0 \neq 0$. Under these assumptions there exists strong limit $\lim_{t \rightarrow \infty} u(t) = v$, $v \in H_m$, and $\|v\| = \|u_0\| = 1$.*

Remark 2. Suppose that the spectrum of A in the interval $[m, m + \epsilon)$ is arbitrary, containing, possibly, countably many eigenvalues λ_j , which possibly form a set dense in the interval $[m, m + \epsilon)$, so that the spectrum of A in $[m, m + \epsilon)$ contains a singular component. Here $\epsilon > 0$ is a small fixed number. Assume that m is an eigenvalue of A , possibly of infinite multiplicity, H_m is the corresponding eigenspace, P_m is the orthogonal projector onto H_m , and $P_mu_0 \neq 0$. Let $H_1 := H_m$ and $H_2 := H_1^\perp$. Then the conclusion of Theorem 2 remains valid.

The idea of the proof is the same as in the proof of Theorem 2. We leave the details of the proof to the reader.

Theorem 3. *If the spectrum $\sigma(A)$ is absolutely continuous on the interval $[m, m + \delta] := \Delta_\delta$, $\delta > 0$, and $E(\Delta_\delta)u_0 \neq 0$, then there does not exist strong limit $\lim_{t \rightarrow \infty} u(t) = v$, $v \in H$.*

Theorem 4. *If $E(\Delta_\delta)u_0 \neq 0$, $m = \inf \sigma(A)$ is an eigenvalue of A , and m is an isolated eigenvalue embedded into absolutely continuous spectrum of A , then there exists strong limit $\lim_{t \rightarrow \infty} u(t) = v$, $v \in H_m$, and $\|v\| = \|u_0\| = 1$.*

2. Proofs

Proof of Theorem 1. If a solution to (2) exists, then $\|u(t)\| = \|u_0\|$. Indeed, multiply (2) by $u(t)$ and get $\frac{d\|u(t)\|^2}{dt} = 0$. This implies the desired conclusion. Therefore, without loss of generality we will assume below that $\|u(t)\| = \|u_0\| = 1$.

Denote $z(t) := \int_0^t (Au(\tau), u(\tau)) d\tau$, so $\dot{z} = (Au(t), u(t))$. From (2) one gets

$$u(t) = e^{z(t)} e^{-tA} u_0, \quad z(t) := \int_0^t (Au(\tau), u(\tau)) d\tau. \quad (4)$$

Apply the operator A to (4) and then multiply by u to get

$$\dot{z} = e^{2z(t)} h(t), \quad h(t) = (Ae^{-2tA}u_0, u_0). \quad (5)$$

From (5) one gets $e^z = (1 - 2 \int_0^t h(\tau) d\tau)^{-\frac{1}{2}}$. This and (4) yield (3). Theorem 1 is proved. \square

Corollary 1. *Formula (3) for $\|u_0\| = 1$ can be rewritten as*

$$u(t) = \frac{\int_m^\infty e^{-st} dE_s u_0}{(\int_m^\infty e^{-2st} d\rho)^{\frac{1}{2}}}, \quad d\rho := d(E_s u_0, u_0). \quad (6)$$

To derive (6) one uses formula (3) and the spectral theorem, in particular, the relation $\int_m^\infty d\rho = \|u_0\|^2 = 1$.

Proof of Theorem 2. Assume for simplicity that $\lambda = m$ is an isolated eigenvalue of A and H_m is the corresponding eigenspace. Decompose H into an orthogonal sum of two subspaces, invariant with respect to A , one of which is H_m . Then the solution to (2) can be written as $u = u_1 + u_2$, where $u_1 \in H_m$ and u_2 is orthogonal to u_1 . One has $\|u(t)\|^2 = \|u_1(t)\|^2 + \|u_2(t)\|^2 = 1$, and $\|u_2(t)\| = o(\|u_1(t)\|)$ as $t \rightarrow \infty$. Therefore, $\lim_{t \rightarrow \infty} \|u_1(t)\| = \lim_{t \rightarrow \infty} \|u(t)\| = 1$. If $\dim H_m = 1$, and the corresponding eigenvector is ϕ , $\|\phi\| = 1$, then there exists strong limit $\lim_{t \rightarrow \infty} u(t) = \phi$. In the general case, Eq. (2) is equivalent to the system of equations:

$$\dot{u}_1 = -Au_1 + bu_1, \quad \dot{u}_2 = -Au_2 + bu_2; \quad b := b(u(t)), \quad (7)$$

$$u_1(0) = u_{01} = E(\Delta_\delta)u_0, \quad u_2 \perp u_1. \quad (8)$$

One has

$$u_j(t) = e^{z(t)-tA}u_{0j}, \quad j = 1, 2, \quad (9)$$

where $z(t)$ is defined in (4). Therefore, $\lim_{t \rightarrow \infty} \frac{\|u_2(t)\|^2}{\|u_1(t)\|^2} = 0$, and

$$\lim_{t \rightarrow \infty} e^{-mt+z(t)} = \frac{\|u_0\|}{\|u_{01}\|}. \quad (10)$$

Consequently, there exists the strong limit:

$$v := \lim_{t \rightarrow \infty} u(t) = u_{01} \frac{\|u_0\|}{\|u_{01}\|}, \quad v \in H_m, \quad (11)$$

and $\|v\| = \|u_0\|$. Theorem 2 is proved. \square

Proof of Theorem 3. Suppose to the contrary that there exists strong limit $\lim_{t \rightarrow \infty} u(t) = v$. Clearly, $v \neq 0$, because $\|u(t)\| = 1$ for all $t \geq 0$, so $\|v\| = 1$. Without loss of generality assume that $u(t) \in E(\Delta_\delta)H$ and A is bounded, because the part A_1 of A in the invariant subspace $E(\Delta_\delta)H$ is bounded. Then the limit

$$\lim_{t \rightarrow \infty} (Au(t), u(t)) = (Av, v) := \lambda$$

exists, and

$$\lim_{t \rightarrow \infty} Au(t) = Av.$$

Therefore $\lim_{t \rightarrow \infty} \dot{u}(t) := w$ exists, and

$$w = -Av + \lambda v.$$

We claim that $w = 0$.

Indeed, if $w \neq 0$, then

$$u(t+h) - u(t) = \int_t^{t+h} \dot{u} d\tau = wh[1 + o(1)], \quad t \rightarrow \infty.$$

This contradicts the Cauchy criterion for the existence of the limit $\lim_{t \rightarrow \infty} u(t) = v$, unless $w = 0$. Thus, $w = 0$ and $Av = \lambda v$, $\|v\| = 1$. Therefore, $\lambda \in \Delta_\delta$ is an eigenvalue of A , contrary to our assumption. Theorem 3 is proved. \square

Remark 3. If the interval $\Delta = [m, m + \delta)$ consists of the points of absolutely continuous spectrum of A , and the projection of the initial data u_0 onto the invariant subspace $E(\Delta)H$ of A is non-zero, then there does not exist strong limit of the solution $u(t)$ to problem (2) as $t \rightarrow \infty$; the trajectory of the solution $u(t)$ does not stay in any fixed finite-dimensional subspace of H , and does not stay in any fixed compact subset of H . It stays on an infinite-dimensional sphere $\|u(t)\| = \|u_0\|$ in H . In this sense the trajectory of the solution $u(t)$ is chaotic.

Proof of Theorem 4. This proof is briefly sketched. If the spectrum of A in the interval $(m, m + \delta)$ is absolutely continuous, then the solution $u(t)$ to (2) can be written as $u = u_m + u'$, where $u_m \in H_m$ and u' is orthogonal to H_m , and $\|u'(t)\| = o(\|u_m(t)\|)$ as $t \rightarrow \infty$. If the spectrum of A is absolutely continuous on $(m, m + \delta)$, then the function $d\rho = \mu(s)ds$, where $\mu \in L^1(\Delta_\delta)$, and the following estimate holds: $\int_m^{m+\delta} e^{-ts} \mu(s) ds = o(e^{-mt})$ as $t \rightarrow \infty$. On the other hand, the part of the solution, which lies in H_m is $e^{-mt}\psi$, where $\psi \in H_m$, $\psi \neq 0$. This part is the main part of the solution as $t \rightarrow \infty$. Dividing the solution by the normalizing factor as in formula (6), one gets in the limit $t \rightarrow \infty$ a normalized element v of H_m . The outline of the proof of Theorem 4 is completed. \square

References

- [1] A. Gorban, I. Karlin, *Invariant Manifolds for Physical and Chemical Kinetics*, Springer, Berlin, 2005.
- [2] A.G. Ramm, *Theory and Applications of Some New Classes of Integral Equations*, Springer-Verlag, New York, 1980.
- [3] A.G. Ramm, Stationary regimes in passive nonlinear networks, in: P. Uslenghi (Ed.), *Nonlinear Electromagnetics*, Acad. Press, New York, 1980, pp. 263–302.
- [4] A.G. Ramm, Attractors of strongly dissipative systems, *Bull. Pol. Acad. Sci.* 57 (1) (2009) 25–31.
- [5] R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, Springer, New York, 1997.