Normal weighted composition operators on the Hardy space $H^2(\mathbb{U})$

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A B S T R A C T

Let φ be an analytic function on the open unit disc $\mathbb{U}$ such that $\varphi(\mathbb{U}) \subseteq \mathbb{U}$, and let ψ be an analytic function on $\mathbb{U}$ such that the weighted composition operator $W_{\psi,\varphi}$ defined by $W_{\psi,\varphi}f = \psi f \circ \varphi$ is bounded on the Hardy space $H^2(\mathbb{U})$. We characterize those weighted composition operators on $H^2(\mathbb{U})$ that are unitary, showing that in contrast to the unweighted case ($\psi \equiv 1$), every automorphism of $\mathbb{U}$ induces a unitary weighted composition operator. A conjugation argument, using these unitary operators, allows us to describe all normal weighted composition operators on $H^2(\mathbb{U})$ for which the inducing map φ fixes a point in $\mathbb{U}$. This description shows both ψ and φ must be linear fractional in order for $W_{\psi,\varphi}$ to be normal (assuming φ fixes a point in $\mathbb{U}$). In general, we show that if $W_{\psi,\varphi}$ is normal on $H^2(\mathbb{U})$ and $\psi \not\equiv 0$, then φ must be either univalent on $\mathbb{U}$ or constant. Descriptions of spectra are provided for the operator $W_{\psi,\varphi} : H^2(\mathbb{U}) \to H^2(\mathbb{U})$ when it is unitary or when it is normal and φ fixes a point in $\mathbb{U}$.

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1. Introduction

Let $H(\mathbb{U})$ denote the collection of all holomorphic functions on the open unit disc $\mathbb{U}$, and let $H^2(\mathbb{U})$ be the classical Hardy space of $\mathbb{U}$, a Hilbert space consisting of those functions in $H(\mathbb{U})$ whose Maclaurin coefficients are square summable. Throughout this paper, φ denotes an analytic selfmap of $\mathbb{U}$; i.e., an element of $H(\mathbb{U})$ mapping $\mathbb{U}$ into $\mathbb{U}$. A result due to Littlewood [8] shows that φ induces a bounded composition operator on $H^2(\mathbb{U})$ defined by $C_\varphi f = f \circ \varphi$. Here, we study weighted composition operators $W_{\psi,\varphi}$ on $H^2(\mathbb{U})$, which result from following composition with φ with multiplication by a weight function $\psi \in H(U)$; thus, $W_{\psi,\varphi}f = \psi f \circ \varphi$. Such weighted composition operators are clearly bounded on $H^2(\mathbb{U})$ when ψ is bounded on $\mathbb{U}$, but the boundedness of ψ on $\mathbb{U}$ is not necessary for $W_{\psi,\varphi}$ to be bounded. For example, if $\varphi(\mathbb{U})$ has closure contained in $\mathbb{U}$, then $\psi \in H^2(\mathbb{U})$ is sufficient to ensure $W_{\psi,\varphi} : H^2(\mathbb{U}) \to H^2(\mathbb{U})$ is bounded. We call $W_{\psi,\varphi}$ the weighted composition operator induced by φ with weight function ψ.

Normal composition operators on the Hardy space $H^2(\mathbb{U})$ of the open unit disc are easily characterized [9, Theorem 2.4]. In fact, merely the assumption that C_φ and C_φ^* commute when applied to the Hardy space functions $f(z) = 1$ and $f(z) = z$ reveals that if $C_\varphi : H^2(\mathbb{U}) \to H^2(\mathbb{U})$ is normal, then $\varphi(z) = cz$ for some constant c, $|c| \leqslant 1$ (see [6, Exercise 8.1.2]). Conversely, if $\varphi(z) = cz$, then C_φ is a diagonal operator relative to the orthonormal basis $(z^n)_{n=0}^\infty$ of $H^2(\mathbb{U})$ and thus C_φ is normal.

For weighted composition operators, the normality question is much more interesting. Indeed, this paper is inspired by Cowen and Ko's characterization of Hermitian weighted composition operators [5], which reveals that with appropriate

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restrictions on a_0 and a_1, the pair $\psi(z) = c/(1 - \overline{a_0}z)$, $\varphi(z) = a_0 + a_1z/(1 - \overline{a_0}z)$ provide a weighted composition operator $W_{\psi,\varphi}$ that is Hermitian whenever c is real.

In this paper, we characterize those weighted composition operators on $H^2(\mathbb{U})$ that are unitary and apply this characterization to describe all normal weighted composition operators on $H^2(\mathbb{U})$ that are induced by a selfmap φ of $\mathbb{U}$ which fixes a point in $\mathbb{U}$. Our characterization of unitary weighted composition operators, which occupies Section 3, shows that every automorphism φ of $\mathbb{U}$ has a companion weight function ψ such that $W_{\psi,\varphi}$ is unitary on $H^2(\mathbb{U})$. In contrast, only the rotation maps $\varphi(z) = \zeta z$, $|\zeta| = 1$, induce unitary composition operators $C_\varphi : H^2(\mathbb{U}) \to H^2(\mathbb{U})$. Our description of all normal weighted composition operators on $H^2(\mathbb{U})$ induced by selfmaps φ of $\mathbb{U}$ fixing a point in $\mathbb{U}$, which appears in Section 4, reveals that for such operators both ψ and φ are linear fractional. In Section 2 below, we establish that if $W_{\psi,\varphi}$ is normal on $H^2(\mathbb{U})$ and $\psi \not\equiv 0$, then either φ is univalent on $\mathbb{U}$ or constant and $\psi(z)$ is nonzero at every $z \in \mathbb{U}$. At the conclusions of Sections 3 and 4, we present spectral characterizations of the unitary and normal operators identified in the section.

In the final section of the paper, we point out that the weight functions ψ that give rise to weighted composition operators $W_{\psi,\varphi}$ that are unitary, Hermitian, or normal (with inducing map fixing a point of $\mathbb{U}$) all have the same form, and then determine when such ψ can pair with a linear-fractional φ to produce a normal operator $W_{\psi,\varphi}$.

2. Preliminaries

Here we collect some background information necessary to our work and then present some simple necessary conditions for $W_{\psi,\varphi} : H^2(\mathbb{U}) \to H^2(\mathbb{U})$ to be normal.

2.1. Reproducing kernels for $H^2(\mathbb{U})$

The Hardy space $H^2(\mathbb{U})$ is a Hilbert space with inner product

$$\langle f, g \rangle = \sum_{n=0}^{\infty} \hat{f}(n) \overline{\hat{g}(n)},$$

where $(\hat{f}(n))$ and $(\hat{g}(n))$ are the sequences of Maclaurin coefficients for f and g respectively. The norm of $f \in H^2(\mathbb{U})$ is given by $(\sum_{n=0}^{\infty} |\hat{f}(n)|^2)^{1/2}$ or, alternatively, by

$$\|f\|_{H^2(\mathbb{U})}^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(e^{it})|^2 \, dt, \tag{1}$$

where $f(e^{it})$ represents the radial limit of f at e^{it}, which exists for a.e. $t \in [0, 2\pi)$.

Let $h \in H^\infty(\mathbb{U})$, the Banach algebra of bounded analytic functions on $\mathbb{U}$ with $\|h\|_\infty = \sup\{|h(z)|: z \in \mathbb{U}\}$. The integral representation of the $H^2(\mathbb{U})$ norm makes it clear that $\|hf\|_{H^2(\mathbb{U})} \leqslant \|h\|_\infty \|f\|_{H^2(\mathbb{U})}$ for each $f \in H^2(\mathbb{U})$. Thus, the *multiplication operator* $M_h : H^2(\mathbb{U}) \to H^2(\mathbb{U})$, defined by $M_h f = hf$, is bounded and linear on $H^2(\mathbb{U})$. When $\psi \in H^\infty(\mathbb{U})$, note that we may write $W_{\psi,\varphi} = M_\psi C_\varphi$ and $\|W_{\psi,\varphi}\| \leqslant \|M_\psi\|\|C_\varphi\| = \|\psi\|_\infty \|C_\varphi\|$.

For each $\beta \in \mathbb{U}$, let $K_\beta = 1/(1 - \bar{\beta}z)$. Then $K_\beta \in H^2(\mathbb{U})$, and it is easy to see that for each function $f \in H^2(\mathbb{U})$,

$$\langle f, K_\beta \rangle = f(\beta);$$

thus, K_β is the *reproducing kernel* at β for $H^2(\mathbb{U})$. We make extensive use of the following simple and well-known lemma.

Lemma 1. *Suppose that $W_{\psi,\varphi} : H^2(\mathbb{U}) \to H^2(\mathbb{U})$ is bounded and $\beta \in \mathbb{U}$. Then*

$$W_{\psi,\varphi}^* K_\beta = \overline{\psi(\beta)} K_{\varphi(\beta)}.$$

Proof. Let f be an arbitrary function in $H^2(\mathbb{U})$. We have

$$\langle f, W_{\psi,\varphi}^* K_\beta \rangle = \langle \psi f \circ \varphi, K_\beta \rangle = \psi(\beta) f(\varphi(\beta)) = \langle f, \overline{\psi(\beta)} K_{\varphi(\beta)} \rangle$$

and the lemma follows. $\square$

2.2. The Denjoy–Wolff point ω

Let φ be an analytic selfmap of $\mathbb{U}$. For n a nonnegative integer, let $\varphi^{[n]}$ denote the n-th iterate of φ so that, e.g., $\varphi^{[0]}$ is the identity function on $\mathbb{U}$ and $\varphi^{[2]} = \varphi \circ \varphi$. If φ is not an elliptic automorphism of $\mathbb{U}$, then there is a (unique) point ω in the closure $\mathbb{U}^-$ of $\mathbb{U}$ such that

$$\omega = \lim_{n \to \infty} \varphi^{[n]}(z)$$

for each $z \in \mathbb{U}$. The point ω, called the *Denjoy–Wolff point* of φ, is also characterized as follows: if $|\omega| < 1$, then $\varphi(\omega) = \omega$ and $|\varphi'(\omega)| < 1$; if $\omega \in \partial \mathbb{U}$, then $\varphi(\omega) = \omega$ and $0 < \varphi'(\omega) \leqslant 1$. If $|\omega| = 1$, then $\varphi(\omega)$ represents the angular (nontangential) limit of φ at ω and $\varphi'(\omega)$ represents the angular derivative of φ at ω. The location of the Denjoy–Wolff point and the behavior of iterate sequences $(\varphi^{[n]}(z))$ as they approach the Denjoy–Wolff point strongly influence properties of the weighted composition operator $W_{\psi,\varphi}$. To see how spectral behavior varies with location of ω, see, e.g., [2, Section 5].

2.3. Cowen's formula for the adjoint of a linear-fractional composition operator

As we have indicated, we will show that when $W_{\psi,\varphi}$ is unitary or when it is normal and φ fixes a point in $\mathbb{U}$, the inducing map φ must be linear fractional and the weight function ψ must be both linear fractional and bounded on $\mathbb{U}$. Thus $W_{\psi,\varphi}^* = (M_\psi C_\varphi)^* = C_\varphi^* M_\psi^*$, and hence Cowen's formula for the adjoint of a linear-fractional composition operator becomes quite important to our work.

When $\varphi(z) = (az + b)/(cz + d)$ is a nonconstant linear-fractional selfmapping of $\mathbb{U}$, Cowen [4] establishes

$$C_\varphi^* = M_g C_\sigma M_h^* \tag{2}$$

where the *Cowen auxiliary functions* g, σ and h are defined as follows:

$$g(z) = \frac{1}{-\bar{b}z + \bar{d}}, \qquad \sigma(z) = \frac{\bar{a}z - \bar{c}}{-\bar{b}z + \bar{d}}, \quad \text{and} \quad h(z) = cz + d.$$

2.4. From the unit disk to the right halfplane and back

The function $T(z) = (1 + z)/(1 - z)$ maps the unit disk $\mathbb{U}$ univalently onto the right halfplane Π. If φ is selfmap of $\mathbb{U}$ with Denjoy–Wolff point 1, then $\Phi := T \circ \varphi \circ T^{-1}$ is a selfmap of Π with Denjoy–Wolff point infinity (meaning $\Phi^{[n]}(w) \to \infty$ as $n \to \infty$ for every $w \in \Pi$).

In the sequel, we will use generic formulas for linear-fractional selfmappings of $\mathbb{U}$ having Denjoy–Wolff point 1. These formulas are easily derived using the correspondence between $\mathbb{U}$ and Π provided by T.

Suppose that φ is a linear-fractional selfmap of $\mathbb{U}$ with Denjoy–Wolff point 1 such that $\varphi(1) = 1$ and $\varphi'(1) = 1$. (Thus φ is of *parabolic type.*) It is easy to see that $\Phi(w) = (T \circ \varphi \circ T^{-1})(w) = w + t$, where $\text{Re}(t) \geqslant 0$ and is nonzero. (Note $\text{Re}(t) = 0$ if and only if φ is an automorphism.) Hence,

$$\varphi(z) = T^{-1}\big(T(z) + t\big) = \frac{(2 - t)z + t}{-tz + (2 + t)}. \tag{3}$$

Suppose that φ is a linear-fractional selfmap of $\mathbb{U}$ such that $\varphi(1) = 1$ and $\varphi'(1) = b < 1$. (Thus φ is of *hyperbolic type.*) It is easy to see that $\Phi(w) = (T \circ \varphi \circ T^{-1})(w) = rw + t$, where $r = 1/b$ and $\text{Re}(t) \geqslant 0$ (and $\text{Re}(t) = 0$ if and only if φ is an automorphism). Hence,

$$\varphi(z) = T^{-1}\big(T(z) + t\big) = \frac{(1 + r - t)z + r + t - 1}{(r - t - 1)z + 1 + r + t}. \tag{4}$$

2.5. Two necessary conditions for normality of $W_{\psi,\varphi}$

Lemma 2. *If $W_{\psi,\varphi}$ is normal then either $\psi \equiv 0$ or ψ never vanishes on $\mathbb{U}$.*

Proof. Suppose $W_{\psi,\varphi}$ is normal and $\psi(\beta) = 0$ for some β in $\mathbb{U}$. Then by Lemma 1, $W_{\psi,\varphi}^* K_\beta = \overline{\psi(\beta)} K_{\varphi(\beta)} \equiv 0$. Since $W_{\psi,\varphi}$ is normal, $\|W_{\psi,\varphi} K_\beta\| = \|W_{\psi,\varphi}^* K_\beta\|$. Therefore, $\|W_{\psi,\varphi} K_\beta\| = 0$ and thus $\frac{\psi(z)}{1 - \bar{\beta}\varphi(z)} = 0$ for every z in $\mathbb{U}$, which implies $\psi \equiv 0$. Thus, if $W_{\psi,\varphi}$ is normal either $\psi \equiv 0$ or ψ is nonzero at each point in $\mathbb{U}$. $\quad\square$

Proposition 3. *Suppose $W_{\psi,\varphi}$ is normal. If φ is not a constant function and ψ is not the zero function then φ is univalent.*

Proof. Suppose φ is not univalent on $\mathbb{U}$ and is nonconstant. Then there exist points a and b in $\mathbb{U}$ such that $a \neq b$ and $\varphi(a) = \varphi(b)$. Since $\psi \not\equiv 0$, from Lemma 2, we conclude that $\psi(a) \neq 0$ and $\psi(b) \neq 0$. Let

$$g = \frac{K_a}{\psi(a)} - \frac{K_b}{\psi(b)}$$

and observe g is a nonzero function in $H^2(\mathbb{U})$. We have $W_{\psi,\varphi}^* g = 0$ by Lemma 1. Since $W_{\psi,\varphi}$ is normal, $\|W_{\psi,\varphi} g\| = \|W_{\psi,\varphi}^* g\| = 0$. But $\|W_{\psi,\varphi} g\| = 0$ implies $g \circ \varphi$ is the zero function. Since φ is nonconstant, g must vanish on a nonempty open subset of $\mathbb{U}$ and hence g must be the zero function, a contradiction. Hence φ is univalent. $\quad\square$

3. Unitary weighted composition operators

Suppose that φ is the constant function $z \mapsto \beta$ for $\beta \in \mathbb{U}$. Then $W_{\psi,\varphi}$ annihilates any function in $H^2(\mathbb{U})$ that vanishes at β and hence $W_{\psi,\varphi}$ cannot be norm-preserving and thus cannot be unitary. Hence if $W_{\psi,\varphi}$ is unitary, we may apply Proposition 3 to conclude that φ must be univalent on $\mathbb{U}$. In fact, φ must be an automorphism of $\mathbb{U}$.

Proposition 4. *Suppose that* $W_{\psi,\varphi} : H^2(\mathbb{U}) \to H^2(\mathbb{U})$ *is unitary. Then* φ *must be an automorphism of* $\mathbb{U}$.

Proof. Because unitary operators are norm preserving, we see

$$1 = \|W_{\psi,\varphi}1\| = \|\psi\|.$$

In addition, if $f(z) = z$, then $\|f\| = 1$ and

$$1 = \|W_{\psi,\varphi}f\| = \|\psi\varphi\|.$$

Since $|\varphi(e^{it})| \leqslant 1$ for a.e. $t \in [0, 2\pi)$ and both $\|\psi\|$ and $\|\psi\varphi\|$ are 1, the integral representation (1) of the norm on $H^2(\mathbb{U})$ shows that $|\varphi(e^{it})| = 1$ a.e. on $\mathbb{U}$ so that φ is an inner function. However, as we have indicated, thanks to Proposition 3, we know φ is univalent. It is not difficult to see that a univalent inner function must be an automorphism of $\mathbb{U}$ (see, e.g., [6, Corollary 3.28, p. 152]), and the proposition follows. $\square$

Now we identify what form the multiplier ψ must take in order that $W_{\psi,\varphi}$ be unitary. Because any automorphism φ of $\mathbb{U}$ must take the value 0 at some β in $\mathbb{U}$, the following proposition shows that if $W_{\psi,\varphi}$ is unitary, then $\psi = ck_{\varphi^{-1}(0)}$, where $|c| = 1$ and $k_{\varphi^{-1}(0)}$ is the normalized reproducing kernel $K_{\varphi^{-1}(0)} / \|K_{\varphi^{-1}(0)}\|$.

Proposition 5. *Suppose the inducing map* φ *satisfies* $\varphi(\beta) = 0$ *for some* $\beta \in \mathbb{U}$. *If* $W_{\psi,\varphi}$ *is unitary, then*

$$\psi = c\frac{K_\beta}{\|K_\beta\|}$$

where $|c| = 1$.

Proof. Suppose that $W_{\psi,\varphi}$ is unitary; then $W_{\psi,\varphi}W_{\psi,\varphi}^*K_\beta = K_\beta$. Applying Lemma 1, we can rewrite the preceding equation as

$$W_{\psi,\varphi}\overline{\psi(\beta)}K_0 = K_\beta.$$

Because $K_0 \equiv 1$, we obtain $\psi\overline{\psi(\beta)} = K_\beta$, so that

$$\psi = \frac{K_\beta}{\overline{\psi(\beta)}}.$$

Evaluating both sides of the preceding equation at β yields $|\psi(\beta)|^2 = K_\beta(\beta) = \|K_\beta\|^2$ and the proposition follows. $\square$

Theorem 6. *The weighted composition operator* $W_{\psi,\varphi}$ *is unitary on* $H^2(\mathbb{U})$ *if and only if* φ *is an automorphism of* $\mathbb{U}$ *and* $\psi = cK_\beta/\|K_\beta\|$ *where* $\varphi(\beta) = 0$ *and* $|c| = 1$.

Proof. If $W_{\psi,\varphi}$ is unitary then φ is an automorphism of $\mathbb{U}$ by Proposition 4 and ψ has the form claimed by Proposition 5. To prove sufficiency, let

$$\varphi(z) = \eta\frac{\beta - z}{1 - \bar{\beta}z} \quad \text{and} \quad \psi = c\frac{K_\beta}{\|K_\beta\|},$$

where $|\eta| = 1$ and $|c| = 1$. The Cowen auxiliary functions of φ are

$$\sigma(z) = \frac{\beta - \bar{\eta}z}{1 - \overline{\eta\beta}z} = \varphi^{-1}(z), \qquad g(z) = \frac{1}{1 - \overline{\eta\beta}z}, \quad \text{and} \quad h(z) = 1 - \bar{\beta}z.$$

Observe that $g \circ \varphi = \|K_\beta\|^2/K_\beta$ and $\psi \circ \sigma = c\|K_\beta\|/g$.

Since $M_h^*M_\psi^* = \bar{c}/\|K_\beta\|$ is a constant, $\sigma = \varphi^{-1}$, and $W_{\psi,\varphi} = M_\psi C_\varphi$ (because ψ is bounded), we see that

$$W_{\psi,\varphi}W_{\psi,\varphi}^* = M_\psi C_\varphi M_g C_\sigma M_h^* M_\psi^* = \frac{\bar{c}}{\|K_\beta\|} \cdot \psi \cdot g \circ \varphi \cdot C_{\sigma \circ \varphi} = I$$

and

$$W_{\psi,\varphi}^* W_{\psi,\varphi} = M_g C_\sigma M_h^* M_\psi^* M_\psi C_\varphi = \frac{\bar{c}}{\|K_\beta\|} \cdot g \cdot \psi \circ \sigma \cdot C_{\varphi \circ \sigma} = I.$$

This completes the proof of the theorem. $\square$

Unitary weighted composition operators $W_{\psi,\varphi}$ divide naturally into three classes based on whether the automorphism φ is (i) elliptic, (ii) hyperbolic, or (iii) parabolic. This observation is the basis for our spectral characterizations of these operators.

Theorem 7. *Suppose that $W_{\psi,\varphi}$ is unitary (so that φ is an automorphism of $\mathbb{U}$).*

(a) *If φ is elliptic, fixing $p \in \mathbb{U}$, then $|\varphi'(p)| = 1$ and the spectrum of $W_{\psi,\varphi}$ is the closure of $\{\psi(p)\varphi'(p)^n \colon n = 0, 1, 2, \ldots\}$.*
(b) *If φ is hyperbolic or parabolic then the spectrum of $W_{\psi,\varphi}$ is the unit circle.*

Proof. Suppose that $W_{\psi,\varphi}$ is unitary; then by Theorem 6 not only is the inducing map φ an automorphism of $\mathbb{U}$, we also have that $\psi = cK_\beta/\|K_\beta\|$, where $\varphi(\beta) = 0$ and $|c| = 1$. Because $W_{\psi,\varphi}$ is unitary, the spectrum of $W_{\psi,\varphi}$ must be contained in the unit circle.

In case (a), $W_{\psi,\varphi}$ is a normal operator whose inducing map φ fixes a point p in $\mathbb{U}$. That the spectrum of $W_{\psi,\varphi} = \{\psi(p)\varphi'(p)^n \colon n = 0, 1, 2, \ldots\}^-$ follows immediately from Proposition 11 of Section 4 below, which characterizes the spectrum of any normal weighted composition operator on $H^2(\mathbb{U})$ induced by a selfmap of $\mathbb{U}$ fixing a point in $\mathbb{U}$. We wish to make clear here the connection between the form of our unitary weighted composition operators in this elliptic case and the form (8) of the normal weighted compositions described in the next section. By doing so, we will confirm that $|\psi(p)\varphi'(p)| = 1$, which must be true since the spectrum of $W_{\psi,\varphi}$ is a subset of the unit circle. We are assuming $\varphi(p) = p$. We can conjugate φ by the self-inverse automorphism $\alpha_p(z) = (p - z)/(1 - \bar{p}z)$ of $\mathbb{U}$, obtaining that $\alpha_p \circ \varphi \circ \alpha_p = \zeta z$ for some unimodular constant ζ; in fact, $\zeta = \varphi'(p)$. Equivalently, $\varphi = \alpha_p \circ (\zeta \alpha_p)$. Since $\varphi(\beta) = 0$, we see $\beta = \alpha_p(\bar{\zeta}p)$. Now observe that $\psi = cK_\beta/\|K_\beta\|$ is also given by

$$\psi(z) = c \frac{1 - |p|^2 \zeta}{|1 - |p|^2 \bar{\zeta}|} \frac{K_p}{K_p \circ \varphi}.$$

Note $\psi(p) = c(1 - |p|^2\zeta)/|1 - |p|^2\bar{\zeta}|$ is a unimodular constant. Thus in case (a) we have

$$\psi = \psi(p) \frac{K_p}{K_p \circ \varphi} \quad \text{and} \quad \varphi = \alpha_p \circ (\zeta \alpha_p).$$

Turning to case (b), let's initially assume that φ is of hyperbolic type, which means that φ has Denjoy–Wolff point $\omega \in \partial\mathbb{U}$ and $\varphi'(\omega) < 1$. We employ the idea of the proof [3, Theorem 4.3] to see that $W_{\psi,\varphi}$ similar to $\zeta W_{\psi,\varphi}$ for every unimodular constant ζ. Thus, the spectrum of $W_{\psi,\varphi}$, which we already knew to be a subset of $\partial\mathbb{U}$, must be all of $\partial\mathbb{U}$. To see that $W_{\psi,\varphi}$ is similar to $\zeta W_{\psi,\varphi}$, take $\zeta = e^{i\theta}$. To establish similarity, one can use the function f identified in the proof of [3, Theorem 4.3]: this function satisfies $f, 1/f \in H^\infty(\mathbb{U})$, and $C_\varphi f = e^{i\theta}f$. It is easy to see that $M_f^{-1} W_{\psi,\varphi} M_f = \zeta W_{\psi,\varphi}$.

Now, we assume that φ is a parabolic automorphism and $W_{\psi,\varphi}$ is normal. We know by Theorem 6 that $\psi = cK_\beta/\|K_\beta\|$, where $|c| = 1$ and $\varphi(\beta) = 0$. Because φ is a parabolic automorphism, φ has Denjoy–Wolff point $\omega \in \partial\mathbb{U}$ and $\varphi'(\omega) = 1$. Without loss of generality, we may assume that $\omega = 1$. (If $\omega \neq 1$, let $U = C_{\bar{\omega}z}$ so that U is unitary and note $UW_{\psi,\varphi}U^* = W_{\tilde{\psi},\tilde{\varphi}}$, where $\tilde{\varphi}(z) = \bar{\omega}\varphi(\omega z)$ is a parabolic automorphism with fixed point 1 and $\tilde{\psi} = K_{\tilde{\beta}}/\|K_{\tilde{\beta}}\|$ where $\tilde{\varphi}(\tilde{\beta}) = 0$.) Since φ is a parabolic automorphism with $\varphi(1) = 1$, We may assume that φ has the form given by (3) with $\text{Re}(t) = 0$ and $t \neq 0$:

$$\varphi(z) = \frac{(2 - t)z + t}{-tz + (2 + t)}. \tag{5}$$

We know the spectrum of $W_{\psi,\varphi}$ is contained in the unit circle. That it equals the unit circle is a consequence of Theorem 19 of [5], or, more accurately, it follows from an argument just like that of [5, Theorem 19]. For completeness we include a proof. Let $\lambda < 0$ be arbitrary. We show that there is a unimodular constant ν such that

$$W_{\psi,\varphi} - \nu e^{\lambda t} I$$

is not bounded below on $H^2(\mathbb{U})$, which, since t nonzero and pure imaginary and $\lambda < 0$ is arbitrary, will complete the proof.

We are assuming φ is given by (5), so that $\varphi(\beta) = 0$ where $\beta = t/(t - 2)$. Its companion weight function has the form

$$\psi = c \frac{K_\beta}{\|K_\beta\|} = \frac{2c}{\sqrt{4 + |t|^2}} \frac{2 + t}{2 + t - tz},$$

where we have used $\bar{t} = -t$ since t is pure imaginary. The proof is based on the observation that $G(z) = \frac{1}{1-z} \exp(\lambda \frac{1+z}{1-z})$ satisfies

$$W_{\psi,\varphi} G = \frac{c(2+t)}{\sqrt{4+|t|^2}} e^{\lambda t} G.$$

Let $\nu = c(2+t)(4+|t|^2)^{-1/2}$ so that ν is a unimodular constant and $W_{\psi,\varphi} G = \nu e^{\lambda t} G$. It appears that the point spectrum of $W_{\psi,\varphi}$ contains the unimodular constant $\nu e^{\lambda t}$ for every $\lambda < 0$. However, G is not in $H^2(\mathbb{U})$. The idea of the proof of [5, Theorem 19] is to modify G as follows

$$G_s(z) = \frac{1}{s-z} \exp\left(\lambda \frac{1+z}{1-z}\right),$$

so that $G_s \in H^2(\mathbb{U})$ for every real number s such that $s > 1$, and then show

$$\left\| W_{\psi,\varphi} \frac{G_s}{\|G_s\|} - \nu e^{\lambda t} \frac{G_s}{\|G_s\|} \right\| \to 0 \quad \text{as } s \to 1^+. \tag{6}$$

Because $S_\lambda(z) := \exp(\lambda \frac{1+z}{1-z})$ is an inner function, $\|G_s\|^2 = \|1/(s-z)\|^2 = s^{-2} \|K_{1/s}\|^2 = 1/(s^2-1)$. A computation shows

$$W_{\psi,\varphi} G_s = e^{\lambda t} 2\nu \frac{1}{(t-st-2)z+2s+st-t} \exp\left(\lambda \frac{1+z}{1-z}\right).$$

Thus, since ν is unimodular, t is pure imaginary, and S_λ is inner, we see the norm on the left of (6) simplifies to

$$\sqrt{s^2-1} \left\| \frac{2}{(t-st-2)z+2s+st-t} - \frac{1}{s-z} \right\| = \sqrt{s^2-1} \left\| \frac{2}{2s+st-t} K_{\frac{2-st+t}{2s-st+t}} - \frac{1}{s} K_{1/s} \right\|,$$

where we used $\bar{t} = -t$ in obtaining the reproducing kernel representation of the fraction on the left within the norm. Expanding

$$\left\langle \frac{2}{2s+st-t} K_{\frac{2-st+t}{2s-st+t}} - \frac{1}{s} K_{1/s}, \frac{2}{2s+st-t} K_{\frac{2-st+t}{2s-st+t}} - \frac{1}{s} K_{1/s} \right\rangle$$

yields (since t is pure imaginary)

$$\frac{4}{|2s+st-t|^2 - |2-st+t|^2} + \frac{1}{s^2-1} - 4\frac{1}{(s-1)} \operatorname{Re}\left(\frac{1}{(2(s+1)-st+t)}\right)$$
$$= \frac{2}{s^2-1} - \frac{8(s+1)}{s-1} \frac{1}{4(s+1)^2 + |t|^2(1-s)^2}.$$

Multiplying the quantity on the right of the preceding equation by s^2-1 and taking the limit as $s \to 1^+$ yields $2-2=0$ and we have established (6), and hence completed our proof that the spectrum of $W_{\psi,\varphi}$ is the unit circle. $\quad\square$

4. Normal $W_{\psi,\varphi}$ with φ fixing a point of $\mathbb{U}$

If $W_{\psi,\varphi} : H^2(\mathbb{U}) \to H^2(\mathbb{U})$ is normal and $\varphi(p) = p$ for some $p \in \mathbb{U}$, then it is easy to prove that the weight function ψ has a simple, linear-fractional form.

Proposition 8. *Suppose that φ has a fixed point $p \in \mathbb{U}$ and $W_{\psi,\varphi}$ is normal. Then*

$$\psi = \psi(p) \frac{K_p}{K_p \circ \varphi}.$$

Proof. We have

$$W_{\psi,\varphi}^* K_p = \overline{\psi(p)} K_p$$

so that K_p is an eigenvector for $W_{\psi,\varphi}^*$ with corresponding eigenvalue $\overline{\psi(p)}$. Since $W_{\psi,\varphi}$ is normal K_p is an eigenvector for $W_{\psi,\varphi}$ with corresponding eigenvalue $\psi(p)$ and we must have

$$\psi(p) K_p = W_{\psi,\varphi} K_p = \psi K_p \circ \varphi, \tag{7}$$

from which the proposition follows. $\quad\square$

Because $K_0 \equiv 1$, the preceding proposition immediately yields:

Corollary 9. *Suppose that $\varphi(0) = 0$. Then $W_{\psi,\varphi}$ is normal if and only if ψ is constant and C_φ is normal.*

Theorem 10. *Suppose that φ has a fixed point $p \in \mathbb{U}$. Then $W_{\psi,\varphi}$ is normal if and only if*

$$\psi = \gamma \frac{K_p}{K_p \circ \varphi} \quad and \quad \varphi = \alpha_p \circ (\delta\alpha_p), \tag{8}$$

where $\alpha_p(z) = (p - z)/(1 - \bar{p}z)$ and δ and γ are constants, with δ satisfying $|\delta| \leqslant 1$.

Proof. Suppose that $W_{\psi,\varphi}$ is normal. Let $\psi_p = \frac{K_p}{\|K_p\|}$. Since α_p is an automorphism taking p to 0 we know from Theorem 6 that W_{ψ_p,α_p} is unitary. It follows that the operator

$$L := \left(W^*_{\psi_p,\alpha_p}\right)(W_{\psi,\varphi})(W_{\psi_p,\alpha_p})$$

is normal. Let σ, g and h be the Cowen auxiliary functions for α_p so that $\sigma = \alpha_p^{-1} = \alpha_p$ and $h(z) = 1 - \bar{p}z$, $g(z) = 1/(1 - \bar{p}z)$. Observe

$$C^*_{\alpha_p} = M_g C_{\alpha_p} M^*_h \quad and \quad M^*_h M^*_{\psi_p} = (M_{\psi_p} M_h)^* = M_{1/\|K_p\|}. \tag{9}$$

Setting $\mu = 1/\|K_p\|$ and using (9), we find

$$L = C^*_{\alpha_p} M^*_{\psi_p} M_\psi C_\varphi M_{\psi_p} C_{\alpha_p} = \mu M_g M_{\psi \circ \alpha_p} M_{\psi_p \circ \varphi \circ \alpha_p} C_{\alpha_p \circ \varphi \circ \alpha_p} = M_q C_{\alpha_p \circ \varphi \circ \alpha_p},$$

where $q \in H^\infty(\mathbb{U})$ is given by $q = \mu g \cdot \psi \circ \alpha_p \cdot \psi_p \circ \varphi \circ \alpha_p$. Since $L = W_{q,\alpha_p \circ \varphi \circ \alpha_p}$ is normal and its inducing map $\alpha_p \circ \varphi \circ \alpha_p$ fixes 0, we may apply Corollary 9 to conclude that q must be a constant map and $C_{\alpha_p \circ \varphi \circ \alpha_p}$ must be normal. As we noted in the Introduction, because $C_{\alpha_p \circ \varphi \circ \alpha_p}$ is normal on $H^2(\mathbb{U})$, there must be a constant δ with $|\delta| \leqslant 1$ such that $\alpha_p \circ \varphi \circ \alpha_p = \delta z$ and we see $\varphi = \alpha_p \circ (\delta\alpha_p)$, as desired. We can confirm that $q = \mu g \cdot \psi \circ \alpha_p \cdot \psi_p \circ \varphi \circ \alpha_p$ is constant and determine its value by a straightforward computation. We know from Proposition 8 that $\psi = \psi(p)K_p/\|K_p\|$; substituting this formula for ψ into $\mu g \cdot \psi \circ \alpha_p \cdot \psi_p \circ \varphi \circ \alpha_p$ and simplifying yields $q \equiv \psi(p)$.

We now must show that if ψ and φ have the forms specified in (8), then $W_{\psi,\varphi}$ is normal. We have

$$\varphi(z) = \alpha_p\big(\delta\alpha_p(z)\big) = \frac{p(1 - \delta) + (\delta - |p|^2)z}{1 - |p|^2\delta + \bar{p}(\delta - 1)z}. \tag{10}$$

Without loss of generality, we take $\gamma = 1$ so that

$$\psi(z) = \frac{K_p(z)}{K_p(\varphi(z))} = \frac{1 - |p|^2}{1 - |p|^2\delta - \bar{p}(1 - \delta)z}. \tag{11}$$

We show that

$$W_{\psi_p,\alpha_p} C_{\delta z} W^*_{\psi_p,\alpha_p} = W_{\psi,\varphi} \tag{12}$$

so that $W_{\psi,\varphi}$ is unitarily equivalent to the normal operator $C_{\delta z}$ and hence is normal. Using (9), we see

$$W^*_{\psi_p,\alpha_p} = \frac{1}{\|K_p\|} M_g C_{\alpha_p},$$

where $g(z) = 1/(1 - \bar{p}z)$. Thus

$$W_{\psi_p,\alpha_p} C_{\delta z} W^*_{\psi_p,\alpha_p} = \frac{K_p}{\|K_p\|^2} g \circ (\delta\alpha_p) C_{\alpha_p \circ (\delta\alpha_p)}. \tag{13}$$

Now

$$g\big(\delta\alpha_p(z)\big) = \frac{1 - \bar{p}z}{1 - |p|^2\delta - \bar{p}(1 - \delta)z};$$

substituting the preceding formula for $g \circ (\delta\alpha_p)$ on the right of (13), using $1/\|K_p\|^2 = 1 - |p|^2$, and (11), we obtain (12), as desired. □

As a corollary of the proof of the preceding theorem, we obtain a spectral characterization of any normal weighted composition operator on $H^2(\mathbb{U})$ induced by a selfmap φ of $\mathbb{U}$ fixing a point in $\mathbb{U}$.

Proposition 11. *Suppose that φ has a fixed point $p \in \mathbb{U}$ and that $W_{\psi,\varphi}$ is normal. Then the spectrum of $W_{\psi,\varphi}$ is the closure of $\{\psi(p)\varphi'(p)^n: n = 0, 1, 2, \ldots\}$.*

Proof. Because $W_{\psi,\varphi}$ is normal and $\varphi(p) = p$ for $p \in \mathbb{U}$, we know from the proof of Theorem 10 that $W_{\psi,\varphi}$ is unitarily equivalent to $\psi(p)C_{\delta z}$, where $\varphi(z) = \alpha_p \circ (\delta\alpha_p)$. Because $\delta = \varphi'(p)$ and the spectrum of $C_{\delta z}$ is the closure of $\{\delta^n: n = 0, 1, 2, \ldots\}$, the unitary equivalence of $W_{\psi,\varphi}$ to $\psi(p)C_{\varphi'(p)z}$ shows that the spectrum of $W_{\psi,\varphi}$ is the closure of $\{\psi(p)\varphi'(p)^n: n \geqslant 0\}$, as desired. $\square$

Continuing with the notation in the proof of the preceding proposition (arising from (8)), we remark that in case $|\delta| = |\varphi'(p)| < 1$, the operator $W_{\psi,\varphi}$ is compact and the spectral characterization of Proposition 11 follows from [7, Theorem 1]. In case $|\delta| = 1$ and $|\gamma| = |\psi(p)| = 1$, the operator $W_{\psi,\varphi}$ is unitary and we obtain the spectral characterization promised by part (a) of Theorem 7.

5. Normality when the inducing function has Denjoy–Wolff point on $\partial\mathbb{U}$

There are normal weighted composition operators $W_{\psi,\varphi}$ where φ is nonautomorphic and has Denjoy–Wolff point on the unit circle. In fact, Cowen and Ko [5] have shown that for every $t > 0$ the parabolic map $\varphi(z) = T^{-1}(T(z) + t)$, where $T(z) = (1 + z)/(1 - z)$, has a companion weight function ψ such that $W_{\psi,\varphi}$ is Hermitian. The weight function ψ in this situation is again linear fractional. The general form of the weight functions ψ that correspond to Hermitian weighted composition operators (characterized in [5]), to unitary weighted composition operators (characterized in Section 3 above), and to the normal weighted composition operators (characterized in Section 4 above) is easily seen to be given by

$$\psi(z) = \rho K_{\sigma(0)}(z) \tag{14}$$

where σ is the Cowen auxiliary function for the linear-fractional inducing map φ and ρ is a constant. The following theorem reveals what is required for normality of a weighted composition operator $W_{\psi,\varphi}$, where φ is linear fractional and ψ has form (14).

Proposition 12. *Suppose*

$$\varphi(z) = \frac{az + b}{cz + d}$$

is a linear-fractional selfmap of $\mathbb{U}$ and $\psi = K_{\sigma(0)}$ where $\sigma(z) = (\bar{a}z - \bar{c})/(-\bar{b}z + \bar{d})$. Then $W_{\psi,\varphi}$ is normal if and only if

$$\frac{|d|^2}{|d|^2 - |b|^2 - (\bar{b}a - \bar{d}c)z}C_{\sigma \circ \varphi} = \frac{|d|^2}{|d|^2 - |c|^2 - (\bar{b}d - c\bar{a})z}C_{\varphi \circ \sigma}. \tag{15}$$

Proof. Recall $C_\varphi^* = M_g C_\sigma M_h^*$, where $g(z) = 1/(-\bar{b}z + d)$, $h(z) = cz + d$, and σ is given in the statement of the proposition. Note that $K_{\sigma(0)}(z) = \frac{d}{cz+d}$ and $K_{\sigma(0)}h = d$. Thus,

$$\left(W_{\psi,\varphi}^* W_{\psi,\varphi}\right)f = M_g C_\sigma M_h^* M_{K_{\sigma(0)}}^* M_{K_{\sigma(0)}} C_\varphi = \bar{d} \cdot g \cdot K_{\sigma(0)} \circ \sigma \cdot f \circ \varphi \circ \sigma,$$

which simplifies to the right-hand side of (15) applied to f. Similarly,

$$\left(W_{\psi,\varphi} W_{\psi,\varphi}^*\right)f = \bar{d} \cdot K_{\sigma(0)} \cdot g \circ \varphi \cdot f \circ \sigma \circ \varphi$$

simplifies to the left-hand side of (15) applied to f. $\square$

Given the results of Section 3 above and from Section 5 of [5], the following result is expected.

Proposition 13. *Suppose that φ is a linear-fractional selfmap of parabolic type and $\psi = K_{\sigma(0)}$, where σ is the Cowen auxiliary function for φ; then $W_{\psi,\varphi}$ is normal.*

Proof. Let ω be the Denjoy–Wolff point of φ so that $|\omega| = 1$, $\varphi(\omega) = \omega$, and $\varphi'(\omega) = 1$. Just as in the proof of Theorem 7, by considering conjugation via $C_{\omega z}$, we can, without loss of generality, assume that $\omega = 1$.

Because φ is parabolic and fixes 1, it has the from given by (3):

$$\varphi(z) = \frac{(2 - t)z + t}{-tz + (2 + t)},$$

where $\mathrm{Re}(t) \geqslant 0$. The condition (15) of Proposition 12 becomes in this situation

$$\frac{|2 + t|^2}{|2 + t|^2 - |t|^2 - 4\mathrm{Re}(t)}C_{\sigma \circ \varphi} = \frac{|2 + t|^2}{|2 + t|^2 - |t|^2 - 4\mathrm{Re}(t)}C_{\varphi \circ \sigma}.$$

Because φ and σ have the same fixed point set, they must commute: $\sigma \circ \varphi = \varphi \circ \sigma$, and the normality of $W_{\psi,\varphi}$ follows. $\quad\square$

Suppose that φ is of hyperbolic type with Denjoy–Wolff point $\omega \in \partial\mathbb{U}$ and $\varphi'(\omega) = b < 1$. Without loss of generality we again assume $\omega = 1$ and hence φ has the form (4):

$$\varphi(z) = \frac{(1 + r - t)z + r + t - 1}{(r - t - 1)z + 1 + r + t},$$

where $r = 1/b$. Suppose a φ of this form induces a normal weighted composition operator under the conditions of Proposition 12. Then, applying both sides of (15) to the constant function 1 and evaluating the result at $z = 0$, we obtain

$$\frac{|1 + r + t|^2}{|1 + r + t|^2 - |r + t - 1|^2} = \frac{|1 + r + t|^2}{|1 + r + t|^2 - |r - t - 1|^2}.$$

It is easy to see that this condition implies $\mathrm{Re}(t) = 0$ so that φ is an automorphism. Thus, no hyperbolic nonautomorphic linear-fractional map (with $\omega \in \partial\mathbb{U}$) can induce a normal weighted composition operator under the conditions of Proposition 12. Some evidence that this is true in general may be found in [1], whose results show that

$W_{\psi,\varphi}$ cannot even be essentially normal if ψ is, say, C^1 on the closure of $\mathbb{U}$ and φ is linear-fractional nonautomorphism with Denjoy–Wolff point $\omega \in \partial\mathbb{U}$ and $\varphi'(\omega) < 1$.

Suppose ψ is C^1 on the closure of $\mathbb{U}$ and φ is a linear-fractional nonautomorphism having Denjoy–Wolff point $\omega \in \partial\mathbb{U}$. Then Lemma 3.3 of [1] shows that $W_{\psi,\varphi}$ is equivalent to $\psi(\omega)C_\varphi$ modulo the compact operators; moreover, if $\varphi'(\omega) < 1$, then Theorem 5.2 of [1] shows that $\psi(\omega)C_\varphi$ is not essentially normal, so that in this situation $W_{\psi,\varphi}$ is not essentially normal.

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