



A Hermitian Cauchy formula on a domain with fractal boundary

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ABSTRACT

Euclidean Clifford analysis is a higher dimensional function theory offering a refinement of classical harmonic analysis. The theory is centered around the concept of monogenic functions, i.e. null solutions of a first order vector valued rotation invariant differential operator called Dirac operator, which factorizes the Laplacian; monogenic functions may thus also be seen as a generalization of holomorphic functions in the complex plane. Hermitian Clifford analysis offers yet a refinement of the Euclidean case; it focusses on the simultaneous null solutions, called Hermitian (or h-) monogenic functions, of two Hermitian Dirac operators which are invariant under the action of the unitary group. In Brackx et al. (2009) [8] a Clifford–Cauchy integral representation formula for h-monogenic functions has been established in the case of domains with smooth boundary, however the approach followed cannot be extended to the case where the boundary of the considered domain is fractal. At present, we investigate an alternative approach which will enable us to define in this case a Hermitian Cauchy integral over a fractal closed surface, leading to several types of integral representation formulae, including the Cauchy and Borel–Pompeiu representations.

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1. Introduction

The Cauchy integral formula for holomorphic functions in the complex plane allows for two generalizations to the case of several complex variables: one may consider a holomorphic kernel and an integral over the separated boundary $\partial_0 \tilde{D}$ of a polydisk $\tilde{D} = \prod_{j=1}^n \tilde{D}_j$ in \mathbb{C}^n , which leads to the formula

$$f(z_1, \dots, z_n) = \frac{1}{(2\pi i)^n} \int_{\partial_0 \tilde{D}} \frac{f(\xi_1, \dots, \xi_n)}{(\xi_1 - z_1) \cdots (\xi_n - z_n)} d\xi_1 \wedge \cdots \wedge d\xi_n, \quad z_j \in \tilde{D}_j$$

or one may take an integral over the smooth boundary ∂D of a bounded domain D in \mathbb{C}^n , in combination with a kernel which is no longer holomorphic but still harmonic, resulting into the Martinelli–Bochner formula, see e.g. [23]:

$$f(z) = \int_{\partial D} f(\xi) U(\xi, z), \quad z \in \mathring{D}$$

with

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$$U(\xi, z) = \frac{(n-1)!}{(2\pi i)^n} \sum_{j=1}^n (-1)^{j-1} \frac{\xi_j^c - z_j^c}{|\xi - z|^{2n}} \widetilde{d\xi_j^c}$$

where \cdot^c denotes the complex conjugate and

$$\widetilde{d\xi_j^c} = d\xi_1^c \wedge \cdots \wedge d\xi_{j-1}^c \wedge d\xi_{j+1}^c \wedge \cdots \wedge d\xi_n^c \wedge d\xi_1 \wedge \cdots \wedge d\xi_n$$

For detailed information on this formula, which reduces to the traditional Cauchy integral formula when $n = 1$, we refer the reader to [22].

An alternative for the generalization of the Cauchy integral formula is offered by Clifford analysis, where functions defined in Euclidean space \mathbb{R}^m and taking values in a Clifford algebra are considered. The theory is centered around the concept of monogenic functions, i.e. null solutions of a first order vector valued differential operator called Dirac operator, which factorizes the Laplacian. The Dirac operator being rotation invariant, the name Euclidean Clifford analysis is used nowadays to refer to this setting. Standard references are [9,14,19,18,20]. In this framework the kernel appearing in the Cauchy formula is monogenic, up to a pointwise singularity, while the integral is taken over the complete boundary:

$$f(X) = \frac{1}{a_m} \int_{\partial D} \frac{\bar{\xi} - \bar{X}}{|\xi - X|^m} d\sigma_{\xi} f(\xi), \quad X \in \mathring{D}$$

where a_m is the area of the unit sphere S^{m-1} in \mathbb{R}^m , $\bar{\cdot}$ denotes the Clifford conjugation and $d\sigma_{\xi}$ is a Clifford algebra valued differential form of order $m-1$. This Clifford–Cauchy integral formula has been a corner stone in the development of the function theory.

More recently Hermitian Clifford analysis has emerged as yet a refinement of the Euclidean setting, for the case of $\mathbb{R}^{2n} \cong \mathbb{C}^n$; here, Hermitian monogenic functions are considered, i.e. functions taking values either in the complex Clifford algebra \mathbb{C}_{2n} or in complex spinor space \mathbb{S} and being simultaneous null solutions of two complex Hermitian Dirac operators, which are invariant under the action of the unitary group. The study of complex Dirac operators (also in other settings) was initiated in [26,25,27]; however, a systematic development of the Hermitian function theory is still in full progress, see e.g. [13,5,6,12,10,1,3,4,2]. As in Euclidean Clifford analysis, a Cauchy integral formula for Hermitian monogenic functions showed to be essential in the further development of the present function theory as well. In [8] such a Cauchy integral formula was established, however not taking the traditional form shown above, a phenomenon which could be expected, since it is known (see [6]) that in some very particular cases Hermitian monogenicity is equivalent with holomorphy in the underlying complex variables. It turned out that a matrix approach, using circulant (2×2) matrix functions, was the key to obtain the desired result, see [7,8,11]. However, if the boundary of the considered domain is a fractal, having Hausdorff dimension between $2n-1$ and $2n$, then the method followed in the cited papers is no longer applicable. In this paper we introduce an alternative way of defining the matricial Hermitian Cauchy integral over a fractal closed surface bounding a Jordan domain, which yields several integral representation formulae, such as the Cauchy, Borel–Pompeiu and Koppelman formulae, for the case of fractal boundaries.

2. Preliminaries

2.1. Basics of Hermitian Clifford analysis

Let (e_1, \dots, e_m) be an orthonormal basis of Euclidean space \mathbb{R}^m and consider the complex Clifford algebra \mathbb{C}_m constructed over \mathbb{R}^m . The non-commutative multiplication in \mathbb{C}_m is governed by the rules:

$$\begin{aligned} e_j^2 &= -1, \quad j = 1, \dots, m \\ e_j e_k + e_k e_j &= 0, \quad j \neq k \end{aligned}$$

The Clifford algebra \mathbb{C}_m is generated additively by elements of the form $e_A = e_{j_1} \dots e_{j_k}$, where $A = \{j_1, \dots, j_k\} \subset \{1, \dots, m\}$ is such that $j_1 < \dots < j_k$, while for $A = \emptyset$, one puts $e_{\emptyset} = 1$, the identity element. Any Clifford number $\lambda \in \mathbb{C}_m$ may thus be written as $\lambda = \sum_A \lambda_A e_A$, $\lambda_A \in \mathbb{C}$, its Hermitian conjugate λ^\dagger being defined by $\lambda^\dagger = \sum_A \lambda_A^c \bar{e}_A$, where the bar denotes the real Clifford algebra conjugation, i.e. the main anti-involution for which $\bar{e}_j = -e_j$, and λ_A^c stands for the complex conjugate of the complex number λ_A .

Euclidean space \mathbb{R}^m is embedded in the Clifford algebra \mathbb{C}_m by identifying (x_1, \dots, x_m) with the real Clifford vector \underline{X} given by $\underline{X} = \sum_{j=1}^m e_j x_j$, for which $\underline{X}^2 = -(\underline{X}, \underline{X}) = -|\underline{X}|^2$. The Fischer dual of \underline{X} is the vector valued first order Dirac operator $\partial_{\underline{X}} = \sum_{j=1}^m e_j \partial_{x_j}$, factorizing the Laplacian: $\Delta_m = -\partial_{\underline{X}}^2$; it underlies the notion of monogenicity of a function, the higher dimensional counterpart of holomorphy in the complex plane. The considered functions are defined on (open subsets of) \mathbb{R}^m and take values in the Clifford algebra \mathbb{C}_m . They are of the form $g = \sum_A g_A e_A$, with g_A complex valued. Whenever a property such as continuity, differentiability, etc., is ascribed to g it is meant that all components g_A show that property. A Clifford algebra valued function g , defined and differentiable in an open region Ω of \mathbb{R}^m , is then called (left) monogenic in Ω iff $\partial_{\underline{X}} g = 0$ in Ω .

The transition from Euclidean Clifford analysis as described above to the Hermitian Clifford setting is essentially based on the introduction of a complex structure J , i.e. a particular $SO(m)$ element, satisfying $J^2 = -\mathbf{1}_m$. Since such an element cannot exist when the dimension of the vector space is odd, we will put $m = 2n$ from now on. In terms of the chosen orthonormal basis, a particular realization of the complex structure is $J[e_{2j-1}] = -e_{2j}$ and $J[e_{2j}] = e_{2j-1}$, $j = 1, \dots, n$. Two projection operators $\pm \frac{1}{2}(\mathbf{1}_{2n} \pm iJ)$ associated to this complex structure J then produce the main objects of Hermitian Clifford analysis by acting upon the corresponding objects in the Euclidean setting, see [5,6]. The vector space \mathbb{C}^{2n} thus decomposes as $W^+ \oplus W^-$ into two isotropic subspaces. The real Clifford vector is now denoted

$$\underline{X} = \sum_{j=1}^n (e_{2j-1}x_{2j-1} + e_{2j}x_{2j})$$

with the corresponding Dirac operator

$$\partial_{\underline{X}} = \sum_{j=1}^n (e_{2j-1}\partial_{x_{2j-1}} + e_{2j}\partial_{x_{2j}})$$

while we will also consider their so-called ‘twisted’ counterparts, obtained through the action of J , i.e.

$$\begin{aligned} \underline{X}| &= \sum_{j=1}^n (e_{2j-1}x_{2j} - e_{2j}x_{2j-1}) \\ \partial_{\underline{X}|} &= \sum_{j=1}^n (e_{2j-1}\partial_{x_{2j}} - e_{2j}\partial_{x_{2j-1}}) \end{aligned}$$

As was the case with $\partial_{\underline{X}}$, a notion of monogenicity may be associated in a natural way to $\partial_{\underline{X}|}$ as well. The projections of the vector variable \underline{X} on the spaces W^\pm then yield the Hermitian Clifford variables \underline{Z} and \underline{Z}^\dagger , given by

$$\underline{Z} = \frac{1}{2}(\underline{X} + i\underline{X}|) \quad \text{and} \quad \underline{Z}^\dagger = -\frac{1}{2}(\underline{X} - i\underline{X}|)$$

and those of the Dirac operator $\partial_{\underline{X}}$ yield (up to a factor) the Hermitian Dirac operators $\partial_{\underline{Z}}$ and $\partial_{\underline{Z}^\dagger}$, given by

$$\partial_{\underline{Z}^\dagger} = \frac{1}{4}(\partial_{\underline{X}} + i\partial_{\underline{X}|}) \quad \text{and} \quad \partial_{\underline{Z}} = -\frac{1}{4}(\partial_{\underline{X}} - i\partial_{\underline{X}|})$$

The Hermitian vector variables and Dirac operators are isotropic, i.e. $(\underline{Z})^2 = (\underline{Z}^\dagger)^2 = 0$ and $(\partial_{\underline{Z}})^2 = (\partial_{\underline{Z}^\dagger})^2 = 0$, whence the Laplacian allows for the decomposition $\Delta_{2n} = 4(\partial_{\underline{Z}}\partial_{\underline{Z}^\dagger} + \partial_{\underline{Z}^\dagger}\partial_{\underline{Z}})$. These objects lie at the core of the Hermitian function theory by means of the following definition (see e.g. [5,13]).

Definition 1. A continuously differentiable function g in $\Omega \subset \mathbb{R}^{2n}$ with values in \mathbb{C}_{2n} is called left Hermitian monogenic (or left h-monogenic for short) in Ω , iff it satisfies in Ω the system $\partial_{\underline{Z}}g = 0 = \partial_{\underline{Z}^\dagger}g$ or, equivalently, the system $\partial_{\underline{X}}g = 0 = \partial_{\underline{X}|}g$.

In a similar way right h-monogenicity is defined. Functions which are both left and right h-monogenic are called two-sided h-monogenic. This definition inspires the statement that h-monogenicity constitutes a refinement of monogenicity, since h-monogenic functions (either left or right) are monogenic w.r.t. both Dirac operators $\partial_{\underline{X}}$ and $\partial_{\underline{X}|}$.

2.2. Transition to a matrix approach

The fundamental solutions of the Dirac operators $\partial_{\underline{X}}$ and $\partial_{\underline{X}|}$ are respectively given by

$$E(\underline{X}) = -\frac{1}{a_{2n}} \frac{\underline{X}}{|\underline{X}|^{2n}}, \quad E|(\underline{X}) = -\frac{1}{a_{2n}} \frac{\underline{X}|}{|\underline{X}|^{2n}}, \quad \underline{X} \in \mathbb{R}^{2n} \setminus \{0\}$$

where a_{2n} denotes the surface area of the unit sphere in \mathbb{R}^{2n} . Introducing the functions $\mathcal{E} = -(E + iE|)$ and $\mathcal{E}^\dagger = (E - iE|)$, explicitly given by

$$\mathcal{E}(\underline{Z}) = \frac{2}{a_{2n}} \frac{\underline{Z}}{|\underline{Z}|^{2n}} \quad \text{and} \quad \mathcal{E}^\dagger(\underline{Z}) = \frac{2}{a_{2n}} \frac{\underline{Z}^\dagger}{|\underline{Z}|^{2n}}$$

these are not the fundamental solutions to the respective Hermitian Dirac operators $\partial_{\underline{Z}}$ and $\partial_{\underline{Z}^\dagger}$. However, introducing the particular circulant (2×2) matrices

$$\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} = \begin{pmatrix} \partial_{\underline{Z}} & \partial_{\underline{Z}^\dagger} \\ \partial_{\underline{Z}^\dagger} & \partial_{\underline{Z}} \end{pmatrix}, \quad \mathcal{E} = \begin{pmatrix} \mathcal{E} & \mathcal{E}^\dagger \\ \mathcal{E}^\dagger & \mathcal{E} \end{pmatrix} \quad \text{and} \quad \delta = \begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix}$$

where δ is the Dirac delta distribution, one obtains that

$$\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \mathcal{E}(\underline{Z}) = \delta(\underline{Z})$$

so that \mathcal{E} may be considered as a fundamental solution of $\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)}$ in a matricial context, see e.g. [7,8,25]. Moreover, the Dirac matrix $\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)}$ in some sense factorizes the Laplacian, since

$$4\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} (\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)})^\dagger = \begin{pmatrix} \Delta_{2n} & 0 \\ 0 & \Delta_{2n} \end{pmatrix} \equiv \Delta$$

where Δ_{2n} is the usual Laplace operator in \mathbb{R}^{2n} . This observation lead to the idea of following a matrix approach in order to establish integral representation formulae in the Hermitian setting, see [8,11]. Moreover, it inspired the following definition.

Definition 2. Let g_1, g_2 be continuously differentiable functions defined in Ω and taking values in \mathbb{C}_{2n} , and consider the matrix function

$$\mathbf{G}_2^1 = \begin{pmatrix} g_1 & g_2 \\ g_2 & g_1 \end{pmatrix}$$

Then \mathbf{G}_2^1 is called left (respectively right) **H**-monogenic in Ω if and only if it satisfies in Ω the system

$$\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \mathbf{G}_2^1 = \mathbf{0} \quad (\text{respectively } \mathbf{G}_2^1 \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} = \mathbf{0})$$

Here $\mathbf{0}$ denotes the matrix with zero entries.

Unless mentioned explicitly, we will only work with left **H**-monogenic functions. Note that the **H**-monogenicity of the matrix function \mathbf{G}_2^1 does not imply the h-monogenicity of its entry functions g_1 and g_2 . However, choosing in particular $g_1 = g$ and $g_2 = 0$, the **H**-monogenicity of the corresponding diagonal matrix, denoted \mathbf{G}_0 , is equivalent to the h-monogenicity of the function g . Moreover, calling a matrix function \mathbf{G}_2^1 harmonic iff it satisfies the equation $\Delta[\mathbf{G}_2^1] = \mathbf{0}$, each **H**-monogenic matrix function \mathbf{G}_2^1 turns out to be harmonic, whence its entries are harmonic functions in the usual sense.

Notions of continuity, differentiability and integrability of \mathbf{G}_2^1 are introduced through the corresponding notions for its entries. In particular, we will need to define in this way the classes $\mathcal{C}^{0,\nu}(\mathbf{E})$ and $\mathcal{L}_p(\mathbf{E})$ of, respectively, Hölder continuous and p -integrable circulant matrix functions over some suitable subset \mathbf{E} of \mathbb{R}^{2n} . However, introducing the non-negative function

$$\|\mathbf{G}_2^1(\underline{X})\| = \max\{|g_1(\underline{X})|, |g_2(\underline{X})|\}$$

these classes of circulant matrix functions may also be defined by means of the traditional conditions

$$\|\mathbf{G}_2^1(\underline{X}) - \mathbf{G}_2^1(\underline{Y})\| \leq c|\underline{X} - \underline{Y}|^\nu, \quad \underline{X}, \underline{Y} \in \mathbf{E}$$

and

$$\int_{\mathbf{E}} \|\mathbf{G}_2^1(\underline{X})\|^p < +\infty$$

respectively. From now on we denote by c a generic positive constant, which can take different values.

For further use, we recall that, for any compact set $\mathbf{E} \subset \mathbb{R}^{2n}$ and for any function $g \in \mathcal{C}^{0,\nu}(\mathbf{E})$, there exists a compactly supported function $\tilde{g} \in \mathcal{C}^\infty(\mathbb{R}^{2n} \setminus \mathbf{E}) \cap \mathcal{C}^{0,\nu}(\mathbb{R}^{2n})$ for which it holds that $\tilde{g}|_{\mathbf{E}} = g$ and

$$|\partial_{x_i} \tilde{g}(\underline{X})| \leq c \operatorname{dist}(\underline{X}, \mathbf{E})^{\nu-1} \quad \text{for } \underline{X} \in \mathbb{R}^{2n} \setminus \mathbf{E}, \quad i = 1, \dots, 2n$$

In fact, this extension theorem is based upon the decomposition into cubes of the open set $\mathbb{R}^{2n} \setminus \mathbf{E}$, the so-called Whitney decomposition, which will be introduced in the next section for the domains under consideration; for further details, we refer the reader to [28]. For our purposes, it suffices to notice that this result may be reformulated in matrix form as follows.

Theorem 1 (Whitney extension theorem). *Let $\mathbf{E} \subset \mathbb{R}^{2n}$ be compact and $\mathbf{G}_2^1 \in \mathcal{C}^{0,\nu}(\mathbf{E})$. Then, there exists a compactly supported matrix function $\tilde{\mathbf{G}}_2^1$ satisfying*

$$(i) \quad \tilde{\mathbf{G}}_2^1|_{\mathbf{E}} = \mathbf{G}_2^1;$$

- (ii) $\tilde{\mathbf{G}}_2^\infty \in \mathcal{C}^1(\mathbb{R}^{2n} \setminus \mathbf{E})$;
- (iii) $\|\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \tilde{\mathbf{G}}_2^1(\underline{X})\| \leq c \operatorname{dist}(\underline{X}, \mathbf{E})^{\nu-1}$, for $\underline{X} \in \mathbb{R}^{2n} \setminus \mathbf{E}$.

Any extension of the matrix function \mathbf{G}_2^1 satisfying the above properties will be called a Whitney type extension of \mathbf{G}_2^1 .

2.3. Some elements of fractal geometry

Let \mathbf{E} be an arbitrary subset of \mathbb{R}^{2n} . Then for any $s \geq 0$ its Hausdorff measure $\mathcal{H}^s(\mathbf{E})$ may be defined by

$$\mathcal{H}^s(\mathbf{E}) = \liminf_{\delta \rightarrow 0} \left\{ \sum_{k=1}^{\infty} (\operatorname{diam} B_k)^s : \mathbf{E} \subset \bigcup_{k=1}^{\infty} B_k, \operatorname{diam} B_k < \delta \right\}$$

the infimum being taken over all countable δ -coverings $\{B_k\}$ of \mathbf{E} with open or closed balls. For $s = 2n$, the Hausdorff measure \mathcal{H}^{2n} coincides, up to a positive multiplicative constant, with the Lebesgue measure \mathcal{L}^{2n} in \mathbb{R}^{2n} .

Now, let \mathbf{E} be compact. The Hausdorff dimension of \mathbf{E} , denoted $\alpha_H(\mathbf{E})$, is then defined as the infimum of all $s \geq 0$ such that $\mathcal{H}^s(\mathbf{E}) < +\infty$. For more details concerning the Hausdorff measure and dimension we refer to [15,16]. Frequently however, see e.g. [24], the so-called box dimension is more appropriate than the Hausdorff dimension to measure the roughness of a given set \mathbf{E} . The box dimension of a compact set $\mathbf{E} \subset \mathbb{R}^{2n}$ is defined as

$$\alpha(\mathbf{E}) = \limsup_{\varepsilon \rightarrow 0} \frac{\log N_{\mathbf{E}}(\varepsilon)}{-\log \varepsilon} \quad (1)$$

where $N_{\mathbf{E}}(\varepsilon)$ stands for the minimal number of ε -balls needed to cover \mathbf{E} . Note that the limit in (1) remains unchanged if $N_{\mathbf{E}}(\varepsilon)$ is replaced by the number of k -cubes, with $2^{-k} \leq \varepsilon < 2^{-k+1}$, intersecting \mathbf{E} . For completeness we recall that a cube Q is called a k -cube if it is of the form

$$[l_1 2^{-k}, (l_1 + 1) 2^{-k}] \times \cdots \times [l_{2n} 2^{-k}, (l_{2n} + 1) 2^{-k}]$$

where k and l_1, \dots, l_{2n} are integers. The box dimension and the Hausdorff dimension of a given compact set \mathbf{E} can be equal, which is for instance the case for the so-called $(2n-1)$ -rectifiable sets (see [17]), but this is not the case in general, where we have that $\alpha_H(\mathbf{E}) \leq \alpha(\mathbf{E})$. The following geometric notion was in [21], and is essential in their method of integrating a form over a fractal boundary.

Definition 3. The compact set \mathbf{E} is said to be d -summable iff the improper integral

$$\int_0^1 N_{\mathbf{E}}(x) x^{d-1} dx$$

converges.

Lemma 1. It holds that

- (i) any d -summable set \mathbf{E} has box dimension $\alpha(\mathbf{E}) \leq d$;
- (ii) if $\alpha(\mathbf{E}) < d$, then \mathbf{E} is d -summable;
- (iii) if \mathbf{E} is d -summable, then it is also $(d + \varepsilon)$ -summable for every $\varepsilon > 0$.

In what follows, we will take $\Omega \subset \mathbb{R}^{2n}$ to be a so-called Jordan domain, i.e. a bounded oriented connected open subset of \mathbb{R}^{2n} , the boundary Γ of which is a compact topological surface. In the case $n = 1$, this notion coincides with the usual one of a Jordan domain in the complex plane. For further use, we also introduce the notation $\Omega^+ \equiv \Omega$, and $\Omega^- \equiv \mathbb{R}^{2n} \setminus \overline{\Omega}$. We will assume that the Hausdorff and box dimensions of the boundary Γ of our Jordan domain Ω satisfy

$$2n - 1 \leq \alpha_H(\Gamma) \leq \alpha(\Gamma) < 2n$$

Note that this includes the case when Γ is fractal in the sense of Mandelbrot, i.e. when $2n - 1 < \alpha_H(\Gamma)$. Under these conditions, there will always exist $d \in [2n - 1, 2n[$ such that Γ is d -summable, see Lemma 1.

As announced in the previous section, we will also need the so-called Whitney decomposition of Ω , of which we will only recall the main lines in the construction; for further details, we refer to [28]. Consider the lattice \mathbb{Z}^{2n} in \mathbb{R}^{2n} as well as the collection of closed unit cubes defined by it, and let \mathcal{M}_1 be the mesh consisting of those unit cubes having a non-empty intersection with Ω . We may then recursively define a chain of meshes \mathcal{M}_k , $k = 2, 3, \dots$, each time bisecting the sides of the cubes of the foregoing mesh. The cubes in the mesh \mathcal{M}_k thus have side length 2^{-k+1} and diameter $|Q| = \sqrt{2n} 2^{-k+1}$. The Whitney decomposition \mathcal{W} of Ω is then obtained by defining, for $k = 2, 3, \dots$,

$$\mathcal{W}^1 = \{Q \in \mathcal{M}_1 \mid \text{all neighboring cubes of } Q \text{ belong to } \Omega\}$$

$$\mathcal{W}^k = \{Q \in \mathcal{M}_k \mid \text{all neighboring cubes of } Q \text{ belong to } \Omega, \text{ and } \nexists Q^* \in \mathcal{W}^{k-1}: Q \subset Q^*\}$$

for which it can be proven that

$$\Omega = \bigcup_{k=1}^{+\infty} \mathcal{W}^k = \bigcup_{k=1}^{+\infty} \bigcup_{Q \in \mathcal{W}^k} Q \equiv \bigcup_{Q \in \mathcal{W}} Q$$

all cubes Q in \mathcal{W} having disjoint interiors. It holds that

$$\text{dist}(\underline{X}, \Gamma) \geq \frac{1}{\sqrt{2n}} |Q| = 2^{-k+1}, \quad \underline{X} \in Q, \quad Q \in \mathcal{W}^k \quad (2)$$

We then have the following relation between the d -summability of the boundary Γ and the Whitney decomposition of Ω .

Lemma 2. (See [21].) If Ω is a Jordan domain of \mathbb{R}^{2n} and its boundary Γ is d -summable, then the expression $\sum_{Q \in \mathcal{W}} |Q|^d$, called the d -sum of the Whitney decomposition \mathcal{W} of Ω , is finite.

3. Hermitian Cauchy integral for domains with fractal boundaries

From now on we reserve the notations \underline{Y} and $\underline{Y}|$ for Clifford vectors associated to points in Ω^\pm . Their Hermitian counterparts are denoted by

$$\underline{V} = \frac{1}{2}(\underline{Y} + i\underline{Y}|), \quad \underline{V}^\dagger = -\frac{1}{2}(\underline{Y} - i\underline{Y}|)$$

By means of the matrix approach sketched above, the following Hermitian Borel–Pompeiu formula was established in [8], for the case of a domain with smooth boundary.

Theorem 2. Let g_1 and g_2 be functions in $C^1(\Omega; \mathbb{C}_{2n})$ and let \mathbf{G}_2^1 be the corresponding circulant matrix function; let Ω be a domain in \mathbb{R}^{2n} with piecewise C^∞ smooth boundary $\partial\Gamma$. It then holds that

$$\mathcal{C}_\Gamma \mathbf{G}_2^1(\underline{Y}) + \mathcal{T}_\Omega \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \mathbf{G}_2^1(\underline{Y}) = \begin{cases} (-1)^{\frac{n(n+1)}{2}} (2i)^n \mathbf{G}_2^1(\underline{Y}), & \underline{Y} \in \Omega^+ \\ 0, & \underline{Y} \in \Omega^- \end{cases}$$

where $\mathcal{C}_\Gamma \mathbf{G}_2^1$ is the Hermitian Cauchy integral given by

$$\mathcal{C}_\Gamma \mathbf{G}_2^1(\underline{Y}) = \int_\Gamma \mathcal{E}(\underline{Z} - \underline{V}) \mathbf{N}_{(\underline{Z}, \underline{Z}^\dagger)} \mathbf{G}_2^1(\underline{X}) d\mathcal{H}^{2n-1}, \quad \underline{Y} \in \Omega^\pm$$

the circulant matrix

$$\mathbf{N}_{(\underline{Z}, \underline{Z}^\dagger)} = \begin{pmatrix} \underline{N} & -\underline{N}^\dagger \\ -\underline{N}^\dagger & \underline{N} \end{pmatrix}$$

containing (up to a constant factor) the Hermitian projections \underline{N} and \underline{N}^\dagger of the unit normal vector $\underline{n}(\underline{X})$ at the point $\underline{X} \in \Gamma$. Furthermore, \mathcal{T}_Ω denotes the Hermitian Téodorescu transform, given for $\mathbf{F}_2^1 \in \mathcal{C}^1(\Omega)$ by

$$\mathcal{T}_\Omega \mathbf{F}_2^1(\underline{Y}) = - \int_\Omega \mathcal{E}(\underline{Z} - \underline{V}) \mathbf{F}_2^1(\underline{X}) dW(\underline{Z}, \underline{Z}^\dagger)$$

where $dW(\underline{Z}, \underline{Z}^\dagger)$ is the associated volume element, given by

$$dV(\underline{X}) = (-1)^{\frac{n(n-1)}{2}} \left(\frac{i}{2}\right)^n dW(\underline{Z}, \underline{Z}^\dagger)$$

reflecting integration in the respective underlying complex planes.

The results mentioned above then inspire the following definition within the present setting.

Definition 4. Let $d \in [2n - 1, 2n)$ and $d - 2n + 1 < \nu \leq 1$. If Ω is a Jordan domain with d -summable boundary Γ , then we define the Hermitian Cauchy integral of a matrix function $\mathbf{G}_2^1 \in \mathcal{C}^{0,\nu}(\Gamma)$ by

$$\mathcal{C}_\Gamma^* \mathbf{G}_2^1(\underline{Y}) = (-1)^{\frac{n(n+1)}{2}} (2i)^n \chi_\Omega(\underline{Y}) \tilde{\mathbf{G}}_2^1(\underline{Y}) - \mathcal{T}_\Omega \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \tilde{\mathbf{G}}_2^1(\underline{Y}), \quad \underline{Y} \in \mathbb{R}^{2n} \setminus \Gamma \quad (3)$$

where

$$\chi_\Omega = \begin{pmatrix} \chi_{\Omega} & 0 \\ 0 & \chi_{\Omega} \end{pmatrix}$$

is the matrix version of the standard characteristic function χ_Ω of Ω and $\tilde{\mathbf{G}}_2^1$ is a Whitney type extension of \mathbf{G}_2^1 .

Clearly, we need to motivate that this definition is legitimate. This is established in the following two propositions.

Proposition 1. The matrix function (3) is well defined for any $\underline{X} \in \mathbb{R}^{2n} \setminus \Gamma$.

Proof. It suffices to show that

$$\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \tilde{\mathbf{G}}_2^1 \in \mathbf{L}_1(\Omega)$$

To this end, let $\mathcal{W} = \bigcup_k \mathcal{W}^k$ be the Whitney decomposition of Ω . Then we have

$$\int_\Omega \|\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \tilde{\mathbf{G}}_2^1(\underline{Y})\| = \sum_{Q \in \mathcal{W}} \int_Q \|\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \tilde{\mathbf{G}}_2^1(\underline{Y})\| \leq c \sum_{Q \in \mathcal{W}} \int_Q \text{dist}(\underline{Y}, \Gamma)^{\nu-1} dV(\underline{Y})$$

the last inequality following from Lemma 1(iii). On account of (2) it thus follows that

$$\int_\Omega \|\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \tilde{\mathbf{G}}_2^1(\underline{Y})\| \leq c \sum_{Q \in \mathcal{W}} |Q|^{\nu-1+2n}$$

the latter sum being finite on account of Lemma 2 and of the fact that $d < \nu - 1 + 2n$. \square

Proposition 2. The matrix function (3) does not depend on the particular choice of the Whitney type extension of \mathbf{G}_2^1 .

Proof. Suppose that $\tilde{\Theta}_2^1$ and $\tilde{\Xi}_2^1$ are two different Whitney type extensions of \mathbf{G}_2^1 . Then, for $\tilde{\Psi}_2^1 = \tilde{\Theta}_2^1 - \tilde{\Xi}_2^1$, it holds that $\tilde{\Psi}_2^1|_\Gamma = \mathbf{0}$. It thus remains to prove that

$$(-1)^{\frac{n(n+1)}{2}} (2i)^n \chi_\Omega(\underline{Y}) \tilde{\Psi}_2^1(\underline{Y}) - \mathcal{T}_\Omega \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \tilde{\Psi}_2^1(\underline{Y}) = \mathbf{0}, \quad \underline{Y} \in \mathbb{R}^{2n} \setminus \Gamma \quad (4)$$

To this end, define

$$\Omega_k = \{\underline{X} \in Q \mid Q \in \mathcal{W}^j, \text{ for some } j \leq k\}$$

It simplifies the argument, and causes no loss of generality, to assume that Ω_k is connected. Observe that the boundary of Ω_k , denoted Γ_k , consists of certain faces of some cubes $Q \in \mathcal{W}^k$. We then have

$$\int_\Omega \mathcal{E}(\underline{Z} - \underline{V}) \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \tilde{\Psi}_2^1(\underline{X}) dW(\underline{Z}, \underline{Z}^\dagger) = \lim_{k \rightarrow \infty} \int_{\Omega_k} \mathcal{E}(\underline{Z} - \underline{V}) \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \tilde{\Psi}_2^1(\underline{X}) dW(\underline{Z}, \underline{Z}^\dagger) \quad (5)$$

Now, take first $\underline{Y} \in \Omega$. In this case, choose k_0 sufficiently large, such that $\underline{Y} \in \Omega_{k_0}$ and $\text{dist}(\underline{Y}, \Gamma_k) \geq \frac{|Q_0|}{2\sqrt{2n}}$ for $k > k_0$, Q_0 being a cube of \mathcal{W}^{k_0} . The Hermitian Borel–Pompeiu formula of Theorem 2, applied to Ω_k , then yields

$$(-1)^{\frac{n(n+1)}{2}} (2i)^n \tilde{\Psi}_2^1(\underline{Y}) + \int_{\Omega_k} \mathcal{E}(\underline{Z} - \underline{V}) \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \tilde{\Psi}_2^1(\underline{X}) dW(\underline{Z}, \underline{Z}^\dagger) = \int_{\Gamma_k} \mathcal{E}(\underline{Z} - \underline{V}) \mathbf{N}_{(\underline{Z}, \underline{Z}^\dagger)}^k \tilde{\Psi}_2^1(\underline{X}) d\mathcal{H}^{2n-1} \quad (6)$$

where $\mathbf{N}_{(\underline{Z}, \underline{Z}^\dagger)}^k$ is the circulant matrix corresponding to the unit normal vector on Γ_k . Next, consider $\underline{X} \in \Gamma_k$, let $Q \in \mathcal{W}^k$ be a cube containing \underline{X} , and take $\underline{P} \in \Gamma$ such that $|\underline{X} - \underline{P}| = \text{dist}(\underline{X}, \Gamma)$. Since $\tilde{\Psi}_2^1|_\Gamma = 0$, we have

$$\|\tilde{\Psi}_2^1(\underline{X})\| = \|\tilde{\Psi}_2^1(\underline{X}) - \tilde{\Psi}_2^1(\underline{P})\| \leq c|\underline{X} - \underline{P}|^\nu \leq c|Q|^\nu$$

If Σ denotes a face of Γ_k and $Q \in \mathcal{W}^k$ is the k -cube containing that face Σ , then we have, for $k > k_0$, that

$$\left\| \int_{\Sigma} \mathcal{E}(\underline{Z} - \underline{V}) \mathbf{N}_{(\underline{Z}, \underline{Z}^\dagger)}^k \tilde{\Psi}_2^1(\underline{X}) d\mathcal{H}^{2n-1} \right\| \leq \frac{c}{|Q_0|^{2n-1}} \int_{\Sigma} \|\tilde{\Psi}_2^1(\underline{X})\| d\mathcal{H}^{2n-1} \leq \frac{c}{|Q_0|^{2n-1}} |Q|^{\nu-1+2n}$$

Since each face of Γ_k belongs to some $Q \in \mathcal{W}^k$, we have, for $k > k_0$,

$$\left\| \int_{\Gamma_k} \mathcal{E}(\underline{Z} - \underline{V}) \mathbf{N}_{(\underline{Z}, \underline{Z}^\dagger)}^k \tilde{\Psi}_2^1(\underline{X}) d\mathcal{H}^{2n-1} \right\| \leq \frac{c}{|Q_0|^{2n-1}} \sum_{Q \in \mathcal{W}^k} |Q|^{\nu-1+2n} \leq \frac{c}{|Q_0|^{2n-1}} \sum_{Q \in \mathcal{W}^k} |Q|^d$$

The finiteness of the d -sum $\sum_{Q \in \mathcal{W}} |Q|^d$ of the Whitney decomposition \mathcal{W} of Ω , see Lemma 2, implies

$$\lim_{k \rightarrow \infty} \int_{\Gamma_k} \mathcal{E}(\underline{Z} - \underline{V}) \mathbf{N}_{(\underline{Z}, \underline{Z}^\dagger)}^k \tilde{\Psi}_2^1(\underline{X}) d\mathcal{H}^{2n-1} = 0$$

Combining (5) with (6) yields (4) for $\underline{Y} \in \Omega$. The same conclusion can be drawn for $\underline{Y} \in \mathbb{R}^{2n} \setminus \overline{\Omega}$, observing that $\text{dist}(\underline{Y}, \Gamma_k) \geq \text{dist}(\underline{Y}, \Gamma)$ for $\underline{Y} \in \mathbb{R}^{2n} \setminus \overline{\Omega}$. \square

Remark 1. It is easily seen that Definition 4 remains valid under the condition that $\nu > m - 1 + 2n$ for any Γ having box dimension $\alpha(\Gamma) = m$.

4. Integral representation formulae

From the definition of the Hermitian Cauchy integral, given in the previous section, several Hermitian integral representation formulae will follow for the case of a Jordan domain with fractal boundary. The first one is the Hermitian Borel–Pompeiu formula, as formulated in the following theorem.

Theorem 3 (Hermitian Borel–Pompeiu formula). *Let Ω be a Jordan domain with d -summable boundary Γ , with $d \in [2n - 1, 2n[$, and take $\mathbf{G}_2^1 \in \mathcal{C}^1(\overline{\Omega})$. Then it holds that*

$$\mathcal{C}_\Gamma^* \mathbf{G}_2^1(\underline{Y}) + \mathcal{T}_\Omega \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \mathbf{G}_2^1(\underline{Y}) = \begin{cases} (-1)^{\frac{n(n+1)}{2}} (2i)^n \mathbf{G}_2^1(\underline{Y}), & \underline{Y} \in \Omega^+ \\ 0, & \underline{Y} \in \Omega^- \end{cases}$$

Proof. Denote by \mathbf{g}_2^1 the trace of \mathbf{G}_2^1 on Γ . Since $\mathbf{G}_2^1 \in \mathcal{C}^1(\overline{\Omega})$, we have that $\mathbf{g}_2^1 \in \mathcal{C}^{0,\nu}(\Gamma)$, for any $\nu \in]0, 1[$. In particular, it is possible to choose $\nu > d + 1 - 2n$, enabling us to use Definition 4, i.e.

$$\mathcal{C}_\Gamma^* \mathbf{G}_2^1(\underline{Y}) = \mathcal{C}_\Gamma^* \mathbf{g}_2^1(\underline{Y}) = (-1)^{\frac{n(n+1)}{2}} (2i)^n \chi_\Omega(\underline{Y}) \tilde{\mathbf{g}}_2^1(\underline{Y}) - \mathcal{T}_\Omega \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \tilde{\mathbf{g}}_2^1(\underline{Y}) \quad (7)$$

where $\tilde{\mathbf{g}}_2^1$ is a Whitney type extension of \mathbf{g}_2^1 . Now, let $\tilde{\mathbf{G}}_2^1$ be a Whitney type extension of $\mathbf{G}_2^1 \in \mathcal{C}^1(\overline{\Omega}) \subset \mathcal{C}^{0,\nu}(\overline{\Omega})$. Then $\tilde{\mathbf{G}}_2^1$ also constitutes a Whitney type extension of $\mathbf{g}_2^1 \in \mathcal{C}^{0,\nu}(\Gamma)$. Indeed, we have:

- (i) $\tilde{\mathbf{G}}_2^1|_\Gamma = \mathbf{g}_2^1$;
- (ii) $\tilde{\mathbf{G}}_2^1 \in \mathcal{C}^1(\mathbb{R}^{2n} \setminus \Gamma)$,

as an obvious consequence of the construction of $\tilde{\mathbf{G}}_2^1$. Moreover, for $\underline{Y} \in \Omega^-$, it holds that

$$\|\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \tilde{\mathbf{G}}_2^1(\underline{Y})\| \leq c \text{dist}(\underline{Y}, \overline{\Omega})^{\nu-1} = c \text{dist}(\underline{Y}, \Gamma)^{\nu-1}$$

while for $\underline{Y} \in \Omega^+$, we have

$$\|\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \tilde{\mathbf{G}}_2^1(\underline{Y})\| = \|\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \mathbf{G}_2^1(\underline{Y})\| \leq c \text{dist}(\underline{Y}, \Gamma)^{\nu-1}$$

since $\mathbf{G}_2^1 \in \mathcal{C}^1(\overline{\Omega})$. Summarizing, we have for $\underline{Y} \in \mathbb{R}^{2n} \setminus \Gamma$:

- (iii) $\|\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \tilde{\mathbf{G}}_2^1(\underline{Y})\| \leq c \text{dist}(\underline{Y}, \Gamma)^{\nu-1}$.

Hence, on account of Proposition 2, we may replace $\tilde{\mathbf{g}}_2^1$ by $\tilde{\mathbf{g}}_2^1$ in the right-hand side of (7), in this way obtaining

$$\mathcal{C}_\Gamma^* \mathbf{G}_2^1(\underline{Y}) + \mathcal{T}_\Omega \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \mathbf{G}_2^1(\underline{Y}) = (-1)^{\frac{n(n+1)}{2}} (2i)^n \chi_\Omega(\underline{Y}) \tilde{\mathbf{G}}_2^1(\underline{Y}) - \mathcal{T}_\Omega \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \tilde{\mathbf{G}}_2^1(\underline{Y}) + \mathcal{T}_\Omega \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \mathbf{G}_2^1(\underline{Y})$$

Since $\tilde{\mathbf{G}}_2^1 = \mathbf{G}_2^1$ in Ω , we thus have

$$\mathcal{C}_\Gamma^* \mathbf{G}_2^1(\underline{Y}) + \mathcal{T}_\Omega \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \mathbf{G}_2^1(\underline{Y}) = (-1)^{\frac{n(n+1)}{2}} (2i)^n \chi_\Omega(\underline{Y}) \tilde{\mathbf{G}}_2^1(\underline{Y}) = \begin{cases} (-1)^{\frac{n(n+1)}{2}} (2i)^n \mathbf{G}_2^1(\underline{Y}), & \underline{Y} \in \Omega^+ \\ 0, & \underline{Y} \in \Omega^- \end{cases}$$

which completes the proof. \square

Observe that in the present setting it still holds that the Téodorescu operator constitutes a right inverse to the Dirac operator, see [11,18]. Explicitly, we have

$$\mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \mathcal{T}_\Omega \mathbf{G}_2^1(\underline{Y}) = \begin{cases} (-1)^{\frac{n(n+1)}{2}} (2i)^n \mathbf{G}_2^1(\underline{Y}), & \underline{Y} \in \Omega^+ \\ 0, & \underline{Y} \in \Omega^- \end{cases}$$

Combination of the above inversion formula with the Hermitian Borel–Pompeiu theorem, yields a Hermitian Koppelman formula in fractal domains, which involves, as particular case, the one proven in [11].

Theorem 4 (Hermitian Koppelman formula). *Let Ω be a Jordan domain with d -summable boundary Γ , with $d \in [2n - 1, 2n[$, and take $\mathbf{G}_2^1 \in \mathcal{C}^1(\overline{\Omega})$. Then*

$$\mathcal{C}_\Gamma^* \mathbf{G}_2^1(\underline{Y}) + \mathcal{T}_\Omega \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \mathbf{G}_2^1(\underline{Y}) + \mathcal{D}_{(\underline{Z}, \underline{Z}^\dagger)} \mathcal{T}_\Omega \mathbf{G}_2^1(\underline{Y}) = \begin{cases} (-1)^{\frac{n(n+1)}{2}} 2^{n+1} i^n \mathbf{G}_2^1(\underline{Y}), & \underline{Y} \in \Omega^+ \\ 0, & \underline{Y} \in \Omega^- \end{cases}$$

Finally, in the case of Hermitian monogenic matrix functions, the Hermitian Borel–Pompeiu formula reduces to a fractal version of the Cauchy integral representation formula.

Theorem 5 (Hermitian Cauchy formula). *Let Ω be a Jordan domain with d -summable boundary Γ , with $d \in [2n - 1, 2n[$, and take $\mathbf{G}_2^1 \in \mathcal{C}^1(\overline{\Omega})$. If \mathbf{G}_2^1 moreover is Hermitian monogenic in Ω , then*

$$\mathcal{C}_\Gamma^* \mathbf{G}_2^1(\underline{Y}) = \begin{cases} (-1)^{\frac{n(n+1)}{2}} (2i)^n \mathbf{G}_2^1(\underline{Y}), & \underline{Y} \in \Omega^+ \\ 0, & \underline{Y} \in \Omega^- \end{cases}$$

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