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Multiplicative matrix-valued functionals and the continuity properties of semigroups corresponding to partial differential operators with matrix-valued coefficients

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ABSTRACT

We define and examine certain matrix-valued multiplicative functionals with local Kato potential terms and use probabilistic techniques to prove that the semigroups of the corresponding self-adjoint partial differential operators with matrix-valued coefficients map from $\mathrm{L}^2(\mathbb{R}^n,\mathbb{C}^d)$ to the space of continuous bounded functions, and that these semigroups have a jointly continuous and spatially bounded integral kernel. These partial differential operators include Yang–Mills type Hamiltonians with "electrical" potentials that are elements of the matrix-valued local Kato class.

1. Main results

Let $\mathbb{R}^n$ and $\mathbb{C}^d$ both be equipped with the corresponding Euclidean metric $\|\bullet\|$. The associated operator norm on $\mathrm{Mat}(\mathbb{C}^d):=\mathrm{Mat}_{d\times d}(\mathbb{C})$ will be denoted with the same symbol. We will use the following notation for any

$$\alpha\in\Omega^1\big(\mathbb{R}^n,\mathrm{Mat}\big(\mathbb{C}^d\big)\big),$$

the smooth 1-forms on $\mathbb{R}^n$ with values in $\mathrm{Mat}(\mathbb{C}^d)$: Any such α can uniquely be written as $\alpha=\sum_{j=1}^n\alpha_j\,\mathrm{d}x^j$ with

$$\alpha_j=\big(\alpha^k_{j,l}\big)^{1\leqslant k\leqslant d}_{1\leqslant l\leqslant d}\in\mathrm{C}^\infty\big(\mathbb{R}^n,\mathrm{Mat}\big(\mathbb{C}^d\big)\big),\quad j=1,\dots,n. \tag{1}$$

Let $\mathscr{U}(d)$ denote the skew-Hermitian elements of $\mathrm{Mat}(\mathbb{C}^d)$, that is, $\mathscr{U}(d)$ is the Lie algebra corresponding to the unitary group $\mathrm{U}(d)$. In this paper, we will be concerned with probabilistic methods for self-adjoint operators in $\mathrm{L}^2(\mathbb{R}^n,\mathbb{C}^d)$ that are formally given by the differential expression

$$\tau(\alpha,V)=-\frac{1}{2}\Delta-\frac{1}{2}\sum_{j=1}^n\alpha_j^2-\frac{1}{2}\sum_{j=1}^n(\partial_j\alpha_j)-\sum_{j=1}^n\alpha_j\partial_j+V, \tag{2}$$

where $\alpha\in\Omega^1(\mathbb{R}^n,\mathscr{U}(d))$ and where $V:\mathbb{R}^n\to\mathrm{Mat}(\mathbb{C}^d)$ is a *potential*, that is, a measurable function with $V(x)=V^*(x)$ for almost every (a.e.) $x\in\mathbb{R}^n$. If $d=1$, then one has $\alpha=\mathrm{i}\tilde{\alpha}$ for some real-valued $\tilde{\alpha}=\sum_{j=1}^n\tilde{\alpha}_j\,\mathrm{d}x^j$, so that $\tau(\alpha,V)$ is nothing

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but the nonrelativistic Hamiltonian corresponding to a charged particle in the magnetic field $\tilde{\alpha} \in \Omega^1_{\mathbb{R}}(\mathbb{R}^n)$ and the electrical potential $V : \mathbb{R}^n \to \mathbb{R}$,

$$\tau(\alpha, V) = -\frac{1}{2}\Delta + \frac{1}{2}\sum_{j=1}^{n}\tilde{\alpha}_j^2 - \frac{\mathrm{i}}{2}\mathrm{div}(\tilde{\alpha}) - \mathrm{i}\sum_{j=1}^{n}\tilde{\alpha}_j\partial_j + V.$$

The following conventions will be used for our probabilistic considerations: For any $x \in \mathbb{R}^n$ we will denote the usual Wiener probability space with

$$\mathscr{P}^x := \big(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geqslant 0}, \mathbb{P}^x\big),$$

where $\Omega = \mathrm{C}([0,\infty), \mathbb{R}^n)$ and where $\mathbb{P}^x$ stands for the Wiener measure on $(\Omega, \mathscr{F})$ which is concentrated on the paths $\omega : [0,\infty) \to \mathbb{R}^n$ with $\omega(0) = x$. The underlying σ-algebra $\mathscr{F}$ and the filtration $\mathscr{F}_*$ will be the ones corresponding to the canonical process

$$X : [0,\infty) \times \Omega \to \mathbb{R}^n, \tag{3}$$

where $\mathscr{F}_*$ will be made right-continuous and complete (locally complete, if Girsanov techniques are used; here we implicitely use the results of Section 5.6 in [8]), whenever necessary. We consider the process X given by (3) as a Brownian motion starting in x under $\mathbb{P}^x$ and we will write "$\underline{\mathrm{d}}$" for Stratonovic differentials, whereas Itô differentials will be written as "d".

Fix $x \in \mathbb{R}^n$ now. If $\alpha \in \Omega^1(\mathbb{R}^n, \mathrm{Mat}(\mathbb{C}^d))$ and $V : \mathbb{R}^n \to \mathrm{Mat}(\mathbb{C}^d)$ is such that

$$\mathbb{P}^x\left\{\int_0^t \big\|V(X_s)\big\|\,\mathrm{d}s < \infty\right\} = 1 \quad \text{for all } t > 0, \tag{4}$$

then the processes

$$\begin{aligned} A^{\alpha,V} &:= \sum_{j=1}^{n}\int_0^{\bullet}\alpha_j(X_s)\,\underline{\mathrm{d}}X_s^j - \int_0^{\bullet}V(X_s)\,\mathrm{d}s : [0,\infty)\times\Omega \to \mathrm{Mat}\big(\mathbb{C}^d\big), \\ B^{\alpha,V} &:= A^{\alpha,V} + \frac{1}{2}\big[A^{\alpha,V}, A^{\alpha,V}\big] : [0,\infty)\times\Omega \to \mathrm{Mat}\big(\mathbb{C}^d\big), \end{aligned} \tag{5}$$

where

$$\big[A^{\alpha,V}, A^{\alpha,V}\big]^j_k := \sum_{l=1}^{d}\big[\big(A^{\alpha,V}\big)^j_l, \big(A^{\alpha,V}\big)^l_k\big] \quad \text{for } j,k = 1,\dots,d$$

is the quadratic covariation, are continuous semi-martingales. For any $l \in \mathbb{N}$ and $t \geqslant 0$ let the simplex $t\Delta_l$ be given by

$$t\Delta_l := \big\{(t_1,\dots,t_l)\ \big|\ 0 \leqslant t_1 \leqslant \dots \leqslant t_l \leqslant t\big\}.$$

Defining a stochastic path ordered exponential[1] by

$$\mathscr{A}_t^{\alpha,V} := \mathbf{1} + \sum_{l=1}^{\infty}\int_{t\Delta_l}\mathrm{d}B_{t_1}^{\alpha,V}\cdots\mathrm{d}B_{t_l}^{\alpha,V}, \tag{6}$$

where the convergence is $\mathbb{P}^x$-a.s. uniformly in compact subsets of $[0,\infty)$ [13], one finds that

$$\mathscr{A}^{\alpha,V} : [0,\infty)\times\Omega \to \mathrm{Mat}\big(\mathbb{C}^d\big)$$

is uniquely determined as the solution of

$$\mathscr{A}_t^{\alpha,V} = \mathbf{1} + \int_0^t \mathscr{A}_s^{\alpha,V}\,\mathrm{d}B_s^{\alpha,V} \tag{7}$$

[1] This notation has to be understood as

$$\mathscr{A}_t^{\alpha,V} = \mathbf{1} + B_t^{\alpha,V} + \int_0^t B_s^{\alpha,V}\,\mathrm{d}B_s^{\alpha,V} + \int_0^t\left(\int_0^s B_r^{\alpha,V}\,\mathrm{d}B_r^{\alpha,V}\right)\mathrm{d}B_s^{\alpha,V} + \cdots.$$

under $\mathbb{P}^x$. It is easily seen that

$$\mathscr{A}_t^{\alpha,V} = \mathbf{1} + \int_0^t \mathscr{A}_s^{\alpha,V}\,\underline{\mathrm{d}}A_s^{\alpha,V}, \tag{8}$$

$$\mathscr{A}_t^{\alpha,V,*} = \mathbf{1} + \int_0^t \big(\underline{\mathrm{d}}A_s^{\alpha,V,*}\big)\mathscr{A}_s^{\alpha,V,*}, \tag{9}$$

$$\mathscr{A}_t^{\alpha,V,-1} = \mathbf{1} - \int_0^t \big(\underline{\mathrm{d}}A_s^{\alpha,V}\big)\mathscr{A}_s^{\alpha,V,-1}. \tag{10}$$

Remark 1.1. If $d = 1$ and $\alpha = \mathrm{i}\tilde{\alpha}$ for some $\tilde{\alpha} \in \Omega^1_{\mathbb{R}}(\mathbb{R}^n)$, then one easily finds

$$\begin{aligned}\mathscr{A}^{\alpha,V} &= \exp\left(\mathrm{i}\sum_{j=1}^n \int_0^{\bullet} \tilde{\alpha}_j(X_s)\underline{\mathrm{d}}X_s^j - \int_0^{\bullet} V(X_s)\,\mathrm{d}s\right)\\ &= \exp\left(\mathrm{i}\sum_{j=1}^n \int_0^{\bullet} \tilde{\alpha}_j(X_s)\,\mathrm{d}X_s^j + \frac{\mathrm{i}}{2}\int_0^{\bullet} \mathrm{div}(\tilde{\alpha})(X_s)\,\mathrm{d}s - \int_0^{\bullet} V(X_s)\,\mathrm{d}s\right),\end{aligned} \tag{11}$$

an expression which is well known from the classical Feynman–Kac–Itô formula. In particular, the identity

$$A_t^{\alpha,V}\big(\omega(s+\bullet)\big) = A_{s+t}^{\alpha,V}(\omega) - A_s^{\alpha,V}(\omega) \quad \text{for all } s,t \geqslant 0,\ \mathbb{P}^x\text{-a.e. } \omega \in \Omega$$

(which follows from approximating the integrals in the definition of $A^{\alpha,V}$ with Riemann type sums as in (84)) and $\mathrm{e}^{z_1+z_2} = \mathrm{e}^{z_1}\mathrm{e}^{z_2}$ directly imply the following relation:

$$\mathscr{A}_{s+t}^{\alpha,V}(\omega) = \mathscr{A}_s^{\alpha,V}(\omega)\mathscr{A}_t^{\alpha,V}\big(\omega(s+\bullet)\big). \tag{12}$$

Although one does not have such an explicit expression for $\mathscr{A}^{\alpha,V}$ for $d > 1$, one can still prove the multiplicative property (12) in the general case:

Lemma 1.2. *The process $\mathscr{A}^{\alpha,V}$ is a multiplicative matrix-valued functional, that is, for any $s,t \geqslant 0$ one has*

$$\mathscr{A}_{s+t}^{\alpha,V} = \mathscr{A}_s^{\alpha,V}\big(\mathscr{A}_t^{\alpha,V} \circ \vartheta_s\big) \quad \mathbb{P}^x\text{-a.s.}, \tag{13}$$

where $\vartheta_s(\omega) = \omega(s+\bullet)$ stands for the shift operator on Ω.

Proof. We fix s and define $\mathscr{A} := \mathscr{A}^{\alpha,V}$ and $A := A^{\alpha,V}$. The following stochastic integrals are all understood with respect to $\mathbb{P}^x$. We will prove that the processes $\mathscr{A}_{s+\bullet}$ and $\mathscr{A}_s(\mathscr{A}_\bullet \circ \vartheta_s)$ both solve the following Stratonovic initial value problem (with respect to the filtration $(\mathscr{F}_{s+t})_{t\geqslant 0}$):

$$U_t = \mathscr{A}_s + \int_0^t U_r\,\underline{\mathrm{d}}_r A_{s+r}. \tag{14}$$

To this end, note that (8) directly implies

$$\mathscr{A}_{s+t} = \mathbf{1} + \int_0^{s+t} \mathscr{A}_r\,\underline{\mathrm{d}}_r A_r = \mathbf{1} + \int_0^s \mathscr{A}_r\,\underline{\mathrm{d}}_r A_r + \int_0^t \mathscr{A}_{r+s}\,\underline{\mathrm{d}}_r A_{r+s} = \mathscr{A}_s + \int_0^t \mathscr{A}_{s+r}\,\underline{\mathrm{d}}_r A_{s+r}. \tag{15}$$

On the other hand, the identity

$$A_r \circ \vartheta_s = A_{s+r} - A_s \quad \mathbb{P}^x\text{-a.s. for all } r \geqslant 0$$

implies the second identity in

$$\mathscr{A}_t \circ \vartheta_s = \mathbf{1} + \left(\int_0^t \mathscr{A}_r\,\underline{\mathrm{d}}_r A_r\right) \circ \vartheta_s = \mathbf{1} + \int_0^t \mathscr{A}_r \circ \vartheta_s\,\underline{\mathrm{d}}_r A_{r+s}, \tag{16}$$

so that the desired equality follows from multiplying the latter equation with $\mathscr{A}_s$ from the left. $\square$

We refer the reader to [14] and the references therein for a detailed study of multiplicative matrix-valued functionals. Matrix-valued Kato functions can be defined as follows:

Definition 1.3. A measurable function $V:\mathbb{R}^n \to \mathrm{Mat}(\mathbb{C}^d)$ is said to belong to the $\mathrm{Mat}(\mathbb{C}^d)$*-valued Kato class of* $\mathbb{R}^n$, if one has

$$\lim_{t\searrow 0}\sup_{x\in\mathbb{R}^n}\mathbb{E}^x\left[\int_0^t \|V(X_s)\|\,\mathrm{d}s\right]=0,$$

and V is said to be in the $\mathrm{Mat}(\mathbb{C}^d)$*-valued local Kato class of* $\mathbb{R}^n$, if $1_K V$ is in the corresponding Kato class for any compact subset $K \subset M$.

We write $\mathcal{K}(\mathbb{R}^n,\mathrm{Mat}(\mathbb{C}^d))$ and $\mathcal{K}_{\mathrm{loc}}(\mathbb{R}^n,\mathrm{Mat}(\mathbb{C}^d))$ for the Kato and the local Kato class, respectively. Note that for a measurable function $V:\mathbb{R}^n \to \mathrm{Mat}(\mathbb{C}^d)$ the condition $V\in\mathcal{K}_{\mathrm{loc}}(\mathbb{R}^n,\mathrm{Mat}(\mathbb{C}^d))$ is equivalent to

$$\varphi V\in\mathcal{K}\big(\mathbb{R}^n,\mathrm{Mat}\big(\mathbb{C}^d\big)\big)\quad\text{for any }\varphi\in\mathsf{C}_0^\infty\big(\mathbb{R}^n\big).$$

For any p such that $p\geqslant 1$ if $m=1$, and $p>m/2$ if $m\geqslant 2$, one has

$$\mathsf{L}^p_{\mathrm{loc}}\big(\mathbb{R}^n,\mathrm{Mat}\big(\mathbb{C}^d\big)\big)\subset\mathcal{K}_{\mathrm{loc}}\big(\mathbb{R}^n,\mathrm{Mat}\big(\mathbb{C}^d\big)\big)\subset\mathsf{L}^1_{\mathrm{loc}}\big(\mathbb{R}^n,\mathrm{Mat}\big(\mathbb{C}^d\big)\big).\tag{17}$$

These inclusions may be found in [1].

Remark 1.4. We will frequently use a simple consequence of the definition of the Kato class: If $V\in\mathcal{K}(\mathbb{R}^n,\mathrm{Mat}(\mathbb{C}^d))$, then the Chapman–Kolmogorov equation for the heat kernel of $\mathbb{R}^n$ shows that for all $t\geqslant 0$,

$$\sup_{x\in\mathbb{R}^n}\mathbb{E}^x\left[\int_0^t \|V(X_s)\|\,\mathrm{d}s\right]<\infty.\tag{18}$$

Using this and the continuity of Brownian motion easily implies the following fact: If $V\in\mathcal{K}_{\mathrm{loc}}(\mathbb{R}^n,\mathrm{Mat}(\mathbb{C}^d))$, then

$$\mathbb{P}^x\left\{\int_0^t \|V(X_s)\|\,\mathrm{d}s<\infty\right\}=1.\tag{19}$$

We can now prove two convergence results for $\mathscr{A}^{\alpha,V}$ that will turn out to be closely related to continuity properties of the semigroup that corresponds to an operator of the form $\tau(\alpha,V)$ as in (2). To this end, a potential V will be called *nonnegative*, $V\geqslant 0$, if all eigenvalues of the matrix $V(x):\mathbb{C}^d\to\mathbb{C}^d$ are nonnegative for a.e. $x\in\mathbb{R}^n$. The following two lemmas extend Lemma C.3 and Lemma C.5 in [3] to the matrix-valued setting:

Proposition 1.5. *Let V be a potential with*

$$0\leqslant V\in\mathcal{K}\big(\mathbb{R}^n,\mathrm{Mat}\big(\mathbb{C}^d\big)\big)$$

and let $\alpha\in\Omega^1(\mathbb{R}^n,\mathscr{U}(d))$ be such that

$$\max_{\substack{i=1,\dots,n\\ j,k=1,\dots,d}}\big|\partial_i\alpha^j_{i,k}\big|\in\mathcal{K}\big(\mathbb{R}^n\big),\qquad \max_{\substack{i=1,\dots,n\\ j,k,l,m=1,\dots,d}}\big|\alpha^j_{i,l}\alpha^k_{i,m}\big|\in\mathcal{K}\big(\mathbb{R}^n\big),\tag{20}$$

where the meaning of the indices in (20) *is as in* (1)*. Then one has*

$$\lim_{t\searrow 0}\sup_{x\in\mathbb{R}^n}\mathbb{E}^x\big[\big\|\mathscr{A}^{\alpha,V}_t-\mathbf{1}\big\|\big]=0.\tag{21}$$

Remark 1.6. If $d=1$, then the estimate $|\mathrm{e}^z-1|\leqslant C|z|\mathrm{e}^{\max\{\mathrm{Re}(z),0\}}$ for all $z\in\mathbb{C}$ combined with (11) and $V\geqslant 0$ directly imply

$$\mathbb{E}^x\big[\big|\mathscr{A}^{\alpha,V}_t-\mathbf{1}\big|\big]\leqslant C\mathbb{E}^x\left[\left|\mathrm{i}\sum_{j=1}^n\int_0^t\tilde{\alpha}_j(X_s)\,\mathrm{d}X^j_s+\frac{\mathrm{i}}{2}\int_0^t\mathrm{div}(\tilde{\alpha})(X_s)\,\mathrm{d}s-\int_0^t V(X_s)\,\mathrm{d}s\right|\right],\tag{22}$$

so that in this case (21) follows immediately from the Itô isometry and the assumptions on (α,V). Since one does not have an explicit expression as (11) for $\mathscr{A}^{\alpha,V}_t(\omega)$ for $d>1$, we have to proceed differently for the general case: We will use the differential equation (8) to rewrite $\mathscr{A}^{\alpha,V}_t(\omega)-\mathbf{1}$, and then use a uniform estimate for $\|\mathscr{A}^{\alpha,V}_t(\omega)\|$ (which is proved in Lemma A.2) in order to derive an estimate that is similar to (22).

Proof of Proposition 1.5. We set $A := A^{\alpha,V}$ and $\mathscr{A} := \mathscr{A}^{\alpha,V}$. Since

$$\mathrm{d}\mathscr{A}^i_j = (\mathscr{A}\,\underline{\mathrm{d}}A)^i_j = \sum_k \mathscr{A}^i_k\,\underline{\mathrm{d}}A^k_j = \sum_k \mathscr{A}^i_k\,\mathrm{d}A^k_j + \sum_c \frac{1}{2}\,\mathrm{d}\big[\mathscr{A}^i_k, A^k_j\big], \tag{23}$$

one has

$$\mathscr{A}^i_j - \delta^i_j = \sum_k \int \mathscr{A}^i_k\,\mathrm{d}A^k_j + \frac{1}{2}\sum_{k,l}\int \mathscr{A}^i_l\,\mathrm{d}\big[A^l_k, A^k_j\big]. \tag{24}$$

Furthermore, by the Itô formula and $[X^i_t, X^j_t] = \delta^{ij}t$, $[X^i_t, t] = 0$ for all $t > 0$, one has

$$A^i_j = \sum_k \int \alpha^i_{k,j}(X)\,\mathrm{d}X^k + \frac{1}{2}\int \sum_k \partial_k \alpha^i_{k,j}(X)\,\mathrm{d}t - \int V^i_j(X)\,\mathrm{d}t \tag{25}$$

and

$$\big[A^i_j, A^k_l\big] = \sum_m \int \alpha^i_{m,j}(X)\alpha^k_{m,l}(X)\,\mathrm{d}t, \tag{26}$$

so that we arrive at

$$\begin{aligned}\mathscr{A}^i_j - \delta^i_j = {}&\sum_{k,l}\int \mathscr{A}^i_k \alpha^k_{l,j}(X)\,\mathrm{d}X^l + \frac{1}{2}\sum_{k,l}\int \mathscr{A}^i_k \partial_l \alpha^k_{l,j}(X)\,\mathrm{d}t\\ &- \sum_k \int \mathscr{A}^i_k V^k_j(X)\,\mathrm{d}t + \frac{1}{2}\sum_{k,l,m}\int \mathscr{A}^i_l \alpha^l_{m,k}(X)\alpha^k_{m,j}(X)\,\mathrm{d}t.\end{aligned} \tag{27}$$

Let $t > 0$. In order to use the Itô isometry, we estimate the stochastic integrals by using Jensen's inequality as follows,

$$\begin{aligned}\mathbb{E}^x\Bigg[\bigg|\int_0^t (\mathscr{A}_s)^i_k \alpha^k_{l,j}(X_s)\,\mathrm{d}X^l_s\bigg|\Bigg]^{2\frac{1}{2}} &\leqslant \mathbb{E}^x\Bigg[\bigg|\int_0^t (\mathscr{A}_s)^i_k \alpha^k_{l,j}(X_s)\,\mathrm{d}X^l_s\bigg|^2\Bigg]^{\frac{1}{2}}\\ &= \mathbb{E}^x\Bigg[\int_0^t \big|(\mathscr{A}_s)^i_k \alpha^k_{l,j}(X_s)\big|^2\,\mathrm{d}s\Bigg]^{\frac{1}{2}}.\end{aligned} \tag{28}$$

By Lemma A.2, there is a $C = C(d) > 0$ such that for all $i, k = 1, \ldots, d$ and $s \geqslant 0$

$$\big|(\mathscr{A}_s)^i_k\big| \leqslant C \quad \mathbb{P}^x\text{-a.s.}, \tag{29}$$

so that

$$\begin{aligned}\mathbb{E}^x\big[\big|\mathscr{A}^i_j - \delta^i_j\big|\big] \leqslant {}& C\sum_{k,l}\mathbb{E}^x\Bigg[\int_0^t \big|\alpha^k_{l,j}(X_s)\big|^2\,\mathrm{d}s\Bigg]^{\frac{1}{2}}\\ &+ \frac{1}{2}C\sum_{k,l}\mathbb{E}^x\Bigg[\int_0^t \big|\partial_l \alpha^k_{l,j}(X_s)\big|\,\mathrm{d}s\Bigg]\\ &+ C\sum_k \mathbb{E}^x\Bigg[\int_0^t \big|V^k_j(X_s)\big|\,\mathrm{d}s\Bigg]\\ &+ \frac{1}{2}C\sum_{k,l,m}\mathbb{E}^x\Bigg[\int_0^t \big|\alpha^l_{m,k}(X_s)\alpha^k_{m,j}(X_s)\big|\,\mathrm{d}s\Bigg]\end{aligned} \tag{30}$$

and the proof is complete by (20). □

If one weakens the Kato assumption on V in the previous proposition to a local Kato assumption, one still has:

Proposition 1.7. *Let V be a potential with*

$$0 \leqslant V \in \mathcal{K}_{\mathrm{loc}}\big(\mathbb{R}^n, \mathrm{Mat}\big(\mathbb{C}^d\big)\big)$$

and let $\alpha \in \Omega^1(\mathbb{R}^n, \mathscr{U}(d))$. Then for any compact $K \subset \mathbb{R}^n$ one has

$$\lim_{t \searrow 0} \sup_{x \in K} \mathbb{E}^x\big[\big\| \mathscr{A}_t^{\alpha,V} - \mathbf{1} \big\|\big] = 0. \tag{31}$$

Proof. For any radius $r > 0$ let $\zeta_{\mathrm{K}_r(0)}$ be the first exit time of X from the open ball $\mathrm{K}_r(0)$. For any $t > 0$ one has

$$\begin{aligned} &\sup_{x \in K} \mathbb{E}^x\big[\big((1 - 1_{\{t<\zeta_{\mathrm{K}_r(0)}\}}) + 1_{\{t<\zeta_{\mathrm{K}_r(0)}\}}\big) \big\| \mathscr{A}_t^{\alpha,V} - \mathbf{1} \big\|\big] \\ &\quad \leqslant 2 \sup_{x \in K} \mathbb{E}^x[1 - 1_{\{t<\zeta_{\mathrm{K}_r(0)}\}}] + \sup_{x \in K} \mathbb{E}^x\big[1_{\{t<\zeta_{\mathrm{K}_r(0)}\}} \big\| \mathscr{A}_t^{\alpha,V} - \mathbf{1} \big\|\big], \end{aligned} \tag{32}$$

where we have used Lemma A.2. Since Levy's maximal inequality (as it is formulated in [16]) implies

$$\sup_{x \in K} \mathbb{E}^x[1 - 1_{\{t<\zeta_{\mathrm{K}_r(0)}\}}] \to 0 \quad \text{as } r \to \infty \text{ for any } t > 0,$$

taking $r \to \infty$ in (32) shows that it is sufficient to prove that for all $r > 0$ one has

$$\sup_{x \in \mathbb{R}^n} \mathbb{E}^x\big[1_{\{t<\zeta_{\mathrm{K}_r(0)}\}} \big\| \mathscr{A}_t^{\alpha,V} - \mathbf{1} \big\|\big] \to 0 \quad \text{as } t \searrow 0. \tag{33}$$

To this end, we first note that (26) shows

$$\big(B^{\alpha,V}\big)^i_j = \sum_k \int \alpha^i_{k,j}(X)\, \underline{\mathrm{d}} X^k - \int V^i_j(X)\, \mathrm{d}t + \frac{1}{2} \sum_{k,l} \int \alpha^i_{l,k}(X) \alpha^k_{l,j}(X)\, \mathrm{d}t. \tag{34}$$

We fix $t > 0$, $r > 0$ and let $\psi \in \mathrm{C}_0^\infty(\mathbb{R}^n)$ be such that $\psi = 1$ in $\mathrm{K}_r(0)$. It follows from (34) that $B_s^{\psi\alpha,\psi V} = B_s^{\alpha,V}$ in $\{t < \zeta_{\mathrm{K}_r(0)}\}$ for all $0 \leqslant s \leqslant t$. As a consequence, the expansion (6) for $\mathscr{A}^{\alpha,V}$ shows

$$\mathbb{E}^x\big[1_{\{t<\zeta_{\mathrm{K}_r(0)}\}} \big\| \mathscr{A}_t^{\psi\alpha,\psi V} - \mathbf{1} \big\|\big] = \mathbb{E}^x\big[1_{\{t<\zeta_{\mathrm{K}_r(0)}\}} \big\| \mathscr{A}_t^{\alpha,V} - \mathbf{1} \big\|\big].$$

Since $\psi\alpha$ and ψV satisfy the assumptions of Proposition 1.5, we have proved (33). □

We now come to the main results of this paper. If $\alpha \in \Omega^1(\mathbb{R}^n, \mathscr{U}(d))$, then the partial differential operator

$$\tau(\alpha, 0)\Psi = -\frac{1}{2}\Delta\Psi - \frac{1}{2}\sum_{j=1}^n \alpha_j^2 \Psi - \frac{1}{2}\sum_{j=1}^n (\partial_j \alpha_j)\Psi - \sum_{j=1}^n \alpha_j \partial_j \Psi, \tag{35}$$

defined initially for all[2] $\Psi \in \mathrm{D}(\tau(\alpha, 0)) = \mathrm{C}_0^\infty(\mathbb{R}^n, \mathbb{C}^d)$, is an essentially self-adjoint nonnegative [9] operator in the Hilbert space $\mathrm{L}^2(\mathbb{R}^n, \mathbb{C}^d)$ of (equivalence classes of) measurable functions $f = (f^1, \ldots, f^d) : \mathbb{R}^n \to \mathbb{C}^d$ such that

$$\|f\|^2_{\mathrm{L}^2(\mathbb{R}^n,\mathbb{C}^d)} := \int_{\mathbb{R}^n} \big\| f(x) \big\|^2 \mathrm{d}x < \infty$$

with scalar product

$$\langle f, g \rangle_{\mathrm{L}^2(\mathbb{R}^n,\mathbb{C}^d)} = \int_{\mathbb{R}^n} \big\langle f(x), g(x) \big\rangle \mathrm{d}x,$$

where $\langle \bullet, \bullet \rangle$ denotes the Euclidean scalar product in $\mathbb{C}^d$. We denote the quadratic form that corresponds to the closure $H(\alpha, 0) \geqslant 0$ of $\tau(\alpha, 0)$ with $q_{\alpha,0}$. One has

$$\mathrm{D}(q_{\alpha,0}) = \left\{ f \in \mathrm{L}^2\big(\mathbb{R}^n, \mathbb{C}^d\big) \;\middle|\; \left(\sum_{j=1}^n \|\partial_j f + \alpha_j f\|^2 \right)^{\frac{1}{2}} \in \mathrm{L}^2\big(\mathbb{R}^n\big) \right\},$$

$$q_{\alpha,0}(f) = \frac{1}{2} \int_{\mathbb{R}^n} \sum_{j=1}^n \big\| \partial_j f(x) + \alpha_j f(x) \big\|^2 \mathrm{d}x, \tag{36}$$

[2] "D(•)" stands for domain of definition.

which follows for example from Proposition 8.13 in [2], if one interprets $\mathrm{d}+\alpha$ as a covariant derivative in $\mathbb{R}^n \times \mathbb{C}^d$. If V is a nonnegative potential with

$$V \in \mathcal{K}_{\mathrm{loc}}\big(\mathbb{R}^n, \mathrm{Mat}\big(\mathbb{C}^d\big)\big) \subset \mathrm{L}^1_{\mathrm{loc}}\big(\mathbb{R}^n, \mathrm{Mat}\big(\mathbb{C}^d\big)\big),$$

then the KLMN-theorem (which we use in the sense of Theorem 10.3.19 in [12]) implies that the quadratic form given by

$$\mathrm{D}(q_{\alpha,V}) := \mathrm{D}(q_{\alpha,0}) \cap \left\{ f \;\Big|\; \int_{\mathbb{R}^n} \big\langle V(x) f(x), f(x)\big\rangle \,\mathrm{d}x < \infty \right\},$$

$$q_{\alpha,V}(f) := q_{\alpha,0}(f) + \int_{\mathbb{R}^n} \big\langle V(x) f(x), f(x)\big\rangle \,\mathrm{d}x$$

is densely defined, closed and nonnegative, and thus uniquely corresponds to a self-adjoint nonnegative operator $H(\alpha, V)$ in $\mathrm{L}^2(\mathbb{R}^n, \mathbb{C}^d)$. Differential operators of this type arise in nonrelativistic quantum mechanics, when one wants to describe the energy of Yang–Mills particles [10,4] (with internal symmetries that are modelled by a subgroup of $\mathrm{U}(k)$), which live on $\mathbb{R}^n$ under the influence of the "electrical" potential V.

Note that under the above assumptions on (α, V), the expressions

$$\mathbb{E}^x\big[\mathscr{A}_t^{\alpha,V} f(X_t)\big], \quad x \in \mathbb{R}^n,\ t > 0,$$

are well defined (this follows from Remark 1.4 and Lemma A.2). As our first main result, we are going to prove the following Feynman–Kac type formula, which will be our main tool in the following:

Theorem 1.8. *Let $\alpha \in \Omega^1(\mathbb{R}^n, \mathscr{U}(d))$ and let V be a potential with*

$$0 \leqslant V \in \mathcal{K}_{\mathrm{loc}}\big(\mathbb{R}^n, \mathrm{Mat}\big(\mathbb{C}^d\big)\big).$$

Then for any $t > 0$, $f \in \mathrm{L}^2(\mathbb{R}^n, \mathbb{C}^d)$ and a.e. $x \in \mathbb{R}^n$ one has

$$\mathrm{e}^{-tH(\alpha,V)} f(x) = \mathbb{E}^x\big[\mathscr{A}_t^{\alpha,V} f(X_t)\big]. \tag{37}$$

The proof of Theorem 1.8 will be given in Section 2.

As a first application of Theorem 1.8, we are going to use Proposition 1.7 to prove the following theorem, which is our second main result:

Theorem 1.9. *Fix the assumptions of Theorem 1.8. Then $\mathrm{e}^{-tH(\alpha,V)} f$ has a bounded continuous representative which is given by*

$$\mathbb{R}^n \to \mathbb{C}^d, \qquad x \mapsto \mathbb{E}^x\big[\mathscr{A}_t^{\alpha,V} f(X_t)\big].$$

In particular, any eigenfunction of $H(\alpha, V)$ can be chosen bounded and continuous.

Remark 1.10. If $n \leqslant 3$ and $V \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{Mat}(\mathbb{C}^d))$, then one has

$$\mathrm{D}\big(H(\alpha, V)\big) \subset \mathrm{H}^2_{\mathrm{loc}}\big(\mathbb{R}^n, \mathbb{C}^d\big), \tag{38}$$

the local second order Sobolev space (this follows for example from Theorem 2.3 in [2]), which proves the continuity of the eigenfunctions in this case. In this sense, the continuity result from Theorem 1.9 extends this continuity to higher dimensions.

Proof of Theorem 1.9. For any function $h : \mathbb{R}^n \to \mathbb{C}^d$ let

$$P_t^{\alpha,V} h(x) := \mathbb{E}^x\big[\mathscr{A}_t^{\alpha,V} h(X_t)\big].$$

If $f \in \mathrm{L}^2(\mathbb{R}^n, \mathbb{C}^d)$, then $P_t^{\alpha,V} f(x)$ is well defined for all $t > 0$, $x \in \mathbb{R}^n$. Due to Lemma A.2, the corresponding semigroup domination

$$\big\|\mathbb{E}^x\big[\mathscr{A}_t^{\alpha,V} f(X_t)\big]\big\| \leqslant \mathbb{E}^x\big[\big\|f(X_t)\big\|\big] \quad \text{for any } x \in \mathbb{R}^n, \tag{39}$$

and the fact that $\mathbb{E}^\bullet[\|f(X_t)\|]$ is bounded, we have that $P_t^{\alpha,V} f$ is bounded for all $t > 0$.

In order to prove the asserted continuity, one can use the boundedness of $P_t^{\alpha,V} f$ and the pointwise semigroup property of $(P_t^{\alpha,V})_{t \geqslant 0}$ (which follows easily from (13)), to see that we can assume that f is bounded. Let us also note that for any $p \in [1, \infty]$ and $t > 0$,

$$P_t^{0,0} : \mathrm{L}^p\big(\mathbb{R}^n, \mathbb{C}^d\big) \to \mathrm{C}\big(\mathbb{R}^n, \mathbb{C}^d\big).$$

Fix some arbitrary $t>0$ and let s be such that $t \geqslant s > 0$. By the above considerations, it is sufficient to prove that for any compact $K \subset \mathbb{R}^n$ one has

$$\sup_{x\in K}\left\|\mathbb{E}^x\left[\tilde{f}(t-s,X_s)\right]-\mathbb{E}^x\left[\mathscr{A}_t^{\alpha,V} f(X_t)\right]\right\| \to 0 \quad \text{as } s\searrow 0, \tag{40}$$

since

$$\tilde{f}:[0,\infty)\times\mathbb{R}^n\to\mathbb{C}^d, \qquad \tilde{f}(u,x):=\mathbb{E}^x\left[\mathscr{A}_u^{\alpha,V} f(X_u)\right]$$

is bounded in x. We set $\mathscr{A}:=\mathscr{A}^{\alpha,V}$. Using the Markov property of the Brownian motion together with (13) shows that for any $x\in\mathbb{R}^n$,

$$\mathbb{E}^x\left[\tilde{f}(t-s,X_s)\right]-\mathbb{E}^x\left[\mathscr{A}_t f(X_t)\right]=\mathbb{E}^x\left[\mathscr{A}_s^{-1}\mathscr{A}_t f(X_t)-\mathscr{A}_t f(X_t)\right]. \tag{41}$$

Noting that by Lemma A.1 one has

$$\left\|\mathscr{A}_s^{-1}\mathscr{A}_t\right\|\leqslant 1 \quad \mathbb{P}^x\text{-a.s.},$$

we can estimate as follows,

$$\left\|\mathbb{E}^x\left[\mathscr{A}_s^{-1}\mathscr{A}_t f(X_t)-\mathscr{A}_t f(X_t)\right]\right\|=\left\|\mathbb{E}^x\left[(1-\mathscr{A}_s)\mathscr{A}_s^{-1}\mathscr{A}_t f(X_t)\right]\right\|\leqslant\|f\|_\infty\mathbb{E}^x\left[\left\|(1-\mathscr{A}_s)\right\|\right].$$

Now (40) follows from Proposition 1.7. □

Our next aim will be to prove that $e^{-tH(\alpha,V)}$ has a jointly continuous integral kernel. To this end, we need the Brownian bridge measure(s) $\mathbb{P}_t^{x,y}$: Let

$$p_t(x,y)=\frac{1}{(2\pi t)^{\frac{n}{2}}}e^{-\frac{\|x-y\|^2}{2t}}$$

stand for the heat kernel of $\mathbb{R}^n$. We fix arbitrary $t>0$, $x,y\in\mathbb{R}^n$ for the following considerations. Let $\Omega_t:=\mathrm{C}([0,t],\mathbb{R}^n)$, let

$$X^{(t)}:[0,t]\times\Omega_t\to\mathbb{R}^n \tag{42}$$

be the canonical process and denote the corresponding σ-algebra and filtration with $\mathscr{F}^{(t)}$ and $(\mathscr{F}_s^{(t)})_{0\leqslant s\leqslant t}$, respectively. The measure $\mathbb{P}_t^x$ stands for the Wiener measure on $(\Omega_t,\mathscr{F}^{(t)})$ which is concentrated on the paths $\omega:[0,t]\to\mathbb{R}^n$ with $\omega(0)=x$. Then the Brownian bridge measure $\mathbb{P}_t^{x,y}$ can be defined as the unique probability measure on $(\Omega_t,\mathscr{F}^{(t)})$ such that

$$\left.\frac{d\mathbb{P}_t^{x,y}}{d\mathbb{P}_t^x}\right|_{\mathscr{F}_s^{(t)}}=\frac{p_{t-s}(X_s^{(t)},y)}{p_t(x,y)} \quad \text{for any } s<t. \tag{43}$$

The process (42) is a well-defined continuous semi-martingale under $\mathbb{P}_t^{x,y}$, which is a Brownian bridge from x to y with terminal time t, so that $\mathbb{P}_t^{x,y}$ is concentrated on the set of paths $\omega:[0,t]\to\mathbb{R}^n$ with $\omega(0)=x$ and $\omega(t)=y$. It is well known (see for example Corollary A.2 in [17]) that the family $\mathbb{P}_t^{x,y}$ disintegrates $\mathbb{P}_t^x$ in the sense that

$$\mathbb{P}_t^x(A)=\int_{\mathbb{R}^n}\mathbb{P}_t^{x,y}(A)p_t(x,y)\,dy \quad \text{for any } A\in\mathscr{F}^{(t)}, \tag{44}$$

and that for any $F\in L^1(\mathbb{P}_t^{y,x})$ one has the following time reversal property:

$$\int_{\Omega_t}F\big(\omega(t-\bullet)\big)\,\mathbb{P}_t^{x,y}(d\omega)=\int_{\Omega_t}F(\omega)\,\mathbb{P}_t^{y,x}(d\omega).$$

The local Kato class is compatible with the Brownian bridge measures in the following sense:

Remark 1.11. If $V\in\mathcal{K}_{\mathrm{loc}}(\mathbb{R}^n,\mathrm{Mat}(\mathbb{C}^d))$, then by Lemma C.8 in [3] one has

$$\mathbb{P}_t^{x,y}\left\{\int_0^t\left\|V\big(X_s^{(t)}\big)\right\|ds<\infty\right\}=1. \tag{45}$$

The following definitions completely follow the construction of $\mathscr{A}^{\alpha,V}$: Let $\alpha \in \Omega^1(\mathbb{R}^n, \mathscr{U}(d))$ and let $V \in \mathcal{K}_{\mathrm{loc}}(\mathbb{R}^n, \mathrm{Mat}(\mathbb{C}^d))$ be a potential. Remark 1.11 and the fact that (42) is a continuous semi-martingale under $\mathbb{P}_t^{x,y}$ show that

$$A^{\alpha,V,(t)} : [0,t] \times \Omega_t \to \mathrm{Mat}(\mathbb{C}^d),$$
$$A_s^{\alpha,V,(t)} := \sum_{j=1}^n \int_0^s \alpha_j\big(X_r^{(t)}\big)\underline{\mathrm{d}}X_r^{(t),j} - \int_0^s V\big(X_r^{(t)}\big)\,\mathrm{d}r$$

is also a continuous semi-martingale under $\mathbb{P}_t^{x,y}$, so that the same is true for

$$B^{\alpha,V,(t)} : [0,t] \times \Omega_t \to \mathrm{Mat}(\mathbb{C}^d), \tag{46}$$

which is defined in analogy to (5). If we furthermore set

$$\mathscr{A}_s^{\alpha,V,(t)} := \mathbf{1} + \sum_{l=1}^{\infty} \int_{s\Delta_l} \mathrm{d}B_{s_1}^{\alpha,V,(t)} \cdots \mathrm{d}B_{s_l}^{\alpha,V,(t)}, \tag{47}$$

where the convergence is $\mathbb{P}_t^{x,y}$-a.s. uniformly in $[0,t]$, we have that

$$\mathscr{A}^{\alpha,V,(t)} : [0,t] \times \Omega_t \to \mathrm{Mat}(\mathbb{C}^d)$$

is uniquely determined as the solution of

$$\mathscr{A}_s^{\alpha,V,(t)} = \mathbf{1} + \int_0^s \mathscr{A}_r^{\alpha,V,(t)}\,\underline{\mathrm{d}}A_r^{\alpha,V,(t)} \tag{48}$$

under $\mathbb{P}_t^{x,y}$. We will use the notation

$$\overrightarrow{\prod_{1 \leqslant j \leqslant n}} M_j := M_1 \cdots M_n \quad \text{for } M_1, \ldots, M_n \in \mathrm{Mat}(\mathbb{C}^d).$$

One has the following Hermitian symmetry:

Lemma 1.12. *Let $\alpha \in \Omega^1(\mathbb{R}^n, \mathscr{U}(d))$, let V be a potential with*

$$0 \leqslant V \in \mathcal{K}_{\mathrm{loc}}\big(\mathbb{R}^n, \mathrm{Mat}(\mathbb{C}^d)\big),$$

and for any $t > 0$ let

$$\mathrm{e}^{-tH(\alpha,V)}(\bullet,\bullet) : \mathbb{R}^n \times \mathbb{R}^n \to \mathrm{Mat}(\mathbb{C}^d),$$
$$\mathrm{e}^{-tH(\alpha,V)}(x,y) := \frac{1}{(2\pi t)^{\frac{n}{2}}}\mathrm{e}^{-\frac{\|x-y\|^2}{2t}}\,\mathbb{E}_t^{x,y}\big[\mathscr{A}_t^{\alpha,V,(t)}\big]. \tag{49}$$

Then $\mathrm{e}^{-tH(\alpha,V)}(x,y)$ is well defined for all $t > 0$, $x, y \in \mathbb{R}^n$ and one has

$$\mathrm{e}^{-tH(\alpha,V)}(y,x) = \mathrm{e}^{-tH(\alpha,V)}(x,y)^*. \tag{50}$$

Remark 1.13. Let $d = 1$. Using that $\mathbb{P}_t^{x,y}$ is equivalent to $\mathbb{P}_t^x$ on $\mathscr{F}_s^{(t)}$ for all $0 \leqslant s < t$, it follows (from taking $s \nearrow t$ and from the fact that $X^{(t)}$ is a continuous semi-martingale under $\mathbb{P}_t^{x,y}$) that for all $j,k = 1,\ldots,n$ one has

$$\big[X^{(t),j}, X^{(t),k}\big]_s = \delta^{jk}s \quad \mathbb{P}_t^{x,y}\text{-a.s. for all } 0 \leqslant s \leqslant t.$$

As a consequence, the Itô formula gives

$$\mathscr{A}_s^{\alpha,V,(t)} = \exp\left(\mathrm{i}\sum_{j=1}^n \int_0^s \tilde{\alpha}_j\big(X_r^{(t)}\big)\,\mathrm{d}X_r^{(t),j} + \frac{\mathrm{i}}{2}\int_0^s \mathrm{div}(\tilde{\alpha})\big(X_r^{(t)}\big)\,\mathrm{d}r - \int_0^s V\big(X_r^{(t)}\big)\,\mathrm{d}r\right) \tag{51}$$

$\mathbb{P}_t^{x,y}$-a.s. for all $0 \leqslant s \leqslant t$. In particular, (50) becomes a simple consequence of the time reversal property of the Brownian bridge in this case. For the general case, we will use a result [6] by Emery, which states that $\mathscr{A}^{\alpha,V,(t)}$ can be approximated by stochastic product integrals.

Proof of Lemma 1.12. The well-definedness of $\mathrm{e}^{-tH(\alpha,V)}(x,y)$ follows from Remark 1.11 and Lemma A.2:

$$\left\|\mathrm{e}^{-tH(\alpha,V)}(x,y)\right\| \leqslant \frac{1}{(2\pi t)^{\frac{n}{2}}}.$$

We set $\mathscr{A}^{(t)} := \mathscr{A}^{\alpha,V,(t)}$ and $B^{(t)} := B^{\alpha,V,(t)}$. The time reversal property of the Brownian bridge measure implies

$$\int_{\Omega_t} \mathscr{A}_t^{(t)}(\omega)\,\mathbb{P}_t^{y,x}(\mathrm{d}\omega) = \int_{\Omega_t} \mathscr{A}_t^{(t)}\big(\omega(t-\bullet)\big)\,\mathbb{P}_t^{x,y}(\mathrm{d}\omega),$$

so that it is sufficient to prove

$$\mathscr{A}_t^{(t),*}\big(\omega(t-\bullet)\big) = \mathscr{A}_t^{(t)}(\omega) \quad \text{for } \mathbb{P}_t^{y,x}\text{-a.e. } \omega \in \Omega_t. \tag{52}$$

We can proceed as follows in order to prove the latter equality: For any partition

$$\sigma = \{0 = t_0 < t_1 < t_2 < \cdots < t_m = t\}$$

of $[0,t]$ we define

$$\mathscr{A}_t^{(t),\sigma} := \big(\mathbf{1} + B_{t_0}^{(t)}\big) \overrightarrow{\prod_{1\leqslant j\leqslant m}} \big(\mathbf{1} + B_{t_j}^{(t)} - B_{t_{j-1}}^{(t)}\big). \tag{53}$$

Analogously to (34) one has

$$\big(B^{(t)}\big)^i_j = \sum_l \int \alpha^i_{l,j}\big(X^{(t)}\big)\,\underline{\mathrm{d}}X^{(t),l} - \int V^i_j\big(X^{(t)}\big)\,\mathrm{d}s + \frac{1}{2}\sum_{k,l}\int \alpha^i_{l,k}\big(X^{(t)}\big)\alpha^k_{l,j}\big(X^{(t)}\big)\,\mathrm{d}s. \tag{54}$$

By [6, p. 256], the family of random variables $(\mathscr{A}_t^{(t),\sigma})_\sigma$ converges in probability (with respect to $\mathbb{P}_t^{y,x}$) to $\mathscr{A}_t^{(t)}$ as $|\sigma| \to 0$. Now the key observation for proving (52) is the following: Since $\alpha_j^* = -\alpha_j$, $j = 1,\dots,n$, and $V = V^*$, approximating the integrals in (54) with Riemann-type sums as in (84) implies

$$B_s^{(t),*}\big(\omega(t-\bullet)\big) = B_t^{(t)}(\omega) - B_{t-s}^{(t)}(\omega) \quad \text{for } \mathbb{P}_t^{y,x}\text{-a.e. } \omega \in \Omega_t,\ 0 \leqslant s \leqslant t. \tag{55}$$

Now (52) follows from (55) and the adjoint version of formula (53). $\square$

Being equipped with this result, we can use Proposition 1.7 to prove our third main result:

Theorem 1.14. *Fix the assumptions of Theorem* 1.8.

(a) *The map* $\mathrm{e}^{-tH(\alpha,V)}(\bullet,\bullet)$ *represents an integral kernel of* $\mathrm{e}^{-tH(\alpha,V)}$ *in the sense that for all* $f \in \mathrm{L}^2(\mathbb{R}^n,\mathbb{C}^d)$ *and a.e.* $x \in \mathbb{R}^n$ *one has*

$$\mathrm{e}^{-tH(\alpha,V)} f(x) = \int_{\mathbb{R}^n} \mathrm{e}^{-tH(\alpha,V)}(x,y) f(y)\,\mathrm{d}y. \tag{56}$$

(b) *The map*

$$(0,\infty)\times\mathbb{R}^n\times\mathbb{R}^n \to \mathrm{Mat}\big(\mathbb{C}^d\big), \qquad (t,x,y) \mapsto \mathrm{e}^{-tH(\alpha,V)}(x,y)$$

is bounded in (x,y) *and jointly continuous in* (t,x,y).

(c) *It holds that*

$$\mathrm{tr}_{\mathrm{L}^2(\mathbb{R}^n,\mathbb{C}^d)}\big(\mathrm{e}^{-tH(\alpha,V)}\big) = \int_{\mathbb{R}^n} \mathrm{tr}_{\mathrm{Mat}(\mathbb{C}^d)}\big(\mathrm{e}^{-tH(\alpha,V)}(x,x)\big)\,\mathrm{d}x, \tag{57}$$

as a number in $[0,\infty]$.

Proof. (a) Let

$$\Pi_t : \Omega \to \Omega_t, \qquad \Pi_t(\omega) = \omega|_{[0,t]}$$

denote the canonical projection. Since $X^{(t)}$ is a continuous semi-martingale under $\mathbb{P}_t^x$ (in fact, a Brownian motion starting in x), the expansion for $\mathscr{A}^{\alpha,V,(t)}$ converges with respect to $\mathbb{P}_t^x$ and one has

$$\mathscr{A}_s^{\alpha,V} = \mathscr{A}_s^{\alpha,V,(t)} \circ \Pi_t \quad \mathbb{P}^x\text{-a.s. for all } 0 \leqslant s \leqslant t. \tag{58}$$

It follows from (58), (44) and $X_t^{(t)} = y$ $\mathbb{P}_t^{x,y}$-a.s. that

$$\begin{aligned}\mathbb{E}^x\big[\mathscr{A}_t^{\alpha,V} f(X_t)\big] &= \mathbb{E}^x\big[\big(\mathscr{A}_t^{\alpha,V,(t)} f\big(X_t^{(t)}\big)\big)\circ \Pi_t\big]\\ &= \mathbb{E}_t^x\big[\mathscr{A}_t^{\alpha,V,(t)} f\big(X_t^{(t)}\big)\big]\\ &= \int_{\mathbb{R}^n} p_t(x,y)\mathbb{E}_t^{x,y}\big[\mathscr{A}_t^{\alpha,V,(t)}\big] f(y)\,\mathrm{d}y.\end{aligned}$$

(b) We set $\mathscr{A}^{(t)} := \mathscr{A}^{\alpha,V,(t)}$ and $A^{(t)} := A^{\alpha,V,(t)}$ for any $t > 0$. The asserted boundedness has already been checked in the proof of Lemma 1.12. For the continuity, let $K \subset \mathbb{R}^n$ be an arbitrary compact subset, and let $\tau_1 \leqslant \tau_2$ be arbitrary positive real numbers. In view of Lemma 1.12, one can go through the same steps as in the proof of Theorem 6.1 in [3] to see that it is sufficient to prove that

$$\lim_{s\searrow 0}\ \sup_{\tau_1\leqslant t\leqslant \tau_2}\ \sup_{x,y\in K}\big\|\Psi(t,s,x,y)\big\| = 0, \tag{59}$$

and that for all $0 < s < \tau_1$,

$$\lim_{r\searrow 0}\ \sup_{\tau_1\leqslant t\leqslant \tilde{t}\leqslant \tau_2,\ |t-\tilde{t}|<r}\ \sup_{x,y\in K,\ \|y-\tilde{y}\|<r}\big\|\Phi(t,\tilde{t},s,x,y,\tilde{y})\big\| = 0, \tag{60}$$

where

$$\begin{aligned}&\Psi : [\tau_1,\tau_2]\times(0,\tau_1)\times K\times K \to \mathrm{Mat}\big(\mathbb{C}^d\big),\\ &\Psi(t,s,x,y) := p_t(x,y)\mathbb{E}_t^{x,y}\big[\mathscr{A}_t^{(t)} - \mathscr{A}_{t-s}^{(t)}\big],\\ &\Phi : [\tau_1,\tau_2]\times[\tau_1,\tau_2]\times(0,\tau_1)\times K\times K\times K \to \mathrm{Mat}\big(\mathbb{C}^d\big),\\ &\Phi(t,\tilde{t},s,x,y,\tilde{y}) := p_{\tilde{t}}(x,\tilde{y})\mathbb{E}_{\tilde{t}}^{x,\tilde{y}}\big[\mathscr{A}_{(t-s)\theta(\tilde{t}-t+s)}^{(\tilde{t})}\big] - p_t(x,y)\mathbb{E}_t^{x,y}\big[\mathscr{A}_{t-s}^{(t)}\big],\end{aligned}$$

and where $\theta : \mathbb{R} \to [0,\infty)$ stands for the Heaviside function.

Proof of (59): One has

$$\begin{aligned}\big\|\Psi(t,s,x,y)\big\| &\leqslant p_t(x,y)\mathbb{E}_t^{x,y}\big[\big\|\mathscr{A}_t^{(t)} - \mathscr{A}_{t-s}^{(t)}\big\|\big]\\ &= p_t(x,y)\mathbb{E}_t^{x,y}\big[\big\|\mathscr{A}_{t-s}^{(t)}\big(\mathscr{A}_{t-s}^{(t),-1}\mathscr{A}_t^{(t)} - \mathbf{1}\big)\big\|\big]\\ &\leqslant p_t(x,y)\mathbb{E}_t^{x,y}\big[\big\|\mathscr{A}_{t-s}^{(t),-1}\mathscr{A}_t^{(t)} - \mathbf{1}\big\|\big],\end{aligned} \tag{61}$$

where we have used that

$$\big\|\mathscr{A}_{t-s}^{(t)}\big\| \leqslant 1 \quad \mathbb{P}_t^{x,y}\text{-a.s.}$$

by Lemma A.2. The time reversal property of the Brownian bridge measure shows

$$\mathbb{E}_t^{x,y}\big[\big\|\mathscr{A}_{t-s}^{(t),-1}\mathscr{A}_t^{(t)} - \mathbf{1}\big\|\big] = \int_{\Omega_t}\big\|\mathscr{A}_{t-s}^{(t),-1}\big(\omega(t-\bullet)\big)\mathscr{A}_t^{(t)}\big(\omega(t-\bullet)\big) - \mathbf{1}\big\|\,\mathbb{P}_t^{y,x}(\mathrm{d}\omega). \tag{62}$$

Using the identity

$$\mathscr{A}_{t-s}^{(t),-1}\big(\omega(t-\bullet)\big)\mathscr{A}_t^{(t)}\big(\omega(t-\bullet)\big) = \mathscr{A}_s^{(t),*}(\omega) \quad \text{for } \mathbb{P}_t^{y,x}\text{-a.e. } \omega\in\Omega_t, \tag{63}$$

which we are going to prove in a moment, and using (43) and (58) we arrive at

$$\begin{aligned}\big\|\Psi(t,s,x,y)\big\| &\leqslant p_t(x,y)\mathbb{E}_t^{y,x}\big[\big\|\mathscr{A}_s^{(t),*} - \mathbf{1}\big\|\big]\\ &= \big(2\pi(t-s)\big)^{-\frac{n}{2}}\mathbb{E}^y\big[\mathrm{e}^{-\frac{\|y-X_s\|^2}{2(t-s)}}\big\|\mathscr{A}_s^{*} - \mathbf{1}\big\|\big]\\ &\leqslant \big(2\pi(t-s)\big)^{-\frac{n}{2}}\mathbb{E}^y\big[\big\|\mathscr{A}_s^{*} - \mathbf{1}\big\|\big].\end{aligned} \tag{64}$$

Now (59) is implied by Proposition 1.7.

It remains to prove (63): Note that if $d = 1$, then this formula follows directly from (51) and $\mathrm{e}^{z_1+z_2} = \mathrm{e}^{z_1}\mathrm{e}^{z_2}$. For the general case, we will (analogously to the proof of Lemma 1.2) use the following trick: We will prove that for fixed t, both sides of (63) solve the same initial value problem with respect to s. To this end, fix some arbitrary $t > 0$, $x, y \in \mathbb{R}^n$, and let the process

$$\tilde{\mathscr{A}}^{(t)} : [0,t]\times\Omega_t \to \mathrm{Mat}\big(\mathbb{C}^d\big)$$

be given by $\tilde{\mathscr{A}}_s^{(t)}(\omega)=\mathscr{A}_{t-s}^{(t)}(\omega(t-\bullet))$. Then, with respect to $\mathbb{P}_t^{y,x}$, one has

$$\tilde{\mathscr{A}}_s^{(t)}(\omega)=\mathbf{1}+\left(\int_0^{t-s}\mathscr{A}_r^{(t)}\,\underline{\mathrm{d}}A_r^{(t)}\right)\big(\omega(t-\bullet)\big). \tag{65}$$

As in (55) one sees

$$A_s^{(t),*}\big(\omega(t-\bullet)\big)=A_t^{(t)}(\omega)-A_{t-s}^{(t)}(\omega)\quad \text{for } \mathbb{P}_t^{y,x}\text{-a.e. } \omega\in\Omega_t, \tag{66}$$

so using the adjoint version of (66) and approximating the Stratonovic integral in (65) with Riemann sums as in (84) easily implies the first identity in

$$\left(\int_0^{t-s}\mathscr{A}^{(t)}\,\underline{\mathrm{d}}A^{(t)}\right)\big(\omega(t-\bullet)\big)=\left(\int_s^{t}\tilde{\mathscr{A}}_r^{(t)}\,\underline{\mathrm{d}}A_r^{(t),*}\right)(\omega)$$
$$=\left(\int_0^{t}\tilde{\mathscr{A}}_r^{(t)}\,\underline{\mathrm{d}}A_r^{(t),*}\right)(\omega)-\left(\int_0^{s}\tilde{\mathscr{A}}_r^{(t)}\,\underline{\mathrm{d}}A_r^{(t),*}\right)(\omega).$$

Thus, $\tilde{\mathscr{A}}^{(t)}$ is uniquely determined as the solution of $\mathrm{d}\tilde{\mathscr{A}}_s^{(t)}=-\tilde{\mathscr{A}}_s^{(t)}\,\underline{\mathrm{d}}A_s^{(t),*}$ with initial value $\tilde{\mathscr{A}}_0^{(t)}(\omega)=\mathscr{A}_t^{(t)}(\omega(t-\bullet))$, which shows

$$\tilde{\mathscr{A}}_s^{(t)}(\omega)=\mathscr{A}_t^{(t)}\big(\omega(t-\bullet)\big)\mathscr{A}_s^{(t),*,-1}(\omega)\quad \text{for } \mathbb{P}_t^{y,x}\text{-a.e. } \omega\in\Omega_t$$

and (63) is proved.

Proof of (60): In view of (60) let $t\leqslant\tilde{t}$. Using (43) and (58) we have

$$\Phi(t,\tilde{t},s,x,y,\tilde{y})=\mathbb{E}^x\Big[(2\pi)^{-\frac{n}{2}}\Big((\tilde{t}-t+s)^{-\frac{n}{2}}\mathrm{e}^{-\frac{\|X_{t-s}-\tilde{y}\|^2}{2(\tilde{t}-t+s)}}-s^{-\frac{n}{2}}\mathrm{e}^{-\frac{\|X_{t-s}-y\|^2}{2s}}\Big)\mathscr{A}_{t-s}\Big], \tag{67}$$

so that Jensen's inequality gives

$$\big\|\Phi(t,\tilde{t},s,x,y,\tilde{y})\big\|^2\leqslant(2\pi)^{-n}\mathbb{E}^x\Big[\Big((\tilde{t}-t+s)^{-\frac{n}{2}}\mathrm{e}^{-\frac{\|X_{t-s}-\tilde{y}\|^2}{2(\tilde{t}-t+s)}}-s^{-\frac{n}{2}}\mathrm{e}^{-\frac{\|X_{t-s}-y\|^2}{2s}}\Big)^2\Big]. \tag{68}$$

Now the proof of Theorem 6.1 in [3] can be copied word by word.

(c) This formula follows directly from the continuity of the integral kernel and well-known algebraic arguments (see for example the proof Proposition 12 in [18]). □

2. Proof of Theorem 1.8

Throughout the proof, we will use the unitarity $\mathscr{A}^{\alpha,0,-1}=\mathscr{A}^{\alpha,0,*}$, which follows from Lemma A.2(a). For any potential $W:\mathbb{R}^n\to\mathrm{Mat}(\mathbb{C}^d)$ that satisfies (4) (with V replaced with W) for all $x\in\mathbb{R}^n$, we define the process

$$\tilde{\mathscr{A}}^{\alpha,W}:[0,\infty)\times\Omega\to\mathrm{Mat}\big(\mathbb{C}^d\big)$$

as the path ordered exponential

$$\tilde{\mathscr{A}}_t^{\alpha,W}=\mathbf{1}+\sum_{l=1}^{\infty}\int_{t\Delta_l}\overrightarrow{\prod_{1\leqslant j\leqslant l}}\big(-\mathscr{A}_{t_j}^{\alpha,0}W(X_{t_j})\mathscr{A}_{t_j}^{\alpha,0,-1}\big)\,\mathrm{d}t_1\cdots\mathrm{d}t_l.$$

Then $\tilde{\mathscr{A}}^{\alpha,W}$ is nothing but the pathwise weak solution [5] of

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathscr{A}}_t^{\alpha,W}=-\tilde{\mathscr{A}}_t^{\alpha,W}\mathscr{A}_t^{\alpha,0}W(X_t)\mathscr{A}_t^{\alpha,0,-1},\quad \tilde{\mathscr{A}}_0^{\alpha,W}=\mathbf{1}, \tag{69}$$

and the Stratonovic product rule implies the following formula for any $x\in\mathbb{R}^n$,

$$\mathscr{A}_t^{\alpha,W}=\tilde{\mathscr{A}}_t^{\alpha,W}\mathscr{A}_t^{\alpha,0}\quad \mathbb{P}^x\text{-a.s.} \tag{70}$$

Furthermore, Gronwall's lemma implies the following inequality for any $x\in\mathbb{R}^n$,

$$\big\|\tilde{\mathscr{A}}_t^{\alpha,W}\big\|\leqslant\mathrm{e}^{\int_0^t\|W(X_s)\|\,\mathrm{d}s}\quad \mathbb{P}^x\text{-a.s.} \tag{71}$$

We fix arbitrary $t>0$ and $f\in\mathrm{L}^2(\mathbb{R}^n,\mathbb{C}^d)$. The remaining proof can be divided into three steps, and it is modelled after the proof of Theorem 1.3 in [7].

Step 1. *Assume that V is a potential in $\mathrm{C}_b(\mathbb{R}^n,\mathrm{Mat}(\mathbb{C}^d))$, the space of continuous bounded functions $\mathbb{R}^n\to\mathrm{Mat}(\mathbb{C}^d)$.*

The operator

$$P_t^{\alpha,V}:\mathrm{L}^2\big(\mathbb{R}^n,\mathbb{C}^d\big)\to \mathrm{L}^2\big(\mathbb{R}^n,\mathbb{C}^d\big),\qquad P_t^{\alpha,V}h(x):=\mathbb{E}^x\big[\mathscr{A}_t^{\alpha,V}h(X_t)\big]$$

is a well-defined bounded linear operator in $\mathrm{L}^2(\mathbb{R}^n,\mathbb{C}^d)$. If $\psi\in\mathrm{C}_0^\infty(\mathbb{R}^n,\mathbb{C}^d)$, then a straightforward calculation, which uses the Itô formula repeatedly, shows that for any $x\in\mathbb{R}^n$, one has the following equality $\mathbb{P}^x$-a.s.,

$$\begin{aligned}\mathscr{A}_t^{\alpha,V}\psi(X_t)&=[\text{a martingale which starts from 0}]+\psi(x)\\&\quad+\int_0^t\mathscr{A}_s^{\alpha,V}\Delta\psi(X_s)\,\mathrm{d}s+\int_0^t\mathscr{A}_s^{\alpha,V}\sum_{j=1}^n\big(\partial_j\alpha_j(X_s)\big)\psi(X_s)\,\mathrm{d}s\\&\quad+2\int_0^t\mathscr{A}_s^{\alpha,V}\sum_{j=1}^n\alpha_j(X_s)\partial_j\psi(X_s)\,\mathrm{d}s+\int_0^t\sum_{j=1}^n\alpha_j^2(X_s)\psi(X_s)\,\mathrm{d}s-\int_0^t\mathscr{A}_s^{\alpha,V}V(X_s)\,\mathrm{d}s,\end{aligned}$$

so that taking $\mathbb{E}^x[\bullet]$ in this equation implies

$$P_t^{\alpha,V}\psi(x)=\psi(x)-\int_0^t P_s^{\alpha,V}H(\alpha,V)\psi(x)\,\mathrm{d}s. \tag{72}$$

This shows $P_t^{\alpha,V}\psi=\mathrm{e}^{-tH(\alpha,V)}\psi$ so that the boundedness of $P_t^{\alpha,V}$ implies $P_t^{\alpha,V}f=\mathrm{e}^{-tH(\alpha,V)}f$, the Feynman–Kac formula.

Step 2. Assume that V is a potential in $\mathrm{L}^\infty(\mathbb{R}^n,\mathrm{Mat}(\mathbb{C}^d))$.

Using Friedrichs mollifiers as in [12, p. 280], one finds a sequence $(V_m)\subset\mathrm{C}_b(\mathbb{R}^n,\mathrm{Mat}(\mathbb{C}^d))$ of potentials with

$$V_m(x)\to V(x)\quad\text{as } m\to\infty,\qquad \big\|V_m(x)\big\|\leqslant C(d)\|V\|_\infty\quad\text{for a.e. } x\in\mathbb{R}^n. \tag{73}$$

It follows from (73) and dominated convergence that for any $\psi\in\mathrm{C}_0^\infty(\mathbb{R}^n,\mathbb{C}^d)$,

$$\big\|H(\alpha,V_m)\psi-H(\alpha,V)\psi\big\|_{\mathrm{L}^2(\mathbb{R}^n,\mathbb{C}^d)}\to 0\quad\text{as } m\to\infty. \tag{74}$$

As a consequence, Theorem VIII 25 and Theorem VIII 20 from [15] show that we may assume

$$\mathrm{e}^{-tH(\alpha,V_m)}f(x)\to\mathrm{e}^{-tH(\alpha,V)}f(x)\quad\text{as } m\to\infty\text{ for a.e. } x\in\mathbb{R}^n. \tag{75}$$

On the other hand, the decomposition (70) combined with Lemma A.1(b) implies

$$\big\|\mathscr{A}_t^{\alpha,V_m}-\mathscr{A}_t^{\alpha,V}\big\|\leqslant\big\|\tilde{\mathscr{A}}_t^{\alpha,V_m}-\tilde{\mathscr{A}}_t^{\alpha,V}\big\|\leqslant\mathrm{e}^{2\int_0^t\|V_m(X_s)\|\,\mathrm{d}s+\int_0^t\|V(X_s)\|\,\mathrm{d}s}\int_0^t\big\|V_m(X_s)-V(X_s)\big\|\,\mathrm{d}s,$$

so that by (73) and dominated convergence,

$$\begin{aligned}&\big\|\mathscr{A}_t^{\alpha,V_m}f(X_t)-\mathscr{A}_t^{\alpha,V}f(X_t)\big\|\leqslant\big\|f(X_t)\big\|\mathrm{e}^{(2C(d)+1)\int_0^t\|V(X_s)\|\,\mathrm{d}s}\int_0^t\big\|V_m(X_s)-V(X_s)\big\|\,\mathrm{d}s\to 0\\&\quad\text{as } m\to\infty,\ \mathbb{P}^x\text{-a.s. for any } x\in\mathbb{R}^n.\end{aligned} \tag{76}$$

Furthermore, (71) and (73) imply[3]

$$\big\|\mathscr{A}_t^{\alpha,V_m}f(X_t)\big\|\leqslant\mathrm{e}^{C(d)\int_0^t\|V(X_s)\|\,\mathrm{d}s}\big\|f(X_t)\big\|\leqslant\mathrm{e}^{C(d)t\|V\|_\infty}\big\|f(X_t)\big\|\in\mathrm{L}^1\big(\mathbb{P}^x\big)$$

so that by (76) we may use dominated convergence to deduce

$$\mathbb{E}^x\big[\mathscr{A}_t^{\alpha,V_m}f(X_t)\big]\to\mathbb{E}^x\big[\mathscr{A}_t^{\alpha,V}f(X_t)\big]\quad\text{as } m\to\infty\text{ for any } x\in\mathbb{R}^n, \tag{77}$$

and the Feynman–Kac formula for essentially bounded potentials follows from combining (75) with the result from step 1.

Step 3. Assume that V is potential with $0\leqslant V\in\mathcal{K}_{\mathrm{loc}}(\mathbb{R}^n,\mathrm{Mat}(\mathbb{C}^d))$.

[3] Note that $\mathbb{E}^x[\|f(X_t)\|]=\mathrm{e}^{t\Delta}\|f(\bullet)\|(x)<\infty$.

Let $U:\mathbb{R}^n \to \mathrm{U}(d)$ be a measurable function with

$$V(x) = U^*(x)\mathrm{diag}\big(v_1(x), \dots, v_d(x)\big)U(x) \quad \text{for a.e. } x \in \mathbb{R}^n,$$

where $v_j : \mathbb{R}^m \to \mathbb{R}$. For any $m \in \mathbb{N}$ we define a potential V_m with $0 \leqslant V_m \in \mathrm{L}^\infty(\mathbb{R}^n, \mathrm{Mat}(\mathbb{C}^d))$ by setting

$$V_m(x) := U^*(x)\mathrm{diag}\big(v_1^{(m)}(x), \dots, v_d^{(m)}(x)\big)U(x),$$

where $v_j^{(m)}(x) := \min\{v_j(x), m\}$. Note that we again have (73) and that by monotone convergence of quadratic forms we may also assume (75) (see [15, Theorem S.14 on p. 373]). On the other hand, (73) shows that one can use the same arguments as in the proof of step 2 to deduce (76). Furthermore, since $V_m \geqslant 0$, it follows from Lemma A.2(a) that

$$\big\|\mathscr{A}_t^{\alpha,V_m} f(X_t)\big\| \leqslant \big\|f(X_t)\big\| \in \mathrm{L}^1\big(\mathbb{P}^x\big),$$

so that we also have (77). Now the general Feynman–Kac formula follows from (75) and step 2.

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Appendix A

We prove two auxiliary results here.

The first assertion gives estimates on the solutions of certain matrix-valued ordinary linear differential equations: Fix $t_0 \geqslant 0$ and let

$$F \in \mathrm{L}^1_{\mathrm{loc}}\big([t_0,\infty), \mathrm{Mat}\big(\mathbb{C}^d\big)\big).$$

Then a standard use of the Banach fixed point theorem shows that there is a unique weak (= absolutely continuous) solution $Y:[t_0,\infty) \to \mathrm{Mat}(\mathbb{C}^d)$ of the ordinary initial value problem

$$\frac{\mathrm{d}}{\mathrm{d}s} Y(s) = Y(s)F(s), \qquad Y(t_0) = \mathbf{1}.$$

We will write $\langle \bullet,\bullet \rangle$ for the Euclidean inner product in $\mathbb{C}^d$ and $\|\bullet\|$ will stand for the induced norm on $\mathbb{C}^d$ and also for the induced operator norm on $\mathrm{Mat}(\mathbb{C}^d)$.

Lemma A.1. (a) *Assume that $F(s)$ is Hermitian and that there exists a real-valued function $c \in \mathrm{L}^1_{\mathrm{loc}}[t_0,\infty)$ such that $F(s) \leqslant c(s)$ for a.e. $s \geqslant t_0$. Then*

$$\big\|Y(t)\big\| \leqslant \mathrm{e}^{\int_{t_0}^t c(r)\,\mathrm{d}r} \quad \text{for any } t \geqslant t_0.$$

(b) *Let $F_1, F_2 \in \mathrm{L}^1_{\mathrm{loc}}([t_0,\infty), \mathrm{Mat}(\mathbb{C}^d))$ and let*

$$Y_1, Y_2 : [t_0,\infty) \to \mathrm{Mat}\big(\mathbb{C}^d\big)$$

be the unique solutions of the ordinary initial value problems

$$\frac{\mathrm{d}}{\mathrm{d}s} Y_j(s) = Y_j(s)F_j(s), \qquad Y_j(t_0) = \mathbf{1} \quad \text{for } j = 1,2.$$

The following inequality holds for all $t \geqslant t_0$,

$$\big\|Y_1(t) - Y_2(t)\big\| \leqslant \mathrm{e}^{2\int_{t_0}^t \|F_1(s)\|\,\mathrm{d}s + \int_{t_0}^t \|F_2(s)\|\,\mathrm{d}s} \int_{t_0}^t \big\|F_1(s) - F_2(s)\big\|\,\mathrm{d}s.$$

Proof. The lemma is included in Proposition B.1 and Proposition B.2 of [7]. We give the short proof for the convenience of the reader.

(a) Let $e_1, \dots, e_k$ be the standard orthonormal basis of $\mathbb{C}^d$. Since $\|Y^*\| = \|Y\|$, we can assume that

$$\frac{\mathrm{d}}{\mathrm{d}s} Y(s) f_j = F(s)Y(s) f_j, \qquad Y(t_0) = \mathbf{1},$$

so

$$\frac{\mathrm{d}}{\mathrm{d}s} \big\|Y(s) f_j\big\|^2 = 2\big\langle F(s)\big(Y(s) f_j\big), Y(s) f_j\big\rangle \leqslant 2c(s)\big\|Y(s) f_j\big\|^2 \quad \text{for a.e. } s \geqslant t_0, \tag{78}$$

and the assertion follows from the Gronwall lemma.

(b) $Y_1(s)$ and $Y_2(s)$ are invertible for any $s \geqslant t_0$ and

$$\frac{\mathrm{d}}{\mathrm{d}s} Y_j^{-1}(s) = -F_j(s) Y_j^{-1}(s).$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}s}\big(Y_1^{-1}(s) Y_2(s)\big) = Y_1^{-1}(s)\big(F_2(s) - F_1(s)\big) Y_2(s) \quad \text{for a.e. } s \geqslant t_0,$$

one obtains the following equality (after integration and multiplication with $Y_1(t)$):

$$Y_2(t) = Y_1(t) + Y_1(t) \int_{t_0}^{t} Y_1^{-1}(s)\big(F_2(s) - F_1(s)\big) Y_2(s)\,\mathrm{d}s.$$

Thus,

$$\big\|Y_1(t) - Y_2(t)\big\| \leqslant \big\|Y_1(t)\big\| \int_{t_0}^{t} \big\|Y_1^{-1}(s)\big\| \big\|F_2(s) - F_1(s)\big\| \big\|Y_2(s)\big\|\,\mathrm{d}s. \tag{79}$$

The claim follows from observing that

$$\big\|Y_j(s)\big\| \leqslant \mathrm{e}^{\int_{t_0}^{t} \|F_j(r)\|\,\mathrm{d}r}, \qquad \big\|Y_j^{-1}(s)\big\| \leqslant \mathrm{e}^{\int_{t_0}^{t} \|F_j(r)\|\,\mathrm{d}r},$$

which follows from the Gronwall lemma. □

Of course, similar results hold if one replaces the time interval $[t_0, \infty)$ with a finite time interval of the form $[t_0, t_1]$.
For the second lemma, we use the notation of (6) and (8).

Lemma A.2. *Let $\alpha \in \Omega^1(\mathbb{R}^n, \mathscr{U}(d))$, let V be a potential with*

$$0 \leqslant V \in \mathcal{K}_{\mathrm{loc}}\big(\mathbb{R}^n, \mathrm{Mat}\big(\mathbb{C}^d\big)\big),$$

and let $x, y \in \mathbb{R}^n$, $t > 0$, $0 \leqslant s \leqslant t$. The following assertions hold:

(a) *One has $\mathscr{A}_t^{\alpha,0,*} = \mathscr{A}_t^{\alpha,0,-1}$ and*

$$\big\|\mathscr{A}_t^{\alpha,V}\big\| \leqslant 1 \quad \mathbb{P}^x\text{-a.s.} \tag{80}$$

(b) *It holds that*

$$\big\|\mathscr{A}_s^{\alpha,V,-1} \mathscr{A}_t^{\alpha,V}\big\| \leqslant 1 \quad \mathbb{P}^x\text{-a.s.}$$

(c) *One has $\mathscr{A}_s^{\alpha,0,(t),*} = \mathscr{A}_s^{\alpha,0,(t),-1}$ and*

$$\big\|\mathscr{A}_s^{\alpha,V,(t)}\big\| \leqslant 1 \quad \mathbb{P}^{x,y}\text{-a.s.}$$

Proof. Firstly, note that under these assumptions on (α, V), the existence of

$$\mathscr{A}^{\alpha,V} : [0,\infty) \times \Omega \to \mathrm{Mat}\big(\mathbb{C}^d\big)$$

as the solution of (8) with respect to $\mathbb{P}^x$, and of

$$\mathscr{A}^{\alpha,V,(t)} : [0,t] \times \Omega_t \to \mathrm{Mat}\big(\mathbb{C}^d\big)$$

as the solution of (48) with respect to $\mathbb{P}_t^{x,y}$ has been established in Section 1. We shall prove (a) and (b). The proof of (c) is similar to the proof of (a).

As we have already remarked in Section 1, $\mathscr{A}^{\alpha,0}$ is invertible and $\mathscr{A}^{\alpha,0,-1}$ is uniquely determined by

$$\mathrm{d}\mathscr{A}^{\alpha,0,-1} = -\big(\underline{\mathrm{d}}A^{\alpha,0}\big)\mathscr{A}^{\alpha,0,-1}, \quad \mathscr{A}_0^{\alpha,0,-1} = \mathbf{1}.$$

Noting that $A^{\alpha,0,*} = -A^{\alpha,0}$ and that $\mathscr{A}^{\alpha,0,*}$ is uniquely determined by

$$\mathrm{d}\mathscr{A}^{\alpha,0,*} = \big(\underline{\mathrm{d}}A^{\alpha,0,*}\big)\mathscr{A}^{\alpha,0,*}, \quad \mathscr{A}_0^{\alpha,0,*} = \mathbf{1},$$

it follows that $\mathscr{A}^{\alpha,0}$ is unitary.

As in the proof of Theorem 1.8, let

$$\tilde{\mathscr{A}}^{\alpha,V}:[0,\infty)\times\Omega\to\mathrm{Mat}(\mathbb{C}^d)$$

be the pathwise weak solution of

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathscr{A}}_t^{\alpha,V}=-\tilde{\mathscr{A}}_t^{\alpha,V}\mathscr{A}_t^{\alpha,0}V(X_t)\mathscr{A}_t^{\alpha,0,-1},\quad \tilde{\mathscr{A}}_0^{\alpha,V}=\mathbf{1}. \tag{81}$$

It follows from Lemma A.1(a) that

$$\big\|\tilde{\mathscr{A}}_t^{\alpha,V}\big\|\leqslant 1\quad \mathbb{P}^x\text{-a.s.}$$

Noting that the Stratonovic product rule implies

$$\mathscr{A}_t^{\alpha,V}=\tilde{\mathscr{A}}_t^{\alpha,V}\mathscr{A}_t^{\alpha,0}\quad \mathbb{P}^x\text{-a.s.,} \tag{82}$$

inequality (80) follows from the fact that $\mathscr{A}^{\alpha,0}$ is unitary.

(b) With the notation of the proof of part (a) one has

$$\big\|\mathscr{A}_s^{\alpha,V,-1}\mathscr{A}_t^{\alpha,V}\big\|=\big\|\mathscr{A}_s^{\alpha,0,-1}\tilde{\mathscr{A}}_s^{\alpha,V,-1}\tilde{\mathscr{A}}_t^{\alpha,V}\mathscr{A}_t^{\alpha,0}\big\|\leqslant\big\|\tilde{\mathscr{A}}_s^{\alpha,V,-1}\tilde{\mathscr{A}}_t^{\alpha,V}\big\|. \tag{83}$$

Noting that for fixed s, the process $\tilde{\mathscr{A}}_s^{\alpha,V,-1}\tilde{\mathscr{A}}_\bullet^{\alpha,V}$ is the unique solution of

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\tilde{\mathscr{A}}_s^{\alpha,V,-1}\tilde{\mathscr{A}}_t^{\alpha,V}\big)=-\big(\tilde{\mathscr{A}}_s^{\alpha,V,-1}\tilde{\mathscr{A}}_t^{\alpha,V}\big)\mathscr{A}_t^{\alpha,0}V(X_t)\mathscr{A}_t^{\alpha,0,-1},$$
$$\tilde{\mathscr{A}}_s^{\alpha,V,-1}\tilde{\mathscr{A}}_t^{\alpha,V}\big|_{t=s}=\mathbf{1},$$

the assertion follows from Lemma A.1. $\square$

Appendix B

For the sake of completeness, we recall the Riemann sum approximation of Stratonovic integrals: Let

$$B,C:[0,\infty)\times\Omega\to\mathbb{R}$$

be continuous semi-martingales with respect to some filtered probability space $(\Omega,\mathscr{F},\mathscr{F}_*,\mathbb{P})$ which satisfies the usual hypothesis. If $0\leqslant t_0\leqslant t_1<\infty$, then the Stratonovic integral over the time interval $[t_0,t_1]$ of B with respect to the integrator C can be approximated as follows:

$$\int_{t_0}^{t_1}B(s)\,\underline{\mathrm{d}}C(s)=\operatorname*{l.i.p.}_{n\to\infty}\sum_{j=1}^{n}\frac{1}{2}\left\{B\left(t_0+\frac{(j-1)(t_1-t_0)}{n}\right)+B\left(t_0+\frac{j(t_1-t_0)}{n}\right)\right\}$$
$$\times\left\{C\left(t_0+\frac{(j-1)(t_1-t_0)}{n}\right)-C\left(t_0+\frac{j(t_1-t_0)}{n}\right)\right\}. \tag{84}$$

Here, l.i.p. stands for the limit in probability with respect to $\mathbb{P}$. In particular, there is a subsequence which converges $\mathbb{P}$-a.s. With obvious adaptions, these considerations carry over to continuous semi-martingales

$$B,C:[0,T]\times\Omega\to\mathbb{R}$$

which are defined on a filtered probability space of the form

$$\big(\Omega,\mathscr{F},(\mathscr{F}_t)_{0\leqslant t\leqslant T},\mathbb{P}\big)$$

with some $0<T<\infty$. A proof of these well-known facts can be found in [11].

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