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Some estimates for Bochner–Riesz operators on the weighted Herz-type Hardy spaces

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ABSTRACT

In this paper, by using the atomic decomposition and molecular characterization of the homogeneous and non-homogeneous weighted Herz-type Hardy spaces $H\dot{K}_q^{\alpha,p}(w_1,w_2)$ $(HK_q^{\alpha,p}(w_1,w_2))$, we obtain some weighted boundedness properties of the Bochner–Riesz operator and the maximal Bochner–Riesz operator on these spaces for $\alpha = n(1/p - 1/q)$, $0 < p \leqslant 1$ and $1 < q < \infty$.

1. Introduction

The Bochner–Riesz operators of order $\delta > 0$ in $\mathbb{R}^n$ are defined initially for Schwartz functions in terms of Fourier transforms by

$$\big(T_R^\delta f\big)\hat{}\,(\xi) = \left(1 - \frac{|\xi|^2}{R^2}\right)_+^\delta \hat{f}(\xi),$$

where $\hat{f}$ denotes the Fourier transform of f. The associated maximal Bochner–Riesz operator is defined by

$$T_*^\delta f(x) = \sup_{R>0} \big|T_R^\delta f(x)\big|.$$

These operators were first introduced by Bochner [3] in connection with summation of multiple Fourier series and played an important role in harmonic analysis. As for their H^p boundedness, Sjölin [21] and Stein, Taibleson and Weiss [22] proved the following theorem (see also [10, page 121]).

Theorem I. *Suppose that $0 < p \leqslant 1$ and $\delta > n/p - (n+1)/2$. Then there exists a constant C independent of f and R such that*

$$\big\|T_R^\delta(f)\big\|_{H^p} \leqslant C\|f\|_{H^p}.$$

In [22], the authors also showed

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Theorem II. *Suppose that* $0 < p < 1$ *and* $\delta = n/p - (n+1)/2$. *Then there exists a constant* C *independent of* f *such that*

$$\sup_{\lambda>0} \lambda^p \left|\left\{x \in \mathbb{R}^n\colon\ T_*^\delta f(x) > \lambda\right\}\right| \leqslant C\|f\|_{H^p}^p.$$

In 1995, Sato [20] considered the weighted case and obtained the following weighted weak type estimate for the maximal operator T_*^δ.

Theorem III. *Let* $w \in A_1$ (*Muckenhoupt weight class*), $0 < p < 1$ *and* $\delta = n/p - (n+1)/2$. *Then there exists a constant* C *independent of* f *such that*

$$\sup_{\lambda>0} \lambda^p \cdot w\left(\left\{x \in \mathbb{R}^n\colon\ T_*^\delta f(x) > \lambda\right\}\right) \leqslant C\|f\|_{H_w^p}^p.$$

In 2006, Lee [9] considered values of δ greater than the critical index $n/p - (n+1)/2$ and proved the following weighted strong type estimate.

Theorem IV. *Let* $w \in A_1$, $0 < p \leqslant 1$ *and* $\delta > n/p - (n+1)/2$. *Then there exists a constant* C *independent of* f *such that*

$$\left\|T_*^\delta(f)\right\|_{L_w^p} \leqslant C\|f\|_{H_w^p}.$$

Using the above H_w^p–L_w^p boundedness of the maximal operator T_*^δ, Lee [9] also obtained the H_w^p boundedness of the Bochner–Riesz operator.

Theorem V. *Let* $w \in A_1$ *with critical index* r_w *for the reverse Hölder condition*, $0 < p \leqslant 1$, $\delta > \max\{n/p - (n+1)/2, [n/p]r_w/(r_w - 1) - (n+1)/2\}$. *Then there exists a constant* C *independent of* f *and* R *such that*

$$\left\|T_R^\delta(f)\right\|_{H_w^p} \leqslant C\|f\|_{H_w^p}.$$

The main purpose of this paper is to discuss some corresponding estimates of Bochner–Riesz operators on the homogeneous and non-homogeneous weighted Herz-type Hardy spaces $H\dot{K}_q^{\alpha,p}(w_1, w_2)$ ($HK_q^{\alpha,p}(w_1, w_2)$). Our main results are stated as follows.

Theorem 1. *Let* $w_1, w_2 \in A_1$ *and* $1 < q < \infty$. *If* $0 < p \leqslant 1$, $\alpha = n(1/p - 1/q)$, $\delta > n/p - (n+1)/2$, *then there exists a constant* C *independent of* f *such that*

$$\left\|T_*^\delta(f)\right\|_{\dot{K}_q^{\alpha,p}(w_1,w_2)} \leqslant C\|f\|_{H\dot{K}_q^{\alpha,p}(w_1,w_2)},$$
$$\left\|T_*^\delta(f)\right\|_{K_q^{\alpha,p}(w_1,w_2)} \leqslant C\|f\|_{HK_q^{\alpha,p}(w_1,w_2)},$$

where $\dot{K}_q^{\alpha,p}(w_1, w_2)$ ($K_q^{\alpha,p}(w_1, w_2)$) *denotes the homogeneous* (*non-homogeneous*) *weighted Herz space.*

Theorem 2. *Let* $w_1, w_2 \in A_1$ *and* $1 < q < \infty$. *If* $0 < p < 1$, $\alpha = n(1/p - 1/q)$, $\delta = n/p - (n+1)/2$, *then there exists a constant* C *independent of* f *such that*

$$\left\|T_*^\delta(f)\right\|_{W\dot{K}_q^{\alpha,p}(w_1,w_2)} \leqslant C\|f\|_{H\dot{K}_q^{\alpha,p}(w_1,w_2)},$$
$$\left\|T_*^\delta(f)\right\|_{WK_q^{\alpha,p}(w_1,w_2)} \leqslant C\|f\|_{HK_q^{\alpha,p}(w_1,w_2)},$$

where $W\dot{K}_q^{\alpha,p}(w_1, w_2)$ ($WK_q^{\alpha,p}(w_1, w_2)$) *denotes the homogeneous* (*non-homogeneous*) *weak weighted Herz space.*

Theorem 3. *Let* $w \in A_1$ *and* $1 < q < \infty$. *If* $0 < p \leqslant 1$, $\alpha = n(1/p - 1/q)$, $\delta > \max\{n/p - (n+1)/2, [n/p]r_w/(r_w - 1) - (n+1)/2\}$, *then there exists a constant* C *independent of* f *and* R *such that*

$$\left\|T_R^\delta(f)\right\|_{H\dot{K}_q^{\alpha,p}(w,w)} \leqslant C\|f\|_{H\dot{K}_q^{\alpha,p}(w,w)},$$
$$\left\|T_R^\delta(f)\right\|_{HK_q^{\alpha,p}(w,w)} \leqslant C\|f\|_{HK_q^{\alpha,p}(w,w)},$$

where r_w *denotes the critical index of* w *for the reverse Hölder condition.*

2. Notations and definitions

First, let us recall some standard definitions and notations. The classical A_p weight theory was first introduced by Muckenhoupt in the study of weighted L^p boundedness of Hardy–Littlewood maximal functions in [19]. A weight w is a locally integrable function on $\mathbb{R}^n$ which takes values in $(0,\infty)$ almost everywhere. $B = B(x_0, r)$ denotes the ball with the center x_0 and radius r. We say that $w \in A_p$, $1 < p < \infty$, if

$$\left(\frac{1}{|B|}\int_B w(x)\,dx\right)\left(\frac{1}{|B|}\int_B w(x)^{-\frac{1}{p-1}}\,dx\right)^{p-1} \leqslant C \quad \text{for every ball } B \subseteq \mathbb{R}^n,$$

where C is a positive constant which is independent of B.

For the case $p = 1$, $w \in A_1$, if

$$\frac{1}{|B|}\int_B w(x)\,dx \leqslant C \cdot \operatorname*{ess\,inf}_{x\in B} w(x) \quad \text{for every ball } B \subseteq \mathbb{R}^n.$$

A weight function w is said to belong to the reverse Hölder class RH_r if there exist two constants $r > 1$ and $C > 0$ such that the following reverse Hölder inequality holds

$$\left(\frac{1}{|B|}\int_B w(x)^r\,dx\right)^{1/r} \leqslant C\left(\frac{1}{|B|}\int_B w(x)\,dx\right) \quad \text{for every ball } B \subseteq \mathbb{R}^n.$$

It is well known that if $w \in A_p$ with $1 < p < \infty$, then $w \in A_r$ for all $r > p$, and $w \in A_q$ for some $1 < q < p$. If $w \in A_p$ with $1 \leqslant p < \infty$, then there exists $r > 1$ such that $w \in RH_r$. It follows from Hölder's inequality that $w \in RH_r$ implies $w \in RH_s$ for all $1 < s < r$. Moreover, if $w \in RH_r$, $r > 1$, then we have $w \in RH_{r+\varepsilon}$ for some $\varepsilon > 0$. We thus write $r_w \equiv \sup\{r > 1\colon\ w \in RH_r\}$ to denote the critical index of w for the reverse Hölder condition.

Given a ball B and $\lambda > 0$, λB denotes the ball with the same center as B whose radius is λ times that of B. For a given weight function w, we denote the Lebesgue measure of B by $|B|$ and the weighted measure of B by $w(B)$, where $w(B) = \int_B w(x)\,dx$.

We give the following results that we will use in the sequel.

Lemma A. *(See [5].) Let $w \in A_p$, $p \geqslant 1$. Then, for any ball B, there exists an absolute constant C such that*

$$w(2B) \leqslant C w(B).$$

In general, for any $\lambda > 1$, we have

$$w(\lambda B) \leqslant C \cdot \lambda^{np} w(B),$$

where C does not depend on B nor on λ.

Lemma B. *(See [5,6].) Let $w \in A_p \cap RH_r$, $p \geqslant 1$ and $r > 1$. Then there exist constants $C_1, C_2 > 0$ such that*

$$C_1\left(\frac{|E|}{|B|}\right)^p \leqslant \frac{w(E)}{w(B)} \leqslant C_2\left(\frac{|E|}{|B|}\right)^{(r-1)/r}$$

for any measurable subset E of a ball B.

Lemma C. *(See [5].) Let $w \in A_q$ and $q > 1$. Then, for all $r > 0$, there exists a constant C independent of r such that*

$$\int_{|x|>r} \frac{w(x)}{|x|^{nq}}\,dx \leqslant C \cdot r^{-nq} w\big(B(0,r)\big).$$

Next we shall give the definitions of the weighted Herz space, weak weighted Herz space and weighted Herz-type Hardy space. In 1964, Beurling [2] first introduced some fundamental form of Herz spaces to study convolution algebras. Later Herz [7] gave versions of the spaces defined below in a slightly different setting. Since then, the theory of Herz spaces has been significantly developed, and these spaces have turned out to be quite useful in harmonic analysis. For instance, they were used by Baernstein and Sawyer [1] to characterize the multipliers on the classical Hardy spaces, and used by Lu and Yang [16] in the study of partial differential equations.

On the other hand, a theory of Hardy spaces associated with Herz spaces has been developed in [4,14]. These new Hardy spaces can be regarded as the local version at the origin of the classical Hardy spaces $H^p(\mathbb{R}^n)$ and are good substitutes for

$H^p(\mathbb{R}^n)$ when we study the boundedness of non-translation invariant operators (see [15]). For the weighted case, in 1995, Lu and Yang introduced the following weighted Herz-type Hardy spaces $H\dot{K}_q^{\alpha,p}(w_1,w_2)$ $(HK_q^{\alpha,p}(w_1,w_2))$ and established their atomic decompositions. In 2006, Lee gave the molecular characterizations of these spaces, he also obtained the boundedness of the Hilbert transform and the Riesz transforms on $H\dot{K}_q^{n(1/p-1/q),p}(w,w)$ and $HK_q^{n(1/p-1/q),p}(w,w)$ for $0<p\leqslant 1$. For the results mentioned above, we refer the readers to the book [18] and the papers [8,12,13,17] for further details.

Let $B_k=\{x\in\mathbb{R}^n\colon\ |x|\leqslant 2^k\}$ and $C_k=B_k\backslash B_{k-1}$ for $k\in\mathbb{Z}$. Denote $\chi_k=\chi_{C_k}$ for $k\in\mathbb{Z}$, $\widetilde{\chi}_k=\chi_k$ if $k\in\mathbb{N}$ and $\widetilde{\chi}_0=\chi_{B_0}$, where χ_{C_k} is the characteristic function of C_k. Given a weight function w on $\mathbb{R}^n$, for $1\leqslant p<\infty$, we denote by $L_w^p(\mathbb{R}^n)$ the space of all functions satisfying

$$\|f\|_{L_w^p(\mathbb{R}^n)}=\left(\int_{\mathbb{R}^n}\left|f(x)\right|^p w(x)\,dx\right)^{1/p}<\infty.$$

Definition 1. Let $\alpha\in\mathbb{R}$, $0<p,q<\infty$ and w_1, w_2 be two weight functions on $\mathbb{R}^n$.

(i) The homogeneous weighted Herz space $\dot{K}_q^{\alpha,p}(w_1,w_2)$ is defined by

$$\dot{K}_q^{\alpha,p}(w_1,w_2)=\left\{f\in L_{loc}^q\left(\mathbb{R}^n\backslash\{0\},w_2\right)\colon\ \|f\|_{\dot{K}_q^{\alpha,p}(w_1,w_2)}<\infty\right\},$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}(w_1,w_2)}=\left(\sum_{k\in\mathbb{Z}}\left(w_1(B_k)\right)^{\alpha p/n}\|f\chi_k\|_{L_{w_2}^q}^p\right)^{1/p}.$$

(ii) The non-homogeneous weighted Herz space $K_q^{\alpha,p}(w_1,w_2)$ is defined by

$$K_q^{\alpha,p}(w_1,w_2)=\left\{f\in L_{loc}^q\left(\mathbb{R}^n,w_2\right)\colon\ \|f\|_{K_q^{\alpha,p}(w_1,w_2)}<\infty\right\},$$

where

$$\|f\|_{K_q^{\alpha,p}(w_1,w_2)}=\left(\sum_{k=0}^{\infty}\left(w_1(B_k)\right)^{\alpha p/n}\|f\widetilde{\chi}_k\|_{L_{w_2}^q}^p\right)^{1/p}.$$

For $k\in\mathbb{Z}$ and $\lambda>0$, we set $E_k(\lambda,f)=|\{x\in C_k\colon\ |f(x)|>\lambda\}|$. Let $\widetilde{E}_k(\lambda,f)=E_k(\lambda,f)$ for $k\in\mathbb{N}$ and $\widetilde{E}_0(\lambda,f)=|\{x\in B(0,1)\colon\ |f(x)|>\lambda\}|$.

Definition 2. Let $\alpha\in\mathbb{R}$, $0<p,q<\infty$ and w_1, w_2 be two weight functions on $\mathbb{R}^n$.

(i) A measurable function $f(x)$ on $\mathbb{R}^n$ is said to belong to the homogeneous weak weighted Herz spaces $W\dot{K}_q^{\alpha,p}(w_1,w_2)$ if

$$\|f\|_{W\dot{K}_q^{\alpha,p}(w_1,w_2)}=\sup_{\lambda>0}\lambda\left(\sum_{k\in\mathbb{Z}}w_1(B_k)^{\alpha p/n}w_2\left(E_k(\lambda,f)\right)^{p/q}\right)^{1/p}<\infty.$$

(ii) A measurable function $f(x)$ on $\mathbb{R}^n$ is said to belong to the non-homogeneous weak weighted Herz spaces $WK_q^{\alpha,p}(w_1,w_2)$ if

$$\|f\|_{WK_q^{\alpha,p}(w_1,w_2)}=\sup_{\lambda>0}\lambda\left(\sum_{k=0}^{\infty}w_1(B_k)^{\alpha p/n}w_2\left(\widetilde{E}_k(\lambda,f)\right)^{p/q}\right)^{1/p}<\infty.$$

Let $\mathscr{S}(\mathbb{R}^n)$ be the class of Schwartz functions and let $\mathscr{S}'(\mathbb{R}^n)$ be its dual space. For $f\in\mathscr{S}'(\mathbb{R}^n)$, the grand maximal function of f is defined by

$$G(f)(x)=\sup_{\varphi\in\mathscr{A}_N}\sup_{|y-x|<t}\left|\varphi_t*f(y)\right|,$$

where $N>n+1$, $\mathscr{A}_N=\{\varphi\in\mathscr{S}(\mathbb{R}^n)\colon\ \sup_{|\alpha|,|\beta|\leqslant N}|x^\alpha D^\beta\varphi(x)|\leqslant 1\}$ and $\varphi_t(x)=t^{-n}\varphi(x/t)$.

Definition 3. Let $0<\alpha<\infty$, $0<p<\infty$, $1<q<\infty$ and w_1, w_2 be two weight functions on $\mathbb{R}^n$.

(i) The homogeneous weighted Herz-type Hardy space $H\dot{K}_q^{\alpha,p}(w_1,w_2)$ associated with the space $\dot{K}_q^{\alpha,p}(w_1,w_2)$ is defined by

$$H\dot{K}_q^{\alpha,p}(w_1,w_2)=\big\{f\in\mathscr{S}'\big(\mathbb{R}^n\big):\ G(f)\in\dot{K}_q^{\alpha,p}(w_1,w_2)\big\}$$

and we define $\|f\|_{H\dot{K}_q^{\alpha,p}(w_1,w_2)}=\|G(f)\|_{\dot{K}_q^{\alpha,p}(w_1,w_2)}$.

(ii) The non-homogeneous weighted Herz-type Hardy space $HK_q^{\alpha,p}(w_1,w_2)$ associated with the space $K_q^{\alpha,p}(w_1,w_2)$ is defined by

$$HK_q^{\alpha,p}(w_1,w_2)=\big\{f\in\mathscr{S}'\big(\mathbb{R}^n\big):\ G(f)\in K_q^{\alpha,p}(w_1,w_2)\big\}$$

and we define $\|f\|_{HK_q^{\alpha,p}(w_1,w_2)}=\|G(f)\|_{K_q^{\alpha,p}(w_1,w_2)}$.

3. The atomic decomposition and molecular characterization

In this article, we will use atomic and molecular decomposition theory for weighted Herz-type Hardy spaces in [8,12,13]. We first characterize weighted Herz-type Hardy spaces in terms of atoms in the following way.

Definition 4. Let $1<q<\infty$, $n(1-1/q)\leqslant\alpha<\infty$ and $s\geqslant[\alpha+n(1/q-1)]$.

(i) A function $a(x)$ on $\mathbb{R}^n$ is called a central (α,q,s)-atom with respect to (w_1,w_2) (or a central $(\alpha,q,s;w_1,w_2)$-atom), if it satisfies:
  (a) $\operatorname{supp} a\subseteq B(0,R)=\{x\in\mathbb{R}^n:\ |x|<R\}$,
  (b) $\|a\|_{L^q_{w_2}}\leqslant w_1(B(0,R))^{-\alpha/n}$,
  (c) $\int_{\mathbb{R}^n}a(x)x^\beta\,dx=0$ for every multi-index β with $|\beta|\leqslant s$.

(ii) A function $a(x)$ on $\mathbb{R}^n$ is called a central (α,q,s)-atom of restricted type with respect to (w_1,w_2) (or a central $(\alpha,q,s;w_1,w_2)$-atom of restricted type), if it satisfies the conditions (b), (c) above and
  (a′) $\operatorname{supp} a\subseteq B(0,R)$ for some $R>1$.

It is worth pointing out that the difference between (a) and (a′) lies in the definitions of homogeneous and non-homogeneous weighted Herz space.

Theorem D. *Let $w_1,w_2\in A_1$, $0<p<\infty$, $1<q<\infty$ and $n(1-1/q)\leqslant\alpha<\infty$. Then we have:*

(i) *$f\in H\dot{K}_q^{\alpha,p}(w_1,w_2)$ if and only if*

$$f(x)=\sum_{k\in\mathbb{Z}}\lambda_k a_k(x),\quad \textit{in the sense of } \mathscr{S}'\big(\mathbb{R}^n\big),$$

where $\sum_{k\in\mathbb{Z}}|\lambda_k|^p<\infty$, each a_k is a central $(\alpha,q,s;w_1,w_2)$-atom. Moreover,

$$\|f\|_{H\dot{K}_q^{\alpha,p}(w_1,w_2)}\approx\inf\Bigg(\sum_{k\in\mathbb{Z}}|\lambda_k|^p\Bigg)^{1/p},$$

where the infimum is taken over all the above decompositions of f.

(ii) *$f\in HK_q^{\alpha,p}(w_1,w_2)$ if and only if*

$$f(x)=\sum_{k=0}^{\infty}\lambda_k a_k(x),\quad \textit{in the sense of } \mathscr{S}'\big(\mathbb{R}^n\big),$$

where $\sum_{k=0}^{\infty}|\lambda_k|^p<\infty$, each a_k is a central $(\alpha,q,s;w_1,w_2)$-atom of restricted type. Moreover,

$$\|f\|_{HK_q^{\alpha,p}(w_1,w_2)}\approx\inf\Bigg(\sum_{k=0}^{\infty}|\lambda_k|^p\Bigg)^{1/p},$$

where the infimum is taken over all the above decompositions of f.

Next we give the definition of central molecule and the molecular characterization of $H\dot{K}_q^{n(1/p-1/q),p}(w,w)$ ($HK_q^{n(1/p-1/q),p}(w,w)$).

Definition 5. For $0 < p \leqslant 1 < q < \infty$, let $w \in A_1$ with critical index r_w for the reverse Hölder condition. Set $s \geqslant [n(1/p-1)]$, $\varepsilon > \max\{sr_w/n(r_w-1) + 1/(r_w-1), 1/p-1\}$, $a = 1-1/p+\varepsilon$ and $b = 1-1/q+\varepsilon$.

(i) A central (p,q,s,ε)-molecule with respect to w (or a central w-(p,q,s,ε)-molecule) is a function $M \in L^q_w(\mathbb{R}^n)$ satisfying:
  (a) $M(x) \cdot w(B(0,|x|))^b \in L^q_w(\mathbb{R}^n)$,
  (b) $\|M\|^{a/b}_{L^q_w} \cdot \|M(\cdot)w(B(0,|\cdot|))^b\|^{1-a/b}_{L^q_w} \equiv \mathcal{N}_w(M) < \infty$,
  (c) $\int_{\mathbb{R}^n} M(x)x^\gamma\,dx = 0$ for every multi-index γ with $|\gamma| \leqslant s$.
(ii) A function $M \in L^q_w(\mathbb{R}^n)$ is called a central (p,q,s,ε)-molecule of restricted type with respect to w (or a central w-(p,q,s,ε)-molecule of restricted type) if it satisfies (a)–(c) and
  (d) $\|M\|_{L^q_w} \leqslant w(B(0,1))^{1/q-1/p}$.

The above $\mathcal{N}_w(M)$ is called the molecular norm of M with respect to w (or w-molecular norm of M).

Theorem E. *Let (p,q,s,ε) be the quadruple in the definition of central w-molecule, let $w \in A_1$ and $\alpha = n(1/p-1/q)$.*

(i) *Every central (p,q,s,ε)-molecule M centered at the origin with respect to w belongs to $H\dot{K}^{\alpha,p}_q(w,w)$ and $\|M\|_{H\dot{K}^{\alpha,p}_q(w,w)} \leqslant C\mathcal{N}_w(M)$, where the constant C is independent of M.*
(ii) *Every central (p,q,s,ε)-molecule of restricted type M with respect to w belongs to $HK^{\alpha,p}_q(w,w)$ and $\|M\|_{HK^{\alpha,p}_q(w,w)} \leqslant C\mathcal{N}_w(M)$, where the constant C is independent of M.*

Throughout this article, we will use C to denote a positive constant, which is independent of the main parameters and not necessarily the same at each occurrence. By $A \sim B$, we mean that there exists a constant $C > 1$ such that $\frac{1}{C} \leqslant \frac{A}{B} \leqslant C$.

4. Proof of Theorem 1

The Bochner–Riesz operators can be expressed as convolution operators

$$T^\delta_R f(x) = (f * \phi_{1/R})(x),$$

where $\phi(x) = [(1-|\cdot|^2)^\delta_+]\hat{}\,(x)$. It is well known that the kernel ϕ can be represented as (see [11,23])

$$\phi(x) = \pi^{-\delta}\Gamma(\delta+1)|x|^{-(\frac{n}{2}+\delta)} J_{\frac{n}{2}+\delta}\big(2\pi|x|\big),$$

where $J_\mu(t)$ is the Bessel function

$$J_\mu(t) = \frac{(\frac{t}{2})^\mu}{\Gamma(\mu+\frac{1}{2})\Gamma(\frac{1}{2})} \int_{-1}^{1} e^{its}\big(1-s^2\big)^{\mu-\frac{1}{2}}\,ds.$$

We shall need the following estimate which can be found in [20].

Lemma 4.1. *Let $0 < p < 1$ and $\delta = n/p-(n+1)/2$. Then the kernel ϕ satisfies the inequality*

$$\sup_{x\in\mathbb{R}^n} \big(1+|x|\big)^{n/p}\big|D^\alpha\phi(x)\big| \leqslant C \quad \textit{for all multi-indices } \alpha.$$

Proof of Theorem 1. First we observe that $\delta > n/p-(n+1)/2$, then we are able to choose a number $0 < p_1 < p$ such that $\delta = n/p_1-(n+1)/2$. Set $s = [n(1/p_1-1)]$. For any central $(\alpha,q,s;w_1,w_2)$-atom a with $\operatorname{supp} a \subseteq B(0,r)$, we are going to prove that $\|T^\delta_*(a)\|_{\dot{K}^{\alpha,p}_q(w_1,w_2)} \leqslant C$, where $C > 0$ is independent of the choice of a. For given $r > 0$, we can find an appropriate number $k_0 \in \mathbb{Z}$ satisfying $2^{k_0-2} < r \leqslant 2^{k_0-1}$. Write

$$\begin{aligned}\big\|T^\delta_*(a)\big\|^p_{\dot{K}^{\alpha,p}_q(w_1,w_2)} &= \sum_{k\in\mathbb{Z}} w_1(B_k)^{\alpha p/n}\big\|T^\delta_*(a)\chi_k\big\|^p_{L^q_{w_2}}\\ &= \sum_{k=-\infty}^{k_0} w_1(B_k)^{\alpha p/n}\big\|T^\delta_*(a)\chi_k\big\|^p_{L^q_{w_2}} + \sum_{k=k_0+1}^{\infty} w_1(B_k)^{\alpha p/n}\big\|T^\delta_*(a)\chi_k\big\|^p_{L^q_{w_2}}\\ &= I_1 + I_2.\end{aligned}$$

Note that $0 < p \leqslant 1$, $\delta > n/p - (n+1)/2$, then $\delta > (n-1)/2$. In this case, it is well known that (see [11,23])

$$T_*^\delta(a)(x) \leqslant C \cdot M(a)(x), \tag{1}$$

where M denotes the Hardy–Littlewood maximal operator. The size condition of central atom a and the above inequality (1) imply

$$I_1 \leqslant C \sum_{k=-\infty}^{k_0} w_1(B_k)^{\alpha p/n} \|a\|_{L^q_{w_2}}^p \leqslant C \sum_{k=-\infty}^{k_0} w_1(B_k)^{\alpha p/n} w_1\big(B(0,r)\big)^{-\alpha p/n}.$$

Since $w_1 \in A_1$, then we know $w \in RH_\mu$ for some $\mu > 1$. When $k \leqslant k_0$, then $B_k \subseteq B_{k_0}$. By Lemma B, we have

$$w_1(B_k) \leqslant C \cdot w_1(B_{k_0})|B_k|^\theta |B_{k_0}|^{-\theta},$$

where $\theta = (\mu - 1)/\mu > 0$. Hence

$$\begin{aligned} I_1 &\leqslant C \sum_{k=-\infty}^{k_0} \big(2^{(k-k_0)\alpha\theta p} w_1(B_{k_0})^{\alpha p/n} w_1\big(B(0,r)\big)^{-\alpha p/n}\big) \\ &\leqslant C \sum_{k=-\infty}^{k_0} 2^{(k-k_0)\alpha\theta p} \\ &= C \sum_{k=-\infty}^{0} 2^{k\alpha\theta p} \\ &\leqslant C. \end{aligned} \tag{2}$$

We now turn to estimate I_2. For any given central $(\alpha, q, s; w_1, w_2)$-atom a with support contained in $B(0,r)$, it is easy to verify that

$$a_1(x) = w\big(B(0,r)\big)^{1/p - 1/p_1} \cdot a(x)$$

is a central $(\alpha_1, q, s; w_1, w_2)$-atom which is supported in $B(0,r)$, where $\alpha_1 = n(1/p_1 - 1/q)$. We now claim that for any $x \in C_k = B_k \backslash B_{k-1}$, the following inequality holds

$$T_*^\delta(a_1)(x) \leqslant C \cdot \frac{r^{n/p_1}}{|x|^{n/p_1}} w_1\big(B(0,r)\big)^{-\alpha_1/n} w_2\big(B(0,r)\big)^{-1/q}. \tag{3}$$

In fact, for any $\varepsilon > 0$, we write

$$a_1 * \phi_\varepsilon(x) = \varepsilon^{-n} \int_{B(0,r)} \phi\left(\frac{x-y}{\varepsilon}\right) a_1(y)\, dy.$$

Let us consider the following two cases.

(i) $0 < \varepsilon \leqslant r$. Note that $\delta = n/p_1 - (n+1)/2$, then by Lemma 4.1, we have

$$\big|a_1 * \phi_\varepsilon(x)\big| \leqslant C \cdot \varepsilon^{n/p_1 - n} \int_{B(0,r)} \frac{|a_1(y)|}{|x-y|^{n/p_1}}\, dy.$$

Observe that when $x \in C_k = B_k \backslash B_{k-1}$, $k > k_0$, then we can easily get $|x| \geqslant 2|y|$, which implies $|x - y| \sim |x|$. We also note that $0 < p_1 < 1$, then $n/p_1 - n > 0$. Thus

$$\big|a_1 * \phi_\varepsilon(x)\big| \leqslant C \cdot r^{n/p_1 - n} \frac{1}{|x|^{n/p_1}} \int_{B(0,r)} \big|a_1(y)\big|\, dy. \tag{4}$$

Denote the conjugate exponent of $q > 1$ by $q' = q/(q-1)$. Using Hölder's inequality, A_q condition and the size condition of a_1, we can get

$$\begin{aligned} \int_{B(0,r)} \big|a_1(y)\big|\, dy &\leqslant \left(\int_{B(0,r)} \big|a_1(y)\big|^q w_2(y)\, dy\right)^{1/q} \left(\int_{B(0,r)} \big(w_2^{-1/q}\big)^{q'} dy\right)^{1/q'} \\ &\leqslant C \|a_1\|_{L^q_{w_2}} \big|B(0,r)\big| w_2\big(B(0,r)\big)^{-1/q} \\ &\leqslant C \big|B(0,r)\big| w_1\big(B(0,r)\big)^{-\alpha_1/n} w_2\big(B(0,r)\big)^{-1/q}. \end{aligned} \tag{5}$$

Substituting the above inequality (5) into (4), we thus obtain

$$\left|a_1 * \phi_\varepsilon(x)\right| \leqslant C \cdot \frac{r^{n/p_1}}{|x|^{n/p_1}} w_1\big(B(0,r)\big)^{-\alpha_1/n} w_2\big(B(0,r)\big)^{-1/q}. \tag{6}$$

(ii) $\varepsilon > r$. Since $0 < p_1 < 1$, then we can find a non-negative integer N such that $\frac{n}{n+N+1} \leqslant p_1 < \frac{n}{n+N}$. It is easy to see that this choice of N implies $[n(1/p_1 - 1)] \geqslant N$. Using the vanishing moment condition of a, Taylor's theorem and Lemma 4.1, we can get

$$\begin{aligned}
\left|a_1 * \phi_\varepsilon(x)\right| &= \varepsilon^{-n}\left|\int_{B(0,r)} \left[\phi\left(\frac{x-y}{\varepsilon}\right) - \sum_{|\gamma|\leqslant N} \frac{D^\gamma \phi(\frac{x}{\varepsilon})}{\gamma!}\left(\frac{y}{\varepsilon}\right)^\gamma\right] a_1(y)\,dy\right| \\
&\leqslant \varepsilon^{-n} \cdot \left(\frac{r}{\varepsilon}\right)^{N+1} \int_{B(0,r)} \sum_{|\gamma|=N+1} \left|\frac{D^\gamma \phi(\frac{x-\theta y}{\varepsilon})}{\gamma!}\right| \left|a_1(y)\right| dy \\
&\leqslant C \cdot \frac{r^{N+1}}{\varepsilon^{n+N+1}} \int_{B(0,r)} \left|\frac{x-\theta y}{\varepsilon}\right|^{-n/p_1} \left|a_1(y)\right| dy,
\end{aligned}$$

where $0 < \theta < 1$. As in the first case (i), we have $|x| \geqslant 2|y|$, which implies $|x - \theta y| \geqslant \frac{1}{2}|x|$. This together with the inequality (5) yield

$$\left|a_1 * \phi_\varepsilon(x)\right| \leqslant C \cdot \frac{r^{n+N+1}}{\varepsilon^{n+N+1-n/p_1}} \frac{1}{|x|^{n/p_1}} w_1\big(B(0,r)\big)^{-\alpha_1/n} w_2\big(B(0,r)\big)^{-1/q}.$$

Observe that $n + N + 1 - n/p_1 \geqslant 0$, then for $\varepsilon > r$, we have $\varepsilon^{n+N+1-n/p_1} \geqslant r^{n+N+1-n/p_1}$. Consequently

$$\left|a_1 * \phi_\varepsilon(x)\right| \leqslant C \cdot \frac{r^{n/p_1}}{|x|^{n/p_1}} w_1\big(B(0,r)\big)^{-\alpha_1/n} w_2\big(B(0,r)\big)^{-1/q}. \tag{7}$$

Summarizing the estimates (6) and (7) derived above and taking the supremum over all $\varepsilon > 0$, we obtain the desired estimate (3). Note that $\alpha = n(1/p - 1/q)$ and $\alpha_1 = n(1/p_1 - 1/q)$. It follows from the inequality (3) that

$$\begin{aligned}
I_2 &\leqslant \sum_{k=k_0+1}^{\infty} w_1(B_k)^{\alpha p/n} w_1\big(B(0,r)\big)^{p(1/p_1-1/p)} \left(\int_{2^{k-1}<|x|\leqslant 2^k} \left|T_*^\delta(a_1)(x)\right|^q w_2(x)\,dx\right)^{p/q} \\
&\leqslant C \sum_{k=k_0+1}^{\infty} r^{np/p_1} w_1(B_k)^{1-p/q} w_1\big(B(0,r)\big)^{-(1-p/q)} w_2\big(B(0,r)\big)^{-p/q} \left(\int_{2^{k-1}<|x|\leqslant 2^k} \frac{w_2(x)}{|x|^{nq/p_1}}\,dx\right)^{p/q} \\
&\leqslant C \sum_{k=k_0+1}^{\infty} \left(\frac{r^{np/p_1}}{2^{knp/p_1}}\right)\left(\frac{w_1(B_k)}{w_1(B(0,r))}\right)^{1-p/q} \left(\frac{w_2(B_k)}{w_2(B(0,r))}\right)^{p/q}.
\end{aligned}$$

When $k > k_0$, then $B_k \supseteq B_{k_0}$. Using Lemma B again, we can get

$$w_i(B_k) \leqslant C \cdot w_i(B_{k_0})|B_k||B_{k_0}|^{-1} \quad \text{for } i = 1 \text{ or } 2.$$

Therefore

$$\begin{aligned}
I_2 &\leqslant C \sum_{k=k_0+1}^{\infty} \left(\frac{2^{k_0 np/p_1}}{2^{knp/p_1}}\right)\left(\frac{2^{kn}}{2^{k_0 n}}\right)^{1-p/q} \left(\frac{2^{kn}}{2^{k_0 n}}\right)^{p/q} \\
&= C \sum_{k=k_0+1}^{\infty} \frac{1}{2^{(k-k_0)(np/p_1-n)}} \\
&= C \sum_{k=1}^{\infty} \frac{1}{2^{k(np/p_1-n)}} \\
&\leqslant C,
\end{aligned} \tag{8}$$

where in the last inequality we have used the fact that $np/p_1 - n > 0$. Combining the above estimate (8) with (2), we get the desired result.

We are now in a position to complete the proof of Theorem 1. For every $f \in H\dot{K}_q^{\alpha,p}(w_1,w_2)$, then by Theorem D, we have the decomposition $f=\sum_{j\in\mathbb{Z}}\lambda_j a_j$, where $\sum_{j\in\mathbb{Z}}|\lambda_j|^p<\infty$ and each a_j is a central $(\alpha,q,s;w_1,w_2)$-atom. Therefore

$$\begin{aligned}\big\|T_*^\delta(f)\big\|^p_{\dot{K}_q^{\alpha,p}(w_1,w_2)} &\leqslant C\sum_{k\in\mathbb{Z}} w_1(B_k)^{\alpha p/n}\left(\sum_{j\in\mathbb{Z}}|\lambda_j|\big\|T_*^\delta(a_j)\chi_k\big\|_{L^q_{w_2}}\right)^p\\ &\leqslant C\sum_{k\in\mathbb{Z}} w_1(B_k)^{\alpha p/n}\left(\sum_{j\in\mathbb{Z}}|\lambda_j|^p\big\|T_*^\delta(a_j)\chi_k\big\|^p_{L^q_{w_2}}\right)\\ &\leqslant C\sum_{j\in\mathbb{Z}}|\lambda_j|^p\\ &\leqslant C\|f\|^p_{H\dot{K}_q^{\alpha,p}(w_1,w_2)}. \qquad\Box\end{aligned}$$

5. Proof of Theorem 2

Proof of Theorem 2. For every $f \in H\dot{K}_q^{\alpha,p}(w_1,w_2)$, by Theorem D, we have the decomposition $f=\sum_{j\in\mathbb{Z}}\lambda_j a_j$, where $\sum_{j\in\mathbb{Z}}|\lambda_j|^p<\infty$ and each a_j is a central $(\alpha,q,[n(1/p-1)];w_1,w_2)$-atom. Without loss of generality, we may assume that $\operatorname{supp} a_j\subseteq B(0,r_j)$ and $r_j=2^j$. For any given $\sigma>0$, we write

$$\begin{aligned}&\sigma^p\cdot\sum_{k\in\mathbb{Z}} w_1(B_k)^{\alpha p/n} w_2\big(\big\{x\in C_k\colon \big|T_*^\delta f(x)\big|>\sigma\big\}\big)^{p/q}\\ &\quad\leqslant \sigma^p\cdot\sum_{k\in\mathbb{Z}} w_1(B_k)^{\alpha p/n} w_2\left(\left\{x\in C_k\colon \sum_{j=k-1}^{\infty}|\lambda_j|\big|T_*^\delta a_j(x)\big|>\sigma/2\right\}\right)^{p/q}\\ &\qquad+\sigma^p\cdot\sum_{k\in\mathbb{Z}} w_1(B_k)^{\alpha p/n} w_2\left(\left\{x\in C_k\colon \sum_{j=-\infty}^{k-2}|\lambda_j|\big|T_*^\delta a_j(x)\big|>\sigma/2\right\}\right)^{p/q}\\ &\quad=J_1+J_2.\end{aligned}$$

Observe that $0<p<1$ and $\delta=n/p-(n+1)/2$, then $\delta>(n-1)/2$. It follows from Chebyshev's inequality and the inequality (1) that

$$\begin{aligned}J_1&\leqslant 2^p\sum_{k\in\mathbb{Z}} w_1(B_k)^{\alpha p/n}\left(\sum_{j=k-1}^{\infty}|\lambda_j|\big\|T_*^\delta(a_j)\chi_k\big\|_{L^q_{w_2}}\right)^p\\ &\leqslant 2^p\sum_{k\in\mathbb{Z}} w_1(B_k)^{\alpha p/n}\left(\sum_{j=k-1}^{\infty}|\lambda_j|^p\big\|T_*^\delta(a_j)\big\|^p_{L^q_{w_2}}\right)\\ &\leqslant C\sum_{k\in\mathbb{Z}} w_1(B_k)^{\alpha p/n}\left(\sum_{j=k-1}^{\infty}|\lambda_j|^p\|a_j\|^p_{L^q_{w_2}}\right).\end{aligned}$$

Changing the order of summation gives

$$J_1\leqslant C\sum_{j\in\mathbb{Z}}|\lambda_j|^p\left(\sum_{k=-\infty}^{j+1} w_1(B_k)^{\alpha p/n} w_1(B_j)^{-\alpha p/n}\right).$$

Note that when $k\leqslant j+1$, then $B_{k-1}\subseteq B_j$. Let θ be the same as in Theorem 1, then by Lemma B, we can get

$$\frac{w_1(B_{k-1})}{w_1(B_j)}\leqslant C\left(\frac{|B_{k-1}|}{|B_j|}\right)^\theta. \tag{9}$$

It follows from Lemma A and the above inequality (9) that

$$\begin{aligned}\sum_{k=-\infty}^{j+1} w_1(B_k)^{\alpha p/n} w_1(B_j)^{-\alpha p/n}&\leqslant C\sum_{k=-\infty}^{j+1} 2^{(k-j-1)\alpha\theta p}\\ &\leqslant C.\end{aligned}$$

Hence

$$J_1 \leqslant C \sum_{j\in\mathbb{Z}} |\lambda_j|^p \leqslant C\|f\|^p_{H\dot{K}^{\alpha,p}_q(w_1,w_2)}. \tag{10}$$

We now turn to deal with J_2. Note that $j \leqslant k-2$, then for any $y \in B_j$ and $x \in C_k = B_k \backslash B_{k-1}$, we have $|x| \geqslant 2|y|$. By using the same arguments as in the proof of Theorem 1, we can get

$$T^\delta_*(a_j)(x) \leqslant C \cdot \left(\frac{2^j}{|x|}\right)^{n/p} w_1(B_j)^{-\alpha/n} w_2(B_j)^{-1/q}.$$

Since $B_j \subseteq B_{k-2}$, then by using Lemma B, we obtain

$$w_i(B_j) \geqslant C \cdot w_i(B_{k-2})|B_j||B_{k-2}|^{-1} \quad \text{for } i = 1 \text{ or } 2.$$

It follows immediately from our assumption $\alpha = n(1/p - 1/q)$ that

$$\begin{aligned} T^\delta_*(a_j)(x) &\leqslant C \cdot \left(\frac{2^j}{2^{k-2}}\right)^{n/p-\alpha-n/q} w_1(B_{k-2})^{-\alpha/n} w_2(B_{k-2})^{-1/q} \\ &\leqslant C \cdot w_1(B_{k-2})^{-\alpha/n} w_2(B_{k-2})^{-1/q}. \end{aligned} \tag{11}$$

Set $A_k = w_1(B_{k-2})^{-\alpha/n} w_2(B_{k-2})^{-1/q}$.

If $\{x \in C_k\colon \sum_{j=-\infty}^{k-2} |\lambda_j||T^\delta_* a_j(x)| > \sigma/2\} = \emptyset$, then the inequality

$$J_2 \leqslant C\|f\|^p_{H\dot{K}^{\alpha,p}_q(w_1,w_2)}$$

holds trivially.

If $\{x \in C_k\colon \sum_{j=-\infty}^{k-2} |\lambda_j||T^\delta_* a_j(x)| > \sigma/2\} \neq \emptyset$, then by the inequality (11), we have

$$\begin{aligned} \sigma &< C \cdot A_k \left(\sum_{j\in\mathbb{Z}} |\lambda_j|\right) \\ &\leqslant C \cdot A_k \left(\sum_{j\in\mathbb{Z}} |\lambda_j|^p\right)^{1/p} \\ &\leqslant C \cdot A_k \|f\|_{H\dot{K}^{\alpha,p}_q(w_1,w_2)}. \end{aligned}$$

Obviously, $\lim_{k\to\infty} A_k = 0$. Then for any fixed $\sigma > 0$, we are able to find a maximal positive integer k_σ such that

$$\sigma < C \cdot A_{k_\sigma} \|f\|_{H\dot{K}^{\alpha,p}_q(w_1,w_2)}.$$

Therefore

$$\begin{aligned} J_2 &\leqslant \sigma^p \cdot \sum_{k=-\infty}^{k_\sigma} w_1(B_k)^{\alpha p/n} w_2(B_k)^{p/q} \\ &\leqslant C\|f\|^p_{H\dot{K}^{\alpha,p}_q(w_1,w_2)} \sum_{k=-\infty}^{k_\sigma} \left(\frac{w_1(B_k)}{w_1(B_{k_\sigma-2})}\right)^{\alpha p/n} \left(\frac{w_2(B_k)}{w_2(B_{k_\sigma-2})}\right)^{p/q}. \end{aligned}$$

Since $B_{k-2} \subseteq B_{k_\sigma-2}$, then by using Lemma B again, we have

$$\frac{w_i(B_{k-2})}{w_i(B_{k_\sigma-2})} \leqslant C\left(\frac{|B_{k-2}|}{|B_{k_\sigma-2}|}\right)^\theta \quad \text{for } i = 1 \text{ or } 2. \tag{12}$$

Applying Lemma A and the above inequality (12), we finally get

$$J_2 \leqslant C\|f\|^p_{H\dot{K}^{\alpha,p}_q(w_1,w_2)} \sum_{k=-\infty}^{k_\sigma} \frac{1}{2^{(k_\sigma-k)n\theta}} \leqslant C\|f\|^p_{H\dot{K}^{\alpha,p}_q(w_1,w_2)}. \tag{13}$$

Combining the above estimate (13) with (10) and taking the supremum over all $\sigma > 0$, we complete the proof of Theorem 2. □

6. Proof of Theorem 3

Proof of Theorem 3. As in Theorem 1, we first choose a number $0 < p_1 < p$ such that $\delta = n/p_1 - (n+1)/2$. Set $s = [n(1/p_1 - 1)]$ and $N = [n(1/p - 1)]$. By using Theorem D and Theorem E, it suffices to show that for every central $(\alpha, q, s; w, w)$-atom f with $\operatorname{supp} f \subseteq B(0, r)$, then $T_R^\delta f$ is a central w-(p, q, N, ε)-molecule. Moreover, its w-molecular norm is uniformly bounded; that is

$$\mathcal{N}_w\big(T_R^\delta f\big) \leqslant C,$$

where the constant C is independent of f and R.

Observe that $\delta > [n/p]r_w/(r_w - 1) - (n+1)/2$, then a simple calculation shows that $Nr_w/n(r_w - 1) + 1/(r_w - 1) < 1/p_1 - 1$, thus we can choose a suitable number $\varepsilon > 0$ satisfying $\max\{Nr_w/n(r_w - 1) + 1/(r_w - 1), 1/p - 1\} < \varepsilon < 1/p_1 - 1$. Let $a = 1 - 1/p + \varepsilon$ and $b = 1 - 1/q + \varepsilon$.

The size condition of central atom f and the inequality (1) imply

$$\big\|T_R^\delta(f)\big\|_{L_w^q} \leqslant \big\|T_*^\delta(f)\big\|_{L_w^q} \leqslant C\|f\|_{L_w^q} \leqslant C \cdot w\big(B(0,r)\big)^{a-b}. \tag{14}$$

On the other hand,

$$\begin{aligned}\big\|T_R^\delta(f) w\big(B\big(0, |\cdot|\big)\big)^b\big\|_{L_w^q}^q &= \int_{|x|\leqslant 2r} \big|T_R^\delta f(x)\big|^q w\big(B\big(0,|x|\big)\big)^{bq} w(x)\,dx + \int_{|x|>2r} \big|T_R^\delta f(x)\big|^q w\big(B\big(0,|x|\big)\big)^{bq} w(x)\,dx \\ &= K_1 + K_2.\end{aligned}$$

Using Lemma A, the inequality (1) and the size condition of f, we obtain

$$\begin{aligned} K_1 &\leqslant w\big(B(0,2r)\big)^{bq} \big\|T_*^\delta(f)\big\|_{L_w^q}^q \\ &\leqslant C \cdot w\big(B(0,r)\big)^{bq} \|f\|_{L_w^q}^q \\ &\leqslant C \cdot w\big(B(0,r)\big)^{bq+1-q/p} \\ &= C \cdot w\big(B(0,r)\big)^{aq}. \end{aligned} \tag{15}$$

Note that when $|x| > 2r$, $y \in B(0,r)$, then we have $|x| > 2|y|$. By using the same arguments as that of Theorem 1 ($w_1 = w_2 = w$), we can deduce

$$T_*^\delta f(x) \leqslant C \cdot \frac{r^{n/p_1}}{|x|^{n/p_1}} w\big(B(0,r)\big)^{-1/p}. \tag{16}$$

If $|x| > 2r$, then $B(0,2r) \subseteq B(0,|x|)$, by Lemma B, we get

$$w\big(B\big(0,|x|\big)\big) \leqslant C \cdot \frac{|x|^n}{(2r)^n} w\big(B(0,2r)\big). \tag{17}$$

It follows from the inequalities (16) and (17) that

$$\begin{aligned} K_2 &\leqslant C \int_{|x|>2r} \frac{r^{nq/p_1}}{|x|^{nq/p_1}} w\big(B(0,r)\big)^{-q/p} \cdot \frac{|x|^{nbq}}{(2r)^{nbq}} w\big(B(0,2r)\big)^{bq} w(x)\,dx \\ &\leqslant C \cdot r^{nq(1/p_1 - b)} w\big(B(0,r)\big)^{-q/p+bq} \int_{|x|>2r} \frac{w(x)}{|x|^{nq(1/p_1-b)}}\,dx. \end{aligned}$$

Observe that $\varepsilon < 1/p_1 - 1$, then we have $1/p_1 - b > 1/q$, which is equivalent to $q(1/p_1 - b) > 1$. Since $w \in A_1$, then $w \in A_{q(1/p_1 - b)}$. Consequently, by Lemma C, we deduce

$$K_2 \leqslant C \cdot w\big(B(0,r)\big)^{-q/p+bq+1} = C \cdot w\big(B(0,r)\big)^{aq}. \tag{18}$$

Hence, by the inequalities (14), (15) and (18), we obtain

$$\begin{aligned} \mathcal{N}_w\big(T_R^\delta f\big) &= \big\|T_R^\delta(f)\big\|_{L_w^q}^{a/b} \cdot \big\|T_R^\delta(f) w\big(B\big(0,|\cdot|\big)\big)^b\big\|_{L_w^q}^{1-a/b} \\ &\leqslant C \cdot w\big(B(0,r)\big)^{(a-b)a/b} w\big(B(0,r)\big)^{(1-a/b)a} \\ &\leqslant C. \end{aligned}$$

It remains to verify the vanishing moments of $T_R^\delta f(x)$. Note that $s \geqslant N$. Therefore, for every multi-index γ with $|\gamma| \leqslant N$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} T_R^\delta f(x)x^\gamma\,dx &= \big(T_R^\delta f(x)x^\gamma\big)\hat{}(0)\\ &= C\cdot D^\gamma\big(\widehat{T_R^\delta f}\big)(0)\\ &= C\cdot D^\gamma(\widehat{\phi_{1/R}}\cdot\widehat{f})(0)\\ &= C\cdot\sum_{|\alpha|+|\beta|=|\gamma|}\big(D^\alpha\widehat{\phi_{1/R}}\big)(0)\big(D^\beta\widehat{f}\big)(0)\\ &= 0.\end{aligned}$$

This completes the proof of Theorem 3. □

Remark. The corresponding results for non-homogeneous weighted Herz-type Hardy spaces can also be proved by atomic and molecular decomposition theory. The arguments are similar, so the details are omitted here.

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