

Multiresolution analysis on product of zero-dimensional Abelian groups<sup>☆</sup>

S.F. Lukomskii

N.G. Chernyshevskii Saratov State University, Russian Federation

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## ABSTRACT

For a product of locally compact zero-dimensional groups  $(G, \dot{+})$ , we describe dilation operators and build the corresponding multiresolution analysis. We prove also that any nonsingular matrix  $\mathcal{A}_{d \times d}$  over the field  $\mathbf{Z}_p$  defines a dilation operator in  $G^d$ .

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## 1. Introduction

In recent years there has been a considerable interest in the problem of constructing wavelet bases on locally compact Abelian groups. In [20–22] these questions were examined on the Cantor dyadic group. V.Yu. Protasov and Yu.A. Farkov [29, 8] gave a characterization of the dyadic compactly supported wavelets on  $\mathbb{R}_+$ , and pointed out an algorithm for their construction. V.Yu. Protasov [28] studied approximative properties of dyadic wavelets as put forward in [29]. Yu.A. Farkov [9,10] pointed out a method for constructing compactly supported orthogonal wavelets on a locally compact Vilenkin group  $G$  with a constant generating sequence, and derived necessary and sufficient conditions for a solution of the refinement equation to generate a multiresolution analysis (MRA in the sequel) of  $L_2(G)$ .

A good deal of studies was devoted to the construction of an MRA on the field  $\mathbb{Q}_p$  of all  $p$ -adic numbers. S.V. Kozyrev [13, 14] found orthonormal  $p$ -adic wavelet bases consisting of eigenfunctions of  $p$ -adic pseudo-differential operators.  $p$ -Adic wavelet basis of S.V. Kozyrev is analog of Haar basis on Vilenkin group [11]. In [2,13,14,17–19,30] it has been noted that  $p$ -adic wavelet analysis has a close connection with the pseudo-differential operator theory.

J.J. Benedetto and R.L. Benedetto [6] have built wavelet bases on a locally compact group containing an open subgroup. In the classic case there exist a general scheme for the construction of wavelets. This scheme is based on the notion of multiresolution analysis (MRA). R.L. Benedetto [7] had doubts that an MRA-theory could be developed in  $L_2(\mathbb{Q}_p)$  because discrete subgroups do not exist in  $\mathbb{Q}_p$ . Trough A.Yu. Khrennikov and V.M. Shelkovich [17] conjectured that the equality

$$\varphi(x) = \sum_{j=0}^{p-1} \varphi(p^{-1}x \dot{-} p^{-1}r)$$

may be considered as a refinement equation for the MRA generating Kozyrev's wavelets. A.Yu. Khrennikov, V.M. Shelkovich, and M.A. Skopina [15,16] considered the refinement equation

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E-mail address: LukomskiiSF@info.sgu.ru.

$$\varphi(x) = \sum_{j=0}^{p^s-1} \beta_j \varphi\left(p^{-1}x \dot{-} \frac{r}{p^s}\right),$$

introduced the concept of a  $p$ -adic MRA with orthogonal refinable function, and described a general scheme for their creation. Riesz bases of wavelets over the field of  $p$ -adic numbers were constructed in [1]. About a wavelet basis in  $L_2(\mathbb{Q}_p)$  and pseudo-differential operators see also [2–4].

The field  $\mathbb{Q}_p$  of all  $p$ -adic numbers is a zero-dimensional locally compact Abelian group, the ring  $\mathbb{Z}_p$  of  $p$ -adic integers is a zero-dimensional compact Abelian group and conversely: any zero-dimensional locally compact Abelian group  $(\mathfrak{G}, +)$  with condition  $pg_n = g_{n+1}$  is the group of all  $p$ -adic numbers, any zero-dimensional compact Abelian group  $(\mathfrak{G}, +)$  with condition  $pg_n = g_{n+1}$  is the group of  $p$ -adic integers (here  $(g_n)$  is a basic system in the group  $\mathfrak{G}$ ). The author [23] put forward a scheme for constructing a Haar system on a compact zero-dimensional Abelian group, introduced [24] the concept of MRA on locally compact zero-dimensional Abelian group with orthogonal refinable function, and described a general scheme for constructing of orthogonal wavelet bases.

The problem of constructing multidimensional MRA is more difficult. During the International Conference “Wavelets and Applications” (2009, St. Petersburg) S.V. Kozyrev has noticed this problem as a cardinal problem of multidimensional  $p$ -adic MRA. In [30] multidimensional 2-adic orthogonal wavelet bases for  $L_2(\mathbb{Q}_2^d)$  was constructed by means of the tensor product of one-dimensional MRA. In this case there exist  $2^d - 1$  wavelets. For a nonempty subset  $E \subset \{1, 2, \dots, d\}$  the corresponding wavelets may be written in the form

$$\psi_E(x_1, x_2, \dots, x_d) = \prod_{j \in E} \psi(x_j) \prod_{j \notin E} \varphi(x_j)$$

where

$$\varphi(x) = \mathbf{1}_{\mathbb{Z}_2}(x), \quad \psi(x) = \varphi\left(\frac{x}{2}\right) - \varphi\left(\frac{x}{2} \dot{-} \frac{1}{2}\right).$$

In [30] it was proved also that spectral analysis of pseudo-differential operators on  $\mathbb{Q}^2$  is a 2-dimensional wavelets analysis. E.J. King and M.A. Skopina [19] constructed MRA and a wavelet basis in  $L_2(\mathbb{Q}_2^2)$ . To construct this basis they used dilation operator with Quincunx matrix  $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$ .

But the general method of constructing of  $p$ -adic multidimensional MRA and multidimensional wavelets does not exist. In the present paper we examine the problem of construction of MRA and orthogonal wavelet bases on product  $\mathbb{Q}_p^d$ . The product  $\mathbb{Q}_p^d$  is not a field of  $q$ -adic numbers, therefore it is impossible to solve this problem within framework  $p$ -adic analysis. But the product  $\mathbb{Q}_p^d$  is a zero-dimensional locally compact abelian group with condition  $pg_n = g_{n+d}$  and conversely: any zero-dimensional locally compact abelian group with condition  $pg_n = g_{n+d}$  is the product  $\mathbb{Q}_p^d$  of groups of all  $p$ -adic numbers [26]. Using this fact we will construct MRA in  $L_2(\mathfrak{G}^d)$ , where  $\mathfrak{G}$  is an arbitrary zero-dimensional group. Taking  $\mathfrak{G} = \mathbb{Q}_p$  we get MRA in  $L_2(\mathbb{Q}_p^d)$ . To construct MRA we need to have a dilation operator. We describe all dilation operators and for any dilation operators we construct the corresponding MRA. Using results of the paper [24] we get an algorithm of constructing of orthogonal wavelet bases in  $L_2(\mathfrak{G}^d)$ . The case  $d = 2$  we will study in detail.

It should be noted (see Section 2) that there exists a natural map  $\theta : \mathfrak{G} \rightarrow [0, +\infty)$  under which we have for the basic chain  $\theta(\mathfrak{G}_n) = [0, 1/m_n]$ . Therefore constructed orthogonal wavelet bases on the product  $\mathfrak{G}^2$  are orthonormal bases in  $L_2([0, +\infty)^2)$ . From it follows that these bases may be of use in image analysis. Using this method of constructing of the dilation operator we can get “intellectual” algorithms of information processing.

## 2. Locally compact zero-dimensional groups, topology and characters

We proceed to give basic notions and facts in the analysis on zero-dimensional groups. A more detailed account may be found in [5].

A topological group in which the connected component of 0 is 0 is usually referred to as a *zero-dimensional group*. If a separable locally compact group  $(\mathfrak{G}, +)$  is zero-dimensional, then topology on it can be generated by means of a descending sequence of subgroups. The converse statement holds for all topological groups (see [5, Chapter 1, Section 3]). So, for a locally compact group, we are going to say ‘zero-dimensional group’ instead of saying ‘a group with topology generated by a sequence subgroups’.

Let  $(\mathfrak{G}, +)$  be a locally compact zero-dimensional Abelian group with topology generated by a countable system of open subgroups

$$\dots \supset \mathfrak{G}_{-n} \supset \dots \supset \mathfrak{G}_{-1} \supset \mathfrak{G}_0 \supset \mathfrak{G}_1 \supset \dots \supset \mathfrak{G}_n \supset \dots$$

where

$$\bigcup_{n=-\infty}^{+\infty} \mathfrak{G}_n = \mathfrak{G}, \quad \bigcap_{n=-\infty}^{+\infty} \mathfrak{G}_n = \{0\}$$

(0 is the null element in the group  $\mathfrak{G}$ ). Given any fixed  $N \in \mathbb{Z}$ , the subgroup  $\mathfrak{G}_N$  is a compact Abelian group with respect to the same operation  $\dot{+}$  under the topology generated by the system of subgroups

$$\mathfrak{G}_N \supset \mathfrak{G}_{N+1} \supset \cdots \supset \mathfrak{G}_n \supset \cdots.$$

As each subgroup  $\mathfrak{G}_n$  is compact, it follows that each quotient group  $\mathfrak{G}_n/\mathfrak{G}_{n+1}$  is finite (say, of order  $p_n$ ). We may always assume that all  $p_n$  are prime numbers, for in fact, by Sylow's theorem (see [12]), the chain of subgroups can be refined so that the quotient groups  $\mathfrak{G}_n/\mathfrak{G}_{n+1}$  will be of prime order. We will name such chain as *basic chain*. In this case, a base of the topology is formed by all possible cosets  $\mathfrak{G}_n \dot{+} g$ ,  $g \in \mathfrak{G}$ .

We further define the numbers  $(m_n)_{n=-\infty}^{+\infty}$  as follows:

$$m_0 = 1, \quad m_{n+1} = m_n \cdot p_n.$$

Clearly, for  $n \geq 1$ ,

$$m_n = p_0 p_1 \cdots p_{n-1}, \quad m_{-n} = \frac{1}{p_{-1} p_{-2} \cdots p_{-n}}.$$

The collection of all such cosets  $\mathfrak{G}_n \dot{+} g$ ,  $n \in \mathbb{Z}$ , along with the empty set forms the semiring  $\mathcal{K}$ . On each coset  $\mathfrak{G}_n \dot{+} g$  we define the measure  $\mu$  by  $\mu(\mathfrak{G}_n \dot{+} g) = \mu \mathfrak{G}_n = 1/m_n$ . So, if  $n \in \mathbb{Z}$  and  $p_n = p$ , we have  $\mu \mathfrak{G}_n \cdot \mu \mathfrak{G}_{-n} = 1$ . The measure  $\mu$  can be extended from the semiring  $\mathcal{K}$  onto the  $\sigma$ -algebra (for example, by using Caratheodory's extension). This gives the translation invariant measure  $\mu$ , which agrees on the Borel sets with the Haar measure on  $\mathfrak{G}$ . Further, let  $\int_{\mathfrak{G}} f(x) d\mu(x)$  be the absolutely convergent integral of the measure  $\mu$ .

Given an  $n \in \mathbb{Z}$ , consider an element  $g_n \in \mathfrak{G}_n \setminus \mathfrak{G}_{n+1}$  and fix it. Then any  $x \in \mathfrak{G}$  has a unique representation of the form

$$x = \sum_{n=-\infty}^{+\infty} a_n g_n, \quad a_n = \overline{0, p_n - 1}, \quad (2.1)$$

the sum (2.1) containing finite number of terms with negative subscripts; that is,

$$x = \sum_{n=N}^{+\infty} a_n g_n, \quad a_n = \overline{0, p_n - 1}, \quad a_N \neq 0. \quad (2.2)$$

We will name system  $(g_n)_{n \in \mathbb{Z}}$  as a *basic system*.

Classical examples of zero-dimensional groups are Vilenkin groups and groups of  $p$ -adic numbers (see [5, Chapter 1, Section 2]). A direct sum of cyclic groups  $Z(p_k)$  of order  $p_k$ ,  $k \in \mathbb{Z}$ , is called a *Vilenkin group*. This means that the elements of a Vilenkin group are infinite sequences  $x = (x_k)_{k=-\infty}^{+\infty}$  such that:

- 1)  $x_k = \overline{0, p_k - 1}$ ;
- 2) only a finite number of  $x_k$  with negative subscripts are different from zero;
- 3) the group operation  $\dot{+}$  is the coordinate-wise addition modulo  $p_k$ ; that is,

$$x \dot{+} y = (x_k \dot{+} y_k), \quad x_k \dot{+} y_k = (x_k + y_k) \bmod p_k.$$

A topology on such group is generated by the chain of subgroups

$$\mathfrak{G}_n = \{x \in \mathfrak{G}: x = (\dots, 0, 0, \dots, 0, x_n, x_{n+1}, \dots), x_v = \overline{0, p_v - 1}, v \geq n\}.$$

It is easy to see that all the subgroups  $\mathfrak{G}_n$  form a descending sequence. As a  $g_n$ , we can take a sequence consisting of all zeros except for a 1 in the  $n$ th spot.

The group  $\mathbb{Q}_p$  of all  $p$ -adic numbers ( $p$  is a prime number) also consists of sequences  $x = (x_k)_{k=-\infty}^{+\infty}$ ,  $x_k = \overline{0, p - 1}$ , only a finite number of  $x_k$  with negative subscripts being different from zero. However, the group operation in  $\mathbb{Q}_p$  is defined differently. Namely, given elements

$$x = (\dots, 0, \dots, 0, x_N, x_{N+1}, \dots) \quad \text{and} \quad y = (\dots, 0, \dots, 0, y_N, y_{N+1}, \dots) \in \mathbb{Q}_p,$$

we again add them coordinate-wise, but whereas in a Vilenkin group  $x_n \dot{+} y_n = (x_n + y_n) \bmod p$  (that is, a 1 is not carried to the next  $(n+1)$ th position), the corresponding  $p$ -adic summation has the property that the 1 occurring as a result of the addition of  $x_n + y_n$  is carried to the next  $(n+1)$ th position. We endow the group  $\mathbb{Q}_p$  with the topology generated by the same system of subgroups  $\mathfrak{G}_n$  as for a Vilenkin group. Similarly, as a  $g_n$ , we may again take the same sequence.

Let  $x = \sum_{n \in \mathbb{Z}} a_n g_n \in G$ . The mapping  $\theta(x) = \sum_{n \in \mathbb{Z}} \frac{a_n}{m_{n+1}}$  establishes the one-to-one correspondence between  $\mathfrak{G}$  and the so-called modified interval  $[0, +\infty)^*$ . The modified interval  $[0, +\infty)^*$  can be interpreted as the interval  $[0, +\infty)$  in which the  $p$ -adic rational points  $\frac{k}{m_n}$  are counted twice: the “left” point  $\frac{k}{m_n} - 0$  corresponds to the element

$$\sum_{j \leq n-1} a_j g_j + \sum_{j=n}^{\infty} (p_j - 1) g_j$$

and the “right” point  $\frac{k}{m_n} + 0$  corresponds to the element

$$\sum_{j \leq n-2} a_j g_j + (a_{n-1} + 1) g_{n-1} + \sum_{j=n}^{\infty} 0 g_j.$$

Under this mapping we have

$$\theta(\mathfrak{G}_n) = \left[ 0, \frac{1}{m_n} - 0 \right] \quad \text{and} \quad \theta(\mathfrak{G}_n + g) = \left[ \frac{k}{m_n} + 0, \frac{k+1}{m_n} - 0 \right].$$

The modified interval  $[0, +\infty)^*$  is very suitable model of zero-dimensional group  $\mathfrak{G}$ .

Let  $X$  be the collection of the characters of a group  $(\mathfrak{G}, +)$ ; it is a group with respect to multiplication too. Also let  $\mathfrak{G}_n^\perp = \{\chi \in X: \forall x \in \mathfrak{G}_n, \chi(x) = 1\}$  be the annihilator of the group  $\mathfrak{G}_n$ . Each annihilator  $\mathfrak{G}_n^\perp$  is a group with respect to multiplication, and the subgroups  $\mathfrak{G}_n^\perp$  form an increasing sequence

$$\cdots \subset \mathfrak{G}_{-n}^\perp \subset \cdots \subset \mathfrak{G}_0^\perp \subset \mathfrak{G}_1^\perp \subset \cdots \subset \mathfrak{G}_n^\perp \subset \cdots \quad (2.3)$$

with

$$\bigcup_{n=-\infty}^{+\infty} \mathfrak{G}_n^\perp = X \quad \text{and} \quad \bigcap_{n=-\infty}^{+\infty} \mathfrak{G}_n^\perp = \{1\},$$

the quotient group  $\mathfrak{G}_{n+1}^\perp / \mathfrak{G}_n^\perp$  having order  $p_n$ . The group of characters  $X$  may be equipped with the topology using the chain of subgroups (2.3), the family of the cosets  $\mathfrak{G}_n^\perp \cdot \chi$ ,  $\chi \in X$ , being taken as a base of the topology. The collection of such cosets, along with the empty set, forms the semiring  $\mathcal{X}$ . Given a coset  $\mathfrak{G}_n^\perp \cdot \chi$ , we define a measure  $\nu$  on it by  $\nu(\mathfrak{G}_n^\perp \cdot \chi) = \nu(\mathfrak{G}_n^\perp) = m_n$  (so that always  $\mu(\mathfrak{G}_n) \nu(\mathfrak{G}_n^\perp) = 1$ ). The measure  $\nu$  can be extended onto the  $\sigma$ -algebra of measurable sets in the standard way (for example, using Caratheodory's extension theorem). One then forms the absolutely convergent integral  $\int_X F(\chi) d\nu(\chi)$  of this measure.

The value  $\chi(g)$  of the character  $\chi$  at an element  $g \in \mathfrak{G}$  will be denoted by  $(\chi, g)$ . The Fourier transform  $\hat{f}$  of an  $f \in L_2(\mathfrak{G})$  is defined as follows

$$\hat{f}(\chi) = \int_{\mathfrak{G}} f(x) \overline{(\chi, x)} d\mu(x) = \lim_{n \rightarrow +\infty} \int_{\mathfrak{G}_{-n}} f(x) \overline{(\chi, x)} d\mu(x),$$

the limit being in the norm of  $L_2(X)$ . For an  $f \in L_2(\mathfrak{G})$ , the inversion formula is valid

$$f(x) = \int_X \hat{f}(\chi) (\chi, x) d\nu(\chi) = \lim_{n \rightarrow +\infty} \int_{\mathfrak{G}_n^\perp} \hat{f}(\chi) (\chi, x) d\nu(\chi);$$

here the limit also signifies the convergence in the norm of  $L_2(\mathfrak{G})$ . If  $f, g \in L_2(\mathfrak{G})$  then the Plancherel formula is valid

$$\int_{\mathfrak{G}} f(x) \overline{g(x)} d\mu(x) = \int_X \hat{f}(\chi) \overline{\hat{g}(\chi)} d\nu(\chi).$$

Endowed with this topology, the group of characters  $X$  is a zero-dimensional locally compact group; there is, however, a dual situation: every element  $x \in \mathfrak{G}$  is a character of the group  $X$ , and  $\mathfrak{G}_n$  is the annihilator of the group  $\mathfrak{G}_n^\perp$ .

### 3. Rademacher functions and dilation operator

Let  $(\mathfrak{G}, +)$  be a locally compact zero-dimensional Abelian group with a basic chain  $(\mathfrak{G}_n)_{n \in \mathbb{Z}}$  and  $\mathfrak{G}_n^\perp$  annihilators of  $\mathfrak{G}_n$ ,  $(g_n)$  a basic system in  $\mathfrak{G}$ .

**Definition 3.1.** A character  $r_n \in \mathfrak{G}_{n+1}^\perp \setminus \mathfrak{G}_n^\perp$  is called the Rademacher function.

**Lemma 3.1** (Properties of Rademacher functions). (See [25].)

- 1)  $r_n(t) = \text{const}$  on any coset  $\mathfrak{G}_{n+1} \dot{+} g$ ,  $r_n(\mathfrak{G}_{n+1}) = 1$  ( $n \in \mathbb{Z}$ ).
- 2)  $r_n(\mathfrak{G}_{n+1} \dot{+} jg_n)$  are distinct roots of unity of order  $p_n$  ( $j = \overline{0, p-1}$ ).
- 3)  $1 + r_n(t) + r_n^2(t) + \dots + r_n^{p_n-1}(t) = \begin{cases} p_n, & t \in \mathfrak{G}_{n+1}, \\ 0, & t \in \mathfrak{G}_n \setminus \mathfrak{G}_{n+1}. \end{cases}$

Below, we will consider a dilation operator on a group  $\mathfrak{G}$ . However, we have been able to define such operator only in the case when  $p_n = p$  for any  $n \in \mathbb{Z}$ . Thus in what follows we will only consider groups  $\mathfrak{G}$  for which  $p_n = p$ . The translation of the argument of a function  $f$  by an element  $g \in \mathfrak{G}$  will be denoted by  $f_{\dot{+}g}$ ; that is,  $f_{\dot{+}g}(x) = f(x \dot{+} g)$ . As regards the operation  $\dot{+}$ , we assume additionally that

$$pg_n = c_1 g_{n+1} \dot{+} c_2 g_{n+2} \dot{+} \dots \dot{+} c_\tau g_{n+\tau}; \quad (3.1)$$

here  $c_1, c_2, \dots, c_\tau = \overline{0, p-1}$  are fixed numbers. It is worth noting, that if  $pg_n = 0$ , then  $\mathfrak{G}$  is a Vilenkin group, and if  $pg_n = g_{n+1}$ , then  $\mathfrak{G}$  is the group  $\mathbb{Q}_p$  of all  $p$ -adic numbers. We set

$$H_n = \left\{ q \in \mathfrak{G} : q = \sum_{j=N}^{n-1} a_j g_j, N \in \mathbb{Z}, a_j = \overline{0, p-1} \right\}.$$

If  $\mathfrak{G}$  is a Vilenkin group, then  $H_n$  is a group. This is not so in the general case (for example, if  $\mathfrak{G}$  is the group of all  $p$ -adic numbers). However, it is worth noting that  $H_n$  is always a countable set.

**Definition 3.2.** We define the mapping  $A: \mathfrak{G} \rightarrow \mathfrak{G}$  by  $Ax := \sum_{n=-\infty}^{+\infty} a_n g_{n-1}$ , where  $x = \sum_{n=-\infty}^{+\infty} a_n g_n \in \mathfrak{G}$ . As any element  $x \in \mathfrak{G}$  can be uniquely expanded as  $x = \sum a_n g_n$ , the mapping  $A: \mathfrak{G} \rightarrow \mathfrak{G}$  is one-to-one onto. The mapping  $A$  is called a dilation operator if  $A(x \dot{+} y) = Ax \dot{+} Ay$  for all  $x, y \in \mathfrak{G}$ .

We note that if  $\mathfrak{G}$  is a Vilenkin group ( $p \cdot g_n = 0$ ) or is the group of all  $p$ -adic numbers ( $p \cdot g_n = g_{n+1}$ ), then  $A$  is an additive operator and hence a dilation operator. Moreover, the operator  $A$  is additive if the condition (3.1) is satisfied. It is also clear that  $A\mathfrak{G}_n = \mathfrak{G}_{n-1}$ .

**Lemma 3.2.** Let  $A$  be a dilation operator. Then

- 1)  $A(\mathfrak{G}_{n+1} \dot{+} \alpha_n g_n \dot{+} \dots \dot{+} \alpha_{n-s} g_{n-s}) = \mathfrak{G}_n \dot{+} \alpha_n g_{n-1} \dot{+} \dots \dot{+} \alpha_{n-s} g_{n-s-1}$ ,
- 2)  $A^{-1}(\mathfrak{G}_n \dot{+} \alpha_{n-1} g_{n-1} \dot{+} \dots \dot{+} \alpha_{n-s} g_{n-s}) = \mathfrak{G}_{n+1} \dot{+} \alpha_{n-1} g_n \dot{+} \dots \dot{+} \alpha_{n-s} g_{n-s+1}$ .

**Lemma 3.3.** Let  $r_0 \in \mathfrak{G}_1^\perp \setminus \mathfrak{G}_0^\perp$  be a Rademacher function. Then functions  $r_n = r_0 A^n$  ( $n \in \mathbb{Z}$ ) are Rademacher functions.

**Proof.** It is evident that  $r_n$  is character. Let us show that  $r_n \in \mathfrak{G}_{n+1}^\perp \setminus \mathfrak{G}_n^\perp$ . By definition  $r_n(\mathfrak{G}_{n+1}) = r_0 A^n(\mathfrak{G}_{n+1}) = r_0(\mathfrak{G}_1) = 1$ . Hence  $r_n \in \mathfrak{G}_{n+1}^\perp$ . Assume that  $r_n \in \mathfrak{G}_n^\perp$ . Then  $r_0 A^n(\mathfrak{G}_n) = r_0(\mathfrak{G}_0) = 1$ . This contradiction concludes the proof.  $\square$

**Lemma 3.4.** Let  $A$  be a dilation operator. Then  $\mathfrak{G}_n^\perp A = \mathfrak{G}_{n+1}^\perp$ ,  $\mathfrak{G}_n^\perp A^{-1} = \mathfrak{G}_{n-1}^\perp$ .

**Proof.** 1) Let us show that  $\mathfrak{G}_n^\perp A \subset \mathfrak{G}_{n+1}^\perp$ . Suppose  $\chi \in \mathfrak{G}_n^\perp A$ . Then  $\chi = \chi_n A$  where  $\chi_n \in \mathfrak{G}_n^\perp$ , that is  $\chi_n(\mathfrak{G}_n) = 1$ . Therefore  $\chi(\mathfrak{G}_{n+1}) = \chi_n(A\mathfrak{G}_{n+1}) = \chi_n(\mathfrak{G}_n) = 1$ . Hence  $\mathfrak{G}_n^\perp A \subset \mathfrak{G}_{n+1}^\perp$ .

2) Let us show that  $\mathfrak{G}_n^\perp A \supset \mathfrak{G}_{n+1}^\perp$ . Suppose  $\chi \in \mathfrak{G}_{n+1}^\perp$ . Then  $(\chi A^{-1})\mathfrak{G}_n = \chi(A^{-1}\mathfrak{G}_n) = \chi(\mathfrak{G}_{n+1}) = 1$ . Denote  $\chi A^{-1} = \chi_n \in \mathfrak{G}_n^\perp$ . Then  $\chi = \chi_n A \in \mathfrak{G}_n^\perp A$ , that is  $\mathfrak{G}_{n+1}^\perp \subset \mathfrak{G}_n^\perp A$ .  $\square$

We will assume later on  $r_n = r_0 A^n$ .

#### 4. Multiresolution analysis on locally compact zero-dimensional groups

Our main objective is to construct orthogonal wavelet bases for  $L_2(\mathfrak{G})$  and  $L_2(\mathfrak{G}^d)$ . For this we will use a multiresolution analysis on the group  $\mathfrak{G}$  as follows [24].

**Definition 4.1.** A family of closed subspaces  $V_n$ ,  $n \in \mathbb{Z}$ , is said to be a multiresolution analysis of  $L_2(\mathfrak{G})$  if the following axioms are satisfied:

- 1)  $V_n \subset V_{n+1}$ ;
- 2)  $\overline{\bigcup_{n \in \mathbb{Z}} V_n} = L_2(\mathfrak{G})$  and  $\bigcap_{n \in \mathbb{Z}} V_n = \{0\}$ ;

- 3)  $f(x) \in V_n \Leftrightarrow f(Ax) \in V_{n+1}$  ( $A$  is a dilation operator);
- 4)  $f(x) \in V_0 \Rightarrow f(x \dot{-} h) \in V_0$  for all  $h \in H_0$  ( $H_0$  is analog of  $\mathbb{Z}$ );
- 5) there exists a function  $\varphi \in L_2(\mathfrak{G})$  such that the system  $(\varphi(x \dot{-} h))_{h \in H_0}$  is an orthonormal basis for  $V_0$ .

A function  $\varphi$  occurring in Axiom 5) is called a *refinable function*.

Next we will follow the conventional approach. Let  $\varphi(x) \in L_2(\mathfrak{G})$ , and suppose that  $(\varphi(x \dot{-} h))_{h \in H_0}$  is an orthonormal system in  $L_2(\mathfrak{G})$ . With the function  $\varphi$  and the dilation operator  $A$ , we define the linear subspaces  $L_j = (\text{span } \varphi(A^j x \dot{-} h))_{h \in H_0}$  and closed subspaces  $V_j = \overline{L_j}$ . If the subspaces  $V_j$  form an MRA, then the function  $\varphi$  is said to *generate* an MRA in  $L_2(\mathfrak{G})$ . We will look up a function  $\varphi \in L_2(\mathfrak{G})$ , which generates an MRA in  $L_2(\mathfrak{G})$ , as a solution of the refinement equation

$$\varphi(x) = \sum_{h \in H} c_h \varphi(Ax \dot{-} h), \quad (4.1)$$

where  $H \subset H_0$  is a finite set.

The next theorem was proved in [24].

**Theorem 4.1.** Let  $\varphi \in L_2(\mathfrak{G})$  be a solution of Eq. (4.1). Suppose that  $|\hat{\varphi}(\chi)| = \mathbf{1}_{\mathfrak{G}_0^\perp}(\chi)$ . Then  $\varphi$  generates an MRA in  $L_2(\mathfrak{G})$ .

We will use this theorem for constructing MRA in  $L_2(\mathfrak{G}^d)$ . As usual,  $W_n$  stands for the orthogonal complement of  $V_n$  in  $V_{n+1}$ , that is,  $V_{n+1} = V_n \otimes W_n$  and  $V_n \perp W_n$  ( $n \in \mathbb{Z}$ , and  $\otimes$  denotes the direct sum). It is readily seen that

- 1)  $f \in W_n \Leftrightarrow f(Ax) \in W_{n+1}$ ;
- 2)  $W_n \perp W_k$  for  $k \neq n$ ;
- 3)  $\bigotimes W_n = L_2(G)$ ,  $n \in \mathbb{Z}$ .

An algorithm of constructing of wavelet bases was specified in [24]. Give this algorithm now. Denote

$$H_0^{(s)} = \{a_{-1}g_{-1} \dot{+} a_{-2}g_{-2} \dot{+} \cdots \dot{+} a_{-s}g_{-s} : a_{-j} = \overline{0, p-1}\}.$$

We need to find a solution  $\varphi$  of the refinement equation

$$\varphi(x) = \sum_{h \in H_0^{(s)}} \beta_h \varphi(Ax \dot{-} h) \quad (4.2)$$

such that  $|\hat{\varphi}(\chi)| = \mathbf{1}_{\mathfrak{G}_0^\perp}(\chi)$ .

The refinement equation (4.2) can be written in terms of the Fourier transform as follows

$$\hat{\varphi}(\chi) = m_0(\chi) \hat{\varphi}(\chi A^{-1}) \quad (4.3)$$

where

$$m_0(\chi) = \frac{1}{p} \sum_{h \in H_0^{(s)}} \beta_h \overline{\hat{\varphi}(\chi, A^{-1}h)}. \quad (4.4)$$

The function  $m_0(\chi)$  is called a mask for Eq. (4.2). We know [24] that

- 1) the mask  $m_0$  is a constant on any coset  $\mathfrak{G}_{-s+1}^\perp \chi$ ,
- 2)  $m_0(\chi) = \hat{\varphi}(1)$  for  $\chi \in \mathfrak{G}_{-s+1}^\perp$ ,
- 3)  $|m_0(\chi)| = 1$  for  $\chi \in \mathfrak{G}_0^\perp \setminus \mathfrak{G}_{-s+1}^\perp$ ,
- 4)  $m_0(\chi) = 0$  for  $\chi \in X \setminus \mathfrak{G}_0^\perp$ .

Since the function  $m_0(\chi)$  is constant on any cosets of the subgroup  $G_{-s+1}^\perp$ , that is, on the sets

$$G_{-s+1}^\perp r_{-s+1}^{\alpha_{-s+1}} r_{-s+2}^{\alpha_{-s+2}} \cdots r_{-1}^{\alpha_{-1}} r_0^{\alpha_0} \subset G_1^\perp,$$

we can pick a character  $\chi_k$  in each of these cosets, where

$$k = \alpha_{-s+1} + \alpha_{-s+2}p + \cdots + \alpha_{-1}p^{s-2} + \alpha_0 p^{s-1}.$$

Knowing a mask it is possible to recover the Fourier transform  $\hat{\varphi}$  from the value of  $\hat{\varphi}(1)$ .

**Step 1.** Build a function  $m_0(\chi)$  which is constant on the cosets of  $G_{-s+1}^\perp$ . We write the equality (4.4) as the system

$$m_0(\chi_k) = \frac{1}{p} \sum_{j=0}^{p^s-1} \beta_j(\chi_k, \overline{A^{-1}h_j}) \quad (k = \overline{0, p^s-1}) \quad (4.5)$$

in the unknowns  $\beta_j$ . Numbers  $j$  and elements  $h_j$  in (4.5) are connected by the next equations

$$j = a_{-1} + a_{-2}p + \dots + a_{-s}p^{s-1}, \quad 0 \leq j \leq p^s - 1, \\ h_j = a_{-1}g_{-1} + a_{-2}g_{-2} + \dots + a_{-s}g_{-s} \in H_0^{(s)}.$$

**Step 2.** Take  $m_0(\chi_k)$  so that  $|m_0(\chi_k)| = 1$  for  $k = \overline{0, p^{s-1}-1}$ ,  $m_0(1) = 1$ , and  $m_0(\chi_k) = 0$  for  $k = \overline{p^{s-1}, p^s-1}$ .

**Step 3.** We set  $\hat{\varphi}(1) = 1$  and build  $\hat{\varphi}(\chi)$  using the equation

$$\hat{\varphi}(\chi) = \hat{\varphi}(1) \prod_{k=0}^{\infty} m_0(\chi A^{-k}).$$

In [17] it was proved that  $|\hat{\varphi}(\chi)| = \mathbf{1}_{G_0^\perp}(\chi)$ , and hence the function  $\varphi$  generates an MRA.

**Step 4.** Define functions  $m_l(\chi) = m_0(\chi r_0^{-l})$ ,  $l = \overline{1, p-1}$  and write  $m_l(\chi)$  in the form

$$m_l(\chi) = \frac{1}{p} \sum_{h \in H_0^{(s)}} \beta_h^{(l)}(\chi, \overline{A^{-1}h})$$

where  $\beta_h^{(l)} = \beta_h(r_0^{-l}, A^{-1}h)$ . It is clear that  $|m_l(G_0^\perp r_0^l)| = 1$  and  $|m_l(G_0^\perp r_0^v)| = 0$  for  $v \neq l$ .

**Step 5.** Define functions

$$\psi_l(x) = \sum_{h \in H_0^{(s)}} \beta_h^{(l)} \varphi(Ax \dot{-} h).$$

The function  $\psi_l(x - h)$ , where  $l = \overline{1, p-1}$ ,  $h \in H_0$ , forms an orthogonal basis for  $W_0$  [24].

**Theorem 4.2.** Functions

$$p^{n/2} \psi_l(A^n x \dot{-} h) \quad (l = \overline{1, p-1}, n \in \mathbb{Z}, h \in H_0)$$

form a complete orthonormal system in  $L_2(\mathfrak{G}^d)$  [24].

**Example 4.1.** Let us consider this algorithm for  $s = 2$ . Let  $\varphi$  be a solution of the refinement equation

$$\varphi(x) = \sum_{h \in H_0^{(2)}} \varphi(Ax \dot{-} h)$$

and  $|\hat{\varphi}(\chi)| = \mathbf{1}_{\mathfrak{G}_0}(\chi)$ . In this case  $H_0^{(2)} = \{a_{-1}g_{-1} + a_{-2}g_{-2}\}$ . Let us denote the element  $h = a_{-1}g_{-1} + a_{-2}g_{-2}$  as  $h_j$ , where  $j = a_{-1} + a_{-2}p$ . We will denote a character  $\chi \in \mathfrak{G}_{-1}^{\perp} r_{-1}^{\alpha_{-1}} r_0^{\alpha_0}$  as  $\chi_k$  where  $k = \alpha_{-1} + \alpha_0 p$ . Let us write for the mask  $m_0(\chi)$  the equations

$$m_0(\chi_k) = \frac{1}{p} \sum_{j=0}^{p^2-1} \beta_j(\chi_k, \overline{A^{-1}h_j}). \quad (4.6)$$

Let us denote  $m_0(\chi_k) = \mu_k$ . We will assume that  $\mu_0 = 1$ ,  $|\mu_k| = 1$  ( $k = \overline{1, p-1}$ ) and  $\mu_k = 0$  when  $k \geq p$ . Since the matrix  $\frac{1}{p}(\chi_k, A^{-1}h_j)_{k,j=0}^{p^2-1}$  is unitary then we find from (4.6)

$$\beta_j = \frac{1}{p} \sum_{k=0}^{p^2-1} \mu_k(\chi_k, A^{-1}h_j) = \frac{1}{p} \sum_{k=0}^{p-1} \mu_k(\chi_k, A^{-1}h_j).$$

Now we will construct  $\hat{\varphi}(\chi)$  using the equations

$$\hat{\varphi}(\chi) = \hat{\varphi}(1) \prod_{k=0}^{\infty} m_0(\chi A^{-k}), \quad \hat{\varphi}(1) = 1.$$

If  $\chi \in \mathfrak{G}_{-1}^{\perp}$ , then  $\chi A^{-k} \in \mathfrak{G}_{-1-k}^{\perp} \subset \mathfrak{G}_{-1}^{\perp}$ . Consequently  $\hat{\varphi}(\mathfrak{G}_{-1}^{\perp}) = 1$ .

If  $\chi \in \mathfrak{G}_{-1}^{\perp} r_{-1}^{\alpha}$  ( $1 \leq \alpha \leq p-1$ ), then  $m_0(\chi) = \mu_{\alpha}$  and  $m_0(\chi A^{-k}) = 1$  when  $k \geq 1$ . Consequently  $\hat{\varphi}(\mathfrak{G}_{-1}^{\perp} r_{-1}^{\alpha}) = \mu_{\alpha}$  ( $1 \leq \alpha \leq p-1$ ).

If  $\chi \notin \mathfrak{G}_0^{\perp}$  then there exists  $n \geq 0$  such that  $\chi \in \mathfrak{G}_{n+1}^{\perp} \setminus \mathfrak{G}_n^{\perp}$ . From it follows  $\chi A^{-n} \in \mathfrak{G}_1^{\perp} \setminus \mathfrak{G}_0^{\perp}$  and  $m_0(\chi A^{-n}) = 0$ . Consequently  $\hat{\varphi}(\chi) = 0$ . Thus we have

$$\hat{\varphi}(\chi) = \begin{cases} \mu_j, & \text{if } \chi \in \mathfrak{G}_{-1}^{\perp} r_{-1}^j \ (j = \overline{0, p-1}), \\ 0, & \text{if } \chi \notin \mathfrak{G}_0^{\perp}. \end{cases}$$

Let us find  $\varphi(x)$ . We have

$$\varphi(x) = \int_X \hat{\varphi}(\chi)(\chi, x) d\nu(\chi) = \sum_{j=0}^{p-1} \mu_j \int_{\mathfrak{G}_{-1}^{\perp} r_{-1}^j} (\chi, x) d\nu(\chi).$$

Let us prove that

$$\int_{\mathfrak{G}_{-1}^{\perp}} (\chi, x) d\nu(\chi) = \frac{1}{p} \mathbf{1}_{\mathfrak{G}_{-1}}(x). \quad (4.7)$$

We will consider elements  $x \in \mathfrak{G}$  as characters of group  $X$ . Denote  $\tilde{x} = x|_{\mathfrak{G}_{-1}^{\perp}}$ . Then  $\tilde{x} = a_{-2}g_{-2} + a_{-3}g_{-3} + \dots + a_{-N}g_{-N} \in H_{-1}$ . Therefore  $H_{-1}$  is a characters group of the compact group  $\mathfrak{G}_{-1}^{\perp}$ , consequently  $H_{-1}$  is orthogonal system and we have

$$\int_{\mathfrak{G}_{-1}^{\perp}} (\chi, h) \overline{(\chi, g)} d\nu(\chi) = \delta_{h,g} = \begin{cases} 0, & h \neq g, \\ \frac{1}{p}, & h = g. \end{cases} \quad (4.8)$$

If  $x \in \mathfrak{G}_{-1}$ , then  $(\chi, x) = 1$  for  $\chi \in \mathfrak{G}_{-1}^{\perp}$ . Therefore

$$\int_{\mathfrak{G}_{-1}^{\perp}} (\chi, x) d\nu(\chi) = \nu(\mathfrak{G}_{-1}^{\perp}) = \frac{1}{p}.$$

If  $x \notin \mathfrak{G}_{-1}$ , then  $x \in H_{-1}$  and  $x \neq 0$ . Consequently we have from (4.8)

$$\int_{\mathfrak{G}_{-1}^{\perp}} (\chi, x)(\chi, 0) d\nu(\chi) = 0.$$

The equality (4.7) is proved.

From the equality (4.7) we obtain

$$\begin{aligned} \int_{\mathfrak{G}_{-1}^{\perp} r_{-1}^j} (\chi, x) d\nu(\chi) &= \int_X (\chi, x) \mathbf{1}_{\mathfrak{G}_{-1}^{\perp} r_{-1}^j}(\chi) d\nu(\chi) \\ &= \int_X (\chi r_{-1}^j, x) \mathbf{1}_{\mathfrak{G}_{-1}^{\perp} r_{-1}^j}(\chi r_{-1}^j) d\nu(\chi) = (r_{-1}^j, x) \int_X (\chi, x) \mathbf{1}_{\mathfrak{G}_{-1}^{\perp}}(\chi) d\nu(\chi) \\ &= \frac{1}{p} (r_{-1}^j, x) \mathbf{1}_{\mathfrak{G}_{-1}}(x). \end{aligned}$$

Finally we obtain

$$\varphi(x) = \frac{1}{p} \mathbf{1}_{\mathfrak{G}_{-1}}(x) \sum_{j=0}^{p-1} \mu_j (r_{-1}, x)^j.$$



Now we will find the functions  $\psi_l$  ( $l = \overline{1, p-1}$ ). Let us denote for the convenience

$$h_j = h_{j_1+j_2p} = j_1g_{-1} \dot{+} j_2g_{-2}, \quad \chi_k = \chi_{k_1+k_2p} = r_{-1}^{k_1} \cdot r_0^{k_2}, \quad \beta_j = \beta_{j_1+j_2p}$$

( $j_1, j_2, k_1, k_2 = \overline{0, p-1}$ ).

By the definition of  $\psi_l$

$$\psi_l(x) = \sum_{j_1=0}^{p-1} \sum_{j_2=0}^{p-1} \beta_{j_1+j_2p} (r_0^l, A^{-1}h_{j_1+j_2p}) \varphi(Ax \dot{-} h_{j_1+j_2p}).$$

Since  $A^{-1}g_{-1} = g_0$ ,  $A^{-1}g_{-2} = g_{-1}$ ,  $(r_{-1}, g_0) = 1$ , we have

$$\begin{aligned} (r_0^l, A^{-1}h_{j_1+j_2p}) &= (r_0, j_1g_0 \dot{+} j_2g_{-1})^l = (r_0, g_0)^{j_1l} (r_0, g_{-1})^{j_2l}, \\ \beta_{j_1+j_2p} &= \frac{1}{p} \sum_{k_1=0}^{p-1} \mu_{k_1}(r_{-1}^{k_1}, j_1g_0 \dot{+} j_2g_{-1}) = \frac{1}{p} \sum_{k_1=0}^{p-1} \mu_{k_1}(r_{-1}, g_{-1})^{k_1j_2}, \\ \mathbf{1}_{\mathfrak{G}_{-1}}(Ax \dot{-} h_{j_1+j_2p}) &= \mathbf{1}_{\mathfrak{G}_0}(x \dot{-} A^{-1}h_{j_1+j_2p}) = \mathbf{1}_{\mathfrak{G}_0 \dot{+} j_1g_0 \dot{+} j_2g_{-1}}(x) = \mathbf{1}_{\mathfrak{G}_0 \dot{+} j_2g_{-1}}(x), \\ \sum_{i=0}^{p-1} \mu_i(r_{-1}, Ax \dot{-} h_{j_1+j_2p})^i &= \sum_{i=0}^{p-1} \mu_i(r_0, x)^i \overline{(r_{-1}, j_1g_{-1} \dot{+} j_2g_{-2})^i}. \end{aligned}$$

Therefore

$$\varphi(Ax \dot{-} h_{j_1+j_2p}) = \frac{1}{p} \mathbf{1}_{\mathfrak{G}_0 \dot{+} j_2g_{-1}}(x) \sum_{i=0}^{p-1} \mu_i(r_0, x)^i \overline{(r_{-1}, j_1g_{-1} \dot{+} j_2g_{-2})^i}.$$

Since  $(r_{-1}, g_{-1}) = (r_0, g_0)$ , then

$$\sum_{j_1=0}^{p-1} \overline{(r_{-1}, g_{-1})^{j_1i}} (r_0, g_0)^{j_1l} = p \delta_{il}$$

and we get finally

$$\psi_l(x) = \mu_l \frac{(r_0, x)^l}{p} \sum_{j=0}^{p-1} \mathbf{1}_{\mathfrak{G}_0 \dot{+} jg_{-1}}(x) \sum_{k=0}^{p-1} \mu_k(r_{-1}, g_{-1})^{kj}. \quad (4.9)$$

We can write wavelets  $\psi_l$  in the form

$$\psi_l(x) = \mu_l(r_0, x)^l \varphi(x). \quad (4.10)$$

Indeed, take  $x \in \mathfrak{G}_0 \dot{+} jg_{-1}$ . Then

$$\frac{1}{p} \sum_{j=0}^{p-1} \mathbf{1}_{\mathfrak{G}_0 \dot{+} jg_{-1}}(x) \sum_{k=0}^{p-1} \mu_k(r_{-1}, g_{-1})^{kj} = \frac{1}{p} \sum_{k=0}^{p-1} \mu_k(r_{-1}, g_{-1})^{kj}. \quad (4.11)$$

On the other hand for  $x \in \mathfrak{G}_0 \dot{+} jg_{-1}$

$$\varphi(x) = \frac{1}{p} \sum_{k=0}^{p-1} \mu_k(r_{-1}, x)^k = \frac{1}{p} \sum_{k=0}^{p-1} \mu_k(r_{-1}, g_{-1})^{kj}. \quad (4.12)$$

Combining (4.11) and (4.12) we obtain (4.10).

If  $\mu_0 = \mu_1 = \dots = \mu_{p-1} = 1$ , then

$$\psi_l(x) = (r_0, x)^l \mathbf{1}_{\mathfrak{G}_0}(x).$$

We will use wavelet functions  $\psi_l$  in Section 6.

## 5. Dilation operators in $\mathfrak{G}^d$

For construction MRA in  $L_2(\mathfrak{G}^d)$  we need to have a dilation operator  $A_d$  in  $\mathfrak{G}^d$ . For definition of the dilation operator in  $\mathfrak{G}^d$  we need to have a basic chain in  $\mathfrak{G}^d$ . We describe an algorithm for construction such basic chain.

Let  $(\mathfrak{G}, \dot{+})$  be a compact zero-dimensional group with a basic chain

$$\cdots \supset \mathfrak{G}_{-n} \supset \cdots \supset \mathfrak{G}_{-1} \supset \mathfrak{G}_0 \supset \mathfrak{G}_1 \supset \cdots \supset \mathfrak{G}_n \supset \cdots.$$

We denote by  $G = \mathfrak{G}^d = \mathfrak{G} \times \mathfrak{G} \times \cdots \times \mathfrak{G}$  the direct sum of  $d$  copies of group  $\mathfrak{G}$ . The base of neighborhood of zero in  $\mathfrak{G}^d$  consists of all products  $\mathfrak{G}_{n_1} \times \mathfrak{G}_{n_2} \times \cdots \times \mathfrak{G}_{n_d}$ . We can take the chain of  $d$ -dimensional cubes

$$\mathfrak{G}_n \times \mathfrak{G}_n \times \cdots \times \mathfrak{G}_n = \mathfrak{G}_n^d$$

as a base of neighborhood of zero in  $\mathfrak{G}^d$ . We note that the chain

$$\cdots \supset \mathfrak{G}_{-n}^d \supset \cdots \supset \mathfrak{G}_{-1}^d \supset \mathfrak{G}_0^d \supset \mathfrak{G}_1^d \supset \cdots \supset \mathfrak{G}_n^d \supset \cdots \quad (5.1)$$

is not a basic chain, since  $(\mathfrak{G}_n^d / \mathfrak{G}_{n+1}^d)^\# = p^d$  is not a prime number. Denote  $\mathfrak{G}_n^d$  as  $G_{nd}$  and refine the chain (5.1) to obtain a basic chain in the following way.

Let  $\mathfrak{G}_{n+1}^d \subset \mathfrak{G}_n^d$ . Take a nonzero element

$$\mathbf{g}_{(n+1)d-1} = (a_{1,0}^{(n)} g_n, a_{1,1}^{(n)} g_n, \dots, a_{1,d-1}^{(n)} g_n) \in G_{nd} \setminus G_{(n+1)d}.$$

Cosets

$$(G_{(n+1)d} \dot{+} j_1 \mathbf{g}_{(n+1)d-1})_{j_1=0}^{p-1}$$

form a group of prime order  $p$ , whence the set

$$G_{(n+1)d-1} = \bigsqcup_{j_1=0}^{p-1} (G_{(n+1)d} \dot{+} j_1 \mathbf{g}_{(n+1)d-1}),$$

where  $\bigsqcup$  stands for the disjoint union, is a group such that

$$G_{(n+1)d} \subset G_{(n+1)d-1} \subset G_{nd}, \quad (G_{(n+1)d-1} / G_{(n+1)d})^\# = p,$$

and

$$(G_{nd} / G_{(n+1)d-1})^\# = p^{d-1}.$$

If  $d-1 > 1$ , we take an element

$$\mathbf{g}_{(n+1)d-2} = (a_{2,0}^{(n)} g_n, a_{2,1}^{(n)} g_n, \dots, a_{2,d-1}^{(n)} g_n) \in G_{nd} \setminus G_{(n+1)d-1}.$$

Similarly to the previous situation, we conclude that the set

$$G_{(n+1)d-2} = \bigsqcup_{j_2=0}^{p-1} (G_{(n+1)d-1} \dot{+} j_2 \mathbf{g}_{(n+1)d-2})$$

is a group such that

$$G_{(n+1)d} \subset G_{(n+1)d-1} \subset G_{(n+1)d-2} \subset G_{nd}, \quad (G_{(n+1)d-2} / G_{(n+1)d-1})^\# = p$$

and

$$(G_{nd} / G_{(n+1)d-2})^\# = p^{d-2}.$$

Continuing this process we obtain for any  $n$  matrix  $\mathcal{A}_n = (a_{l,v}^{(n)})$  ( $a_{l,v}^{(n)} = \overline{0, p-1}$ ;  $v = \overline{0, d-1}$ ;  $l = \overline{1, d}$ ) the set of subgroups  $G_{(n+1)d-l}$  and the set of elements  $\mathbf{g}_{(n+1)d-l} \in G_{nd} \setminus G_{(n+1)d-l+1}$  such that

$$\mathbf{g}_{(n+1)d-l} = (a_{l,0}^{(n)} g_n, a_{l,1}^{(n)} g_n, \dots, a_{l,d-1}^{(n)} g_n), \quad (5.2)$$

$$G_{(n+1)d-l} = \bigsqcup_{j_l=0}^{p-1} (G_{(n+1)d-l+1} \dot{+} j_l \mathbf{g}_{(n+1)d-l}),$$

$$(G_{(n+1)d-l} / G_{(n+1)d-l+1})^\# = p. \quad (5.3)$$

The first row of the matrix  $\mathcal{A}_n = (a_{l,v}^{(n)})$  we can choose arbitrarily. The  $l$ th row of the matrix  $\mathcal{A}_n$  we choose such that

$$(a_{l,0}^{(n)}, a_{l,1}^{(n)}, \dots, a_{l,d-1}^{(n)}) \neq ((a_{1,0}^{(n)}j_1 + a_{2,0}^{(n)}j_2 + \dots + a_{l-1,0}^{(n)}j_{l-1}) \bmod p, \dots, (a_{1,d-1}^{(n)}j_1 + a_{2,d-1}^{(n)}j_2 + \dots + a_{l-1,d-1}^{(n)}j_{l-1}) \bmod p) \quad (5.4)$$

for all  $j_1, j_2, \dots, j_{l-1} = \overline{0, p-1}$ . Thus we obtain the nested sequence of subgroups

$$\dots \supset G_{-n} \supset \dots \supset G_{-1} \supset G_0 \supset G_1 \supset \dots \supset G_n \supset \dots \quad (5.5)$$

and the sequence of elements  $\mathbf{g}_n \in G_n \setminus G_{n+1}$ , such that

$$(G_n/G_{n+1})^\# = p, \quad n \in \mathbb{Z}.$$

**Theorem 5.1.** *The topology and measure generated by the basic chain (5.5) coincide with the topology and measure generated by the chain  $(G_{nd}) = (\mathfrak{G}_n^d)$ .*

**Proof.** 1) Any coset  $G_{nd+l} + \mathbf{h} \subset G_{nd}$  is a finite union of cosets

$$G_{(n+1)d} + (j_0 \mathbf{g}_n, j_1 \mathbf{g}_n, \dots, j_{d-1} \mathbf{g}_n);$$

therefore topologies generated by chains  $(G_n)$  and  $(G_{nd})$  coincide.

2) Cosets  $G_{nd+l} + \mathbf{h}$  together with empty set form a semiring  $\mathfrak{M}_d$ . Define a measure  $\mu$  on  $\mathfrak{M}_d$  as follows:

$$\mu(G_{nd+l} + \mathbf{h}) = \mu G_{nd+l} = \frac{1}{p^{nd+l}}.$$

Let  $\mu^*$  be the corresponding outer measure. Cosets  $G_{nd} + \mathbf{g}$  together with the empty set form a semiring  $\mathfrak{N}_d$ . Define a measure  $m$  on  $\mathfrak{N}_d$  by

$$m(G_{nd} + \mathbf{g}) = m G_{nd} = \frac{1}{p^{nd}}.$$

Let  $m^*$  be the corresponding outer measure.

It is enough to prove that  $m^* = \mu^*$ . It is evident that  $\mathfrak{N}_d \subset \mathfrak{M}_d$ , thus measures  $m$  and  $\mu$  coincide on  $\mathfrak{N}_d$ . Therefore,  $\mu^* E \leq m^* E$  for any  $E \in G$ . On the other hand, any coset  $G_{nd+l} + \mathbf{h} \in \mathfrak{M}_d$  is a finite union of  $p_n^{d-l}$  disjoint cosets  $G_{(n+1)d} + \mathbf{g} \in \mathfrak{N}_d$  and

$$\begin{aligned} \mu(G_{nd+l} + \mathbf{h}) &= p^{d-l} \cdot \frac{1}{p^{(n+1)d}} \\ &= p^{d-l} m(G_{(n+1)d} + \mathbf{g}) = \sum m(G_{(n+1)d} + \mathbf{g}). \end{aligned}$$

It means that any covering of the set  $E$  by cosets  $G_{nd+l} + \mathbf{h} \in \mathfrak{M}_d$  is a covering by sets  $G_{(n+1)d} \in \mathfrak{N}_d$  with preservation of the sum of measures of covering cosets. Therefore,  $\mu^* E \geq m^* E$  and consequently  $\mu^* E = m^* E$ .  $\square$

It follows from this theorem that the set of subgroup  $(G_{nd+l})$  is a basic chain, and system  $(\mathbf{g}_n)$  is a basic system. Let us observe, that any sequence  $(\tilde{\mathbf{g}}_n)$  for which  $\tilde{\mathbf{g}}_n \in G_{n+1} \setminus G_n$  is a basic system. Using this algorithm we can obtain all basic chains in  $\mathfrak{G}^d$ .

**Theorem 5.2.** *Let*

$$\mathfrak{G}_{n+1}^d = \tilde{G}_{(n+1)d} \subset \tilde{G}_{(n+1)d-1} \subset \tilde{G}_{(n+1)d-2} \subset \dots \subset \tilde{G}_{nd+1} \subset \tilde{G}_{nd} = \mathfrak{G}_n^d$$

*be a chain of distinct enclosed subgroups,  $\tilde{G}_{(n+1)d} = G_{(n+1)d}$ ,  $\tilde{G}_{nd} = G_{nd}$ . Then there exists a matrix  $\mathcal{A}_n = (a_{l,v}^{(n)})$  ( $a_{l,v}^{(n)} = \overline{0, p-1}$ ;  $v = \overline{0, d-1}$ ;  $l = \overline{1, d}$ ) such that*

$$\tilde{G}_{(n+1)d-l} = \bigsqcup_{j=0}^{p-1} (\tilde{G}_{(n+1)d-l+1} + j \mathbf{g}_{(n+1)d-l}), \quad (5.6)$$

$$\mathbf{g}_{(n+1)d-l} = (a_{l,0}^{(n)} \mathbf{g}_n, a_{l,1}^{(n)} \mathbf{g}_n, \dots, a_{l,d-1}^{(n)} \mathbf{g}_n). \quad (5.7)$$

**Proof.** Let  $\tilde{G}$  be a subgroup such that  $G_{(n+1)d} \subset \tilde{G} \subset G_{nd}$ . Since  $(G_{nd}/G_{(n+1)d})^\# = p^d$ , it follows that  $(\tilde{G}/G_{(n+1)d})^\# = p^\gamma$  ( $1 \leq \gamma < d$ ). Therefore  $(\tilde{G}_{nd+l}/\tilde{G}_{nd+l+1})^\# = p$  for any  $l = \overline{0, d-1}$ . Since  $(\tilde{G}_{(n+1)d-1}/G_{(n+1)d})^\# = p$  is a prime number, it follows that for any element  $\mathbf{g} \in \tilde{G}_{(n+1)d-1} \setminus G_{(n+1)d}$  the factor group  $\tilde{G}_{(n+1)d-1}/G_{(n+1)d}$  is a collection of cosets

$$G_{(n+1)d} \dot{+} j\mathbf{g} \quad (j = 0, 1, \dots, p-1).$$

We can choose the element  $\mathbf{g} \in \tilde{G}_{(n+1)d-1} \setminus G_{(n+1)d}$  in the form

$$\mathbf{g} = (a_{1,0}^{(n)}g_n, a_{1,1}^{(n)}g_n, \dots, a_{1,d-1}^{(n)}g_n)$$

where  $a_{1,i}^{(n)} = \overline{0, p-1}$  and  $(a_{1,0}^{(n)}, a_{1,1}^{(n)}, \dots, a_{1,d-1}^{(n)}) \neq (0, 0, \dots, 0)$ . From it follows that

$$\tilde{G}_{(n+1)d-1} = \bigsqcup_{j=0}^{p-1} (G_{(n+1)d} \dot{+} j\mathbf{g}_{(n+1)d-1}) = G_{(n+1)d-1}.$$

Similarly we obtain (5.6) and (5.7) for any  $l = \overline{1, d}$ .  $\square$

We will assume that matrices  $\mathcal{A}_n$  are the same for any  $n \in \mathbb{Z}$  i.e.  $\mathcal{A}_n = \mathcal{A}$  for all  $n \in \mathbb{Z}$ .

Using this basic system  $(\mathbf{g}_n)$  we can define an operator  $A_d$  in the following way

$$A_d \mathbf{x} = \sum_{n \in \mathbb{Z}} a_n \mathbf{g}_{n-1} \quad \text{if } \mathbf{x} = \sum_{n \in \mathbb{Z}} a_n \mathbf{g}_n \quad (a_n = \overline{0, p-1}).$$

**Theorem 5.3.** If the operation  $\dot{+}$  in a group  $\mathfrak{G}$  satisfies the condition (3.1) then the operator  $A_d$  is additive, consequently  $A_d$  is a dilation operator.

**Proof.** We will prove that there exists a vector  $(\alpha_1^*, \alpha_2^*, \dots, \alpha_{\tau d}^*)$  ( $\alpha_j^* = \overline{0, p-1}$ ), such that

$$p\mathbf{g}_n = \alpha_1^* \mathbf{g}_{n+1} \dot{+} \alpha_2^* \mathbf{g}_{n+2} \dot{+} \dots \dot{+} \alpha_{\tau d}^* \mathbf{g}_{n+\tau d}. \quad (5.8)$$

By the condition (3.1)

$$p\mathbf{g}_n = c_1 \mathbf{g}_{n+1} \dot{+} c_2 \mathbf{g}_{n+2} \dot{+} \dots \dot{+} c_\tau \mathbf{g}_{n+\tau}.$$

Therefore

$$\begin{aligned} p\mathbf{g}_{nd+l} &= p\mathbf{g}_{(n+1)d-d+l} = p(a_{d-l,0}^{(n)}g_n, a_{d-l,1}^{(n)}g_n, \dots, a_{d-l,d-1}^{(n)}g_n) \\ &= (a_{d-l,0}^{(n)}c_1 \mathbf{g}_{n+1}, a_{d-l,1}^{(n)}c_1 \mathbf{g}_{n+1}, \dots, a_{d-l,d-1}^{(n)}c_1 \mathbf{g}_{n+1}) \\ &\quad \dot{+} (a_{d-l,1}^{(n)}c_2 \mathbf{g}_{n+2}, a_{d-l,1}^{(n)}c_2 \mathbf{g}_{n+2}, \dots, a_{d-l,d-1}^{(n)}c_2 \mathbf{g}_{n+2}) \\ &\quad \vdots \\ &\quad \dot{+} (a_{d-l,0}^{(n)}c_\tau \mathbf{g}_{n+\tau}, a_{d-l,1}^{(n)}c_\tau \mathbf{g}_{n+\tau}, \dots, a_{d-l,d-1}^{(n)}c_\tau \mathbf{g}_{n+\tau}) \\ &= c_1 \mathbf{g}_{(n+2)d-d+l} \dot{+} c_2 \mathbf{g}_{(n+3)d-d+l} \dot{+} \dots \dot{+} c_\tau \mathbf{g}_{(n+\tau+1)d-d+l} \\ &= 0 \cdot \mathbf{g}_{nd+l+1} \dot{+} 0 \cdot \mathbf{g}_{nd+l+2} \dot{+} \dots \dot{+} 0 \cdot \mathbf{g}_{nd+l+(d-1)} \dot{+} c_1 \mathbf{g}_{(n+1)d+l} \\ &\quad \dot{+} 0 \cdot \mathbf{g}_{(n+1)d+l+1} \dot{+} 0 \cdot \mathbf{g}_{(n+1)d+l+2} \dot{+} \dots \dot{+} 0 \cdot \mathbf{g}_{(n+1)d+l+(d-1)} \dot{+} c_2 \mathbf{g}_{(n+2)d+l} \\ &\quad \vdots \\ &\quad \dot{+} 0 \cdot \mathbf{g}_{(n+\tau-1)d+l+1} \dot{+} 0 \cdot \mathbf{g}_{(n+\tau-1)d+l+2} \dot{+} \dots \dot{+} 0 \cdot \mathbf{g}_{(n+\tau-1)d+l+(d-1)} \dot{+} c_\tau \mathbf{g}_{(n+\tau)d+l}. \end{aligned}$$

Thus (5.8) is fulfilled with  $(\alpha_1^*, \alpha_2^*, \dots, \alpha_{\tau d}^*) = (\underbrace{0, 0, \dots, 0}_{d-1}, \underbrace{c_1, 0, \dots, 0}_{d-1}, \underbrace{0, c_2, \dots, 0}_{d-1}, \dots, \underbrace{0, 0, \dots, 0}_{d-1}, c_\tau)$ .  $\square$

**Definition 5.1.** We will call the matrix  $\mathcal{A}$  a matrix of dilation operator  $A_d$ .

Now we want to get a condition under which a matrix  $\mathcal{A}$  is the matrix of some dilation operator. Let  $\mathbf{Z}_p$  be a residue-class field i.e.  $\mathbf{Z}_p = \{0, 1, \dots, p-1\}$  with operations  $m \dot{+} n = (m+n) \bmod p$ ,  $m \cdot n = \underbrace{m \dot{+} m \dot{+} \dots \dot{+} m}_n$ .

**Theorem 5.4.** Let  $\mathcal{A} = (a_{i,j})_{i=1,j=0}^{d-1,d}$  be a nonsingular matrix over the field  $\mathbf{Z}_p$ . Then  $\mathcal{A}$  is the matrix of a dilation operator in  $\mathfrak{G}^d$ .

**Proof.** Since  $\mathcal{A}$  is a nonsingular matrix over the field  $\mathbf{Z}_p$  then conditions (5.4) are fulfilled. Hence  $\mathcal{A}$  is a matrix of dilation operator  $A_d$ .  $\square$

## 6. MRA on product of zero-dimensional groups

Using Theorems 4.1, 5.1–5.3 we obtain

**Theorem 6.1.** Let  $(\mathfrak{G}, \dot{+})$  be a locally compact zero-dimensional abelian group with a basic chain  $(\mathfrak{G}_n)_{n \in \mathbb{Z}}$  and a basic system  $\mathbf{g}_n \in \mathfrak{G}_n \setminus \mathfrak{G}_{n+1}$  ( $n \in \mathbb{Z}$ ) such that

- 1)  $(\mathfrak{G}_n / \mathfrak{G}_{n+1})^\# = p$ ,  $p$  is a prime number;
- 2) there exist numbers  $c_1, c_2, \dots, c_\tau \in \overline{0, p-1}$ , such that

$$p\mathbf{g}_n = c_1\mathbf{g}_{n+1} \dot{+} c_2\mathbf{g}_{n+2} \dot{+} \dots \dot{+} c_\tau\mathbf{g}_{n+\tau}.$$

Let  $G = \mathfrak{G}^d$  ( $d \geq 2$ ) be a direct sum of  $d$  copies of groups  $\mathfrak{G}$ , basic chain  $(G_{(n+1)d-l})$  and basic system  $\mathbf{g}_{(n+1)d-l}$  be defined by Eqs. (5.2) and (5.3). Let  $\varphi \in L_2(\mathfrak{G}^d)$  be a solution of the equation

$$\varphi(\mathbf{x}) = \sum_{\mathbf{h} \in H} c_{\mathbf{h}} \varphi(A_d(\mathbf{x} \dot{-} \mathbf{h}))$$

where  $H \subset H_0$  is a finite set,  $A_d$  is  $d$ -dimensional dilation operator. Suppose that  $\hat{\varphi}(\chi) = \mathbf{1}_{G_0^\perp}(\chi)$ . Then  $\varphi$  generates MRA on  $L_2(G)$ .

To construct wavelet functions on  $\mathfrak{G}^d$  we need to have a Rademacher function on  $\mathfrak{G}^d$ .

**Theorem 6.2.** Let  $\mathcal{A} = (a_{i,j})$  be a nonsingular matrix of dilation operator  $A_d$ . Assume that numbers  $\gamma_j \in \overline{0, d-1}$  ( $j = 0, 1, \dots, p-1$ ) satisfy the congruences system

$$\gamma_0 a_{l,0} + \gamma_1 a_{l,1} + \dots + \gamma_{d-1} a_{l,d-1} \equiv 0 \pmod{p} \quad (l = \overline{1, d-1}), \quad (6.1)$$

$$\gamma_0 a_{d,0} + \gamma_1 a_{d,1} + \dots + \gamma_{d-1} a_{d,d-1} \equiv 1 \pmod{p}. \quad (6.2)$$

Then

- 1) the function

$$\mathbf{r}_0(\mathbf{t}) := r_0(\gamma_0 t^{(0)}) \cdot r_0(\gamma_1 t^{(1)}) \cdot \dots \cdot r_0(\gamma_{d-1} t^{(d-1)}) \quad (6.3)$$

is a Rademacher function on  $G = \mathfrak{G}^d$ .

- 2)  $\mathbf{r}_0(G_1 \dot{+} \mathbf{g}_0) = r_0(\mathbf{g}_0)$ , where  $r_0$  is one-dimensional Rademacher function.

**Proof.** Since the matrix  $\mathcal{A} = (a_{i,j})$  is nonsingular then a solution of congruences system (6.1)–(6.2) exists. We need to prove that  $\mathbf{r}_0(G_1) = 1$  and  $\mathbf{r}_0(G_1 \dot{+} j\mathbf{g}_0) \neq 1$  when  $j = \overline{1, d-1}$ .

Let  $\mathbf{x}_1 \in G_1$ . By the construction of subgroups  $G_{nd+l}$  we have

$$\mathbf{x}_1 = \mathbf{x}_2 \dot{+} j_{d-1}(a_{d-1,0}\mathbf{g}_0, a_{d-1,1}\mathbf{g}_0, \dots, a_{d-1,d-1}\mathbf{g}_0)$$

where  $\mathbf{x}_2 \in G_2$ . By analogy

$$\mathbf{x}_2 = \mathbf{x}_3 \dot{+} j_{d-2}(a_{d-2,0}\mathbf{g}_0, a_{d-2,1}\mathbf{g}_0, \dots, a_{d-2,d-1}\mathbf{g}_0),$$

where  $\mathbf{x}_3 \in G_3$ , consequently

$$\begin{aligned} \mathbf{x}_1 = & \mathbf{x}_3 \dot{+} j_{d-1}(a_{d-1,0}\mathbf{g}_0, a_{d-1,1}\mathbf{g}_0, \dots, a_{d-1,d-1}\mathbf{g}_0) \\ & \dot{+} j_{d-2}(a_{d-2,0}\mathbf{g}_0, a_{d-2,1}\mathbf{g}_0, \dots, a_{d-2,d-1}\mathbf{g}_0). \end{aligned}$$

Finally we obtain

$$\begin{aligned}\mathbf{x}_1 &= \mathbf{x}_d \dot{+} j_{d-1}(a_{d-1,0}g_0, a_{d-1,1}g_0, \dots, a_{d-1,d-1}g_0) \\ &\quad \dot{+} j_{d-2}(a_{d-2,0}g_0, a_{d-2,1}g_0, \dots, a_{d-2,d-1}g_0) \\ &\quad \vdots \\ &\quad \dot{+} j_1(a_{1,0}g_0, a_{1,1}g_0, \dots, a_{1,d-1}g_0), \quad \mathbf{x}_d \in G_d.\end{aligned}$$

By the definition of  $\mathbf{r}_0$  we have

$$\begin{aligned}\mathbf{r}_0(\mathbf{x}_1) &= r_0(\gamma_0 x^{(0)}) r_0(\gamma_1 x^{(1)}) \cdots r_0(\gamma_{d-1} x^{(d-1)}) \\ &= \prod_{l=1}^{d-1} r_0(j_l(\gamma_0 a_{1,0} + \gamma_1 a_{l,1} + \cdots + \gamma_{d-1} a_{l,d-1}) g_0).\end{aligned}$$

Using (6.1) we obtain  $\mathbf{r}_0(\mathbf{x}) = 1$ .

Let  $\mathbf{x} \in G_1 \dot{+} j\mathbf{g}_0$ ,  $j \neq 0$ . Using (6.2) we have

$$\begin{aligned}\mathbf{r}_0(G_1 \dot{+} j\mathbf{g}_0) &= \mathbf{r}_0(j\mathbf{g}_0) \\ &= r_0(j\gamma_0 a_{d,0}g_0) \cdot r_0(j\gamma_1 a_{d,1}g_0) \cdots r_0(j\gamma_{d-1} a_{d,d-1}g_0) \\ &= r_0(jg_0(\gamma_0 a_{d,0} + \gamma_1 a_{d,1} + \cdots + \gamma_{d-1} a_{d,d-1})) \neq 1.\end{aligned}$$

If  $j = 1$  and  $\gamma_0 a_{d,0} + \gamma_1 a_{d,1} + \cdots + \gamma_{d-1} a_{d,d-1} \equiv 1 \pmod{p}$ , then  $\mathbf{r}_0(G_1 \dot{+} \mathbf{g}_0) = r_0(g_0)$ . This completes the proof of Theorem 6.2.  $\square$

**Example 6.1.** We will use Theorem 6.1 to construct wavelet bases in  $L_2(\mathfrak{G}^d)$ . Let  $\mathcal{A} = (a_{i,k})_{i=1,k=0}^{d-1}$  be a matrix of the dilation operator  $A_d$ ,

$$\mathbf{g}_{(n+1)d-l} = (a_{l,0}\mathbf{g}_n, a_{l,1}\mathbf{g}_n, \dots, a_{l,d-1}\mathbf{g}_n) \in G_{(n+1)d-l} \setminus G_{(n+1)d-l+1}$$

a basic system,  $(G_{(n+1)d-l})_{n \in \mathbb{Z}, l=1, \dots, d}$  a basic chain corresponding to the matrix  $\mathcal{A}$ . Suppose integers  $\gamma_0, \gamma_1, \dots, \gamma_{d-1}$  satisfy system (6.1)–(6.2),  $\mathbf{r}_0(\mathbf{x})$  is defined by Eq. (6.3). By the definition of operator  $A_d$  and Rademacher function  $\mathbf{r}_n$  we have  $(\mathbf{r}_n, \mathbf{x}) = (\mathbf{r}_0 A_d^n \mathbf{x})$ , therefore  $(\mathbf{r}_n, \mathbf{g}_n) = (\mathbf{r}_0, A_d^n \mathbf{g}_n) = (\mathbf{r}_0, \mathbf{g}_0) = (r_0, g_0)$ .

We will find a refinable function  $\varphi(\mathbf{x})$  for which  $\hat{\varphi}(\chi) = \mathbf{1}_{G_0^\perp}(\chi)$  as a solution of the refinable equation

$$\varphi(\mathbf{x}) = \sum_{\mathbf{h} \in H_0^{(2)}} C_{\mathbf{h}} \varphi(A_d(\mathbf{x} \dot{-} \mathbf{h}))$$

where  $H_0^{(2)} = \{(\alpha_{-1}\mathbf{g}_{-1} + \alpha_{-2}\mathbf{g}_{-2})\}$ ,  $\alpha_j = \overline{0, p-1}$ . Using Example 4.1 we find the refinable function

$$\varphi(\mathbf{x}) = \frac{1}{p} \mathbf{1}_{G_{-1}}(\mathbf{x}) \sum_{j=0}^{p-1} \mu_j(\mathbf{r}_{-1}, \mathbf{x})^j \quad (\mu_0 = 1; |\mu_j| = 1, j = \overline{1, p-1}),$$

and wavelet functions

$$\psi_l(\mathbf{x}) = \frac{\mu_l}{p} (\mathbf{r}_0, \mathbf{x})^l \mathbf{1}_{G_{-1}}(\mathbf{x}) \sum_{k_1=0}^{p-1} \mu_{k_1}(\mathbf{r}_{-1}, \mathbf{x})^{k_1} \quad (l = \overline{1, p-1}), \quad (6.4)$$

where

$$\begin{aligned}G_{-1} &= \bigcup_{k=0}^{p-1} (G_0 \dot{+} k(a_{1,0}g_1, a_{1,1}g_1, \dots, a_{1,d-1}g_1)), \quad G_0 = \mathfrak{G}_0^d, \\ (\mathbf{r}_0, \mathbf{x}) &= r_0(\gamma_0 x^{(0)}) r_0(\gamma_1 x^{(1)}) \cdots r_0(\gamma_{d-1} x^{(d-1)}).\end{aligned}$$

It follows from (4.9) that we can write wavelet functions  $\psi_l$  in the form

$$\psi_l(\mathbf{x}) = \frac{\mu_l}{p} (\mathbf{r}_0, \mathbf{x})^l \sum_{k=0}^{p-1} \mathbf{1}_{G_0 \dot{+} k\mathbf{g}_{-1}}(\mathbf{x}) \sum_{j=0}^{p-1} (\mathbf{r}_0, \mathbf{g}_0)^{jk} \mu_j.$$

If

$$\mathbf{x} \in G_0 \dot{+} k\mathbf{g}_{-1} = \bigsqcup_{j_d=0}^{p-1} (G_1 \dot{+} j_d(a_{d,0}g_0, a_{d,1}g_0, \dots, a_{d,d-1}g_0)) \dot{+} k\mathbf{g}_{-1}$$

then  $\mathbf{x} \in G_1 \dot{+} j_d\mathbf{g}_0 \dot{+} k\mathbf{g}_{-1}$  where

$$\mathbf{g}_0 = (a_{d,0}g_0, a_{d,1}g_0, \dots, a_{d,d-1}g_0), \quad \mathbf{g}_{-1} = (a_{1,0}g_{-1}, a_{1,1}g_{-1}, \dots, a_{1,d-1}g_{-1}).$$

Therefore

$$(\mathbf{r}_0, \mathbf{x}) = (\mathbf{r}_0, \mathbf{g}_0)^{j_d} (\mathbf{r}_0, \mathbf{g}_{-1})^k.$$

Thus if  $\mathbf{x} \in G_1 \dot{+} j_d\mathbf{g}_0 \dot{+} k\mathbf{g}_{-1}$  then

$$\psi_l(\mathbf{x}) = \frac{\mu_l}{p} (\mathbf{r}_0, \mathbf{g}_0)^{j_d l} (\mathbf{r}_0, \mathbf{g}_{-1})^{kl} \sum_{j=0}^{p-1} \mu_j (\mathbf{r}_0, \mathbf{g}_0)^{kj}. \quad (6.5)$$

We have  $(\mathbf{r}_0, \mathbf{g}_0) = (r_0, g_0) = e^{\frac{2\pi i}{p}}$  for any group  $(\mathfrak{G}, \dot{+})$ . The value  $(\mathbf{r}_0, \mathbf{g}_{-1})$  depends on group operation  $\dot{+}$ . If  $(\mathfrak{G}, \dot{+})$  is a Vilenkin group ( $pg_n = 0$ ), then  $(r_0, g_{-1}) = (r_0, g_0) = e^{\frac{2\pi i}{p}}$ , hence

$$(\mathbf{r}_0, \mathbf{g}_{-1}) = r_0(\gamma_0 a_{1,0}g_{-1}) r_0(\gamma_1 a_{1,1}g_{-1}) \cdots r_0(\gamma_{d-1} a_{1,d-1}g_{-1}) = 1.$$

Substituting  $(\mathbf{r}_0, \mathbf{g}_0)$  and  $(\mathbf{r}_0, \mathbf{g}_{-1})$  in (6.5) we get

$$\psi_l(\mathbf{x}) = \frac{\mu_l}{p} \exp\left(\frac{2\pi i}{p} j_d l\right) \sum_{j=0}^{p-1} \mu_j \exp\left(\frac{2\pi i}{p} k j\right).$$

If  $(\mathfrak{G}, \dot{+}) = \mathbb{Q}_p$  ( $pg_n = g_{n+1}$ ), then  $(r_0, g_{-1}) = e^{\frac{2\pi i}{p^2}}$ ,  $(r_0, g_0) = e^{\frac{2\pi i}{p}}$ , hence

$$\begin{aligned} (\mathbf{r}_0, \mathbf{g}_{-1}) &= r_0(\gamma_0 a_{1,0}g_{-1}) r_0(\gamma_1 a_{1,1}g_{-1}) \cdots r_0(\gamma_{d-1} a_{1,d-1}g_{-1}) \\ &= e^{\frac{2\pi i}{p^2} (\gamma_0 a_{1,0} + \gamma_1 a_{1,1} + \cdots + \gamma_{d-1} a_{1,d-1})} \end{aligned}$$

and we get more complicated formula for  $\psi_l$ .

By Theorem 4.2 the system

$$p^{\frac{n}{2}} \psi_l(A_d^n(\mathbf{x} \dot{+} \mathbf{h})) \quad (n \in \mathbb{Z}, \mathbf{h} \in H_0) \quad (6.6)$$

is an orthonormal bases in  $L_2(G) = L_2(\mathfrak{G}^d)$ . The set  $H_0$  in (6.6) is the set of shifts

$$H_0 = \{\alpha_{-1}\mathbf{g}_{-1} \dot{+} \alpha_{-2}\mathbf{g}_{-2} \dot{+} \cdots \dot{+} \alpha_{-N}\mathbf{g}_{-N}\}$$

where

$$\mathbf{g}_{-md-l} = (a_{l,0}g_{-m}, a_{l,1}g_{-m}, \dots, a_{l,d-1}g_{-m}).$$

**Example 6.2.** Let us consider the case  $p = d = 2$ ,  $pg_n = g_{n+1}$  in detail. Under this conditions  $(\mathfrak{G}, \dot{+})$  is the field of 2-adic numbers  $\mathbb{Q}_2$ ,  $\mathfrak{G}_0 = \mathbb{Z}_2$ . We have 3 possibilities for the basic chain in  $\mathfrak{G}^2$ .

$$\begin{aligned} \text{A1)} \quad G_{2(n+1)-1} &= \bigsqcup_{j=0}^1 (\mathfrak{G}_{n+1} \times \mathfrak{G}_{n+1} \dot{+} j(g_n, 0)) = \mathfrak{G}_n \times \mathfrak{G}_{n+1}, \\ G_{2(n+1)-2} &= \bigsqcup_{j=0}^1 (\mathfrak{G}_{2(n+1)-1} \dot{+} j(0, g_n)) = \mathfrak{G}_n \times \mathfrak{G}_n, \\ \text{A2)} \quad G_{2(n+1)-1} &= \bigsqcup_{j=0}^1 (\mathfrak{G}_{n+1} \times \mathfrak{G}_{n+1} \dot{+} j(0, g_n)) = \mathfrak{G}_{n+1} \times \mathfrak{G}_n, \\ G_{2(n+1)-2} &= \bigsqcup_{j=0}^1 (\mathfrak{G}_{2(n+1)-1} \dot{+} j(g_n, 0)) = \mathfrak{G}_n \times \mathfrak{G}_n, \end{aligned}$$

$$A3) \quad G_{2(n+1)-1} = \bigsqcup_{j=0}^1 (\mathfrak{G}_{n+1} \times \mathfrak{G}_{n+1} \dot{+} j(g_n, g_n)),$$

$$G_{2(n+1)-2} = \bigsqcup_{j=0}^1 (\mathfrak{G}_{2(n+1)-1} \dot{+} j(0, g_n)).$$

In case A1) we have the basic system as follows

$$\mathbf{g}_{2(n+1)-1} = (g_n, 0), \quad \mathbf{g}_{2(n+1)-2} = (0, g_n).$$

We can write the dilation operator  $A_2$  in the form  $A_2(x^{(0)}, x^{(1)}) = (Ax^{(1)}, x^{(0)})$ , where  $A$  is one-dimensional dilation operator. Indeed

$$A_2(\mathbf{g}_{2(n+1)-1}) = A_2(g_n, 0) = (A0, g_n) = (0, g_n) = \mathbf{g}_{2(n+1)-2},$$

$$A_2(\mathbf{g}_{2(n+1)-2}) = A_2(0, g_n) = (Ag_n, 0) = (g_{n-1}, 0) = \mathbf{g}_{2n-1}.$$

We will find a scaling function  $\varphi(\mathbf{x}) = \varphi(x^{(0)}, x^{(1)})$  with condition  $|\hat{\varphi}(\chi)| = \mathbf{1}_{G_0}(\chi)$  as a solution of refinement equation

$$\varphi(\mathbf{x}) = \sum_{\mathbf{h} \in H_0^{(1)}} c_{\mathbf{h}} \varphi(\mathbf{x} \dot{-} \mathbf{h}),$$

where  $H_0^{(1)} = \{\alpha_{-1}\mathbf{g}_{-1}\}$ . In [23] it was proved, that

$$\varphi(\mathbf{x}) = \mathbf{1}_{G_0}(\mathbf{x}),$$

$$\psi_l(\mathbf{x}) = \mathbf{r}_l^l(\mathbf{x}) \mathbf{1}_{G_0}(\mathbf{x}) \quad (l = 1).$$

Since

$$\mathbf{r}_0(\mathbf{x}) \begin{cases} 1, & \mathbf{x} \in G_{+1} = \mathfrak{G}_0 \times \mathfrak{G}_1, \\ -1, & \mathbf{x} \in \mathfrak{G}_0 \times (\mathfrak{G}_0 \setminus \mathfrak{G}_1), \end{cases}$$

we can write the wavelet function  $\psi_1(\mathbf{x})$  in the form

$$\psi_1(x^{(0)}, x^{(1)}) = r_0(x^{(1)}) \mathbf{1}_{G_0}(x^{(0)}, x^{(1)}).$$

The wavelet  $\psi_1$  may be obtained from separable MRA [30,27]. Using separable MRA, we find 3 wavelet functions

$$\psi_{\{1\}}(x^{(0)}, x^{(1)}) = r_0(x^{(1)}) \mathbf{1}_{\mathfrak{G}_0 \times \mathfrak{G}_0}(x^{(0)}, x^{(1)}) = \psi_1(x^{(0)}, x^{(1)}),$$

$$\begin{aligned} \psi_{\{0\}}(x^{(0)}, x^{(1)}) &= r_0(x^{(0)}) \mathbf{1}_{\mathfrak{G}_0 \times \mathfrak{G}_0}(x^{(0)}, x^{(1)}) \\ &= \frac{1}{\sqrt{2}} (\psi_1(A_2(x^{(0)}, x^{(1)})) + \psi_1(A_2(x^{(0)}, x^{(1)}) - (g_{-1}, 0))), \end{aligned}$$

$$\begin{aligned} \psi_{\{0,1\}}(x^{(0)}, x^{(1)}) &= r_0(x^{(0)}) r_0(x^{(1)}) \mathbf{1}_{\mathfrak{G}_0 \times \mathfrak{G}_0}(x^{(0)}, x^{(1)}) \\ &= \frac{1}{\sqrt{2}} (\psi_1(A_2(x^{(0)}, x^{(1)})) - \psi_1(A_2(x^{(0)}, x^{(1)}) - (g_{-1}, 0))) \end{aligned}$$

which may be written over unique wavelet  $\psi_1$ . The dilation operator for wavelet system  $\psi_{\{1\}}, \psi_{\{0\}}, \psi_{\{0,2\}}$  is  $\tilde{A}_2(x^{(0)}, x^{(1)}) = (Ax^{(0)}, Ax^{(1)})$ . It is evident  $A_2^2 = \tilde{A}_2$ .

In the case A3) we take the basic system as follows

$$\mathbf{g}_{2(n+1)-1} = (g_n \dot{-} g_{n+1}, g_n \dot{+} g_{n+1}) \in G_{2(n+1)-1} \setminus G_{2(n+1)},$$

$$\mathbf{g}_{2(n+1)-2} = (g_n, g_{g_{n+1}}) \in G_{2n} \setminus G_{2(n+1)-1},$$

and define the dilation operator  $A_2$  in the standard way. Using the equation  $2g_n = g_{n+1}$  we can write the operator  $A_2$  in the following way

$$A_2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A & A \\ -A & A \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A(x \dot{+} y) \\ A(-x \dot{+} y) \end{pmatrix}.$$

We will find a scaling function  $\varphi(\mathbf{x}) = \varphi(x^{(0)}, x^{(1)})$  with condition  $|\hat{\varphi}(\chi)| = \mathbf{1}_{G_0}(\chi)$  as solution of refinable equation



$$\varphi(\mathbf{x}) = \sum_{\mathbf{h} \in H_0^{(1)}} c_{\mathbf{h}} \varphi(\mathbf{x} \dot{-} \mathbf{h}).$$

We know that

$$\varphi(\mathbf{x}) = \mathbf{1}_{G_0}(\mathbf{x}), \quad \psi_l(\mathbf{x}) = \mathbf{r}_0^l(\mathbf{x}) \mathbf{1}_{G_0}(\mathbf{x}) \quad (l = 1).$$

Using the definition of  $G_0$  and  $G_1$  we find

$$\psi_1(x^{(0)}, x^{(1)}) = \varphi(A_2 \mathbf{x}) - \varphi(A_2 \mathbf{x} \dot{-} (g_1, g_1)).$$

This wavelet was constructed by E.J. King and M.A. Skopina in [19]. The matrix  $\begin{pmatrix} A & A \\ -A & A \end{pmatrix}$  is an analog of Quincunx matrix.

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