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Green's function for first-order multipoint boundary value problems and applications to the existence of solutions with constant sign

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ABSTRACT

We consider a first-order linear differential equation subject to boundary value conditions which take into account the values of the function at multiple points in the interval of interest. For this problem, we calculate the Green's function which allows to express in integral form the exact expression of the unique solution to the multipoint boundary value problem under the appropriate conditions. From this study, some results are derived concerning the existence of solutions with a constant sign (that is, some comparison results for first-order multipoint boundary value problems).

1. Introduction

In this paper, we study a class of linear differential equations subject to multipoint boundary value conditions, whose solution (under conditions guaranteeing existence and uniqueness) is calculated explicitly by means of the Green's function. The analysis of the sign of the Green's function allows to deduce sufficient conditions for the existence of solutions with a constant sign to the multipoint boundary value problem.

Ref. [8] is related to the study of periodicity of solutions to differential equations with sublinear impulses. On the other hand, in [7], B. Liu proves existence and uniqueness results for first-order multipoint boundary value problems. The work [1] is focused on the approximation of solutions to m-point nonlocal boundary value problems for second-order differential equations, while three-point boundary value problems are considered in [5] via the monotone method [4]. Other references devoted to multipoint boundary value problems for second-order functional differential equations are [10,11], and a class of nonlocal boundary value problems for impulsive second-order differential equations on an infinite interval is studied in [12]. In [2,6,9], we find multipoint boundary value problems for higher-order ordinary differential equations. Finally, for the study of differential equations with nonlinear multipoint boundary conditions, we refer to [3].

2. Problem of interest and expression of the exact solution

In this paper, we consider the multipoint boundary value problem for first-order linear differential equations

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$$\begin{cases} u'(t)+a(t)u(t)=\sigma(t), & t\in J=[0,T], \\ \lambda u(r_0)=\sum_{i=1}^{m}\lambda_i u(r_i), \end{cases} \tag{1}$$

where $m \geqslant 1$, $\lambda, \lambda_1, \dots, \lambda_m \in \mathbb{R}$, $a, \sigma : [0,T] \to \mathbb{R}$, and $0 \leqslant r_0 < r_1 < \cdots < r_m \leqslant T$ (hence $T > 0$). We denote $\lambda_0 = -\lambda$, in order to unify the notation in the multipoint boundary condition, obtaining

$$\begin{cases} u'(t)+a(t)u(t)=\sigma(t), & t\in J=[0,T], \\ \sum_{i=0}^{m}\lambda_i u(r_i)=0. \end{cases} \tag{2}$$

Definition 2.1. A solution to problem (2) is a function $u \in C^1(J)$ satisfying the conditions in (2).

Next, we calculate the Green's function to obtain the exact expression of the solution to the multipoint boundary value problem for first-order linear differential equations (2), under certain restrictions on the constants. This type of results are useful to study the behavior of the solutions to related nonlinear differential equations.

First, we consider the case where $r_0 > 0$, that is, the multipoint boundary condition does not consider the value of the function at the starting point. In our notation, the sums corresponding to an upper limit less than the lower limit are considered to be null.

Theorem 2.2. *Let $J = [0,T]$, $m \geqslant 1$, $\lambda, \lambda_1, \dots, \lambda_m \in \mathbb{R}$, $a, \sigma \in C(J)$, $0 < r_0 < r_1 < \cdots < r_m \leqslant T$ and denote*

$$H := \sum_{i=0}^{m} \lambda_i e^{-\int_0^{r_i} a(u)\,du}.$$

If $H \neq 0$, there exists a unique solution to (2) *given by*

$$u(t) = \int_0^T G(t,s)\sigma(s)\,ds,$$

where

$$G(t,s) = e^{-\int_0^t a(u)\,du} \begin{cases} \left(\frac{-1}{H}\sum_{i=j+1}^{m}\lambda_i e^{-\int_s^{r_i} a(u)\,du} + e^{\int_0^s a(u)\,du}\right), & \text{if } s\in(r_j, r_{j+1}],\ s\leqslant t\ (j=-1,\dots,k), \\ \frac{-1}{H}\sum_{i=j+1}^{m}\lambda_i e^{-\int_s^{r_i} a(u)\,du}, & \text{if } s\in(r_j, r_{j+1}],\ s> t\ (j=k,\dots,p), \end{cases} \tag{3}$$

for $t \in (r_k, r_{k+1}]$, with $k = -1, \dots, p$, where $r_{p+1} = T$. Note that, in (3)*, j may also take the values $-1, \dots, p$.*

On the other hand, if $H = 0$, problem (2) *is solvable only for*

$$\sum_{i=0}^{m} \lambda_i \int_0^{r_i} \sigma(s) e^{-\int_s^{r_i} a(u)\,du}\,ds = 0,$$

in which case, there is an infinite number of solutions.

Proof. Using that the solution to the initial value problem

$$\begin{cases} v'(t)+a(t)v(t)=\sigma(t), & t\in J, \\ v(0)=v_0, \end{cases} \tag{4}$$

is

$$v(t) = v_0 e^{-\int_0^t a(u)\,du} + \int_0^t \sigma(s) e^{-\int_s^t a(u)\,du}\,ds, \tag{5}$$

we get

$$v(r_i) = v_0 e^{-\int_0^{r_i} a(u)\,du} + \int_0^{r_i} \sigma(s) e^{-\int_s^{r_i} a(u)\,du}\,ds, \quad i = 0, \dots, m.$$

Hence, the boundary condition $\sum_{i=0}^{m} \lambda_i v(r_i) = 0$ is written as

$$\sum_{i=0}^{m} \lambda_i \left(v_0 e^{-\int_0^{r_i} a(u)\,du} + \int_0^{r_i} \sigma(s) e^{-\int_s^{r_i} a(u)\,du}\, ds \right) = 0,$$

which implies

$$v_0 = \frac{-1}{\sum_{i=0}^{m} \lambda_i e^{-\int_0^{r_i} a(u)\,du}} \sum_{i=0}^{m} \lambda_i \int_0^{r_i} \sigma(s) e^{-\int_s^{r_i} a(u)\,du}\, ds$$
$$= \frac{-1}{H} \sum_{i=0}^{m} \lambda_i \int_0^{r_i} \sigma(s) e^{-\int_s^{r_i} a(u)\,du}\, ds,$$

provided that $H \neq 0$, hence the solution sought is

$$u(t) = \frac{-1}{H} \sum_{i=0}^{m} \lambda_i \int_0^{r_i} \sigma(s) e^{-\int_s^{r_i} a(u)\,du}\, ds\, e^{-\int_0^{t} a(u)\,du} + \int_0^{t} \sigma(s) e^{-\int_s^{t} a(u)\,du}\, ds$$
$$= \frac{-1}{H} \sum_{i=0}^{m} \lambda_i \left(\sum_{l=0}^{i-1} \int_{r_l}^{r_{l+1}} \sigma(s) e^{-\int_s^{r_i} a(u)\,du}\, ds + \int_0^{r_0} \sigma(s) e^{-\int_s^{r_i} a(u)\,du}\, ds \right) e^{-\int_0^{t} a(u)\,du} + \int_0^{t} \sigma(s) e^{-\int_s^{t} a(u)\,du}\, ds,$$

where the sums corresponding to an upper limit less than the lower limit are null. Denoting by $r_{-1} = 0$, we get

$$u(t) = \frac{-1}{H} \sum_{i=0}^{m} \lambda_i \sum_{l=-1}^{i-1} \int_{r_l}^{r_{l+1}} \sigma(s) e^{-\int_s^{r_i} a(u)\,du}\, ds\, e^{-\int_0^{t} a(u)\,du} + \int_0^{t} \sigma(s) e^{-\int_s^{t} a(u)\,du}\, ds$$
$$= \frac{-1}{H} \sum_{j=-1}^{m-1} \int_{r_j}^{r_{j+1}} \sigma(s) \sum_{i=j+1}^{m} \lambda_i e^{-\int_s^{r_i} a(u)\,du}\, ds\, e^{-\int_0^{t} a(u)\,du} + \int_0^{t} \sigma(s) e^{-\int_s^{t} a(u)\,du}\, ds.$$

For $t \in (r_k, r_{k+1}]$, $-1 \leqslant k \leqslant m-1$ if $r_m = T$, and $-1 \leqslant k \leqslant m$ if $r_m < T$, then

$$\int_0^{t} \sigma(s) e^{-\int_s^{t} a(u)\,du}\, ds = \sum_{l=-1}^{k-1} \int_{r_l}^{r_{l+1}} \sigma(s) e^{-\int_s^{t} a(u)\,du}\, ds + \int_{r_k}^{t} \sigma(s) e^{-\int_s^{t} a(u)\,du}\, ds.$$

Therefore, for $t \in (r_k, r_{k+1}]$, $-1 \leqslant k \leqslant m-1$ if $r_m = T$, and $-1 \leqslant k \leqslant m$ if $r_m < T$,

$$u(t) = \sum_{j=-1}^{k-1} \int_{r_j}^{r_{j+1}} \sigma(s) \left(\frac{-1}{H} \sum_{i=j+1}^{m} \lambda_i e^{-\int_s^{r_i} a(u)\,du} e^{-\int_0^{t} a(u)\,du} + e^{-\int_s^{t} a(u)\,du} \right) ds$$
$$+ \int_{r_k}^{t} \sigma(s) \left(\frac{-1}{H} \sum_{i=k+1}^{m} \lambda_i e^{-\int_s^{r_i} a(u)\,du} e^{-\int_0^{t} a(u)\,du} + e^{-\int_s^{t} a(u)\,du} \right) ds$$
$$+ \int_{t}^{r_{k+1}} \sigma(s) \frac{-1}{H} \sum_{i=k+1}^{m} \lambda_i e^{-\int_s^{r_i} a(u)\,du} e^{-\int_0^{t} a(u)\,du}\, ds + \sum_{j=k+1}^{m-1} \int_{r_j}^{r_{j+1}} \sigma(s) \frac{-1}{H} \sum_{i=j+1}^{m} \lambda_i e^{-\int_s^{r_i} a(u)\,du} e^{-\int_0^{t} a(u)\,du}\, ds,$$

that is,

$$u(t) = \sum_{j=-1}^{k-1} \int_{r_j}^{r_{j+1}} \sigma(s) \left(\frac{-1}{H} \sum_{i=j+1}^{m} \lambda_i e^{-\int_s^{r_i} a(u)\,du} + e^{\int_0^{s} a(u)\,du} \right) e^{-\int_0^{t} a(u)\,du}\, ds$$
$$+ \int_{r_k}^{t} \sigma(s) \left(\frac{-1}{H} \sum_{i=k+1}^{m} \lambda_i e^{-\int_s^{r_i} a(u)\,du} + e^{\int_0^{s} a(u)\,du} \right) e^{-\int_0^{t} a(u)\,du}\, ds$$

$$+\int_t^{r_{k+1}}\sigma(s)\frac{-1}{H}\sum_{i=k+1}^{m}\lambda_i e^{-\int_s^{r_i}a(u)\,du}e^{-\int_0^t a(u)\,du}\,ds+\sum_{j=k+1}^{m-1}\int_{r_j}^{r_{j+1}}\sigma(s)\frac{-1}{H}\sum_{i=j+1}^{m}\lambda_i e^{-\int_s^{r_i}a(u)\,du}e^{-\int_0^t a(u)\,du}\,ds$$

$$=\int_0^{r_m}G(t,s)\sigma(s)\,ds,$$

where G is given in (3). Note that, if $r_0>0$, it is obvious that, for $t\in[0,r_0]$,

$$u(t)=\int_0^t\sigma(s)\left(\frac{-1}{H}\sum_{i=0}^{m}\lambda_i e^{-\int_s^{r_i}a(u)\,du}+e^{\int_0^s a(u)\,du}\right)e^{-\int_0^t a(u)\,du}\,ds+\int_t^{r_0}\sigma(s)\frac{-1}{H}\sum_{i=0}^{m}\lambda_i e^{-\int_s^{r_i}a(u)\,du}e^{-\int_0^t a(u)\,du}\,ds$$

$$+\sum_{j=0}^{m-1}\int_{r_j}^{r_{j+1}}\sigma(s)\frac{-1}{H}\sum_{i=j+1}^{m}\lambda_i e^{-\int_s^{r_i}a(u)\,du}e^{-\int_0^t a(u)\,du}\,ds,$$

which coincides with the previous expression with $k=-1$.

Summarizing, if $r_m=T$ $(p=m-1)$, then

$$G(t,s)=e^{-\int_0^t a(u)\,du}\begin{cases}(\frac{-1}{H}\sum_{i=j+1}^{m}\lambda_i e^{-\int_s^{r_i}a(u)\,du}+e^{\int_0^s a(u)\,du}), & \text{if } s\in(r_j,r_{j+1}],\ j=-1,\dots,k,\ s\leqslant t,\\ \frac{-1}{H}\sum_{i=j+1}^{m}\lambda_i e^{-\int_s^{r_i}a(u)\,du}, & \text{if } s\in(r_j,r_{j+1}],\ j=k,\dots,m-1,\ s>t\end{cases}$$

for $t\in(r_k,r_{k+1}]$, with $k=-1,\dots,m-1$, which coincides with expression in (3). Note that, for $k=-1$, we get $t\in[0,r_0]$ and

$$G(t,s)=e^{-\int_0^t a(u)\,du}\begin{cases}(\frac{-1}{H}\sum_{i=0}^{m}\lambda_i e^{-\int_s^{r_i}a(u)\,du}+e^{\int_0^s a(u)\,du}), & \text{if } s\leqslant t,\\ \frac{-1}{H}\sum_{i=j+1}^{m}\lambda_i e^{-\int_s^{r_i}a(u)\,du}, & \text{if } s>t.\end{cases}$$

On the other hand, if $r_m<T$ $(p=m)$ then we have to study the value of the solution for $t\in(r_m,r_{m+1}]=(r_m,T]$. For $t\leqslant r_m$, the Green's function coincides with the expression previously obtained, and, for $t\in(r_m,T]$, we have

$$u(t)=\sum_{j=-1}^{m-1}\int_{r_j}^{r_{j+1}}\sigma(s)\left(\frac{-1}{H}\sum_{i=j+1}^{m}\lambda_i e^{-\int_s^{r_i}a(u)\,du}+e^{\int_0^s a(u)\,du}\right)e^{-\int_0^t a(u)\,du}\,ds+\int_{r_m}^{t}\sigma(s)e^{-\int_s^t a(u)\,du}\,ds$$

$$=\int_0^{r_{m+1}}G(t,s)\sigma(s)\,ds,$$

where, for $t\in(r_m,T]$,

$$G(t,s)=\begin{cases}(\frac{-1}{H}\sum_{i=j+1}^{m}\lambda_i e^{-\int_s^{r_i}a(u)\,du}+e^{\int_0^s a(u)\,du})e^{-\int_0^t a(u)\,du}, & \text{if } s\in(r_j,r_{j+1}],\ j=-1,\dots,m-1\\ \quad\text{(obviously } s\leqslant t),\\ e^{-\int_s^t a(u)\,du}, & \text{if } s\in(r_m,t],\\ 0, & \text{if } s\in(t,T],\end{cases}$$

which coincides with expression (3) taking $k=m$, since $\sum_{i=m+1}^{m}\lambda_i e^{-\int_s^{r_i}a(u)\,du}=0$.

Finally, if $H=0$, problem (2) is solvable for

$$\sum_{i=0}^{m}\lambda_i\int_0^{r_i}\sigma(s)e^{-\int_s^{r_i}a(u)\,du}\,ds=0,$$

in which case, there is an infinite number of solutions given by (5) for any value of $v_0\in\mathbb{R}$. □

Remark 2.3. The solution to (2) written in integral form is equal to $u(t)=\int_0^{r_m}G(t,s)\sigma(s)\,ds$, if $r_m=T$, and hence $p=m-1$ in Theorem 2.2. On the other hand, it is equal to $u(t)=\int_0^{r_{m+1}}G(t,s)\sigma(s)\,ds$, if $r_m<T$, so that $p=m$ in the statement of Theorem 2.2.

Remark 2.4. Despite the definition of $G(t,s)$ is not needed for $t=0$, it is clear from the procedure followed that $G(0,s)=G(0^+,s)$, for every s, that is, the expression obtained for $t\in(0,r_0]$ $(k=-1)$ is also valid for $t=0$.

Next, we consider the case where $r_0=0$. Revising the proof of Theorem 2.2, we easily deduce the expression of the Green's function, since $r_{-1}=r_0=0$ reduces to consider nonnegative values of k and j.

Theorem 2.5. *Let $J=[0,T]$, $m\geqslant 1$, $\lambda,\lambda_1,\dots,\lambda_m\in\mathbb{R}$, $a,\sigma\in C(J)$, $0=r_0<r_1<\cdots<r_m\leqslant T$ and denote $H:=\lambda_0+\sum_{i=1}^m\lambda_i e^{-\int_0^{r_i}a(u)\,du}$. If $H\neq 0$, there exists a unique solution to* (2) *given by*

$$u(t)=\int_0^T G(t,s)\sigma(s)\,ds,$$

where

$$G(t,s)=e^{-\int_0^t a(u)\,du}\begin{cases}\left(\frac{-1}{H}\sum_{i=j+1}^m\lambda_i e^{-\int_s^{r_i}a(u)\,du}+e^{\int_0^s a(u)\,du}\right), & \text{if } s\in(r_j,r_{j+1}],\ s\leqslant t\ (j=0,\dots,k),\\ \frac{-1}{H}\sum_{i=j+1}^m\lambda_i e^{-\int_s^{r_i}a(u)\,du}, & \text{if } s\in(r_j,r_{j+1}],\ s>t\ (j=k,\dots,p),\end{cases}\tag{6}$$

for $t\in(r_k,r_{k+1}]$, with $k=0,\dots,p$, where $r_{p+1}=T$, and the expression obtained for $t\in(0,r_1]$ $(k=0)$ is also valid for $t=0$. Note that, in (3), *j may also take the values $0,\dots,p$.*

On the other hand, if $H=0$, problem (2) *is solvable only for*

$$\sum_{i=1}^m\lambda_i\int_0^{r_i}\sigma(s)e^{-\int_s^{r_i}a(u)\,du}\,ds=0,$$

in which case, there is an infinite number of solutions.

Remark 2.6. Using a unified notation, in Theorems 2.2 and 2.5, function G can be written as

$$G(t,s)=\frac{e^{\int_t^s a(u)\,du}}{H}\begin{cases}\sum_{i=0}^j\lambda_i e^{-\int_0^{r_i}a(u)\,du}, & \text{if } s\in(r_j,r_{j+1}],\ s\leqslant t\ (j=q,\dots,k),\\ -\sum_{i=j+1}^m\lambda_i e^{-\int_0^{r_i}a(u)\,du}, & \text{if } s\in(r_j,r_{j+1}],\ s>t\ (j=k,\dots,p),\end{cases}\tag{7}$$

for $t\in(r_k,r_{k+1}]$, with $k=q,\dots,p$, where $r_q=0$ and $r_{p+1}=T$.

Following the unified notation of Remark 2.6, we have the following corollary for $a(t)=M$ constant.

Corollary 2.7. *Let $J=[0,T]$, $m\geqslant 1$, $\lambda,\lambda_1,\dots,\lambda_m\in\mathbb{R}$, $\sigma\in C(J)$, $0\leqslant r_0<r_1<\cdots<r_m\leqslant T$ and denote $H:=\sum_{i=0}^m\lambda_i e^{-Mr_i}$. If $H\neq 0$, there exists a unique solution to problem* (8):

$$\begin{cases}u'(t)+Mu(t)=\sigma(t), & t\in J=[0,T],\\ \sum_{i=0}^m\lambda_i u(r_i)=0,\end{cases}\tag{8}$$

given by $u(t)=\int_0^T G(t,s)\sigma(s)\,ds$, where

$$G(t,s)=\frac{e^{M(s-t)}}{H}\begin{cases}\sum_{i=0}^j\lambda_i e^{-Mr_i}, & \text{if } s\in(r_j,r_{j+1}],\ s\leqslant t\ (j=q,\dots,k),\\ -\sum_{i=j+1}^m\lambda_i e^{-Mr_i}, & \text{if } s\in(r_j,r_{j+1}],\ s>t\ (j=k,\dots,p),\end{cases}\tag{9}$$

for $t\in(r_k,r_{k+1}]$, with $k=q,\dots,p$, where $r_q=0$ and $r_{p+1}=T$. Note that, in (9), *j may also take the values $q,\dots,p$.*

On the other hand, if $H=0$, problem (8) *is solvable only for*

$$\sum_{i=0}^m\lambda_i\int_0^{r_i}\sigma(s)e^{-M(r_i-s)}\,ds=0,$$

in which case, there is an infinite number of solutions.

3. Existence of solutions with a constant sign

In this section, we start analyzing the sign of the Green's function G obtained in Theorems 2.2 and 2.5.

Theorem 3.1. *Let $J=[0,T]$, $m\geqslant 1$, $\lambda,\lambda_1,\dots,\lambda_m\in\mathbb{R}$, $a\in C(J)$, $0=r_0<r_1<\dots<r_m\leqslant T$ and denote $H:=\sum_{i=0}^m\lambda_i e^{-\int_0^{r_i}a(u)\,du}=\lambda_0+\sum_{i=1}^m\lambda_i e^{-\int_0^{r_i}a(u)\,du}=-\lambda+\sum_{i=1}^m\lambda_i e^{-\int_0^{r_i}a(u)\,du}$. Suppose that one of the following assertions holds*:

(I) $H>0$ (*that is,* $\lambda<\sum_{i=1}^m\lambda_i e^{-\int_0^{r_i}a(u)\,du}$) *and* $\sum_{i=j+1}^m\lambda_i e^{-\int_{r_{j+1}}^{r_i}a(u)\,du}\leqslant 0$ *for every* $j=0,\dots,m-1$; *or*

(II) $H<0$ (*that is,* $\lambda>\sum_{i=1}^m\lambda_i e^{-\int_0^{r_i}a(u)\,du}$) *and* $\sum_{i=j+1}^m\lambda_i e^{-\int_{r_{j+1}}^{r_i}a(u)\,du}\geqslant 0$ *for every* $j=0,\dots,m-1$.

Then $G(t,s)\geqslant 0$, *for* $(t,s)\in[0,T]\times[0,T]$.

Proof. It is easy to check that

$$\frac{1}{H}\sum_{i=j+1}^m\lambda_i e^{-\int_s^{r_i}a(u)\,du}=\frac{1}{H}\sum_{i=j+1}^m\lambda_i e^{-\int_{r_{j+1}}^{r_i}a(u)\,du}e^{-\int_s^{r_{j+1}}a(u)\,du}\leqslant 0,$$

for all $s\in(r_j,r_{j+1}]$, and $j=0,\dots,m-1$. Hence

$$\frac{-1}{H}\sum_{i=j+1}^m\lambda_i e^{-\int_s^{r_i}a(u)\,du}\geqslant 0,\quad \forall s\in(r_j,r_{j+1}],\ \forall j=0,\dots,m-1.$$

Note that the condition on the sign of $\sum_{i=j+1}^m\lambda_i e^{-\int_{r_{j+1}}^{r_i}a(u)\,du}$ is only checked for $j=0,\dots,m-1$ even if $r_m<T$, since it is trivially satisfied for $j=m$ (it is equal to zero). The proof is complete. $\square$

Remark 3.2. Note that Theorem 3.1 makes no sense for the case $r_0>0$, since, in this case, it would be necessary to prove the nonpositivity of

$$\frac{1}{H}\sum_{i=j+1}^m\lambda_i e^{-\int_s^{r_i}a(u)\,du}$$

for $s\in(r_j,r_{j+1}]$, also for $j=-1$. However, it is impossible that $\frac{1}{H}\sum_{i=0}^m\lambda_i e^{-\int_s^{r_i}a(u)\,du}$ is nonpositive on $(0,r_0]$, since at $s=0$ it is equal to 1.

Remark 3.3. Taking into account expression (7), the study of the sign of the expressions $\sum_{i=j+1}^m\lambda_i e^{-\int_{r_{j+1}}^{r_i}a(u)\,du}$ in conditions (I) and (II) in Theorem 3.1 can be replaced by the study of the sign of the expressions $\sum_{i=j+1}^m\lambda_i e^{-\int_0^{r_i}a(u)\,du}$.

Theorem 3.4 *(Sign of the solution for $r_0=0$). Let $J=[0,T]$, $m\geqslant 1$, $\lambda,\lambda_1,\dots,\lambda_m\in\mathbb{R}$, $a,\sigma\in C(J)$, $0=r_0<r_1<\dots<r_m\leqslant T$, denote $H:=\lambda_0+\sum_{i=1}^m\lambda_i e^{-\int_0^{r_i}a(u)\,du}$, and consider the Green's function G given in Theorem* 2.5. *Assume that one of the conditions* (I) *or* (II) *in Theorem* 3.1 *holds.*

- *If $\sigma\geqslant 0$, then the solution to* (2) *is nonnegative.*
- *If $\sigma\leqslant 0$, then the solution to* (2) *is nonpositive.*

Remark 3.5. For $r_0=0$, we study the implications of condition (I) in Theorem 3.1. We study the validity of

$$\sum_{i=j+1}^m\lambda_i e^{-\int_{r_{j+1}}^{r_i}a(u)\,du}\leqslant 0$$

for $j=m-1$, which is reduced to $\lambda_m\leqslant 0$. For $j=m-2$, it is reduced to

$$\lambda_{m-1}\leqslant -\lambda_m e^{-\int_{r_{m-1}}^{r_m}a(u)\,du},$$

for $j=m-3$ to

$$\begin{aligned}\lambda_{m-2}&\leqslant -\lambda_{m-1}e^{-\int_{r_{m-2}}^{r_{m-1}}a(u)\,du}-\lambda_m e^{-\int_{r_{m-2}}^{r_m}a(u)\,du}\\&=\big(-\lambda_{m-1}-\lambda_m e^{-\int_{r_{m-1}}^{r_m}a(u)\,du}\big)e^{-\int_{r_{m-2}}^{r_{m-1}}a(u)\,du},\end{aligned}$$

and so on (while $j\geqslant 0$).

On the other hand, for condition (II) included in Theorem 3.1, we study the validity of $\sum_{i=j+1}^{m} \lambda_i e^{-\int_{r_{j+1}}^{r_i} a(u)\,du} \geqslant 0$ for every $j = 0, \dots, m-1$. This is reduced, for $j = m-1$, to $\lambda_m \geqslant 0$. For $j = m-2$, it is reduced to

$$\lambda_{m-1} \geqslant -\lambda_m e^{-\int_{r_{m-1}}^{r_m} a(u)\,du},$$

for $j = m-3$ to

$$\lambda_{m-2} \geqslant -\lambda_{m-1} e^{-\int_{r_{m-2}}^{r_{m-1}} a(u)\,du} - \lambda_m e^{-\int_{r_{m-2}}^{r_m} a(u)\,du},$$

and so on (while $j \geqslant 0$).

Example 3.6. For $0 = r_0 < r_1 = T$, and the boundary condition $-\lambda_0 u(0) = \lambda_1 u(T)$, we get $H := \lambda_0 + \lambda_1 e^{-\int_0^T a(u)\,du} = -\lambda + \lambda_1 e^{-\int_0^T a(u)\,du}$. Conditions for the nonnegative character of G given by Theorem 3.1 are $H > 0$ and $\lambda_1 \leqslant 0$, or $H < 0$ and $\lambda_1 \geqslant 0$.

In other words, we can consider

$$\frac{\lambda_1}{\lambda_0 + \lambda_1 e^{-\int_0^T a(u)\,du}} \leqslant 0. \tag{10}$$

If $\lambda_1 \neq 0$ (otherwise we have an initial value problem), this inequality is equivalent to

$$\frac{\lambda_0}{\lambda_1} + e^{-\int_0^T a(u)\,du} \leqslant 0,$$

that is,

$$\frac{\lambda_0}{\lambda_1} \leqslant -e^{-\int_0^T a(u)\,du}. \tag{11}$$

The Green's function in this case is given by

$$G(t,s) = e^{-\int_0^t a(u)\,du} \begin{cases} \left(\frac{-1}{H} \lambda_1 e^{-\int_s^T a(u)\,du} + e^{\int_0^s a(u)\,du}\right), & \text{if } 0 \leqslant s \leqslant t \leqslant T, \\ \frac{-1}{H} \lambda_1 e^{-\int_s^T a(u)\,du}, & \text{if } 0 \leqslant t < s \leqslant T. \end{cases} \tag{12}$$

On the other hand, if $\lambda_1 = 0$, the inequality (10) is trivially satisfied if $\lambda_0 \neq 0$ ($H \neq 0$), in which case, we may derive results on the existence of solutions with constant sign for the initial value problem associated. Note that, if $H = 0$ ($\lambda_0 = 0$), there is an infinite number of solutions.

Consider the linear equation $u'(t) + Mu(t) = \sigma(t)$, $t \in [0,T]$, that is, a is a constant function, hence, for $\lambda_1 \neq 0$, the condition for the nonnegative character of the Green's function is

$$\frac{\lambda}{\lambda_1} \geqslant e^{-MT}.$$

For instance, for equation $u'(t) + Mu(t) = \sigma(t)$, $t \in [0,T]$, and the boundary conditions $\lambda u(0) = e^{MT} u(T)$, where $\lambda \geqslant 1$, or $\lambda u(0) = -e^{MT} u(T)$, where $\lambda \leqslant -1$, then $G \geqslant 0$.

On the other hand, if $a(t) = t$, $t \in [0,T]$, (11) is written as $\frac{\lambda}{\lambda_1} \geqslant e^{-\frac{T^2}{2}}$. Therefore, for equation $u'(t) + tu(t) = \sigma(t)$, $t \in [0,1]$, and the boundary conditions $\lambda u(0) = \sqrt{e} u(1)$ with $\lambda \geqslant 1$, or $\lambda u(0) = -\sqrt{e} u(1)$ with $\lambda \leqslant -1$, then $G \geqslant 0$. In the more general case $a(t) = Mt + N$, $t \in [0,T]$, (11) is reduced to $\frac{\lambda}{\lambda_1} \geqslant e^{-T(\frac{MT}{2}+N)}$.

If $a(t) = t^n$, $t \in [0,1]$ ($T = 1$), (11) is $\frac{\lambda}{\lambda_1} \geqslant e^{-\frac{1}{n+1}}$. For equation $u'(t) + t^n u(t) = \sigma(t)$, $t \in [0,1]$, and the boundary conditions $\lambda u(0) = e^{\frac{1}{n+1}} u(1)$ where $\lambda \geqslant 1$, or $\lambda u(0) = -e^{\frac{1}{n+1}} u(1)$ where $\lambda \leqslant -1$, then $G \geqslant 0$.

Finally, if $a(t) = \sin(\pi t)$, $t \in [0,T]$, (11) is written as $\frac{\lambda}{\lambda_1} \geqslant e^{\frac{1}{\pi}(\cos(\pi T)-1)}$.

Example 3.7. For $0 = r_0 < r_1 < r_2 = T$, and the boundary condition $-\lambda_0 u(0) = \lambda_1 u(r_1)$ (here $p = m = 1$), the value of $H := \lambda_0 + \lambda_1 e^{-\int_0^{r_1} a(u)\,du}$ is similar to that in Example 3.6, except that T is replaced by r_1. It is clear that the condition for the nonnegative character of G (see also Theorem 3.1) is $\frac{\lambda_1}{H} \leqslant 0$. In this case, function G is given by

$$G(t,s) = e^{-\int_0^t a(u)\,du} \begin{cases} \left(\frac{-1}{H} \lambda_1 e^{-\int_s^{r_1} a(u)\,du} + e^{\int_0^s a(u)\,du}\right), & \text{if } t \in [0,T],\ s \in [0,r_1],\ s \leqslant t, \\ \frac{-1}{H} \lambda_1 e^{-\int_s^{r_1} a(u)\,du}, & \text{if } 0 \leqslant t < s \leqslant r_1, \\ e^{\int_0^s a(u)\,du}, & \text{if } r_1 \leqslant s \leqslant t \leqslant T, \\ 0, & \text{if } r_1 \leqslant t < s \leqslant T. \end{cases} \tag{13}$$

As a particular case, if $a(t)=M$, $t\in[0,T]$, and $\lambda_1\neq 0$, we check $\frac{\lambda}{\lambda_1}\geqslant e^{-Mr_1}$ for the nonnegative character of G. In this case, for equation $u'(t)+Mu(t)=\sigma(t)$, $t\in[0,T]$, and the boundary conditions $\lambda u(0)=e^{Mr_1}u(T)$ where $\lambda\geqslant 1$, or $\lambda u(0)=-e^{Mr_1}u(T)$ where $\lambda\leqslant -1$, then $G\geqslant 0$.

If $a(t)=t$, $t\in[0,1]$ $(T=1)$ and $r_1=\frac{\sqrt{2}}{2}$, we check $\frac{\lambda}{\lambda_1}\geqslant\frac{1}{\sqrt[4]{e}}$. Therefore, for equation $u'(t)+tu(t)=\sigma(t)$, $t\in[0,1]$, and the boundary conditions $\lambda u(0)=\sqrt[4]{e}u(\frac{\sqrt{2}}{2})$ where $\lambda\geqslant 1$, or $\lambda u(0)=-\sqrt[4]{e}u(\frac{\sqrt{2}}{2})$ where $\lambda\leqslant -1$, then $G\geqslant 0$.

If $a(t)=t^n$, $t\in[0,1]$ $(T=1)$ and $r_1=\frac{\sqrt{2}}{2}$, we check $\frac{\lambda}{\lambda_1}\geqslant e^{-\frac{1}{2^{(n+1)/2}(n+1)}}$. Hence, for equation $u'(t)+t^n u(t)=\sigma(t)$, $t\in[0,1]$, and the boundary conditions $\lambda u(0)=e^{\frac{1}{2^{(n+1)/2}(n+1)}}u(\frac{\sqrt{2}}{2})$ where $\lambda\geqslant 1$, or $\lambda u(0)=-e^{\frac{1}{2^{(n+1)/2}(n+1)}}u(\frac{\sqrt{2}}{2})$ where $\lambda\leqslant -1$, then $G\geqslant 0$.

Example 3.8. If we consider $0=r_0<r_1<r_2\leqslant T$, and the boundary condition $-\lambda_0 u(0)=\lambda_1 u(r_1)+\lambda_2 u(r_2)$ (here $m=2$), hence the condition for the nonnegative character of G is, according to the proof of Theorem 3.1,

$$\frac{1}{H}\sum_{i=j+1}^{2}\lambda_i e^{-\int_{r_{j+1}}^{r_i}a(u)\,du}\leqslant 0,\quad \forall j=0,1, \tag{14}$$

where $H:=-\lambda+\lambda_1 e^{-\int_0^{r_1}a(u)\,du}+\lambda_2 e^{-\int_0^{r_2}a(u)\,du}$. That is, we have to check conditions

$$\frac{\lambda_2}{H}\leqslant 0,\quad\text{and}\quad \frac{\lambda_1+\lambda_2 e^{-\int_{r_1}^{r_2}a(u)\,du}}{H}\leqslant 0, \tag{15}$$

both if $r_2<T$ or $r_2=T$ (since for $r_2<T$ and $j=2$ condition (14) is trivially satisfied). Condition (15) is translated into

$$\lambda_2\leqslant 0,\qquad \lambda_1+\lambda_2 e^{-\int_{r_1}^{r_2}a(u)\,du}\leqslant 0,\qquad -\lambda+\lambda_1 e^{-\int_0^{r_1}a(u)\,du}+\lambda_2 e^{-\int_0^{r_2}a(u)\,du}>0,$$

or

$$\lambda_2\geqslant 0,\qquad \lambda_1+\lambda_2 e^{-\int_{r_1}^{r_2}a(u)\,du}\geqslant 0,\qquad -\lambda+\lambda_1 e^{-\int_0^{r_1}a(u)\,du}+\lambda_2 e^{-\int_0^{r_2}a(u)\,du}<0.$$

For $a(t)=M$, $t\in[0,T]$, these are rewritten as

$$\lambda_2\leqslant 0,\qquad \lambda_1\leqslant -\lambda_2 e^{-M(r_2-r_1)},\qquad \lambda<\lambda_1 e^{-Mr_1}+\lambda_2 e^{-Mr_2},$$

or

$$\lambda_2\geqslant 0,\qquad \lambda_1\geqslant -\lambda_2 e^{-M(r_2-r_1)},\qquad \lambda>\lambda_1 e^{-Mr_1}+\lambda_2 e^{-Mr_2}.$$

On the other hand, if $a(t)=t$, $t\in[0,T]$, and $0=r_0<r_1<r_2\leqslant T$, we get

$$\lambda_2\leqslant 0,\qquad \lambda_1\leqslant -\lambda_2 e^{-\frac{r_2^2}{2}+\frac{r_1^2}{2}},\qquad \lambda<\lambda_1 e^{-\frac{r_1^2}{2}}+\lambda_2 e^{-\frac{r_2^2}{2}},$$

or

$$\lambda_2\geqslant 0,\qquad \lambda_1\geqslant -\lambda_2 e^{-\frac{r_2^2}{2}+\frac{r_1^2}{2}},\qquad \lambda>\lambda_1 e^{-\frac{r_1^2}{2}}+\lambda_2 e^{-\frac{r_2^2}{2}}.$$

For $r_1=1$ and $r_2=2$, the conditions are

$$\lambda_2\leqslant 0,\qquad \lambda_1\leqslant -\lambda_2 e^{-\frac{3}{2}},\qquad \lambda<\lambda_1 e^{-\frac{1}{2}}+\lambda_2 e^{-2},$$

or

$$\lambda_2\geqslant 0,\qquad \lambda_1\geqslant -\lambda_2 e^{-\frac{3}{2}},\qquad \lambda>\lambda_1 e^{-\frac{1}{2}}+\lambda_2 e^{-2}.$$

For $a(t)=Mt+N$, $t\in[0,T]$, (15) is written as

$$\frac{\lambda_2}{H}\leqslant 0,\quad\text{and}\quad \frac{\lambda_1+\lambda_2 e^{-M\frac{r_2^2}{2}+M\frac{r_1^2}{2}-Nr_2+Nr_1}}{H}\leqslant 0, \tag{16}$$

where $H:=-\lambda+\lambda_1 e^{-M\frac{r_1^2}{2}-Nr_1}+\lambda_2 e^{-M\frac{r_2^2}{2}-Nr_2}$.

Finally, for $a(t)=\sin(\pi t)$, $t\in[0,T]$, (15) is written as

$$\frac{\lambda_2}{H}\leqslant 0,\quad\text{and}\quad \frac{\lambda_1+\lambda_2 e^{\frac{1}{\pi}(\cos\pi r_2-\cos\pi r_1)}}{H}\leqslant 0, \tag{17}$$

where $H:=-\lambda+\lambda_1 e^{\frac{1}{\pi}(\cos\pi r_1-1)}+\lambda_2 e^{\frac{1}{\pi}(\cos\pi r_2-1)}$.

Example 3.9. Analogously, if $0 = r_0 < r_1 < r_2 < r_3 \leqslant T$, and the boundary condition is $-\lambda_0 u(0) = \lambda_1 u(r_1) + \lambda_2 u(r_2) + \lambda_3 u(r_3)$ (here $m = 3$), we have $H := -\lambda + \lambda_1 e^{-\int_0^{r_1} a(u)\,du} + \lambda_2 e^{-\int_0^{r_2} a(u)\,du} + \lambda_3 e^{-\int_0^{r_3} a(u)\,du}$ and the conditions for the nonnegative character of G are

$$\frac{\lambda_3}{H} \leqslant 0, \qquad \frac{\lambda_2 + \lambda_3 e^{-\int_{r_2}^{r_3} a(u)\,du}}{H} \leqslant 0, \quad \text{and} \quad \frac{\lambda_1 + \lambda_2 e^{-\int_{r_1}^{r_2} a(u)\,du} + \lambda_3 e^{-\int_{r_1}^{r_3} a(u)\,du}}{H} \leqslant 0. \tag{18}$$

If $a(t) = M$, $t \in [0, T]$, then $H := -\lambda + \lambda_1 e^{-Mr_1} + \lambda_2 e^{-Mr_2} + \lambda_3 e^{-Mr_3}$, and sufficient conditions for the nonnegative character of G are

$$\frac{\lambda_3}{H} \leqslant 0, \qquad \frac{\lambda_2 + \lambda_3 e^{-M(r_3 - r_2)}}{H} \leqslant 0, \quad \text{and} \quad \frac{\lambda_1 + \lambda_2 e^{-M(r_2 - r_1)} + \lambda_3 e^{-M(r_3 - r_1)}}{H} \leqslant 0. \tag{19}$$

Next, we show a result on the nonpositive character of the function G.

Theorem 3.10. *Let $J = [0, T]$, $m \geqslant 1$, $\lambda, \lambda_1, \ldots, \lambda_m \in \mathbb{R}$, $a \in C(J)$, and $0 \leqslant r_0 < r_1 < \cdots < r_m = T$.*

(i) *If*

$$\frac{\sum_{i=j+1}^m \lambda_i e^{-\int_0^{r_i} a(u)\,du}}{\sum_{i=0}^m \lambda_i e^{-\int_0^{r_i} a(u)\,du}} \geqslant 1, \quad \textit{for every } j = 0, \ldots, m-1$$

(*where the denominator is non-null*), *then $G(t, s) \leqslant 0$, for $(t, s) \in [0, T] \times [0, T]$.*

(ii) *If the following assertion holds*

$$\frac{\sum_{i=0}^j \lambda_i e^{-\int_0^{r_i} a(u)\,du}}{\sum_{i=j+1}^m \lambda_i e^{-\int_0^{r_i} a(u)\,du}} \leqslant 0, \quad \textit{for every } j = 0, \ldots, m-1,$$

where the denominators are non-null, then $G(t, s) \leqslant 0$, for $(t, s) \in [0, T] \times [0, T]$.

Proof. To check the nonpositivity of G, since

$$\frac{-1}{H} \sum_{i=j+1}^m \lambda_i e^{-\int_s^{r_i} a(u)\,du} + e^{\int_0^s a(u)\,du} > \frac{-1}{H} \sum_{i=j+1}^m \lambda_i e^{-\int_s^{r_i} a(u)\,du},$$

where $H := \sum_{i=0}^m \lambda_i e^{-\int_0^{r_i} a(u)\,du} = -\lambda + \sum_{i=1}^m \lambda_i e^{-\int_0^{r_i} a(u)\,du}$, it suffices to prove that

$$e^{\int_0^s a(u)\,du} \leqslant \frac{1}{H} \sum_{i=j+1}^m \lambda_i e^{-\int_s^{r_i} a(u)\,du}, \quad \text{for all } s \in (r_j, r_{j+1}], \text{ and } j = -1, \ldots, m-1,$$

that is,

$$\frac{\sum_{i=j+1}^m \lambda_i e^{-\int_0^{r_i} a(u)\,du}}{\sum_{i=0}^m \lambda_i e^{-\int_0^{r_i} a(u)\,du}} \geqslant 1, \quad \text{for all } j = -1, \ldots, m-1,$$

which is trivially satisfied for $j = -1$.

On the other hand, if $\sum_{i=j+1}^m \lambda_i e^{-\int_0^{r_i} a(u)\,du} \neq 0$, the previous inequalities can be written as

$$\frac{\sum_{i=j+1}^m \lambda_i e^{-\int_0^{r_i} a(u)\,du}}{\sum_{i=0}^j \lambda_i e^{-\int_0^{r_i} a(u)\,du} + \sum_{i=j+1}^m \lambda_i e^{-\int_0^{r_i} a(u)\,du}} = \frac{1}{1 + \frac{\sum_{i=0}^j \lambda_i e^{-\int_0^{r_i} a(u)\,du}}{\sum_{i=j+1}^m \lambda_i e^{-\int_0^{r_i} a(u)\,du}}} \geqslant 1, \quad \text{for all } j = -1, \ldots, m-1,$$

which coincide with condition in (ii) (it is trivially fulfilled for $j = -1$).

Note that

$$\frac{\sum_{i=0}^j \lambda_i e^{-\int_0^{r_i} a(u)\,du}}{\sum_{i=j+1}^m \lambda_i e^{-\int_0^{r_i} a(u)\,du}} \leqslant 0, \quad \text{for every } j = 0, \ldots, m-1,$$

is equivalent to

$$\frac{\sum_{i=0}^{j}\lambda_i e^{-\int_{r_0}^{r_i}a(u)\,du}}{\sum_{i=j+1}^{m}\lambda_i e^{-\int_{r_0}^{r_i}a(u)\,du}} \leqslant 0, \quad \text{for every } j=0,\dots,m-1. \qquad \square$$

Remark 3.11. Conditions (i), (ii) in Theorem 3.10 are only checked for $j=0,\dots,m-1$, even if $r_0>0$, since they are trivially satisfied for $j=-1$.

Remark 3.12. Note that, in the statement of Theorem 3.10, it is not possible to consider $r_m<T$, since, in such a case, we should prove the inequality in (i) $\frac{\sum_{i=j+1}^{m}\lambda_i e^{-\int_0^{r_i}a(u)\,du}}{\sum_{i=0}^{m}\lambda_i e^{-\int_0^{r_i}a(u)\,du}} \geqslant 1$ also for $j=m$, which is not true.

Corollary 3.13. *Assume that $J=[0,T]$, $m\geqslant 1$, $\lambda,\lambda_1,\dots,\lambda_m\in\mathbb{R}$, $a\in C(J)$, and $0\leqslant r_0<r_1<\cdots<r_m=T$. If one of the following conditions holds*:

- $\sum_{i=0}^{m}\lambda_i e^{-\int_0^{r_i}a(u)\,du}>0$ *and* $\sum_{i=0}^{j}\lambda_i e^{-\int_0^{r_i}a(u)\,du}\leqslant 0$, *for every* $j=0,\dots,m-1$, *or*
- $\sum_{i=0}^{m}\lambda_i e^{-\int_0^{r_i}a(u)\,du}<0$ *and* $\sum_{i=0}^{j}\lambda_i e^{-\int_0^{r_i}a(u)\,du}\geqslant 0$, *for every* $j=0,\dots,m-1$,

then $G(t,s)\leqslant 0$, *for* $(t,s)\in[0,T]\times[0,T]$.

Proof. Derived directly from condition (i) in Theorem 3.10. $\square$

Remark 3.14. In these conditions, the lower limit of the integral can be equivalently chosen as r_0 instead of 0.

Theorem 3.15 *(Sign of the solution for $r_m=T$). Let $J=[0,T]$, $m\geqslant 1$, $\lambda,\lambda_1,\dots,\lambda_m\in\mathbb{R}$, $a,\sigma\in C(J)$, $0\leqslant r_0<r_1<\cdots<r_m=T$, denote $H:=\lambda_0+\sum_{i=1}^{m}\lambda_i e^{-\int_0^{r_i}a(u)\,du}$, and consider the Green's function G given in Theorems* 2.2 *and* 2.5 (*with* $p=m-1$)*. Assume that one of the conditions* (i) *or* (ii) *in Theorem* 3.10 *holds* (*or any in Corollary* 3.13)*. Under these hypotheses*:

- *If* $\sigma\geqslant 0$, *then the solution to* (2) *is nonpositive.*
- *If* $\sigma\leqslant 0$, *then the solution to* (2) *is nonnegative.*

Example 3.16. For $0\leqslant r_0<r_1=T$, and the boundary condition $-\lambda_0 u(r_0)=\lambda_1 u(T)$ (here $m=1$), we get that the nonpositivity conditions for G are (see Corollary 3.13)

- $\lambda_0+\lambda_1 e^{-\int_{r_0}^{r_1}a(u)\,du}>0$ and $\lambda_0\leqslant 0$, or
- $\lambda_0+\lambda_1 e^{-\int_{r_0}^{r_1}a(u)\,du}<0$ and $\lambda_0\geqslant 0$.

According to (ii) in Theorem 3.10, the condition is $\frac{\lambda_0 e^{-\int_0^{r_0}a(u)\,du}}{\lambda_1 e^{-\int_0^{r_1}a(u)\,du}}=\frac{\lambda_0}{\lambda_1}e^{\int_{r_0}^{r_1}a(u)\,du}\leqslant 0$, that is, $\frac{\lambda_0}{\lambda_1}\leqslant 0$. The Green's function is given in this case by (3) or (6), where $m=1$, $p=0$, and $r_1=T$. For instance, if $r_0>0$, G has expression

$$G(t,s)=e^{-\int_0^t a(u)\,du}\begin{cases}\left(\frac{-1}{H}\left(\lambda_0 e^{-\int_s^{r_0}a(u)\,du}+\lambda_1 e^{-\int_s^{T}a(u)\,du}\right)+e^{\int_0^s a(u)\,du}\right), & \text{if } s\in[0,r_0],\ s\leqslant t,\\ \left(\frac{-1}{H}\lambda_1 e^{-\int_s^{T}a(u)\,du}+e^{\int_0^s a(u)\,du}\right), & \text{if } s\in(r_0,T],\ s\leqslant t,\\ \frac{-1}{H}\left(\lambda_0 e^{-\int_s^{r_0}a(u)\,du}+\lambda_1 e^{-\int_s^{T}a(u)\,du}\right), & \text{if } s\in[0,r_0],\ s>t,\\ \frac{-1}{H}\lambda_1 e^{-\int_s^{T}a(u)\,du}, & \text{if } s\in(r_0,T],\ s>t,\end{cases} \tag{20}$$

where $H:=\lambda_0 e^{-\int_0^{r_0}a(u)\,du}+\lambda_1 e^{-\int_0^{T}a(u)\,du}$.

Example 3.17. For $0\leqslant r_0<r_1<r_2=T$, and the boundary condition $-\lambda_0 u(r_0)=\lambda_1 u(r_1)+\lambda_2 u(T)$ (here $m=2$), the conditions for nonpositivity of G are, according to Corollary 3.13,

$$\lambda_0+\lambda_1 e^{-\int_{r_0}^{r_1}a(u)\,du}+\lambda_2 e^{-\int_{r_0}^{r_2}a(u)\,du}>0, \qquad \lambda_0\leqslant 0, \qquad \lambda_0+\lambda_1 e^{-\int_{r_0}^{r_1}a(u)\,du}\leqslant 0,$$

or

$$\lambda_0+\lambda_1 e^{-\int_{r_0}^{r_1}a(u)\,du}+\lambda_2 e^{-\int_{r_0}^{r_2}a(u)\,du}<0, \qquad \lambda_0\geqslant 0, \qquad \lambda_0+\lambda_1 e^{-\int_{r_0}^{r_1}a(u)\,du}\geqslant 0.$$

On the other hand, by condition (ii) in Theorem 3.10, other conditions for the nonpositivity of G are

$$\frac{\lambda_0 e^{-\int_0^{r_0} a(u)\,du}}{\lambda_1 e^{-\int_0^{r_1} a(u)\,du} + \lambda_2 e^{-\int_0^{r_2} a(u)\,du}} \leqslant 0,$$

and

$$\frac{\lambda_0 e^{-\int_0^{r_0} a(u)\,du} + \lambda_1 e^{-\int_0^{r_1} a(u)\,du}}{\lambda_2 e^{-\int_0^{r_2} a(u)\,du}} \leqslant 0,$$

that is,

$$\frac{\lambda_0}{\lambda_1 e^{-\int_{r_0}^{r_1} a(u)\,du} + \lambda_2 e^{-\int_{r_0}^{r_2} a(u)\,du}} \leqslant 0, \qquad \frac{\lambda_0 + \lambda_1 e^{-\int_{r_0}^{r_1} a(u)\,du}}{\lambda_2 e^{-\int_{r_0}^{r_2} a(u)\,du}} \leqslant 0$$

assuming that the denominators are non-null.

The Green's function is given by (3) or (6), where $m = 2$, $p = 1$, and $r_2 = T$. For instance, if $r_0 = 0$, G has expression

$$G(t,s) = e^{-\int_0^t a(u)\,du} \begin{cases} (\frac{-1}{H} \sum_{i=1}^2 \lambda_i e^{-\int_s^{r_i} a(u)\,du} + e^{\int_0^s a(u)\,du}), & \text{if } s \in [r_0, r_1],\ s \leqslant t, \\ (\frac{-1}{H} \lambda_2 e^{-\int_s^{r_2} a(u)\,du} + e^{\int_0^s a(u)\,du}), & \text{if } s \in (r_1, T],\ s \leqslant t, \\ \frac{-1}{H} \sum_{i=1}^2 \lambda_i e^{-\int_s^{r_i} a(u)\,du}, & \text{if } s \in [r_0, r_1],\ s > t, \\ \frac{-1}{H} \lambda_2 e^{-\int_s^{r_2} a(u)\,du}, & \text{if } s \in (r_1, T],\ s > t, \end{cases} \tag{21}$$

where $H := \lambda_0 e^{-\int_0^{r_0} a(u)\,du} + \lambda_1 e^{-\int_0^{r_1} a(u)\,du} + \lambda_2 e^{-\int_0^{r_2} a(u)\,du}$. On the other hand, if $r_0 > 0$, we have to add to (21) the definition of $G(t,s)$, for $t \in [0, r_0]$, that is,

$$G(t,s) = e^{-\int_0^t a(u)\,du} \begin{cases} (\frac{-1}{H} \sum_{i=0}^2 \lambda_i e^{-\int_s^{r_i} a(u)\,du} + e^{\int_0^s a(u)\,du}), & \text{if } s \in [0, r_0],\ s \leqslant t, \\ \frac{-1}{H} \sum_{i=0}^2 \lambda_i e^{-\int_s^{r_i} a(u)\,du}, & \text{if } s \in [0, r_0],\ s > t. \end{cases} \tag{22}$$

Example 3.18. For $0 \leqslant r_0 < r_1 < r_2 < r_3 = T$, and the boundary condition $-\lambda_0 u(r_0) = \lambda_1 u(r_1) + \lambda_2 u(r_2) + \lambda_3 u(T)$ (here $m = 3$), the conditions for nonpositivity of G are, according to Corollary 3.13, one of the following choices

$$\sum_{i=0}^3 \lambda_i e^{-\int_{r_0}^{r_i} a(u)\,du} > 0, \qquad \lambda_0 \leqslant 0, \qquad \lambda_0 + \lambda_1 e^{-\int_{r_0}^{r_1} a(u)\,du} \leqslant 0, \qquad \lambda_0 + \lambda_1 e^{-\int_{r_0}^{r_1} a(u)\,du} + \lambda_2 e^{-\int_{r_0}^{r_2} a(u)\,du} \leqslant 0,$$

or

$$\sum_{i=0}^3 \lambda_i e^{-\int_{r_0}^{r_i} a(u)\,du} < 0, \qquad \lambda_0 \geqslant 0, \qquad \lambda_0 + \lambda_1 e^{-\int_{r_0}^{r_1} a(u)\,du} \geqslant 0, \qquad \lambda_0 + \lambda_1 e^{-\int_{r_0}^{r_1} a(u)\,du} + \lambda_2 e^{-\int_{r_0}^{r_2} a(u)\,du} \geqslant 0,$$

and the Green's function can be obtained similarly to previous examples.

4. Conclusion

In this paper, we have obtained the exact expression of the solution to a first-order linear differential equation with multipoint boundary value conditions by the calculus of the associated Green's function G. We have also deduced, from the study of the sign of G, the existence of solutions with a constant sign, showing some particular cases. These results are helpful to study some related nonlinear problems [4].

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