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Multiple solutions for quasilinear elliptic problems via critical points in open sublevels and truncation principles

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ABSTRACT

We study a quasilinear elliptic problem depending on a parameter λ of the form

$$-\Delta_p u = \lambda f(u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

We present a novel variational approach that allows us to obtain multiplicity, regularity and a priori estimate of solutions by assuming certain growth and sign conditions on f prescribed only near zero. More precisely, we describe an interval of parameters λ for which the problem under consideration admits at least three nontrivial solutions: two extremal constant-sign solutions and one sign-changing solution. Our approach is based on an abstract localization principle of critical points of functionals of the form $\mathbb{E} = \Phi - \lambda\Psi$ on open sublevels $\Phi^{-1}(]-\infty, r[)$, combined with comparison principles and the sub-supersolution method. Moreover, variational and topological arguments, such as the mountain pass theorem, in conjunction with truncation techniques are the main tools for the proof of sign-changing solutions.

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be a bounded domain with a C^2-boundary $\partial\Omega$, and let $W_0^{1,p}(\Omega)$ denote the usual Sobolev space of functions with generalized homogeneous boundary values endowed with the norm

$$\|u\| := \left(\int_\Omega |\nabla u(x)|^p \, dx \right)^{1/p}. \tag{1}$$

Throughout this paper we assume $p > N$, which implies that $W_0^{1,p}(\Omega)$ is compactly embedded into $C^0(\overline{\Omega})$ with the norm of the embedding operator denoted by c and given by

$$c := \sup_{u \in W_0^{1,p}(\Omega),\ u \neq 0} \frac{\|u\|_{C^0(\overline{\Omega})}}{\|u\|} < +\infty. \tag{2}$$

In this paper we consider the following parameter-dependent quasilinear elliptic boundary value problem

$$\begin{cases} -\Delta_p u = \lambda f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplace operator, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, and λ is a positive parameter.

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The main goal of this paper is to show that there are finite open intervals $\Lambda_k \subset \mathbb{R}_+$, $k = 1, 2$, such that for any $\lambda \in \Lambda_1$, problem (3) admits at least two nontrivial solutions of constant sign, while for $\lambda \in \Lambda_2 \subset \Lambda_1$, there exist at least three nontrivial solutions with two of them of constant sign and a third one that is sign-changing. It should be noted that in recent years a number of papers have been published dealing with multiple solutions, and, in particular, with multiple constant-sign and sign-changing solutions; see, e.g., [1–11], and the references therein. In order to show the existence of multiple solutions, usually certain growth conditions of different nature on the nonlinearity $s \mapsto f(s)$ are required.

Unlike in the above references, the novelty of this paper is to show the existence of multiple solutions in the case that $s \mapsto f(s)$ satisfies certain growth condition only in some neighborhood of $s = 0$. More precisely, let λ_k, $k = 1, 2$, denote the first and second eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$, and let $F : \mathbb{R} \to \mathbb{R}$ denote the primitive of f given by

$$F(s) := \int_0^s f(t)\, dt, \quad \forall\, s \in \mathbb{R},$$

then we make the following assumptions on f near zero: (for $a, b \in \mathbb{R} \cup \{\pm\infty\}$ we denote by $]a, b[$ the open interval in $\mathbb{R}$)

(f_1) $\lim_{t\to 0} \frac{f(t)}{|t|^{p-2}t} = L \in]0, +\infty[$.

(f_{2,λ_k}) *There exists a positive number ρ_0 such that*

$$\frac{\max_{|s|\le\rho_0} F(s)}{\rho_0^p} < \frac{1}{c^p \lambda_k |\Omega|} \lim_{s\to 0} \frac{F(s)}{|s|^p}, \tag{4}$$

where $|\Omega|$ stands for the Lebesgue measure of Ω.

Remark 1.1. In view of assumption (f_1), it is easy to verify that $\lim_{s\to 0} \frac{F(s)}{|s|^p} = \frac{L}{p}$. Moreover, to show that the class of functions satisfying (f_1) and (f_{2,λ_k}) is non empty, consider the case $N = 1$, $p = 2$, $\Omega =]0, 1[$, $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(t) = \frac{t}{1 + t + t^2}$$

for every $t \in \mathbb{R}$, and recall that $c^2 \le 1/4$, while $\lambda_k = k^2\pi^2$, with $k = 1, 2$.

As will be seen in Section 2, a crucial role in the existence proof of constant-sign solutions, i.e., a positive and a negative solution of problem (3), is played by the following version of an abstract critical point theorem obtained in [12, Theorem 1.1] which we recall for convenience.

Theorem 1.1. *Let X be a reflexive Banach space, $\Phi : X \to \mathbb{R}$ and $\Psi : X \to \mathbb{R}$ two continuously Gâteaux differentiable functionals such that Φ is coercive, continuous and sequentially weakly lower semicontinuous (w.l.s.c.), while Ψ is sequentially weakly upper semicontinuous. Let $r > \inf_X \Phi$ and put*

$$\varphi(r) := \inf_{v\in\Phi^{-1}(]-\infty,r[)} \frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u) - \Psi(v)}{r - \Phi(v)}.$$

Then, for every $\lambda \in \left]0, \frac{1}{\varphi(r)}\right[$ the functional $\mathbb{E} := \Phi - \lambda\Psi$ has a critical point $u_\lambda \in \Phi^{-1}(]-\infty, r[)$ such that $\mathbb{E}(u_\lambda) \le \mathbb{E}(v)$ for every $v \in \Phi^{-1}(]-\infty, r[)$.

2. Nontrivial constant-sign solutions

Let us recall that a solution of (3) is any function $u \in W_0^{1,p}(\Omega)$ satisfying

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x) \cdot \nabla\varphi(x)\, dx = \lambda \int_\Omega f(u(x))\varphi(x)\, dx, \quad \forall \varphi \in W_0^{1,p}(\Omega). \tag{5}$$

Thanks to (2) and the continuity of f, if $u \in W_0^{1,p}(\Omega)$ is a solution of (3), then $\Delta_p u \in L^\infty(\Omega)$ and the nonlinear regularity theory [13, Theorem 1.5.6] assures that $u \in C^{1,\gamma}(\overline{\Omega})$ for some $\gamma \in]0, 1[$ and $u \in C_0^1(\overline{\Omega})$. In addition, if u is nonnegative, then from (f_1), follows that there exists a constant $\tilde{c}_\lambda > 0$ such that $\Delta_p u \le \tilde{c}_\lambda u^{p-1}$. Hence, applying Vázquez's strong maximum principle [14], one has that if $u \neq 0$ then $u \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$, that is the interior of the positive cone $C_0^1(\overline{\Omega})_+ := \{u \in C_0^1(\overline{\Omega}) : u(x) \ge 0, \forall x \in \overline{\Omega}\}$, with respect to the Banach space $C_0^1(\overline{\Omega}) := \{u \in C^1(\overline{\Omega}) : u(x) = 0, \forall x \in \partial\Omega\}$. In particular, it is well known that

$$\mathrm{int}(C_0^1(\overline{\Omega})_+) = \left\{u \in C_0^1(\overline{\Omega}) : u(x) > 0\, \forall x \in \Omega, \text{ and } \frac{\partial u}{\partial n}(x) < 0\, \forall x \in \partial\Omega\right\},$$

where $n = n(x)$ is the outer unit normal at $x \in \partial\Omega$.

Inequality (4) of (f_{2,λ_k}) gives rise to the definition of the following intervals

$$\Lambda_k = \left]\frac{\lambda_k}{p\lim\limits_{s\to 0}\frac{F(s)}{|s|^p}}, \frac{\rho_0^p}{pc^p|\Omega|\max\limits_{|s|\le\rho_0}F(s)}\right[. \tag{6}$$

The existence of positive and negative solutions is given by the following.

Theorem 2.1. *Assume hypotheses* (f_1) *and* (f_{2,λ_1}). *Then, for every* $\lambda \in \Lambda_1$, *problem* (3) *admits at least one solution* $v_+ \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$ *and one solution* $v_- \in -\mathrm{int}(C_0^1(\overline{\Omega})_+)$ *such that* $\|v_\pm\|_{C^0(\overline{\Omega})} < \rho_0$.

Proof. Put $X := W_0^{1,p}(\Omega)$ and

$$\tilde{f}(t) := \begin{cases} f(t) & \text{if } t \ge 0 \\ 0 & \text{if } t < 0, \end{cases} \qquad \tilde{F}(s) := \int_0^s \tilde{f}(t)\,dt, \quad \forall s \in \mathbb{R},$$

$$\Phi(u) := \frac{1}{p}\|u\|^p, \qquad \Psi(u) := \int_\Omega \tilde{F}(u(x))\,dx, \qquad \mathbb{E}(u) := \Phi - \lambda\Psi(u)$$

for every $u \in X$ and $\lambda > 0$. In view of (2), for $r := \frac{\rho_0^p}{pc^p}$ one has

$$\Phi^{-1}(]-\infty, r[) \subseteq \left\{u \in C^0(\overline{\Omega}) : \|u\|_{C^0(\overline{\Omega})} < \rho_0\right\}. \tag{7}$$

From this, we obtain the following estimate

$$\varphi(r) \le \frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u)}{r} \le pc^p|\Omega|\frac{\max\limits_{|s|\le\rho_0}\tilde{F}(s)}{\rho_0^p} \le pc^p|\Omega|\frac{\max\limits_{|s|\le\rho_0}F(s)}{\rho_0^p},$$

which implies $\Lambda_1 \subseteq \left]0, \frac{1}{\varphi(r)}\right[$. Fix $\lambda \in \Lambda_1$ and apply Theorem 1.1 to conclude the existence of a $v_+ \in \Phi^{-1}(]-\infty, r[)$ such that $\mathbb{E}(v_+) \le \mathbb{E}(u)$ for every $u \in \Phi^{-1}(]-\infty, r[)$, that is, v_+ is a local minimum of $\mathbb{E}$, and due to (7), we see that $\|v_+\|_{C^0(\overline{\Omega})} < \rho_0$. We claim that $v_+ \ne 0$. By assumption (f_1) and Remark 1.1, we readily see that for any $\lambda \in \Lambda_1$ the inequality $\frac{L}{\lambda_1} > \frac{1}{\lambda}$ holds true, and thus there are two positive numbers δ and α such that

$$\frac{f(t)}{|t|^{p-2}t} > L - \alpha > \frac{\lambda_1}{\lambda}, \quad \forall t \in]-\delta, \delta[\setminus\{0\}. \tag{8}$$

Let φ_1 denote the positive eigenfunction, related to the first eigenvalue λ_1, such that $\|\varphi_1\|_p = 1$. It is well known that $\varphi_1 \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$. Thus, for $\varepsilon > 0$ small enough one gets $\|\varepsilon\varphi_1\|_{C^0(\overline{\Omega})} < \delta$ which in view of (8) yields

$$\mathbb{E}(v_+) \le \mathbb{E}(\varepsilon\varphi_1) < \frac{\varepsilon^p}{p}[\lambda_1 - \lambda(L-\alpha)] < 0 = \mathbb{E}(0), \tag{9}$$

namely 0 is not a local minimum of $\mathbb{E}$, and $v_+ \ne 0$ solves the problem

$$-\Delta_p u = \lambda\tilde{f}(u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

Note that

$$\tilde{f}(v_+(x)) = \begin{cases} f(v_+(x)) & \text{if } v_+(x) \ge 0 \\ 0 & \text{if } v_+(x) < 0, \end{cases} \tag{10}$$

and $\|v_+\|_{C^0(\overline{\Omega})} < \rho_0$. Since v_+ is a (weak) solution of the above problem,

$$\int_\Omega |\nabla v_+|^{p-2}\nabla v_+ \cdot \nabla\varphi\,dx = \lambda\int_\Omega \tilde{f}(v_+)\varphi\,dx, \quad \forall\,\varphi \in W_0^{1,p}(\Omega).$$

Put $s^+ = \max\{s, 0\}$, $s^- = \max\{-s, 0\}$ and $\varphi = v_+^-$, we obtain $\|v_+^-\| = 0$, and thus $v_+ \ge 0$, which by (10) shows that v_+ is a solution of (3).

Reasoning in a similar way, the existence of a negative solution v_-, with $\|v_-\|_{C^0(\overline{\Omega})} < \rho_0$ can be obtained too. Finally, $v_+ \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$ and $v_- \in -\mathrm{int}(C_0^1(\overline{\Omega})_+)$, as pointed out at the beginning of this section. $\square$

Remark 2.1. The mere existence of v_+ and v_- can still be assured if assumption (f_1) is replaced by the slightly more general condition

$$\liminf_{s\to 0} \frac{F(s)}{|s|^p} > 0.$$

Moreover, bearing in mind [15, formula (6b)], it is possible to obtain a precise estimate of the intervals Λ_k, $k = 1, 2$.

3. Extremal constant-sign solutions

In this section we are going to prove that for each $\lambda \in \Lambda_1$ there are a smallest positive and a greatest negative solution of (3).

Theorem 3.1. *Under the assumptions of Theorem 2.1, for every $\lambda \in \Lambda_1$, problem (3) admits the smallest positive solution $u_+ = u_+(\lambda) \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$ within $[0, v_+]$ and the greatest negative solution $u_- = u_-(\lambda) \in -\mathrm{int}(C_0^1(\overline{\Omega})_+)$ within $[v_-, 0]$, such that $\|u_\pm\|_{C^0(\bar{\Omega})} < \rho_0$.*

Proof. Fix $\lambda \in \Lambda_1$, and let v_+ and v_- be the positive and negative solutions of (3), respectively, as obtained in Theorem 2.1. Since $v_+, \varphi_1 \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$, for $\varepsilon > 0$ small enough we can obtain $\varepsilon\varphi_1 \le v_+$. Arguing as in (8), and choosing ε even smaller if needed such that $\varepsilon \in]0, \delta/\|\varphi_1\|_{C^0(\overline{\Omega})}[$, we obtain

$$\begin{aligned} -\Delta_p(\varepsilon\varphi_1) - \lambda f(\varepsilon\varphi_1) &= \lambda_1(\varepsilon\varphi_1)^{p-1} - \lambda \frac{f(\varepsilon\varphi_1)}{(\varepsilon\varphi_1)^{p-1}}(\varepsilon\varphi_1)^{p-1} \\ &\le \lambda_1(\varepsilon\varphi_1)^{p-1} - \lambda(L-\alpha)(\varepsilon\varphi_1)^{p-1} \le 0, \end{aligned}$$

which proves that $\varepsilon\varphi_1$ is a subsolution of (3), and thus $\varepsilon\varphi_1, v_+$ form an ordered pair of sub-supersolution. Applying [16, Theorem 3.22], (see also [17]) there exists a smallest and a greatest solution of (3) within the ordered interval $[\varepsilon\varphi_1, v_+]$. Apparently v_+ is the greatest solution. We denote by $u_\varepsilon \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$ the smallest solution of (3) within $[\varepsilon\varphi_1, v_+]$. Let $\{\varepsilon_n\}$ be a decreasing sequence of positive numbers such that $\varepsilon_1 = \varepsilon$, $\varepsilon_n \downarrow 0_+$, and denote by $u_n \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$ the smallest solution of (3) within the interval $[\varepsilon_n\varphi_1, v_+]$. Since $\{u_n(x)\}$ is non increasing, we can define

$$u_+(x) := \lim_{n\to+\infty} u_n(x) \quad \text{for every } x \in \overline{\Omega}. \tag{11}$$

It is obvious that $0 \le u_+ \le v_+$. Let us verify that

$$u_+ \quad \text{is a non zero solution of problem (3)}. \tag{12}$$

Since every u_n is a solution of (3), one has

$$\int_\Omega |\nabla u_n(x)|^{p-2}\nabla u_n(x)\cdot\nabla\varphi(x)\,dx = \lambda\int_\Omega f(u_n(x))\varphi(x)\,dx, \quad \forall\varphi \in W_0^{1,p}(\Omega). \tag{13}$$

Testing (13) with $\varphi = u_n$ one readily gets $\|\nabla u_n\|_p^p \le \lambda\rho_0|\Omega|\max_{t\in[0,\rho_0]} f(t)$, i.e., $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. In view of (11), and because $W_0^{1,p}(\Omega) \hookrightarrow\hookrightarrow C^0(\overline{\Omega})$, we see that

$$u_n \rightharpoonup u_+ \quad \text{in } W_0^{1,p}(\Omega), \qquad u_n \to u_+ \quad \text{in } C^0(\overline{\Omega}). \tag{14}$$

Taking the test function $\varphi = u_n - u_+$ in (13), we get

$$\lim_{n\to+\infty}\int_\Omega |\nabla u_n(x)|^{p-2}\nabla u_n(x)\cdot\nabla(u_n - u_+)(x)\,dx = 0,$$

which, together with (14) and the S_+-property of $-\Delta_p$, yields

$$u_n \to u_+ \quad \text{strongly in } W_0^{1,p}(\Omega). \tag{15}$$

From (15), (14), passing to the limit in (13), one has that u_+ solves (3).

By contradiction, assume that $u_+ = 0$. Put $\tilde{u}_n := u_n/\|\nabla u_n\|_p$ for every $n \in \mathbb{N}$. Obviously, $\tilde{u}_n \in W_0^{1,p}(\Omega)$ and $\|\tilde{u}_n\| = 1$. Passing to a subsequence if necessary (still denoted by $\tilde{u}_n$), there exists some $\tilde{u} \in W_0^{1,p}(\Omega)$ such that

$$\tilde{u}_n \rightharpoonup \tilde{u} \quad \text{in } W_0^{1,p}(\Omega), \qquad \tilde{u}_n \to \tilde{u} \quad \text{in } C^0(\overline{\Omega}). \tag{16}$$

Dividing by $\|\nabla u_n\|_p^{p-1}$ in (13) one gets

$$\int_\Omega |\nabla\tilde{u}_n(x)|^{p-2}\nabla\tilde{u}_n(x)\cdot\nabla\varphi(x)\,dx = \lambda\int_\Omega \frac{f(u_n(x))}{u_n^{p-1}(x)}\tilde{u}_n^{p-1}(x)\varphi(x)\,dx, \tag{17}$$

for every $\varphi \in W_0^{1,p}(\Omega)$. In particular, for $\varphi = \tilde{u}_n - \tilde{u}$ we obtain

$$\int_\Omega |\nabla \tilde{u}_n(x)|^{p-2} \nabla \tilde{u}_n(x) \cdot \nabla(\tilde{u}_n - \tilde{u})(x)\, dx = \lambda \int_\Omega \frac{f(u_n(x))}{u_n^{p-1}(x)} \tilde{u}_n^{p-1}(x)(\tilde{u}_n - \tilde{u})(x)\, dx. \tag{18}$$

Applying (14), (16), (f_1), and the assumption $u_+ = 0$, the right-hand side of (18) tends to zero. Hence, again by the S_+-property of $-\Delta_p$ and (16), it follows that $\tilde{u}_n \to \tilde{u}$ in $W_0^{1,p}(\Omega)$, so that $\|\tilde{u}\| = 1$, and thus $\tilde{u} \neq 0$. Passing to the limit in (17), in view of (f_1), we obtain, for every $\varphi \in W_0^{1,p}(\Omega)$,

$$\int_\Omega |\nabla \tilde{u}(x)|^{p-2} \nabla \tilde{u}(x) \cdot \nabla \varphi(x)\, dx = \lambda L \int_\Omega \tilde{u}^{p-1}(x)\varphi(x)\, dx.$$

This implies that $\tilde{u}$ is a non-trivial eigenfunction of $(-\Delta_p, W_0^{1,p}(\Omega))$ related to the eigenvalue $\lambda L > \lambda_1$. Hence, $\tilde{u}$ must change sign (see [18]), against the fact that it is the limit of nonnegative functions. This proves (12). The arguments given at the beginning of Section 2 assure that $u_+ \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$.

Finally, let us verify that u_+ is in fact the smallest positive solution of (3) within $[0, v_+]$. Indeed, if u is any positive solution of (3) such that $0 \le u \le v_+$, then $u \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$ and, for some $n \in \mathbb{N}$ one has that $\varepsilon_n \varphi_1 \le u \le v^+$. Recalling (11) and that u_n was constructed as the smallest solution of (3) within $[\varepsilon_n\varphi_1, v_+]$, one has that $u_+ \le u_n \le u$ and the proof is complete for the part regarding u_+. Similar arguments show the existence of $u_- \in -\mathrm{int}(C_0^1(\overline{\Omega})_+)$, being the greatest negative solution of (3) within $[v_-, 0]$. $\square$

Remark 3.1. We point out that Theorem 3.1 gives a more precise additional extremality information with respect to the conclusion of Theorem 2.1.

4. Variational characterization of the extremal solutions

In this section we are going to variationally characterize the extremal constant-sign solutions u_+ and u_- obtained in Theorem 3.1. For this purpose we introduce the following truncation functions

$$\tau_+(x,s) := \begin{cases} 0 & \text{if } s \in]-\infty, 0] \\ s & \text{if } s \in]0, u_+(x)] \\ u_+(x) & \text{if } s \in [u_+(x), +\infty[\end{cases}$$

$$\tau_-(x,s) := \begin{cases} u_-(x) & \text{if } s \in]-\infty, u_-(x)] \\ s & \text{if } s \in]u_-(x), 0[\\ 0 & \text{if } s \in [0, +\infty[\end{cases}$$

$$\tau_0(x,s) := \begin{cases} u_-(x) & \text{if } s \in]-\infty, u_-(x)] \\ s & \text{if } s \in]u_-(x), u_+(x)[\\ u_+(x) & \text{if } s \in [u_+(x), +\infty[, \end{cases}$$

and, for every $\lambda > 0$, the associated truncated functionals on $W_0^{1,p}(\Omega)$

$$\mathbb{E}_\pm(u) := \frac{1}{p}\|\nabla u\|_p^p - \lambda \int_\Omega (F \circ \tau_\pm)(x, u(x))\, dx,$$

$$\mathbb{E}_0(u) := \frac{1}{p}\|\nabla u\|_p^p - \lambda \int_\Omega (F \circ \tau_0)(x, u(x))\, dx.$$

Our main goal here is to show that u_+ and u_- are local minimizers of $\mathbb{E}_0$.

Lemma 4.1. *Assume that the assumptions of Theorem 3.1 hold. Then, for every $\lambda \in \Lambda_1$, the function $u_+ = u_+(\lambda)$ is a global minimizer of $\mathbb{E}_+$ and a local minimizer of $\mathbb{E}_0$, and $u_- = u_-(\lambda)$ is a global minimizer of $\mathbb{E}_-$ and a local minimizer of $\mathbb{E}_0$.*

Proof. Let us begin by observing that

$$\text{if } v \text{ is a critical point of } \mathbb{E}_+, \quad \text{then } 0 \le v \le u_+ \tag{19}$$

which, by the definition of τ_+, implies that any critical point of $\mathbb{E}_+$ is a solution of (3) that belongs to $[0, u_+]$. To prove (19), let $v \in W_0^{1,p}(\Omega)$ be a critical point of $\mathbb{E}_+$, i.e., v satisfies

$$\int_\Omega |\nabla v|^{p-2} \nabla v \cdot \nabla \varphi\, dx = \lambda \int_\Omega f(\tau_+(x, v))\varphi\, dx, \quad \forall\, \varphi \in W_0^{1,p}(\Omega). \tag{20}$$

Since u_+ is a positive solution of (3), by using (20) and the special test function $\varphi = (v - u_+)^+$ as well as the definition of $\mathbb{E}_+$ and τ_+, one has

$$\int_\Omega [|\nabla v(x)|^{p-2}\nabla v(x) - |\nabla u_+(x)|^{p-2}\nabla u_+(x)] \cdot \nabla (v - u_+)^+(x)\, dx$$
$$= \lambda \int_{\{v>u_+\}} [f(\tau_+(x, v(x))) - f(u_+(x))](v - u_+)(x)\, dx = 0.$$

Hence, $\nabla(v - u_+) = 0$ a.e. in $\{v > u_+\}$, i.e. $\nabla(v - u_+)^+ = 0$ a.e. in Ω and $\|(v - u_+)^+\| = 0$, which implies $v \le u_+$. Testing (20) with $\varphi = -v^-$, one gets $\int_\Omega |\nabla v^-|^p\, dx = 0$, and thus $v^- = 0$, which proves (19).

Since $\mathbb{E}_+$ is coercive and w.l.s.c., it has a global minimizer $z_+ \in W_0^{1,p}(\Omega)$, i.e.,

$$\mathbb{E}_+(z_+) = \inf_{W_0^{1,p}(\Omega)} \mathbb{E}_+. \tag{21}$$

Thus, z_+ is a critical point of $\mathbb{E}_+$ and, by (19), $0 \le z_+ \le u_+$. Because $u_+ \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$, for $\sigma > 0$ small enough one gets $\sigma\varphi_1 \le u_+$ and, reasoning as in (8) and (9), $\mathbb{E}_+(\sigma\varphi_1) < 0$, that is $z_+ \ne 0$ is a nontrivial solution of (3), belonging to $[0, u_+]$, which by the minimality property of u_+ assures that $z_+ = u_+$, that is, u_+ is a global minimum of $\mathbb{E}_+$. Since $u_+ \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$, there exists a neighborhood U of u_+ with respect to the topology of $C_0^1(\overline{\Omega})$ such that $U \subset C_0^1(\overline{\Omega})_+$ and $\mathbb{E}_0(u_+) = \mathbb{E}_+(u_+) \le \mathbb{E}_+(u) = \mathbb{E}_0(u)$, for every $u \in U$. In other words, u_+ is a local minimum of $\mathbb{E}_0$ with respect to the topology of $C_0^1(\overline{\Omega})$ and, bearing in mind [13, p. 655–656] (see also [19]), it turns out to be a $W_0^{1,p}(\Omega)$-local minimum. The assertion for u_- can be obtained analogously. □

5. Sign-changing solution

In this section we will see that by restricting the parameter range for λ to Λ_2 (note $\Lambda_2 \subset \Lambda_1$), a sign-changing solution also exists.

Theorem 5.1. *Assume hypotheses* (f_1) *and* $(\mathrm{f}_{2,\lambda_2})$. *Then, for every* $\lambda \in \Lambda_2$ *problem* (3) *has a solution* $u_+ = u_+(\lambda) \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$, *a solution* $u_- = u_-(\lambda) \in -\mathrm{int}(C_0^1(\overline{\Omega})_+)$ *and a nontrivial sign-changing solution* $u_0 = u_0(\lambda) \in C_0^1(\overline{\Omega})$, *whose norms in* C^0 *are less than* ρ_0.

Proof. Let us fix $\lambda \in \Lambda_2 \subset \Lambda_1$ and consider $u_+ \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$ and $u_- \in -\mathrm{int}(C_0^1(\overline{\Omega})_+)$ given by Theorem 3.1. Arguing as in (19), one has that

$$\text{if } v \text{ is a critical point of } \mathbb{E}_0, \quad \text{then } u_- \le v \le u_+. \tag{22}$$

Hence, every critical point of $\mathbb{E}_0$ is a solution of (3) that belongs to the interval $[u_-, u_+]$. It is easy to verify that $\mathbb{E}_0$ is coercive and w.l.s.c., with $\inf_{W_0^{1,p}(\Omega)} \mathbb{E}_0 < 0$. Thus, there exists $z_0 \in W_0^{1,p}(\Omega)$ such that $z_0 \ne 0$ and $\mathbb{E}_0(z_0) = \inf_{W_0^{1,p}(\Omega)} \mathbb{E}_0$, that is z_0 is a critical point of $\mathbb{E}_0$, and thus z_0 is a solution of (3) with $z_0 \in [u_-, u_+]$. Now, distinguish two cases:

(A) $z_0 \ne u_-$ and $z_0 \ne u_+$. Then, $z_0 \in]u_-, u_+[\setminus\{0\}$ is a critical point of $\mathbb{E}_0$. In view of (22) and the extremality properties of u_- and u_+, z_0 must be sign-changing and we conclude taking $u_0 = z_0$.

(B) Either $z_0 = u_-$ or $z_0 = u_+$. Let, for instance, $z_0 = u_+$. Thus, u_+ is a global minimum of $\mathbb{E}_0$, while u_- is a local minimum for the same functional (see Lemma 4.1). If u_- is a non-strict local minimum the proof is done because then $\mathbb{E}_0$ admits infinitely many local minima at the level $\mathbb{E}_0(u_-)$ that, by (22) and the extremality properties of u_- and u_+, must be sign-changing solutions. Therefore, we may assume that u_- is a strict local minimizer. In this case, there exists $\rho \in]0, \|u_- - u_+\|[$ such that

$$\mathbb{E}_0(u_+) \le \mathbb{E}_0(u_-) < \inf_{\|u-u_-\|=\rho} E_0(u). \tag{23}$$

Obviously, $\mathbb{E}_0$ satisfies the Palais–Smale condition and, applying the mountain pass theorem [20], it has a third critical point $u_0 \in W_0^{1,p}(\Omega)$ such that

$$\inf_{\|u-u_-\|=\rho} E_0(u) \le \mathbb{E}_0(u_0) = \inf_{\gamma\in\Gamma} \max_{t\in[-1,1]} \mathbb{E}_0(\gamma(t)), \tag{24}$$

where $\Gamma = \{\gamma \in C([-1, 1], W_0^{1,p}(\Omega)) : \gamma(-1) = u_-, \gamma(1) = u_+\}$. In order to exclude that $u_0 = 0$ we will prove that

$$\mathbb{E}_0(u_0) < 0. \tag{25}$$

For this goal, we will use the variational characterization of λ_2 (see [21])

$$\lambda_2 = \inf_{\gamma\in\Gamma_0} \max_{t\in[-1,1]} \|\nabla\gamma(t)\|_p^p, \tag{26}$$

where $\Gamma_0 = \{\gamma \in C^0([-1,1], S) : \gamma(-1) = -\varphi_1, \gamma(1) = \varphi_1\}$, while $S = W_0^{1,p}(\Omega) \cap \partial B_1^{L^p(\Omega)}$, being $\partial B_1^{L^p(\Omega)} = \{u \in L^p(\Omega) : \|u\|_p = 1\}$, is considered with the $W_0^{1,p}(\Omega)$-topology.

Since $\lambda \in \Lambda_2$, in particular, $\lambda > \lambda_2/L$ and there exist $\beta, \delta > 0$ such that

$$\frac{f(t)}{|t|^{p-2}t} > L - \beta > \frac{\lambda_2}{\lambda} \quad \forall t \in]-\delta, \delta[\setminus\{0\}. \tag{27}$$

By (26), for $\beta' \in]0, \lambda(L-\beta) - \lambda_2[$, there exists $\gamma \in \Gamma_0$ such that

$$\max_{t\in[-1,1]} \|\nabla\gamma(t)\|_p^p < \lambda_2 + \frac{\beta'}{2}. \tag{28}$$

Let $S_C = S \cap C_0^1(\overline{\Omega})$ be endowed with the $C_0^1(\bar{\Omega})$-topology and $\Gamma_{0,C} = \{\gamma \in C^0([-1,1], S_C) : \gamma(-1) = -\varphi_1, \gamma(1) = \varphi_1\}$. Because S_C is dense in S, $\Gamma_{0,C}$ is dense in Γ_0, and for $0 < r \le (\lambda_2 + \beta')^{1/p} - (\lambda_2 + \frac{\beta'}{2})^{1/p}$ there exists $\gamma_0 \in \Gamma_{0,C}$ such that $\max_{t\in[-1,1]} \|\nabla\gamma(t) - \nabla\gamma_0(t)\|_p < r$, and

$$\max_{t\in[-1,1]} \|\nabla\gamma_0(t)\|_p^p < \lambda_2 + \beta'. \tag{29}$$

Moreover, since $u_+, -u_- \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$, there exists $\delta' > 0$ such that

$$u_+ + B_{\delta'}^{C_0^1(\overline{\Omega})} \subset \operatorname{int}(C_0^1(\overline{\Omega})_+), \qquad -u_- + B_{\delta'}^{C_0^1(\overline{\Omega})} \subset \operatorname{int}(C_0^1(\overline{\Omega})_+), \tag{30}$$

where $B_{\delta'}^{C_0^1(\overline{\Omega})} = \{u \in C_0^1(\overline{\Omega}) : \|u\|_{C_0^1(\overline{\Omega})} \le \delta'\}$. Obviously $\gamma_0 : [-1,1] \to C_0^1(\bar{\Omega})$ is continuous and there exists $M > 0$ with $\max_{t\in[-1,1]} \|\gamma_0(t)\|_{C^1(\overline{\Omega})} \le M$. Fix $\varepsilon_1 \in \left]0, \min\{\delta/M, \delta'/M\}\right[$ and pick $\varepsilon \in]0, \varepsilon_1[$. Then, $\varepsilon\gamma_0$ is a path in $C_0^1(\overline{\Omega})$ joining $-\varepsilon\varphi_1$ and $\varepsilon\varphi_1$. Moreover, for every $t \in [-1,1]$ one has

$$\varepsilon|\gamma_0(t)(x)| \le \varepsilon\|\gamma_0(t)\|_{C_0^1(\overline{\Omega})} \le \varepsilon_1 M < \delta, \quad \forall x \in \Omega, \tag{31}$$

as well as $\varepsilon\|\gamma_0(t)\|_{C_0^1(\overline{\Omega})} \le \varepsilon_1 M < \delta'$, that is $\pm\varepsilon\gamma_0(t) \in B_{\delta'}^{C_0^1(\overline{\Omega})}$. By (30) it follows that $u_+ - \varepsilon\gamma_0(t), -u_- + \varepsilon\gamma_0(t) \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$, and thus

$$u_- \le \varepsilon\gamma_0(t) \le u_+. \tag{32}$$

Hence, putting together (29), (32), (31), (27) and recalling that $\gamma_0([-1,1]) \subset \partial B_1^{L^p(\Omega)}$, one has

$$\begin{aligned} \mathbb{E}_0(\varepsilon\gamma_0(t)) &= \frac{\varepsilon^p}{p}\|\nabla\gamma_0(t)\|_p^p - \lambda\int_\Omega F(\tau_0(x, \varepsilon\gamma_0(t)(x)))\,dx \\ &\le \frac{\varepsilon^p}{p}[\lambda_2 + \beta' - \lambda(L-\beta)] < 0 \quad \forall t \in [-1,1]. \end{aligned} \tag{33}$$

Now set

$$c_+ = \mathbb{E}_+(\varepsilon\varphi_1), \qquad m_+ = \mathbb{E}_+(u_+) \qquad E_+^{c_+} = \{u \in W_0^{1,p}(\Omega) : \mathbb{E}_+(u) \le c_+\}.$$

Since $u_+ - \varepsilon\varphi_1 \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ and u_+ is the smallest positive solution of (3), $\varepsilon\varphi_1$ is not a critical point of $\mathbb{E}_+$ and $m_+ < c_+$. For every $\mu \in]m_+, c_+]$ one has that μ is not a critical value of $\mathbb{E}_+$. In fact, by contradiction, if $w_+ \in W_0^{1,p}(\Omega)$ is a critical point of $\mathbb{E}_+$ with $\mathbb{E}_+(w_+) = \mu \in]m_+, c_+]$, then, due to (19), $0 \le w_+ \le u_+$ and $w_+ \ne 0$ because $c_+ = \mathbb{E}_+(\varepsilon\varphi_1) = \mathbb{E}_0(\varepsilon\varphi_1) = \mathbb{E}_0(\varepsilon\gamma_0(1)) < 0$, in view of (33). Hence $w_+ = u_+$ and $\mu = m_+$ that is a contradiction. It is simple to verify that $\mathbb{E}_+$ satisfies the Palais–Smale condition, so that we can apply the second deformation lemma [13, p. 366] to the C^1 function $\mathbb{E}_+$, and obtain $\eta \in C^0([0,1] \times E_+^{c_+}, E_+^{c_+})$ such that $\eta(0,u) = u$ and $\eta(1,u) = u_+$ for every $u \in E_+^{c_+}$, as well as $E_+(\eta(t,u)) \le E_+(u)$ for every $t \in [0,1]$ and $u \in E_+^{c_+}$. Let us define the path $\gamma_+ : [0,1] \to W_0^{1,p}(\Omega)$ by putting $\gamma_+(t) := \eta(t, \varepsilon\varphi_1)^+ = \max\{\eta(t, \varepsilon\varphi_1), 0\}$ for every $t \in [0,1]$. Clearly $\gamma_+ \in C^0([0,1], W_0^{1,p}(\Omega))$ joining $\varepsilon\varphi_1$ and u_+. Moreover, for every $t \in [0,1]$ one has

$$\begin{aligned} \mathbb{E}_0(\gamma_+(t)) &= \frac{1}{p}\int_{\{\eta(t,\varepsilon\varphi_1)>0\}} |\nabla\eta(t, \varepsilon\varphi_1)(x)|^p\,dx - \lambda\int_{\{\eta(t,\varepsilon\varphi_1)>0\}} F(\tau_+(x, \eta(t, \varepsilon\varphi_1)(x)))\,dx \\ &\le \mathbb{E}_+(\eta(t, \varepsilon\varphi_1)) \le \mathbb{E}_+(\varepsilon\varphi_1) < 0, \end{aligned} \tag{34}$$

where we use the definitions of $\mathbb{E}_0$, τ_0, τ_+ and $\mathbb{E}_+$ as well the properties of η. Reasoning in the same way with the functional $\mathbb{E}_-$ it is possible to construct a continuous path $\gamma_- : [0,1] \to W_0^{1,p}(\Omega)$ joining $-\varepsilon\varphi_1$ and u_- such that

$$\mathbb{E}_0(\gamma_-(t)) < 0, \quad \forall t \in [0,1]. \tag{35}$$

The union of γ_-, $\varepsilon\gamma_0$ and γ_+ produces a path $\gamma \in \Gamma$ such that, because of (35), (33) and (34) and the continuity of $\mathbb{E}_0$, $\max_{t\in[-1,1]} \mathbb{E}_0(\gamma(t)) < 0$, which proves (25). Keeping in mind that u_0 is a nontrivial solution of (3), distinct from u_- and u_+ such that, by (22), $u_- \le u_0 \le u_+$, the extremality properties of u_- and u_+ assures that u_0 must change sign. Moreover, the regularity theory implies that $u_0 \in C_0^1(\overline{\Omega})$, which completes the proof. $\square$

Example 5.1. For every $\lambda \in \left]4\pi^2, \frac{\pi^4}{1-\exp(-\pi^4/4)}\right[$ the following Dirichlet problem

$$\begin{cases} -u'' = \lambda u \, \exp\left(\dfrac{u^4}{\pi^4} - u^2\right) & \text{in }]0,1[\\ u(0) = u(1) = 0 \end{cases} \tag{36}$$

satisfies the conclusion of Theorem 5.1 with $\rho_0 = \pi^2$.

Corollary 5.1. *Assume* (f_1) *and the following condition:*

$$\lim_{\rho\to+\infty} \frac{\max_{|s|\le\rho} F(s)}{\rho^p} = 0. \tag{37}$$

Then, for every $\lambda > \lambda_2/L$ problem (3) has a solution $u_+ = u_+(\lambda) \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$, a solution $u_- = u_-(\lambda) \in -\mathrm{int}(C_0^1(\overline{\Omega})_+)$ and a nontrivial sign-changing solution $u_0 = u_0(\lambda) \in C_0^1(\overline{\Omega})$.

Proof. Fix $\lambda > \lambda_2/L$. From (37) there exists $\rho_0 = \rho_0(\lambda) > 0$ such that $\frac{\lambda_2}{L} < \lambda < \frac{\rho_0^p}{pc^p|\Omega| \max_{|s|\le\rho_0} F(s)}$ and the conclusion follows at once by Theorem 5.1, if we observe that $\lambda \in \Lambda_2$. $\square$

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