J. Math. Anal. Appl. 396 (2012) 640–655

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Unitary dimension reduction for a class of self-adjoint extensions with applications to graph-like structures

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ARTICLE INFO

Article history:
Received 4 September 2011
Available online 6 July 2012
Submitted by Junping Shi

Keywords:
Self-adjoint extension
Weyl function
Boundary triplet
Quantum graph
Metric graph

ABSTRACT

We consider a class of self-adjoint extensions using the boundary triplet technique. Assuming that the associated Weyl function has the special form $M(z) = (m(z)\mathrm{Id} - T)n(z)^{-1}$ with a bounded self-adjoint operator T and scalar functions m, n we show that there exists a class of boundary conditions such that the spectral problem for the associated self-adjoint extensions in gaps of a certain reference operator admits a unitary reduction to the spectral problem for T. As a motivating example we consider differential operators on equilateral metric graphs, and we describe a class of boundary conditions that admit a unitary reduction to generalized discrete Laplacians.

1. Introduction

The present work is motivated by the study of the relationship between discrete operators on graphs and differential operators on metric graphs (quantum graphs); see [1–5]. Let us recall the basic notions and introduce an illustrative example.

Let G be a countable graph, the sets of the vertices and of the edges of G will be denoted by $\mathcal{V}$ and $\mathcal{E}$, respectively, and multiple edges and self-loops are allowed. For an edge $e \in \mathcal{E}$ we denote by $\iota e \in \mathcal{V}$ its initial vertex and by $\tau e \in \mathcal{V}$ its terminal vertex. For a vertex v, the number of outgoing edges and the number of ingoing edges will be denoted by $\operatorname{outdeg} v$ and $\operatorname{indeg} v$, respectively, and the degree of v is $\deg v := \operatorname{indeg} v + \operatorname{outdeg} v$. In what follows, we assume that there are no isolated vertices, i.e. $\deg v \geq 1$ for all $v \in \mathcal{V}$. Introduce the discrete Hilbert space

$$l^2(G) := \left\{ f : \mathcal{V} \to \mathbb{C} : \ \|f\|^2 = \sum_{v\in\mathcal{V}} \deg v |f(v)|^2 < +\infty \right\}$$

and the transition operator Δ in $l^2(G)$,

$$(\Delta f)(v) = \frac{1}{\deg v} \left(\sum_{e:\iota e = v} f(\tau e) + \sum_{e:\tau e = v} f(\iota e) \right). \tag{1}$$

Numerous works treat the relationship between the properties of Δ and G; see e.g. [6] and references therein.

Let us now introduce a continuous Laplacian on G. Consider the Hilbert space $\mathcal{H} := \bigoplus_{e\in\mathcal{E}} \mathcal{H}_e$, $\mathcal{H}_e = L^2(0, 1)$, and the operator Λ, $\Lambda(f_e) = (-f_e'')$, acting on the functions $f = (f_e) \in H^2(0, 1)$ satisfying the so-called standard boundary

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doi:10.1016/j.jmaa.2012.07.005

conditions:

$$f_e(1) = f_b(0) \quad \text{for all } b, e \in \mathcal{E} \text{ with } \iota b = \tau e \ (= \text{continuity at each vertex}),$$
$$\sum_{e:\iota e = v} f_e'(0) - \sum_{e:\tau e = v} f_e'(1) = 0.$$

It is known that Λ is self-adjoint and that its spectrum is closely related with the spectrum of Δ: denoting $\sigma_D = \{(\pi n)^2 : n \in \mathbb{N}\}$ one has the relationship

$$\operatorname{spec}_j \Lambda \setminus \sigma_D = \{z \notin \sigma_D : \cos\sqrt{z} \in \operatorname{spec}_j \Delta\}, \quad j \in \{\text{p}, \text{pp}, \text{disc}, \text{ess}, \text{ac}, \text{sc}\}. \tag{2}$$

For some particular configurations the above relationship was used (implicitly) first in the physics literature; see e.g. [7,8] and the historical remarks in [9, Section III.2]. Concerning mathematically rigorous results, for $j \in \{\text{p}, \text{disc}, \text{ess}\}$ the above equalities (2) were proved, for example, in [10] for finite graphs and in [11] for infinite ones. In [12] the result was obtained for the first time for all types of spectra using a completely different machinery, and the work [13] used the results of [12] to prove a similar result for continuous Laplacians with more general boundary conditions. We note that all the spectral components for Δ can be non-trivial; see e.g. [14–17] for respective examples. We refer e.g. to [18–31] for generalizations to more general differential operators and for the analysis of particular configurations. The aim of the present paper is to improve the relation (2). If Ω is a Borel set in $\mathbb{R}$ and A is a selfadjoint operator, denote by A_Ω the part of A in Ω, i.e. $A_\Omega = A 1_\Omega(A)$ considered as an operator in $\operatorname{ran} 1_\Omega(A)$; here $1_\Omega(A)$ is the spectral projector of A onto Ω. A simple corollary of Theorem 17 below is the following.

Proposition 1. *Denote $\eta(z) := \cos\sqrt{z}$, then for any interval $J \subset \mathbb{R} \setminus \sigma_D$ the operator Λ_J is unitarily equivalent to the operator $\eta^{-1}\big(\Delta_{\eta(J)}\big)$.*

It was noted by the author in [27] that the operator Λ can be studied at an abstract level using the language of boundary triplets and self-adjoint extensions [32,33,12]. Let S be a closed densely defined symmetric operator in a separable Hilbert space $\mathcal{H}$ with the domain $\operatorname{dom} S$. Assume that S has equal deficiency indices, i.e. $\dim\ker(S^* + i) = \dim\ker(S^* - i)$. A *boundary triplet* for S consists of a Hilbert space $\mathcal{G}$ and two linear maps $\Gamma, \Gamma' : \operatorname{dom} S \to \mathcal{G}$ satisfying the following two conditions [32]:

- $\langle f, S^* g\rangle - \langle S^* f, g\rangle = \langle \Gamma f, \Gamma' g\rangle - \langle \Gamma' f, \Gamma g\rangle$ for all $f, g \in \operatorname{dom} S^*$,
- the application $(\Gamma, \Gamma') : \operatorname{dom} S^* \ni f \mapsto (\Gamma f, \Gamma' f) \in \mathcal{G} \oplus \mathcal{G}$ is surjective.

We will consider the two distinguished self-adjoint extensions of S:

$$H^0 := S^*|_{\ker \Gamma} \quad \text{and} \quad H := S^*|_{\ker \Gamma'}. \tag{3}$$

An essential role in the analysis of the self-adjoint extensions is played by the so-called Weyl function $M(z)$ which is defined as follows. For $z \notin \operatorname{spec} H^0$ consider the operator $\gamma(z) := \big(\Gamma|_{\ker(S^*-z)}\big)^{-1}$ which is a linear topological isomorphism between $\mathcal{G}$ and $\ker(S^* - z) \subset \mathcal{H}$, then the map $\mathbb{C} \setminus \operatorname{spec} H^0 \ni z \mapsto \gamma(z) \in \mathcal{L}(\mathcal{G}, \mathcal{H})$ (called γ-field) is holomorph. The operator function $\mathbb{C} \setminus \operatorname{spec} H^0 \ni z \mapsto M(z) := \Gamma'\gamma(z) \in \mathcal{L}(\mathcal{G})$ is called the *Weyl* function associated with the boundary triplet [33]. Outside $\operatorname{spec} H^0 \cup \operatorname{spec} H$ the Krein resolvent formula holds, $(H^0 - z)^{-1} - (H - z)^{-1} = \gamma(z) M(z)^{-1} \gamma(\bar{z})^*$, and we have the relation [33,12]

$$\operatorname{spec}_j H \setminus \operatorname{spec} H^0 = \big\{z \notin \operatorname{spec} H^0 : 0 \in \operatorname{spec}_j M(z)\big\}, \quad j \in \{\text{p}, \text{disc}, \text{ess}\}. \tag{4}$$

Numerous papers were devoted to the question whether one can explain the relation (4) and to recover, for example, the singular or the absolutely continuous spectrum of H in terms of the spectral properties of M, see e.g. [34–36,12,33,37] and references therein. Our main result contributes this direction and concerns Weyl functions of a special form.

Theorem 2. *Assume that the Weyl function M has the form*

$$M(z) = \frac{m(z)\mathrm{Id} - T}{n(z)} \tag{5}$$

where

- *T is a bounded self-adjoint operator in $\mathcal{G}$,*
- *m and n are scalar functions which are holomorph outside $\operatorname{spec} H^0$.*

Assume that there exists a spectral gap $J := (a_0, b_0) \subset \mathbb{R} \setminus \operatorname{spec} H^0$ such that m and n admit a holomorph continuation to J, are both real-valued in J, that $n \neq 0$ in J, and that $m(J) \cap \operatorname{spec} T \neq \emptyset$, then

(a) *there exists an interval K containing $m^{-1}(\operatorname{spec} T) \cap J$ such that $m : K \to m(K)$ is a bijection; denote by μ the inverse function;*
(b) *the operator H_J is unitarily equivalent to $\mu(T_{m(J)})$.*

As was shown in [27], the analysis of the above operator Λ can be put into the framework of boundary triplets: the associated Weyl function in suitable coordinates has the requested form $M(z) = \big(\Delta - \cos\sqrt{z}\,\mathrm{Id}\big)\sqrt{z}/\sin\sqrt{z}$, and Proposition 1 becomes a simple corollary of Theorem 2. We recall these constructions and generalize the above example in Section 3.

Theorem 2 shows that the spectral analysis of H in the interval J is equivalent to the spectral analysis of the operator T on a "smaller" space $\mathcal{G}$, and this fact can be considered as a dimension reduction. Note that for $n = \text{const} \neq 0$ Theorem 2 is actually proved in [34]: it is not stated explicitly, but the proof of Theorem 4.4 in [34] contains the result, and we are adapting their scheme of proof to the case of non-constant n. The main difference comes from the fact that for constant n the function m is strictly increasing, while this is no more true for general n, which brings some additional difficulties. Note that the results of [34] are suitable for the analysis of operators that can be represented as direct sums of operators with deficiency indices $(1, 1)$, but this does not cover the above example with the continuous graph Laplacian.

We emphasize that the condition $m(J) \cap \operatorname{spec} T \neq \emptyset$ in Theorem 2 is just to avoid some pathologies in the notation and this does not bring any restriction. If $m(J) \cap \operatorname{spec} T = \emptyset$, then by (4) the operator H has no spectrum in J, and the assertion (b) still holds formally, as both operators are defined on the zero space.

Note that as an obvious corollary of Theorem 2 we have the following assertion obtained already in the author's joint work [12, Theorem 3.16] by a different method:

Corollary 3. *For any $x \in J$ and any $j \in \{\mathrm{p}, \mathrm{pp}, \mathrm{disc}, \mathrm{ess}, \mathrm{ac}, \mathrm{sc}\}$ the assertions*

- $x \in \operatorname{spec}_j H$,
- $m(x) \in \operatorname{spec}_j T$

are equivalent.

2. Proof of the unitary equivalence

This section is devoted to the proof of Theorem 2.

2.1. Operator-valued measures

In what follows, by $\mathcal{B}(\mathbb{R})$ we denote the algebra of Borel subsets of $\mathbb{R}$, and by $\mathcal{B}_b(\mathbb{R})$ its subalgebra consisting of the bounded Borel subsets. If $\mathcal{H}$ and $\mathcal{H}'$ are Hilbert spaces, then $\mathcal{L}(\mathcal{H}, \mathcal{H}')$ stands for the space of bounded linear operators from $\mathcal{H}$ to $\mathcal{H}'$, and $\mathcal{L}(\mathcal{H}) := \mathcal{L}(\mathcal{H}, \mathcal{H})$. A mapping $\Sigma : \mathcal{B}_b(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$ is called an *operator-valued measure* (in $\mathcal{H}$) if it is σ-additive with respect to the strong convergence and if $\Sigma(B) = \Sigma(B)^* \geq 0$ for all $B \in \mathcal{B}_b(\mathbb{R})$. An operator-valued measure Σ is called *bounded* if it extends by σ-additivity to a map $\mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$. A bounded operator-valued measure Σ is called *orthogonal* if it satisfies two additional conditions: $\Sigma(B_1 \cap B_2) = \Sigma(B_1)\Sigma(B_2)$ for all $B_1, B_2 \in \mathcal{B}(\mathbb{R})$ and $\Sigma(\mathbb{R}) = \mathrm{Id}$.

Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces, $K : \mathcal{H}_2 \to \mathcal{H}_1$ be a bounded linear operator, and Σ_1 be a bounded operator-valued spectral measure in $\mathcal{H}_1$, then the mapping $\Sigma_2 : \mathcal{B}(\mathbb{R}) \ni B \mapsto \Sigma_2(B) := K^*\Sigma_1(B)K \in \mathcal{L}(\mathcal{H}_2)$ is a bounded operator-valued measure in $\mathcal{H}_2$ which is called a *dilation* of Σ_1. This dilation is *orthogonal* if the above representation holds with a unitary operator K and is called *minimal* if the closed linear span of the subspaces $\Sigma_1(B)\operatorname{ran} K$, $B \in \mathcal{B}(\mathbb{R})$, coincides with $\mathcal{H}_1$. If a bounded operator-valued measure is an orthogonal dilation of another bounded operator-valued measure, then these two measures are called *unitarily equivalent*. Note that the spectral measure of a self-adjoint operator is always an orthogonal operator-valued measure. The following assertion is well known; see e.g. [38, Chapter 4] or [39].

Theorem 4 (*Generalized Naimark's Dilation Theorem*)**.** *Any bounded operator-valued measure Σ can be represented as a minimal dilation of an orthogonal operator-valued measure Σ^0, and Σ^0 is called a* minimal orthogonal operator-valued measure associated with Σ*. If a bounded operator-valued measure can be represented as a minimal orthogonal dilation of two different orthogonal operator-valued measures, then these two orthogonal operator-valued measures are unitarily equivalent.*

Let us recall some tools that allow one to obtain some information on the spectral measures for self-adjoint extensions using the Weyl functions.

Let $\mathbb{C}_+ := \{z \in \mathbb{C} : \Im z > 0\}$ and $\mathcal{H}$ be a Hilbert space. A map $\mathbb{C}_+ \ni z \mapsto F(z) \in \mathcal{L}(\mathcal{H})$ is called an (operator-valued) *Herglotz function* on $\mathcal{H}$ if $\Im F(z) \geq 0$ for all $z \in \mathbb{C}_+$. To each Herglotz function F on $\mathcal{H}$ one can associate a uniquely defined bounded operator-valued measure (bounded Herglotz measure), in $\mathcal{H}$, which we denote by Σ_F^0, and two non-negative operators C_0 and C_1 on $\mathcal{H}$ such that

$$F(z) = C_0 + C_1 z + \int_{\mathbb{R}} \frac{1 + tz}{t - z}\,\Sigma_F^0(dt) \quad \text{for all } z \in \mathbb{C}_+.$$

One can introduce another operator-valued measure Σ_F (unbounded Herglotz measure) associated with F by the equality

$$\Sigma_F(B) := \int_B (1 + t^2)\,\Sigma_F^0(dt), \quad B \in \mathcal{B}_b(\mathbb{R}).$$

This operator-valued measure is unbounded in general, but it can be recovered from the values F by the explicit Stieltjes inversion formula

$$\Sigma_F\left((a,b)\right) = \operatorname*{s-lim}_{\delta\to 0+}\operatorname*{s-lim}_{\varepsilon\to 0+}\frac{1}{\pi}\int_{a+\delta}^{b-\delta}\Im F(x+i\varepsilon)\,dx; \tag{6}$$

see [40,41]. Note that the Weyl function $M(z)$ defined by a boundary triplet is always a Herglotz function and satisfies $M(\bar z) = M(z)^*$; see e.g. [33], [25, Proposition 1.21]. The following fact is known [36, Section 3].

Proposition 5. *Let S be a closed densely defined symmetric operator in a Hilbert space $\mathcal{H}$ with equal deficiency indices, and let $(\mathcal{G},\Gamma,\Gamma')$ be an associated boundary triplet. Let M be the associated Weyl function and H^0 be the restriction of S^* to $\ker\Gamma$. Assume that S is simple (i.e. has no invariant subspaces on which it is self-adjoint), then the spectral measure for H^0 is a minimal orthogonal operator-valued measure associated with the bounded operator-valued Herglotz measure Σ_M^0 associated with M.*

The following proposition combines the above results and provides a step toward the proof of Theorem 2.

Proposition 6. *Let the assumptions of Theorem 2 be fulfilled, and let the assertion* (a) *of Theorem 2 hold. Set $N(z) := -M(z)^{-1}$ and let Σ_N^0 be the associated bounded Herglotz measure. Define its restriction $\Sigma_{N,J}^0$ onto J by $\Sigma_{N,J}^0(B) = \Sigma_N^0(B\cap J)$. If $\Sigma_{N,J}^0$ is a minimal dilation of the spectral measure E_R of the operator $R = \mu\left(T_{m(J)}\right)$, then the operators H_J and R are unitarily equivalent.*

Proof. (a) Assume first that S is a simple operator. Introduce the new boundary triplet $(\mathcal{G},\widetilde\Gamma,\widetilde\Gamma')$ with $\widetilde\Gamma := -\Gamma'$ and $\widetilde\Gamma' := \Gamma$. The associated Weyl function is $N(z) := -M(z)^{-1}$, and is hence also a Herglotz one, and the operator H becomes then the restriction of S^* to $\ker\widetilde\Gamma$. By Proposition 5 one can represent Σ_N^0 as a minimal dilation of the spectral measure E_H of H, $\Sigma_N^0(B) = K^*E_H(B)K$, $K\in\mathcal{L}(\mathcal{G},\mathcal{H})$, then

$$\Sigma_{N,J}^0(B) = \Sigma_N^0(B\cap J) = K^*E_H(B\cap J)K = L^*E_{H,J}(B)L,$$

where $E_{H,J}$ defined by $E_{H,J}(B) = E_H(B\cap J)$ is considered as an orthogonal measure in $\mathcal{H}' := \operatorname{ran} E_H(J)$, and $L = \Pi K$ with $\Pi : \mathcal{H}\to\mathcal{H}'$ being the orthogonal projector. Therefore, $E_{H,J}$ is another minimal orthogonal measure associated with $\Sigma_{N,J}^0$; hence E_R and $E_{H,J}$ are unitarily equivalent by Naimark's theorem (Theorem 4). This means that there exists a unitary U such that $E_{H,J}(B) = U^*E_R(B)U$ for all $B\subset J$, and

$$H_J = \int_J t\,E_{H,J}(dt) = U^*\int_J t\,E_R(dt)\,U = U^*RU.$$

(b) If the operator S is not simple, one can decompose the Hilbert space $\mathcal{H}$ and the operator S into a direct sum $\mathcal{H} = \mathcal{H}_0\oplus\mathcal{K}$, $S = S_0\oplus L$, such that L is a self-adjoint operator in $\mathcal{K}$ and S_0 is a closed densely defined *simple* symmetric operator in $\mathcal{H}_0$ whose deficiency indices are equal to those for S. Moreover, $(\mathcal{G},\bar\Gamma,\bar\Gamma')$, where $\bar\Gamma$ and $\bar\Gamma'$ are the restrictions of Γ and Γ' respectively to $\operatorname{dom} S_0^*$, is a boundary triplet for S_0 with the same Weyl function $M(z)$. Moreover, one has $H^0 = A^0\oplus L$ and $H = A\oplus L$, where A^0 is the restriction of S_0^* to $\ker\bar\Gamma$ and A is the restriction of S_0^* to $\ker\bar\Gamma'$. One has $J\subset\mathbb{R}\setminus\operatorname{spec} A^0$ and $J\subset\mathbb{R}\setminus\operatorname{spec} L$, which means that H_J is unitarily equivalent to A_J. Finally, applying the part (a) to the operators S_0, A and A^0 one shows that A_J is unitarily equivalent to R. $\square$

2.2. Technical estimates

In this section, we use the notation and the assumptions introduced in Theorem 2 and Proposition 6. The aim of this section is to calculate the bounded Herglotz measure Σ_N^0 associated to N in terms of the spectral measure for the operator R.

Denote

$$S_T := [\inf\operatorname{spec} T, \sup\operatorname{spec} T],\quad K := m^{-1}(S_T)\cap J. \tag{7}$$

The following assertion was proved in [25, Lemma 3.13].

Lemma 7. *For any $x\in K$ one has $m'(x)\neq 0$.*

We will prove the following.

Lemma 8. *The set K is connected.*

Let $(a,b)\subset J$. By the Stieltjes inversion formula (6) one has

$$\Sigma_N^0\left((a,b)\right) = \operatorname*{s-lim}_{\delta\to 0+}\operatorname*{s-lim}_{\varepsilon\to 0+}\frac{1}{2\pi i}\int_{a+\delta}^{b-\delta}\left(N(x+i\varepsilon)-N(x-i\varepsilon)\right)\,dx. \tag{8}$$

On the other hand, there holds

$$N(x+i\varepsilon)-N(x-i\varepsilon) = \int_{\mathbb{R}} \left(\frac{n(x+i\varepsilon)}{\lambda-m(x+i\varepsilon)} - \frac{n(x-i\varepsilon)}{\lambda-m(x-i\varepsilon)}\right) E_T(d\lambda)$$
$$= \int_{S_T} \left(\frac{n(x+i\varepsilon)}{\lambda-m(x+i\varepsilon)} - \frac{n(x-i\varepsilon)}{\lambda-m(x-i\varepsilon)}\right) E_T(d\lambda), \tag{9}$$

where E_T is the spectral measure associated with T.

For a Borel subset I of J denote

$$k_I(\lambda,\varepsilon) = \frac{1}{2\pi i}\int_I \left(\frac{n(x+i\varepsilon)}{\lambda-m(x+i\varepsilon)} - \frac{n(x-i\varepsilon)}{\lambda-m(x-i\varepsilon)}\right) dx. \tag{10}$$

Our main technical estimate is the following proposition.

Proposition 9. *Assume that $I=[a,b]\subset J$. For some $\varepsilon_0>0$ there holds*

$$\sup_{\substack{\lambda\in S_T\\ \varepsilon\in(0,\varepsilon_0)}} \left|k_I(\lambda,\varepsilon)\right| < +\infty \tag{11}$$

and for any $\lambda\in S_T$ one has

$$\lim_{\varepsilon\to 0+} k_I(\lambda,\varepsilon) = \begin{cases} 0, & \lambda\notin m\left([a,b]\right), \\ \dfrac{1}{2}\,\mu'(\lambda)n\left(\mu(\lambda)\right), & \lambda\in\left\{m(a),m(b)\right\}, \\ \mu'(\lambda)n\left(\mu(\lambda)\right), & \lambda\in m\left((a,b)\right). \end{cases} \tag{12}$$

Here μ is the inverse to $K\ni x\mapsto m(x)\in m(K)$; this inverse exists by Lemmas 7 and 8.

To prove Proposition 9 let us make some preliminary steps.

Lemma 10. *Let $I\subset J$ be a closed segment such that $m'(x)\neq 0$ for $x\in I$. Then, for some $\varepsilon_0>0$ and for all $x\in I$, $\lambda\in\mathbb{R}$ and $0<|\varepsilon|<\varepsilon_0$ there holds*

$$\frac{1}{\lambda-m(x+i\varepsilon)} = \frac{1}{\lambda-m(x)-i\varepsilon m'(x)}\cdot\left(1+\varepsilon\, g(x,\lambda,\varepsilon)\right), \tag{13}$$

where

$$\sup_{\substack{x\in I,\,\lambda\in\mathbb{R}\\ 0<|\varepsilon|<\varepsilon_0}} \left|g(x,\lambda,\varepsilon)\right| < +\infty.$$

Proof. There holds

$$\frac{1}{\lambda-m(x+i\varepsilon)} = \frac{f(x,\lambda,\varepsilon)}{\lambda-m(x)-i\varepsilon m'(x)} \tag{14}$$

with

$$f(x,\lambda,\varepsilon) = \frac{\lambda-m(x)-i\varepsilon m'(x)}{\lambda-m(x+i\varepsilon)} = 1+\frac{m(x+i\varepsilon)-m(x)-i\varepsilon m'(x)}{\lambda-m(x+i\varepsilon)}. \tag{15}$$

Due to the analyticity of m, there exists $C>0$ such that

$$\left|m(x)+i\varepsilon m'(x)-m(x+i\varepsilon)\right| \le C\varepsilon^2 \quad \text{for all } x\in I,\ |\varepsilon|<\varepsilon_0. \tag{16}$$

On the other hand, denoting $k=\inf_{x\in I}|m'(x)|>0$, one has $\left|\lambda-m(x)-i\varepsilon m'(x)\right|\ge k|\varepsilon|$. Therefore, one can find $c>0$ such that

$$\left|\lambda-m(x+i\varepsilon)\right| \ge c|\varepsilon| \quad \text{for all } \lambda\in\mathbb{R},\ x\in I,\ |\varepsilon|\le\varepsilon_0. \tag{17}$$

Using (16) and (17) one obtains, with $b=C/c>0$,

$$\left|\frac{m(x+i\varepsilon)-m(x)-i\varepsilon m'(x)}{\lambda-m(x+i\varepsilon)}\right| \le b\varepsilon \quad \text{for all } x\in I,\ \lambda\in\mathbb{R},\ 0<|\varepsilon|<\varepsilon_0. \quad \square$$

Lemma 11. *The result of Proposition 9 holds under the additional assumption*

$$m'(x) \neq 0 \quad \text{for all } x \in I.$$

Proof. Let us take the same ε_0 as in Lemma 10. Using the representation (13) one can write

$$k_I(\lambda,\varepsilon) = \frac{1}{2\pi i}\int_a^b \left[\frac{n(x+i\varepsilon)\cdot(1+\varepsilon\, g(x,\lambda,\varepsilon))}{\lambda - m(x) - i\varepsilon m'(x)} - \frac{n(x-i\varepsilon)\cdot(1-\varepsilon\, g(x,\lambda,-\varepsilon))}{\lambda - m(x) + i\varepsilon m'(x)}\right] dx. \tag{18}$$

As n is holomorph, one can write $n(x+i\varepsilon) = n(x) + \varepsilon p(x,\varepsilon)$ with

$$\sup_{\substack{x\in I\\ |\varepsilon|<\varepsilon_0}} \big|p(x,\varepsilon)\big| < +\infty.$$

Substituting this representation into (18) one obtains

$$\begin{aligned} k_I(\lambda,\varepsilon) &= \underbrace{\frac{1}{2\pi i}\int_a^b n(x)\left(\frac{1}{\lambda - m(x) - i\varepsilon m'(x)} - \frac{1}{\lambda - m(x) + i\varepsilon m'(x)}\right) dx}_{=:I_1(\lambda,\varepsilon)} \\ &\quad + \underbrace{\frac{1}{2\pi i}\int_a^b \frac{\varepsilon r(x,\lambda,\varepsilon)}{\lambda - m(x) - i\varepsilon m'(x)}\, dx}_{=:I_2(\lambda,\varepsilon)} + \underbrace{\frac{1}{2\pi i}\int_a^b \frac{\varepsilon r(x,\lambda,-\varepsilon)}{\lambda - m(x) + i\varepsilon m'(x)}\, dx}_{=:I_3(\lambda,\varepsilon)} \end{aligned} \tag{19}$$

with

$$r(x,\lambda,\varepsilon) := p(x,\varepsilon)\,(1+\varepsilon g(x,\lambda,\varepsilon)) + n(x) g(x,\lambda,\varepsilon).$$

One has obviously

$$\sup_{\substack{x\in I,\ \lambda\in\mathbb{R}\\ 0<|\varepsilon|<\varepsilon_0}} \big|r(x,\lambda,\varepsilon)\big| =: C < +\infty$$

Denoting

$$k = \inf_{x\in[a,b]} |m'(x)| > 0$$

one can estimate, for all $\lambda\in\mathbb{R}$ and $0<|\varepsilon|<1$,

$$\left|\frac{\varepsilon r(x,\lambda,\varepsilon)}{\lambda - m(x) + i\varepsilon m'(x)}\right| \le \frac{R}{k}. \tag{20}$$

Therefore, one has

$$\big|I_{2,3}(\lambda,\varepsilon)\big| \le \frac{R|b-a|}{2\pi k} \quad \text{for all } \lambda\in\mathbb{R} \text{ and } 0<|\varepsilon|<1.$$

Let us study the expression for I_1. By elementary transformations one obtains

$$I_1(\lambda,\varepsilon) = \frac{1}{\pi}\int_a^b \frac{\varepsilon m'(x) n(x)}{(\lambda - m(x))^2 + (\varepsilon m'(x))^2}\, dx.$$

Denoting $N := \sup_{x\in I}\big|n(x)\big|$ one obtains

$$\begin{aligned} |I_1| &\le \frac{N}{\pi}\int_a^b \frac{\big|m'(x)\big|}{(\lambda - m(x))^2 + \varepsilon^2 k^2}\, dx \\ &= \frac{N}{\pi}\left|\int_{m(a)}^{m(b)} \frac{\varepsilon}{(\lambda - y)^2 + \varepsilon^2 k^2}\, dy\right| \le \frac{N}{\pi}\int_{-\infty}^{+\infty} \frac{\varepsilon}{y^2 + \varepsilon^2 k^2}\, dy = \frac{N}{k}. \end{aligned}$$

The estimate (11) is proved.

To show the equalities (12) let us study first the limits of I_2 and I_3. By (20) and due to the boundedness of (a,b) one obtains by virtue of the Lebesgue dominated convergence

$$\lim_{\varepsilon\to 0+} I_2(\lambda,\varepsilon) = \int_a^b \lim_{\varepsilon\to 0+} \frac{\varepsilon r(x,\lambda,\varepsilon)}{\lambda - m(x) + i\varepsilon m'(x)}\, dx,$$

note that for x satisfying $\lambda \neq m(x)$ (which can be violated for at most one point of $[a,b]$) one has

$$\lim_{\varepsilon\to 0+} \frac{\varepsilon r(x,\lambda,\varepsilon)}{\lambda - m(x) + i\varepsilon m'(x)} = 0.$$

Therefore, $\lim_{\varepsilon\to 0+} I_2(\lambda,\varepsilon) = 0$. By the same arguments, $\lim_{\varepsilon\to 0+} I_3(\lambda,\varepsilon) = 0$.

To study the limit of I_1 we assume without loss of generality that $m'(x) > 0$ on I (otherwise one changes the signs of T, m and n). Introduce a new variable $y = m(x)$; by the implicit function theorem one has $x = \varphi(y)$ and $\varphi'(y) = \big(m'(x)\big)^{-1}$. This gives

$$I_1(\lambda,\varepsilon) = \frac{1}{\pi}\int_{m(a)}^{m(b)} \frac{\varepsilon n\,(\varphi(y))}{(\lambda-y)^2 + \frac{\varepsilon^2}{\varphi'(y)^2}}\,dy.$$

Introducing another new variable $z = \frac{y-\lambda}{\varepsilon}$ one arrives at

$$I_1(\lambda,\varepsilon) = \frac{1}{\pi}\int_{\frac{m(a)-\lambda}{\varepsilon}}^{\frac{m(b)-\lambda}{\varepsilon}} \frac{n\,(\varphi(\varepsilon z+\lambda))}{z^2 + \frac{1}{\varphi'(\varepsilon z+\lambda)^2}}\,dy. \tag{21}$$

One has

$$\sup_{\frac{m(a)-\lambda}{\varepsilon}\le z\le \frac{m(b)-\lambda}{\varepsilon}} \big|n\,(\varphi(\varepsilon z+\lambda))\big| = \sup_{a\le x\le b}\big|n(x)\big| \le N$$

and

$$\inf_{\frac{m(a)-\lambda}{\varepsilon}\le z\le \frac{m(b)-\lambda}{\varepsilon}} \frac{1}{\varphi'(\varepsilon z+\lambda)^2} = \inf_{a\le x\le b} m'(x)^2 = k^2 > 0,$$

therefore,

$$\left|\frac{n\big(\varphi(\varepsilon z+\lambda)\big)}{z^2 + \frac{1}{\varphi'(\varepsilon z+\lambda)^2}}\right| \le \frac{N}{z^2+\mu^2} \in L^1(\mathbb{R}).$$

Hence one has due to the Lebesgue dominated convergence

$$\lim_{\varepsilon\to 0+} I_1(\lambda,\varepsilon) = \frac{1}{\pi}\int_{\lim\limits_{\varepsilon\to 0+}\frac{m(a)-\lambda}{\varepsilon}}^{\lim\limits_{\varepsilon\to 0+}\frac{m(b)-\lambda}{\varepsilon}} \lim_{\varepsilon\to 0+} \frac{n\,(\varphi(\varepsilon z+\lambda))}{z^2 + \frac{1}{\varphi'(\varepsilon z+\lambda)^2}}\,dy.$$

Recall that (for $a \neq 0$)

$$\int_{-\infty}^{0} \frac{dt}{a^2+t^2} = \int_{0}^{+\infty} \frac{dt}{a^2+t^2} = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{dt}{a^2+t^2} = \frac{\pi}{2|a|}.$$

Clearly, for any $c \in J$

$$\lim_{\varepsilon\to 0+} \frac{m(c)-\lambda}{\varepsilon} = \begin{cases} +\infty, & \lambda < m(c)\\ 0 & \lambda = m(c)\\ -\infty, & \lambda > m(c)\end{cases}$$

and that for $m(a) \le \lambda \le m(b)$ there holds

$$\lim_{\varepsilon\to 0+} \frac{n\,(\varphi(\varepsilon z+\lambda))}{z^2 + \frac{1}{\varphi'(\varepsilon z+\lambda)^2}} = \frac{n\,(\varphi(\lambda))}{z^2 + \frac{1}{\varphi'(\lambda)^2}}.$$

It remains to note that $\mu(x) = \varphi(x)$ for $x \in m(I\cap K)$. The equalities (12) are hence obtained. $\square$

Lemma 12. *Let L be a connected subset of K with $m(L)\cap \operatorname{spec} T \neq \emptyset$; then the functions m' and n are either both strictly positive or both strictly negative in L.*

Proof. Take $\lambda \in \operatorname{spec} T$ such that $\lambda \in m(L)$. As $\Im N(x+i\varepsilon) > 0$ for $\varepsilon > 0$, one has

$$\frac{1}{2i}\left(\frac{n(x+i\varepsilon)}{\lambda - m(x+i\varepsilon)} - \frac{n(x-i\varepsilon)}{\lambda - m(x-i\varepsilon)}\right) \ge 0$$

for all $x \in \mathbb{R}$. Integrating this inequality on any $[a,b] \subset L$ such that $\lambda \in m\,([a,b])$ and passing to the limit as $\varepsilon \to 0+$ we obtain, by Lemma 11, $n\,(\mu(\lambda))\,\mu'(\lambda) \ge 0$. Let $\lambda = m(y)$, $y \in L$; then $0 \le n\,(\mu\,(m(y)))\,\mu'\,(m(y)) = \frac{n(y)}{m'(y)}$. On the other hand,

$n(y) \neq 0$ by assumption and $m'(y) \neq 0$ by Lemma 7; hence the inequality is strict; hence $m'(y)$ and $n(y)$ are either both negative or both positive. As the two functions m' and n are continuous and do not vanish in the connected set L, they have the same sign in whole L. $\square$

Now we are able to show that K has a rather simple structure given in Lemma 8.

Proof of Lemma 8. If the set K is not connected, then there are two different values $x_1, x_2 \in J$ with $m(x_1) = m(x_2) = \tau$ with $\tau \in \{\inf \operatorname{spec} T, \sup \operatorname{spec} T\}$ (automatically $\tau \in \operatorname{spec} T$). Due to analyticity of m and without loss of generality one can assume that $\tau = \sup \operatorname{spec} T$, that $x_1 < x_2$ and that $m(x) > \tau$ for $x_1 < x < x_2$. Then $m'(x_1) > 0$ and $m'(x_2) < 0$. By Lemma 12, one has $n(x_1) > 0$ and $n(x_2) < 0$, therefore, n has to vanish in at least one point of the interval $(x_1, x_2) \subset J$, which is impossible. $\square$

Now we can prove the complete version of Proposition 9.

Proof of Proposition 9. By Lemma 8, there exists a bounded open interval Ω containing $m^{-1}(S_T) \cap J$ such that $m'(x) \neq 0$ for $x \in \Omega$. Denote $L := I \cap \bar{\Omega}$ and $P := \overline{I \setminus L}$. One has $k_I(\lambda, \varepsilon) = k_P(\lambda, \varepsilon) + k_L(\lambda, \varepsilon)$.

Consider the term k_P. As $m(P) \cap S_T = \emptyset$ by construction, the subintegral expression in (10) does not show any singularity for small ε, i.e., for any $\varepsilon_0 > 0$ there exists $C > 0$ such that

$$\left| \frac{n(x+i\varepsilon)}{\lambda - m(x+i\varepsilon)} - \frac{n(x-i\varepsilon)}{\lambda - m(x-i\varepsilon)} \right| \leq C$$

for all $x \in P$, $\lambda \in S_T$ and $0 < \varepsilon < \varepsilon_0$, and

$$|k_P(\lambda, \varepsilon)| \leq C|P| \quad \text{for all } \lambda \in S_T \text{ and } 0 < \varepsilon < \varepsilon_0.$$

Furthermore, the Lebesgue dominated convergence and the equality

$$\lim_{\varepsilon \to 0+} \frac{n(x+i\varepsilon)}{\lambda - m(x+i\varepsilon)} = \lim_{\varepsilon \to 0+} \frac{n(x-i\varepsilon)}{\lambda - m(x-i\varepsilon)} = \frac{n(x)}{\lambda - m(x)}$$

implies $\lim_{\varepsilon \to 0+} k_P(\lambda, \varepsilon) = 0$ for all $\lambda \in S_T$.

To analyze the second term k_L, we remark that, by construction, L is a closed interval and $m'(x) \neq 0$ for $x \in L$; hence Lemma 11 is applicable. $\square$

2.3. Spectral measures and proof of Theorem 2

From now on we introduce the operator

$$\widetilde{T} := T_{m(J)}$$

and the orthogonal projector

$$P : \mathcal{G} \to \widetilde{\mathcal{G}} := \operatorname{ran} E_T\,(m(J))\,.$$

Recall that we consider $\widetilde{T}$ as a self-adjoint operator in $\widetilde{\mathcal{G}}$.

Proposition 13. *Let μ be the inverse function to $K \ni x \mapsto m(x) \in m(K) \equiv m(J)$; then the operator $n\left(\mu(\widetilde{T})\right)\mu'(\widetilde{T})$ is bounded, and for any bounded Borel set $B \subset J$ there holds*

$$\Sigma_N(B) = P^* n\left(\mu(\widetilde{T})\right)\mu'(\widetilde{T}) E_{\widetilde{T}}\,(m(B))\,P, \tag{22}$$

$$\Sigma_N^0(B) = P^* n\left(\mu(\widetilde{T})\right)\mu'(\widetilde{T})\left(1 + \mu(\widetilde{T})^2\right)^{-1} E_{\widetilde{T}}\,(m(B))\,P. \tag{23}$$

Proof. By the σ-additivity it is sufficient to consider open intervals $B = (a, b)$.

(a) Assume first $\bar{B} = [a, b] \subset J$. Applying (11) and the Fubini theorem to the expression (8) for Σ_0 one obtains

$$\Sigma_N(B) = \operatorname*{s-lim}_{\delta \to 0+} \operatorname*{s-lim}_{\varepsilon \to 0+} \int_{S_T} k_{[a+\delta, b-\delta]}(\lambda, \varepsilon)\, E_T(d\lambda).$$

Take any $h \in \mathcal{H}$. Using again (11) and the Lebesgue dominated convergence one obtains, by virtue of (12),

$$\begin{aligned}\operatorname*{s-lim}_{\varepsilon \to 0+} \int_{S_T} k_{[a+\delta,b-\delta]}(\lambda, \varepsilon)\, dE_T(\lambda)h &= \int_{S_T} \operatorname*{s-lim}_{\varepsilon \to 0+} k_{[a+\delta,b-\delta]}(\lambda, \varepsilon)\, dE_T(\lambda)h \\ &= \widetilde{f}(T)E_T\left(m\left((a+\delta, b-\delta)\right)\right)h \\ &\quad + \frac{1}{2}\left[\widetilde{f}\left(m(a+\delta)\right)E_T\left(\{m(a+\delta)\}\right) + \widetilde{f}\left(m(b-\delta)\right)E_T\left(\{m(b-\delta)\}\right)\right]h \end{aligned} \tag{24}$$

where

$$\widetilde{f}(x) = \begin{cases} n\left(\mu(x)\right)\mu'(x), & \text{for } x \in S_T \cap m(J), \\ 0, & \text{otherwise.} \end{cases}$$

Hence, noting that the function $\widetilde{f}$ is a priori bounded on $m(B)$ and passing to the limit as $\delta \to 0+$ we obtain

$$\Sigma_N(B) := \widetilde{f}(T)E_T\left(m(B)\right). \tag{25}$$

On the other hand, there holds

$$E_T\left(m(B)\right) = P^* E_{\widetilde{T}}\left(m(B)\right)P, \qquad \widetilde{f}(T) := P^* n\left(\mu(\widetilde{T})\right)\mu'(\widetilde{T})P, \qquad PP^* = \mathrm{Id}_{\widetilde{\mathcal{G}}},$$

which transforms (25) into (22).

(b) Let $B = (a, b) \subset J$ be an arbitrary open interval. In this case the boundedness of $\widetilde{f}$ on $m(B)$ is a priori not guaranteed; hence one can have troubles when passing to the limit in (24). To deal with this case consider the sequence $B_n = (a + 1/n, b - 1/n)$. One has obviously $\bar{B}_n \subset J$; hence for any $h \in \operatorname{dom} L$, $L = \widetilde{f}(T)$, we have

$$\lim_{n\to+\infty} E_T\left(m(B_n)\right)Lh = E_T\left(m(B)\right)Lh.$$

On the other hand, by (a), one has

$$\operatorname*{s-lim}_{n\to+\infty} LE_T\left(m(B_n)\right) = \operatorname*{s-lim}_{n\to+\infty} \Sigma_N(B_n) = \Sigma_N(B).$$

Therefore, for all $h \in \operatorname{dom} L$ we have $LE_T\left(m(B)\right)h = \Sigma_N(B)h$, which is extended by continuity to all $h \in \mathcal{H}$ and shows the boundedness of L.

(c) We have

$$\begin{aligned} \Sigma_N^0(B) &= \int_B \frac{\Sigma_N(dt)}{1+t^2} = P^* \int_B \frac{n\left(\mu(\widetilde{T})\right)\mu'(\widetilde{T})E_{\widetilde{T}}\left(m(dt)\right)}{1+t^2} P \\ &= P^* n\left(\mu(\widetilde{T})\right)\mu'(\widetilde{T}) \int_{m(B)} \frac{E_{\widetilde{T}}(dy)}{1+\mu(y)^2} P \\ &= P^* n\left(\mu(\widetilde{T})\right)\mu'(\widetilde{T}) \left(1+\mu(\widetilde{T})^2\right)^{-1} E_{\widetilde{T}}\left(m(B)\right)P. \quad \square \end{aligned}$$

Now we are in position to conclude the proof of the main result.

Proof of Theorem 2. Recall that we have $R = \mu(\widetilde{T})$, and, therefore, $\widetilde{T} = m(R)$. Note first that the assertion (a) holds with K defined in (7); it satisfies the requested conditions due to Lemmas 8 and 12.

To proceed with the assertion (b), let us prove first the equality

$$\Sigma_N(B) = P^* n(R)\left(m'(R)\right)^{-1} E_R(B)\, P^* \quad \text{for all Borel sets } B \subset J. \tag{26}$$

By the σ-additivity and the regularity arguments used in the proof of Proposition 13 it is sufficient to study the case when B is an open interval such that $\bar{B} \subset J$. We have $E_{\widetilde{T}}\left(m(B)\right) = E_{m(R)}\left(m(B)\right) = E_R(B)$. Substituting this equality in (22) and using the identity $\mu'(x) = \left[m'\left(\mu(x)\right)\right]^{-1}$, we obtain the requested equality (26). Analogously, from (23) we deduce for $B \in \mathcal{B}(\mathbb{R})$, $B \subset J$,

$$\Sigma_N^0(B) = P^* n(R)\left(m'(R)\right)^{-1}(1+R^2)^{-1}E_R(B)P. \tag{27}$$

Now consider the operator-valued measure $B \mapsto \Sigma_{N,J}^0(B) := \Sigma_N^0(B \cap J)$ on $\mathcal{G}$. One can rewrite (27) as

$$\Sigma_{N,J}^0(B) = D^* E_R(B) D,$$

where

$$D = \left[n(R)m'(R)^{-1}(1+R^2)^{-1}\right]^{1/2} P.$$

Note that the operator $n(R)m'(R)^{-1}$ is positive due to Lemma 12; hence $\ker D^* = 0$ and $\overline{\operatorname{ran} D} = \widetilde{\mathcal{G}}$. Therefore, $\Sigma_{N,J}^0$ is a minimal dilation of the orthogonal measure $E_{R,J}$, and the operators H_J and R are unitarily equivalent by Proposition 6. Theorem 2 is proved. $\square$

3. Graph-like structures

In this section, we are going to discuss a class of examples in which Weyl functions of the form (5) appear. We are interested in the case $n \neq \text{const}$; examples with $n = \text{const}$ can be found e.g. in [34, Section 4] or [12, Subsection 1.4.4]. We introduce first a rather general abstract construction and then discuss its realizations by quantum graphs.

3.1. Gluing along graphs

A part of the constructions of this subsection already appeared in [13,29]. Let G be a graph as in the introduction. For $v \in \mathcal{V}$ we denote $E^\iota_v := \{e \in \mathcal{E} : \iota e = v\} \subset \mathcal{E}$ and $E^\tau_v := \{e \in \mathcal{E} : \tau e = v\} \subset \mathcal{E}$ and denote by E_v the *disjoint* union of these two sets, $E_v := E^\iota_v \sqcup E^\tau_v$.

Let now $\mathcal{K}$ be a Hilbert space and L be a closed densely defined symmetric operator in $\mathcal{K}$ with the deficiency indices $(2, 2)$. Consider a boundary triplet $(\mathbb{C}^2, \pi, \pi')$ for L,

$$\pi f = \begin{pmatrix} \pi_\iota f \\ \pi_\tau f \end{pmatrix}, \qquad \pi' f = \begin{pmatrix} \pi'_\iota f \\ \pi'_\tau f \end{pmatrix},$$

and let L^0 be the restriction of L^* to $\ker \pi$. Denote by $\gamma(z)$ the associated γ-field and by $m(z)$ the corresponding Weyl function, which is in this case just a 2×2 matrix function,

$$m(z) = \begin{pmatrix} m_{\iota\iota}(z) & m_{\iota\tau}(z) \\ m_{\tau\iota}(z) & m_{\tau\tau}(z) \end{pmatrix}.$$

We are going to interpret the operator L and its boundary triplet as a description of an object having two ends, ι and τ, e.g. $\Gamma_\iota f$ and $\Gamma'_\iota f$ are interpreted as the boundary values of f at τ. Our aim is to replace each edge of G by a copy of this object and glue these copies together by suitable boundary conditions at the vertices. To make this construction more evident and to provide it with a geometric interpretation let us consider two examples.

Example 14. Our main example is a Sturm–Liouville operator; see [27, Section 4] for the details of the construction. Let $l > 0$ and let $V \in L^2(0, l)$ be a real-valued potential. Consider the operator

$$L := -\frac{d^2}{dx^2} + V$$

with the domain $H^2_0(0, l) = \{f \in H^2(0, l) : f(0) = f(l) = f'(0) = f'(l) = 0\}$. Its adjoint L^* is given by the same differential expression on the domain $H^2(0, l)$, and as a boundary triplet one can take

$$\pi f = \begin{pmatrix} f(0) \\ f(l) \end{pmatrix}, \qquad \pi'(f) := \begin{pmatrix} f'(0) \\ -f'(l) \end{pmatrix}. \tag{28}$$

The associated γ-field is given by

$$\gamma(z) \begin{pmatrix} \xi_\iota \\ \xi_\tau \end{pmatrix}(x) = \frac{\xi_\tau - \xi_\iota c(l; z)}{s(l; z)}\, s(x; z) + \xi_\iota c(x; z)$$

and the Weyl function is

$$m(z) = \frac{1}{s(l; z)} \begin{pmatrix} -c(l; z) & 1 \\ 1 & -s'(l; z) \end{pmatrix}, \tag{29}$$

where s and c are the solutions of the differential equation $-y''(t) + V(t)y(t) = zy(t)$ satisfying the boundary conditions $s(0; z) = c'(0; z) = 0$ and $s'(0; z) = c(0; z) = 1$. Note that the associated operator L^0 is just the above Sturm–Liouville operator with the Dirichlet boundary conditions at 0 and l. Its spectrum σ_D consists of simple eigenvalues ν_n, $n \in \mathbb{N}$, $\nu_{n+1} > \nu_n$, which are the zeros of the function $\nu \mapsto s(l; \nu)$. $\square$

Example 15. Let L^0 be the Laplace–Beltrami operator on a closed manifold M, $2 \le \dim M \le 3$. Take two points $x_1, x_2 \in M$ and denote by L the restriction of L^0 to the functions $f \in \operatorname{dom} L^0$ with $f(x_1) = f(x_2) = 0$. Then L is a closed symmetric operator with deficiency indices $(2, 2)$, and one can construct an associated boundary triplet and the Weyl function as follows; see [12, Section 1.4.3]. Let

$$F(x, y) = \begin{cases} \dfrac{1}{2\pi} \log \dfrac{1}{d(x, y)}, & \dim M = 2, \\ \dfrac{1}{4\pi\, d(x, y)}, & \dim M = 3, \end{cases}$$

where $d(x, y)$ is the geodesic distance between $x, y \in M$. Any function $f \in \operatorname{dom} L^*$ has the asymptotic behavior

$$f(x) = a_j(f) F(x, x_j) + b_j(f) + o(1), \quad x \to x_j,\ a_j(f), b_j(f) \in \mathbb{C},\ j = 1, 2;$$

hence as a boundary triplet one can take $(\mathbb{C}^2, \Gamma, \Gamma')$ with

$$\Gamma f = \begin{pmatrix} a_1(f) \\ a_2(f) \end{pmatrix}, \qquad \Gamma' f = \begin{pmatrix} b_1(f) \\ b_2(f) \end{pmatrix}.$$

Note that the original operator L^0 is just the restriction of L^* to $\ker \Gamma$, and its spectrum is discrete. The Weyl function m for the above boundary triplet has the form

$$m(z) = \begin{pmatrix} G^r(x_1, x_1; z) & G(x_1, x_2; z) \\ G(x_2, x_1; z) & G^r(x_2, x_2; z) \end{pmatrix},$$

where G is the Green function of L^0, i.e. the integral kernel of the resolvent $(L^0 - z)^{-1}$, and G^r is the regularized Green function, defined as the difference $G^r(x, y; z) := G(x, y; z) - F(x, y)$ and extended to the diagonal $x = y$ by continuity. □

To introduce rigorously the gluing of copies of L along the edges of G, let us consider the Hilbert space $\mathcal{H} := \bigoplus_{e\in\mathcal{E}} \mathcal{H}_e$, $\mathcal{H}_e = \mathcal{K}$, and the symmetric operator $S = \oplus_{e\in\mathcal{E}} L_e$, $L_e = L$. Clearly, S is closed densely defined in $\mathcal{H}$, has equal deficiency indices, and $S^* = \bigoplus_{e\in\mathcal{E}} L_e^*$. As a boundary triplet for S one can take $(\widetilde{\mathcal{G}}, \widetilde{\Gamma}, \widetilde{\Gamma}')$ with

$$\widetilde{\mathcal{G}} := \bigoplus_{e\in\mathcal{E}} \mathbb{C}^2, \qquad \widetilde{\Gamma}(f_e) = (\pi f_e), \qquad \widetilde{\Gamma}'(f_e) = (\pi' f_e),$$

where π and π' are defined by (28). This construction does not take into account the combinatorial structure of the graph G, and we prefer to modify it by regrouping all the components with respect to the vertices. More precisely, for any $v \in \mathcal{V}$ denote $\mathcal{G}_v := \mathbb{C}^{\deg v}$ and set $\mathcal{G} := \bigoplus_{v\in\mathcal{V}} \mathcal{G}_v$. For $\phi \in \mathcal{G}$ we will write $\phi = (\phi_v)_{v\in\mathcal{V}}$, $\phi_v = (\phi_{v,e})_{e\in E_v} \in \mathcal{G}_v$, or simply $\phi = (\phi_{v,e})$. The scalar product of $\phi, \psi \in \mathcal{G}$ is hence defined as

$$\langle \phi, \psi \rangle_{\mathcal{G}} = \sum_{v\in\mathcal{V}} \langle \phi_v, \psi_v \rangle_{\mathcal{G}_v} = \sum_{v\in\mathcal{V}} \sum_{e\in E_v} \overline{\phi_{e,v}} \psi_{e,v}.$$

As a boundary triplet for S we take now $(\mathcal{G}, \Gamma, \Gamma')$ with

$$\Gamma f = (\Gamma_v f)_{v\in\mathcal{V}}, \quad \Gamma_v f = (\Gamma_{v,e} f)_{e\in E_v}, \ \Gamma_{v,e} = \begin{cases} \pi_\iota f_e & \text{if } v = \iota e, \\ \pi_\tau f_e & \text{if } v = \tau e, \end{cases}$$

and Γ' is defined analogously. Let us calculate the Weyl function for this boundary triplet. Let $\xi = (\xi_{v,e}) \in \mathcal{G}$ and $z \notin \operatorname{spec} L^0$. The function $f \in \ker(S^* - z)$ with $\Gamma f = \xi$ has the form $f = (f_e)$,

$$f_e = \gamma(z) \begin{pmatrix} \xi_{\iota e,e} \\ \xi_{\tau e,e} \end{pmatrix}, \qquad \begin{pmatrix} \Gamma'_{\iota e,e} f \\ \Gamma'_{\tau e,e} f \end{pmatrix} = \pi' \gamma(z) \begin{pmatrix} \xi_{\iota e,e} \\ \xi_{\tau e,e} \end{pmatrix} = m(z) \begin{pmatrix} \xi_{\iota e,e} \\ \xi_{\tau e,e} \end{pmatrix}.$$

Therefore,

$$(M(z)\xi)_{v,e} = \Gamma'_{v,e} f = \begin{cases} m_{\iota\iota}(z)\xi_{v,e} + m_{\iota\tau}(z)\xi_{v_e,e}, & \text{if } v = \iota e, \\ m_{\tau\tau}(z)\xi_{v,e} + m_{\tau\iota}(z)\xi_{v_e,e}, & \text{if } v = \tau e, \end{cases} \tag{30}$$

where

$$v_e = \begin{cases} \tau e & \text{for } v = \iota e, \\ \iota e & \text{for } v = \tau e. \end{cases}$$

Note that if the symmetry conditions

$$m_{\iota\iota}(z) = m_{\tau\tau}(z) \quad \text{and} \quad m_{\iota\tau}(z) = m_{\tau\iota}(z) \tag{31}$$

are satisfied, then the above expression for $M(z)$ can be simplified to

$$M(z) = m_{\iota\iota}(z)\mathrm{Id} + m_{\iota\tau}(z)\, D, \tag{32}$$

where D is the self-adjoint operator in $\mathcal{G}$ acting as

$$(D\xi)_{v,e} = \xi_{v_e,e}.$$

The restriction H^0 of S^* to $\ker \Gamma$ is just the direct sum of the copies of L^0,

$$H^0 = \bigoplus_{e\in\mathcal{E}} L^0;$$

hence $\operatorname{spec} H^0 = \operatorname{spec} L^0$ and any spectral gap of L^0 is also a spectral gap for H^0.

Now impose gluing boundary conditions at each vertex $v \in \mathcal{V}$ by

$$A_v \Gamma_v f = B_v \Gamma'_v f \tag{33}$$

where A_v, B_v are $\deg v \times \deg v$ matrices such that $A_v B_v^* = B_v A_v^*$ and $\det(A_v A_v^* + B_v B_v^*) > 0$ (these conditions are usually called Rofe-Beketov ones, [40, Section 125, Theorem 4]). One can rewrite these conditions in the equivalent normalized form

$$(1 - U_v)\Gamma_v = i(1 + U_v)\Gamma'_v f, \quad U_v \in \mathcal{U}(\deg v) \tag{34}$$

or

$$P_v \Gamma'_v f = C_v P \Gamma_v f, \qquad (1 - P_v)\Gamma_v f = 0, \tag{35}$$

where P_v is the orthogonal projector from $\mathbb{C}^{\deg v}$ to

$$\mathcal{L}_v := \ker(1 + U_v)^{\perp}$$

and C_v is a self-adjoint operator in $\mathcal{L}_v$ defined as

$$C_v = -i(1 - P_v U_v P_v^*)(1 + P_v U_v P_v^*)^{-1}.$$

The equivalent boundary conditions (33), (34), (35) define a self-adjoint operator (see e.g. [12, Section 1]) and we denote this operator by H. Note that in general H is not transversal to H^0 as one has $\operatorname{dom} H \cap \operatorname{dom} H^0 = \ker P\Gamma' \cap \ker \Gamma \neq \operatorname{dom} S$, $P := \bigoplus_{v\in\mathcal{V}} P_v$, so let us proceed as in [25, Theorem 1.32].

Denote by $\widetilde{S}$ the restriction of S^* to $\ker P\Gamma' \cap \ker \Gamma$, then $\widetilde{S}^*$ is the restriction of S^* to $\ker(1 - P)\Gamma$, and as a boundary triplet for $\widetilde{S}$ one can take $(\mathcal{G}_P, \Gamma_P, \Gamma'_P)$ defined by

$$\mathcal{G}_P = \operatorname{ran} P = \bigoplus_{v\in\mathcal{V}} \mathcal{L}_v, \qquad \Gamma_P = P\Gamma P^*, \qquad \Gamma'_P := P\Gamma' P^*$$

($\mathcal{G}_P$ is considered with the scalar product induced by the inclusion $\mathcal{G}_P \subset \mathcal{G}$), and the associated Weyl function M_P takes the form

$$M_P(z) := PM(z)P^*.$$

Now H becomes the restriction of $\widetilde{S}^*$ to the vectors f satisfying

$$\Gamma'_P f := C\Gamma_P f, \quad C := \bigoplus_{v\in\mathcal{V}} C_v,$$

and the operator H^0 is still the restriction of $\widetilde{S}^*$ to $\ker \Gamma_P$. The following theorem shows that the spectral analysis of H can be reduced in certain cases to the spectral analysis of the discrete operator D_P on $\mathcal{G}_P$,

$$D_P := PDP^*.$$

Theorem 16. *Assume that the symmetry conditions (31) hold and that there is $\theta \in \mathbb{C}$, such that $|\theta| = 1$, $\theta \neq -1$, and*

$$\bigcup_{v\in\mathcal{V}} \operatorname{spec} U_v \setminus \{-1\} = \{\theta\}. \tag{36}$$

Set

$$\alpha := -\frac{i(1-\theta)}{1+\theta}, \qquad \eta_\alpha(z) := \frac{\alpha - m_{\iota\iota}(z)}{m_{\iota\tau}(z)}.$$

Assume now that there exists an interval $J \subset \mathbb{R}\setminus\operatorname{spec} L^0$ such that $m_{\iota\tau}(z) \neq 0$ for $z \in J$. Then the operators H_J and $\eta_\alpha^{-1}\big((D_P)_{\eta_\alpha(J)}\big)$ are unitarily equivalent.

Proof. Let us show that the assumptions of Theorem 2 are satisfied. First of all, as mentioned above, due to (31) and (32) one has $M_P(z) := m_{\iota\iota}(z)\mathrm{Id}_P + m_{\iota\tau}(z)D_P$. On the other hand, under the assumption (36) all the operators C_v are just the multiplications by α; hence H is the restriction of $\widetilde{S}^*$ to $\ker(\Gamma'_P - \alpha\Gamma_P)$. Now introduce another boundary triplet $(\mathcal{G}_P, \Gamma_{P,\alpha}, \Gamma'_{P,\alpha})$ for $\widetilde{S}$ by $\Gamma_{P,\alpha} = \Gamma_P$ and $\Gamma'_{P,\alpha} = \Gamma'_P - \alpha\Gamma_P$. The associated Weyl function is

$$M_{P,\alpha}(z) = M_P(z) - \alpha\mathrm{Id} = (m_{\iota\iota}(z) - \alpha)\,\mathrm{Id} + m_{\iota\tau}D_P = \frac{\eta_\alpha(z)\,\mathrm{Id} - D_P}{-m_{\iota\tau}(z)^{-1}}.$$

As $H = \widetilde{S}^*_{\ker \Gamma'_{P,\alpha}}$, the result follows from Theorem 2. □

In Example 14, the symmetry conditions (31) are satisfied if the potential V is symmetric, i.e. if $V(x) \equiv V(l - x)$; cf. [27, Section 4]. In Example 15 these conditions hold, e.g. if there exists an isometry g of M such that $g(x_1) = x_2$. If M is a two-dimensional sphere, then the condition (31) holds for arbitrary x_1 and x_2; we refer to the paper [42] studying various systems of coupled spheres. Note also that the operator D_P can be viewed as a generalized Laplacian on the graph G; see [13,29]. We will also see below that the transition operator (1) is a particular case of D_P for a suitable projector P.

3.2. Quantum graph case

Consider now in greater detail the constructions of Section 3.1 for the Sturm–Liouville operator L from Example 14.

Let, as previously, $l > 0$, $V \in L^2(0, l)$ be a real-valued potential and fix $\alpha : \mathcal{V} \to \mathbb{R}$. Denote by H the self-adjoint operator acting in $\mathcal{H} := \bigoplus_{e\in\mathcal{E}} L^2(0, l)$ as

$$H(f_e) \mapsto (-f''_e + Vf_e) \tag{37}$$

on the functions $f = (f_e) \in \bigoplus_{e\in\mathcal{E}} H^2(0, l)$ satisfying the boundary conditions

$$\begin{aligned}&\text{the value } f_e(v) =: f(v) \text{ is the same for all } e \in E_v,\\ &\sum_{e:\iota e=v} f_e'(v) = \alpha(v) f(v), \quad v \in \mathcal{V},\end{aligned} \tag{38}$$

where we denote

$$f_e(v) = \begin{cases} f_e(0) & \text{if } \iota e = v,\\ f_e(l) & \text{if } \tau e = v,\end{cases} \qquad f_e'(v) = \begin{cases} f_e'(0) & \text{if } \iota e = v,\\ -f_e'(l) & \text{if } \tau e = v.\end{cases}$$

Recall that by σ_D we denote the spectrum of the operator $f \mapsto -f'' + Vf$ on $[0, l]$ with the Dirichlet boundary conditions.

Theorem 17. *Assume that H is defined by (37) and (38), that the potential V is symmetric, $V(x) \equiv V(l - x)$, and that*

$$\alpha(v) = \alpha \deg v \tag{39}$$

for some $\alpha \in \mathbb{R}$. Then, for any interval $J \subset \mathbb{R} \setminus \sigma_D$ the operator H_J is unitarily equivalent to $\eta_\alpha^{-1}\left(\Delta_{\eta_\alpha(J)}\right)$, where Δ is the operator in $l^2(G)$ given by (1) and

$$\eta_\alpha(z) = c(l; z) + \alpha s(l; z). \tag{40}$$

Proof. The operator H has the structure requested in Section 3.1: it represents copies of the same operator L from Example 14 coupled through boundary conditions at each vertex of the graph. One can rewrite the boundary conditions (38) in the normalized form (34) with

$$U_v = \frac{2}{\deg v + i\alpha(v)} J_{\deg v} - I_{\deg v},$$

where I_n and J_n are respectively the $n \times n$ identity matrix and the $n \times n$ matrix whose all entries are 1 [43]. The value -1 is an eigenvalue of U_v of multiplicity $\deg v - 1$, and the orthogonal projector P_v onto $\ker(U_v + 1)^\perp$ is just the orthogonal projector onto the one-dimensional space spanned by the vector p_v, where p_v is the vector of length $\deg v$ whose all entries are 1, i.e., in the matrix form,

$$P_v = \frac{1}{\deg v} J_{\deg v}.$$

Finally we see that the equalities (39) give the representation (36).

As noted above, the symmetry of the potential V guarantees that the conditions (31) hold. Theorem 16 and the formulas (29) show that H_J is unitarily equivalent to $\eta_\alpha^{-1}\left((D_P)_{\eta_\alpha(J)}\right)$. On the other hand, consider the unitary transformation

$$\Theta : l^2(G) \to \mathcal{G}_P, \qquad (\Theta\xi)_v = \xi(v) p_v. \tag{41}$$

Applying D_P to $\Theta\xi$ we obtain

$$\begin{aligned}(D_P\Theta\xi)_{v,e} &= (PDP^*\Theta\xi)_{v,e} = \frac{1}{\deg v}\sum_{e\in E_v}\left(DP^*\Theta\xi\right)_{v,e}\\ &= \frac{1}{\deg v}\sum_{e\in E_v}(\Theta\xi)_{v_e,e} = \frac{1}{\deg v}\sum_{e\in E_v}\xi(v_e),\end{aligned}$$

i.e. $D_P\Theta = \Theta\Delta$; hence D_P and Δ are unitarily equivalent. □

Taking in this theorem $l = 1$, $V = 0$, $\alpha = 0$ we obtain $\eta_0(z) = \cos\sqrt{z}$, which gives Proposition 1.

Let us mention some other cases where the unitary dimension reduction is possible.

Theorem 18. *Let $V \in L^2(0, l)$ be arbitrary and the condition (39) hold. Assume that the ratio $\kappa := \frac{\operatorname{outdeg} v}{\deg v}$ is the same for all $v \in \mathcal{V}$. Then H_J is unitarily equivalent to $\eta_\alpha^{-1}\left(\Delta_{\eta_\alpha(J)}\right)$ with $\eta_\alpha(z) = \kappa c(l; z) + (1 - \kappa)s'(l; z) + \alpha s(l; z)$.*

Proof. Note that we still have $m_{\iota\tau} = m_{\tau\iota}$. Take the same unitary transformation (41) and calculate $M_P\Theta$:

$$\begin{aligned}(PM(z)P^*\Theta)\xi_{v,e} &= \frac{1}{\deg v}\left\{\sum_{e:\iota e=v}\left[m_{\iota\iota}(z)(\Theta\xi)_{v,e} - m_{\iota\tau}(z)(\Theta\xi)_{v_e,e}\right] + \sum_{e:\tau e=v}\left[m_{\tau\tau}(z)(\Theta\xi)_{v,e} - m_{\tau\iota}(z)(\Theta\xi)_{v_e,e}\right]\right\}\\ &= \frac{1}{\deg v}\left[\left(\operatorname{outdeg} v\cdot m_{\iota\iota}(z) + \operatorname{indeg} v\cdot m_{\iota\iota}(z)\right)\xi(v) + m_{\iota\tau}(z)\sum_{e\in E_v}\xi(v_e)\right];\end{aligned}$$

hence

$$M_P(z)\Theta = \frac{\Theta\Delta - \big(\kappa c(l;z) + (1-\kappa)s'(l;z)\big)\Theta}{s(l;z)},$$

and the rest of the proof is similar to that of Theorem 16. $\square$

One can extend the above results to the case with magnetic fields following the constructions of [27,29]. Namely, let $(a_e)_{e\in\mathcal{E}}$ be a family of magnetic potentials, $a_e \in C^1([0,l])$. Denote by $\widetilde{H}$ the self-adjoint operator in $\mathcal{H} := \bigoplus_{e\in\mathcal{E}} L^2(0,l)$ as

$$(g_e) \mapsto \big((i\partial + a_e)^2 g_e + V g_e\big), \quad \partial g_e := g_e',$$

on the functions $g = (g_e) \in \bigoplus_{e\in\mathcal{E}} H^2(0,l)$ satisfying the magnetic analogue of the boundary conditions (38),

$$\text{the value } g_e(v) =: g(v) \text{ is the same for all } e \in E_v,$$
$$\sum_{e:\iota e = v} \big[g_e'(v) - i a_e(v) g_e(v)\big] = \alpha(v) g(v), \quad v \in \mathcal{V}.$$

Applying the unitary transformation

$$g_e(t) = \exp\left(\int_0^t a_e(s)ds\right) f_e(t)$$

and introducing the parameters

$$\beta_e = \int_0^l a_e(s)ds$$

one sees that $\widetilde{H}$ is unitarily equivalent to the operator H acting as $(f_e) \mapsto (-f_e'' + V f_e)$ with the boundary conditions

$$\text{the value } e^{i\beta_{v,e}} f_e(v) =: f(v) \text{ is the same for all } e \in E_v,$$
$$\sum_{e:\iota e = v} e^{i\beta_{v,e}} f_e'(v) = \alpha(v) g(v), \quad v \in \mathcal{V}, \text{ with } \beta_{v,e} = \begin{cases} 0 & \text{if } v = \iota e, \\ \beta_e & \text{if } v = \tau e. \end{cases}$$

By a minor modification of the preceding constructions one can show that Theorems 17 and 18 hold in the same form if one replaces the operator Δ by its magnetic version Δ_β,

$$\Delta_\beta f(v) = \frac{1}{\deg v}\left(\sum_{e:\iota e = v} e^{-i\beta_e} f(\tau e) + \sum_{e:\tau e = v} e^{i\beta_e} f(\iota e)\right).$$

In particular, the above construction can be applied to the example considered in [25] i.e. to the two-dimensional lattice with a uniform magnetic field. The respective operator Δ_β is the discrete magnetic Laplacian, and using this correspondence one can show that the quantum graph Hamiltonian has a singular continuous spectrum; we refer to [25] for precise constructions and explicit expressions for the Weyl function.

Let us now comment on the dimension reduction for boundary conditions different from (38).

Example 19 (δ'-*Coupling*)**.** Another popular class of boundary conditions is the so-called δ' coupling [43],

$$\sum_{e\in E_v} f_e'(v) = 0, \qquad f_e(v) - f_b(v) = \frac{\beta(v)}{\deg v}\big(f_e'(v) - f_b'(v)\big), \quad e, b \in E_v, \ v \in \mathcal{V},$$

where $\beta(v)$ are non-zero real constants. These boundary conditions can be rewritten in the normalized form (34) with

$$U(v) = -\frac{\deg v + i\beta(v)}{\deg v - i\beta(v)} I_{\deg v} + \frac{2}{\deg v - i\beta(v)} J_{\deg v},$$

and the condition (36) is fulfilled if $\beta(v) = \beta \deg v$ for some $\beta \in \mathbb{R}\setminus\{0\}$. Hence for an even potential V Theorem 16 applies, and for any interval $J \subset \mathbb{R}\setminus\sigma_D$ the operator H_J is unitarily equivalent to $\eta_{1/\beta}^{-1}\big((D_P)_{\eta_{1/\beta}(J)}\big)$ with $\eta_{1/\beta}$ defined by (40) and $P = \bigoplus P_v$, where P_v is the orthogonal projector in $\mathbb{C}^{\deg v}$ onto the subspace $p_v^\perp$. Such operator D_P appeared already in [22] in a slightly different problem. $\square$

Example 20 (δ_s' *Coupling*)**.** One can also consider the so-called δ_s' coupling given by the following boundary conditions [43]:

$$f_e'(v) = f_b'(v) =: f'(v), \quad e, b \in E_v, \qquad \sum_{e\in E_v} f_e(v) = \alpha(v) f'(v), \quad v \in \mathcal{V}. \tag{42}$$

To treat this case it is better to modify the boundary triplet for the initial operator L: instead of (28) one can define

$$\pi f = \begin{pmatrix} -f'(0) \\ f'(l) \end{pmatrix}, \qquad \pi' f = \begin{pmatrix} f(0) \\ f(l) \end{pmatrix},$$

then the associated Weyl function is

$$m(z) = \frac{1}{c'(l;z)} \begin{pmatrix} s'(l;z) & 1 \\ 1 & c(l;z) \end{pmatrix}.$$

Note that the reference operator L^0 is now the Neumann operator on $[0, l]$. Denote by σ_N its spectrum. With this new boundary triplet the boundary conditions (42) become similar to the Kirchoff boundary conditions (38); they can be rewritten in the normalized form (34) with

$$U_v = \frac{1}{\deg v - i\alpha(v)} J_{\deg v} - I_{\deg v}.$$

Assuming now that V is symmetric and that (36) holds and proceeding as in Theorem 17 one can show that for any interval $J \subset \mathbb{R} \setminus \sigma_N$ the operator H_J is unitarily equivalent to $\eta_\alpha^{-1}\big((-\Delta)_{\eta_\alpha(J)}\big)$ with $\eta_\alpha(z) = c(l;z) + \alpha c'(l;z)$. □

In the above examples, we considered second order differential operators only. We believe that, with some suitable modifications, similar relationships should exist for other type of operators, like the averaging operator [44] or the fourth order or mixed order operators appearing in the description of beams [45,20]. We hope to clarify the situation in subsequent works.

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