J. Math. Anal. Appl. 401 (2013) 501–509

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Operator monotone functions, Jacobi operators and orthogonal polynomials

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ARTICLE INFO

Article history:
Received 30 July 2012
Available online 21 December 2012
Submitted by Michael J. Schlosser

Keywords:
Orthogonal polynomials
Löwner theorem
Jacobi operator
Pick function
Nevanlinna function

ABSTRACT

We reveal a connection between operator monotone functions and orthogonal polynomials. Especially, we express an operator monotone function with a Jacobi operator, and show that it is a limit of rational operator monotone functions. Further we prove that the 'principal inverse' of an orthogonal polynomial is operator monotone and hence it has a holomorphic extension to the open upper half plane, namely a Pick function (or Nevanlinna function).

1. Introduction

A real continuous function $f(t)$ defined on an interval I is called an *operator monotone function* on I and denoted by $f \in \mathbf{P}(I)$ if for bounded selfadjoint operators A, B whose spectra lie in I, $A \leqq B$ implies $f(A) \leqq f(B)$. It is fundamental that

$$-\frac{1}{t} \in \mathbf{P}(-\infty, 0) \cap \mathbf{P}(0, \infty). \tag{1}$$

The fact $t^a \in \mathbf{P}[0, \infty)$ for $0 < a \leqq 1$ is called the Löwner–Heinz inequality. It is also known that $\log t \in \mathbf{P}(0, \infty)$ and $\tan t \in \mathbf{P}(-\frac{\pi}{2}, \frac{\pi}{2})$. We [14] have recently shown that the 'principal inverse' of the gamma function is operator monotone.

If $f \in \mathbf{P}(I)$ and f is continuous on the closure $\bar{I}$ of I, then $f(t) \in \mathbf{P}(\bar{I})$. From now on, we therefore constrain I to be an open interval and assume $f \in \mathbf{P}(I)$. The following theorem is due to Löwner [11].

Let $f(t)$ be a non-constant function defined on I. Then $f(t) \in \mathbf{P}(I)$ if and only if $f(t)$ has a holomorphic extension $f(z)$ to the open upper half plane Π_+ so that $f(\Pi_+) \subseteqq \Pi_+$.

Recall that a holomorphic function $h(z)$ on Π_+ with the range in Π_+ is called a *Pick function* or a *Nevanlinna function*. By Herglotz's theorem and Nevanlinna's theorem, a Pick function $h(z)$ admits a unique representation

$$h(z) = \alpha + \beta z + \int_{-\infty}^{\infty} \left(\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right) d\sigma(\lambda) \quad (z \in \Pi_+), \tag{2}$$

where α is real, $\beta \geqq 0$ and σ is a Borel measure induced by a right continuous non-decreasing function such that

$$\int_{-\infty}^{\infty} \frac{1}{1 + \lambda^2} d\sigma(\lambda) < \infty.$$

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doi:10.1016/j.jmaa.2012.12.016

Thus $f(t) \in \mathbf{P}(I)$ has a holomorphic extension $f(z)$ on Π_+ which admits a unique representation (2). Since $\sigma(I) = 0$ arises from the Stieltjes inversion formula, $f(t)$ therefore admits a unique representation

$$f(t) = \alpha + \beta t + \int_{-\infty}^{\infty} \left(\frac{1}{\lambda - t} - \frac{\lambda}{1+\lambda^2} \right) d\sigma(\lambda) \quad (t \in I),$$

where $\int_{-\infty}^{\infty} \frac{1}{1+\lambda^2} d\sigma(\lambda) < \infty$ and $\sigma(I) = 0$.

This indicates that $f(t) \in \mathbf{P}(-\infty, \infty)$ if and only if $f(t) = \alpha + \beta t$. It is evident that $f(t) \in \mathbf{P}(a, b)$ if and only if $f(\frac{b-a}{2}t + \frac{b+a}{2}) \in \mathbf{P}(-1, 1)$ and that $f(t) \in \mathbf{P}(a, \infty)$ if and only if $f(-\frac{2}{t-1} - 1 + a) \in \mathbf{P}(-1, 1)$. So we further constrain I to be $(-1, 1)$. Refer to [1,5,12] for details on operator monotone functions.

Tridiagonal matrices

$$J_\infty = \begin{pmatrix} b_0 & a_0 & 0 & \cdots \\ a_0 & b_1 & a_1 & \\ 0 & a_1 & b_2 & \ddots \\ \vdots & & \ddots & \ddots \end{pmatrix}, \quad J_n = \begin{pmatrix} b_0 & a_0 & 0 & \cdots & 0 \\ a_0 & b_1 & a_1 & & \vdots \\ 0 & a_1 & b_2 & \ddots & \\ \vdots & & \ddots & \ddots & a_{n-1} \\ 0 & \cdots & & a_{n-1} & b_n \end{pmatrix}, \tag{3}$$

where $a_j > 0$, $b_j \in \mathbf{R}$, are called a Jacobi operator and Jacobi matrix, respectively. In this paper we denote both of them by J for simplicity. $\{\mathbf{e}_k\}_{k=0}^{\infty}$ and $\{\mathbf{e}_k\}_{k=0}^{n}$ stand for the conventional orthonormal bases in ℓ^2 and $\mathbf{C}^{n+1}$ respectively. For a bounded selfadjoint operator T and a vector $\mathbf{x}$, it is known [4,5] that $\langle (T-z)^{-1}\mathbf{x}, \mathbf{x}\rangle$ is a Pick function, where $\langle , \rangle$ is the inner product. The connection between a Jacobi operator and orthonormal polynomials $\{p_n\}_{n=0}^{\infty}$ is well-known (e.g. [4,13]). The objective of this paper is to reveal a connection among operator monotone functions, Jacobi operators and orthonormal polynomials: in particular, we show that $f(t) \in \mathbf{P}(-1, 1)$ if and only if there is a contractive Jacobi operator J such that

$$f(t) = f(0) + f'(0)\langle t(1-tJ)^{-1}\mathbf{e}_0, \mathbf{e}_0\rangle \quad (|t| < 1).$$

This result naturally provides that a bounded selfadjoint operator with a cyclic vector is unitarily equivalent to a Jacobi operator. It also leads to the fact that an operator monotone function is a limit of a sequence of rational operator monotone functions. We finally show that the "principal inverse" of $p_n(t)$ is operator monotone and hence its holomorphic extension to Π_+ is a univalent Pick function.

2. Jacobi operators

The objective of this section is to express an operator monotone function with a Jacobi operator. While the following fact has been shown in [7] (cf. [1]), we derive it simply from (2).

Lemma 1. $f(t) \in \mathbf{P}(-1, 1)$ *admits a unique representation*

$$f(t) = f(0) + \int_{-1}^{1} \frac{t}{1-\lambda t} d\sigma(\lambda), \tag{4}$$

where σ is a finite Borel measure on $[-1, 1]$. *Conversely any function of this form is in* $\mathbf{P}(-1, 1)$.

Proof. By the Löwner theorem, $f(t)$ has a holomorphic extension $f(z)$ that is a Pick function. Since $f(t)$ is real valued, by reflection $f(z)$ is holomorphically extendable to $\Pi_+ \cup \Pi_- \cup (-1, 1)$, where Π_- is the open lower half plane. Therefore $-f(\frac{1}{z})$ is holomorphic on $\mathbf{C} \setminus [-1, 1]$, and it is a Pick function. By (2) we obtain

$$-f\left(\frac{1}{z}\right) = \alpha + \beta z + \int_{-\infty}^{\infty} \left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2} \right) d\sigma(\lambda)$$

for $z \in \Pi_+$. Since the Stieltjes inversion formula gives $\sigma(-\infty, -1) = \sigma(1, \infty) = 0$, we get

$$-f\left(\frac{1}{z}\right) = \alpha + \beta z + \int_{-1}^{1} \left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2} \right) d\sigma(\lambda).$$

Tending z to the real axis yields

$$-f\left(\frac{1}{t}\right) = \alpha + \beta t + \int_{-1}^{1} \left(\frac{1}{\lambda - t} - \frac{\lambda}{1+\lambda^2} \right) d\sigma(\lambda)$$

for $t \in (-\infty, -1) \cup (1, \infty)$. Since $f(\frac{1}{t}) \to f(0)$ and $\int_{-1}^{1} \frac{1}{\lambda - t} d\sigma(\lambda) \to 0$ as $t \to \infty$, we have

$$\beta = 0, \qquad -f(0) = \alpha - \int_{-1}^{1} \frac{\lambda}{1+\lambda^2} d\sigma(\lambda).$$

We therefore obtain

$$-f\left(\frac{1}{t}\right) = -f(0) + \int_{-1}^{1} \frac{1}{\lambda - t} d\sigma(\lambda) \quad (t \in (-\infty, -1) \cup (1, \infty)).$$

By replacing $1/t$ with t we get

$$f(t) = f(0) + \int_{-1}^{1} \frac{t}{1 - \lambda t} d\sigma(\lambda) \quad (-1 < t < 1),$$

because this equality is valid for $t = 0$ as well. Conversely suppose $f(t)$ is expressed as (4). Then the integrand is in $\mathbf{P}(-1, 1)$ for each λ; indeed, for $\lambda \neq 0$, from (1) it follows that

$$\frac{t}{1 - \lambda t} = -\frac{1}{\lambda} + \frac{1}{\lambda^2} \frac{1}{\frac{1}{\lambda} - t} \in \mathbf{P}(-1, 1).$$

For a selfadjoint operator A, by the Fubini theorem, we get

$$f(A) = f(0)I + \int_{-1}^{1} A(1 - \lambda A)^{-1} \sigma(\lambda).$$

This entails $f(A) \leqq f(B)$ for $A \leqq B$, that is to say $f(t) \in \mathbf{P}(-1, 1)$. □

Let σ be a finite Borel measure on $[-1, 1]$, and construct orthonormal polynomials $\{p_n(t)\}_{n=0}^{\infty}$ with positive leading coefficients $\{\alpha_n\}$ by the Gram–Schmidt method. Then the three-term recurrence formula

$$\begin{aligned} tp_n(t) &= a_n p_{n+1}(t) + b_n p_n(t) + a_{n-1} p_{n-1}(t) \quad (n = 1, 2, \ldots) \\ tp_0(t) &= a_0 p_1(t) + b_0 p_0(t) \end{aligned} \tag{5}$$

holds, where

$$0 < \frac{\alpha_n}{\alpha_{n+1}} = a_n = \int_{-1}^{1} tp_n(t) p_{n+1}(t) d\sigma(t), \qquad b_n = \int_{-1}^{1} tp_n(t)^2 d\sigma(t). \tag{6}$$

Let us consider the Jacobi operator (3) associated to these $\{a_n\}$ and $\{b_n\}$. By (3) and (5)

$$(J_n - t) \begin{bmatrix} p_0(t) \\ p_1(t) \\ \vdots \\ p_n(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -a_n p_{n+1}(t) \end{bmatrix}, \tag{7}$$

from which it follows that the eigenvalues of J_n are coincident with the zeros of $p_{n+1}(t)$, because $p_0(t)$ is a non-zero constant. Moreover, it is well-known that zeros of $p_n(t)$ are in $[-1, 1]$ and simple, and that zeros of $p_n(t)$ and $p_{n+1}(t)$ interlace each other.

From (6), by the Schwarz inequality, it follows that $|a_n| \leqq 1$ and $|b_n| \leqq 1$. For any $\mathbf{x} = (x_0, x_1, \ldots) \in \ell^2$ we have

$$\|J\mathbf{x}\|^2 = \sum |a_n x_{n+1} + b_n x_n + a_{n-1} x_{n-1}|^2 \leqq 9\|\mathbf{x}\|^2.$$

Hence J is bounded, namely $\|J\| \leqq 3$. This result is known (e.g., see p. 22 of [4]). We now give a more precise estimate.

Lemma 2. *Let J be a Jacobi operator corresponding to a finite Borel measure on $[-1, 1]$. Then $\|J\| \leqq 1$.*

Proof. Assume $J = J_n$. Then it is clear that $\|J\| \leqq 1$, for J_n is a Hermitian matrix with all eigenvalues in $[-1, 1]$. Assume next $J = J_\infty$. Consider a bounded selfadjoint operator

$$\tilde{J}_n = \begin{pmatrix} J_n & 0 \\ 0 & 0 \end{pmatrix}$$

on ℓ^2 for each n. Since $\|\tilde{J}_n\| \leqq 1$ and

$$\begin{aligned} \|J\mathbf{x} - \tilde{J}_n\mathbf{x}\|^2 &= |a_n x_{n+1}|^2 + \sum_{k=n+1}^{\infty} |a_k x_{k+1} + b_k x_k + a_{k-1} x_{k-1}|^2 \\ &\leqq 3 \sum_{k=n}^{\infty} (|x_{k+1}|^2 + |x_k|^2 + |x_{k-1}|^2) \to 0 \quad \text{as } n \to \infty, \end{aligned}$$

we obtain $\|J\| \leqq 1$. □

Lemma 3. *Let T be a contractive selfadjoint operator on Hilbert space. Then for any vector* $\mathbf{x}$

$$\langle t(1-tT)^{-1}\mathbf{x}, \mathbf{x}\rangle \in \mathbf{P}(-1, 1). \tag{8}$$

Proof. Let $T = \int_{-1}^{1} \lambda dE_\lambda$ be the spectral decomposition of T. Then

$$\begin{aligned}\langle t(1-tT)^{-1}\mathbf{x}, \mathbf{x}\rangle &= \sum_{n=0}^{\infty}\langle T^n\mathbf{x}, \mathbf{x}\rangle t^{n+1} = \sum_{n=0}^{\infty}\left(\int_{-1}^{1}\lambda^n d\langle E_\lambda \mathbf{x}, \mathbf{x}\rangle\right)t^{n+1}\\ &= \int_{-1}^{1}\frac{t}{1-t\lambda}d\langle E_\lambda\mathbf{x}, \mathbf{x}\rangle \quad (|t| < 1).\end{aligned}$$

By the second statement of Lemma 1 we get (8). □

While the following is known (e.g., see p. 19 of [4]), for completeness we give another simple proof.

Lemma 4. *Let J and $\tilde{J}$ be Jacobi operators on ℓ^2 as (3). Then $J = \tilde{J}$ if*

$$\langle J^n\mathbf{e}_0, \mathbf{e}_0\rangle = \langle \tilde{J}^n\mathbf{e}_0, \mathbf{e}_0\rangle \quad (n = 1, 2, \ldots). \tag{9}$$

Proof. From (9) it follows that

$$\left\langle \sum_{k=0}^{n} c_k J^k\mathbf{e}_0, \sum_{k=0}^{n} d_k J^k\mathbf{e}_0\right\rangle = \left\langle \sum_{k=0}^{n} c_k \tilde{J}^k\mathbf{e}_0, \sum_{k=0}^{n} d_k \tilde{J}^k\mathbf{e}_0\right\rangle$$

for $c_k, d_k \in \mathbf{C}$. Since $\mathbf{e}_0$ is cyclic, this induces a unitary operator U on ℓ^2 such that $U\sum_{k=0}^{n} c_k J^k\mathbf{e}_0 = \sum_{k=0}^{n} c_k\tilde{J}^k\mathbf{e}_0$. This implies $U\mathbf{C}^{n+1} = \mathbf{C}^{n+1}$, because $\{\sum_{k=0}^{n} c_k J^k\mathbf{e}_0 : c_k \in \mathbf{C}\} = \mathbf{C}^{n+1}$, where $\mathbf{C}^n$ is embedded in ℓ^2. Observe that $U^*\mathbf{e}_0 = \mathbf{e}_0$ follows from $U\mathbf{e}_0 = \mathbf{e}_0$. This implies that $\mathbf{C}^1$ reduces U. Similarly we can inductively verify that $U\mathbf{e}_n = \lambda_n\mathbf{e}_n$ for $|\lambda_n| = 1$, namely U is a diagonal operator. Since $UJ = \tilde{J}U$, putting

$$J = \begin{pmatrix} b_0 & a_0 & 0 & \cdots \\ a_0 & b_1 & a_1 & \\ 0 & a_1 & b_2 & \ddots \\ \vdots & & \ddots & \ddots \end{pmatrix}, \qquad \tilde{J} = \begin{pmatrix} \tilde{b}_0 & \tilde{a}_0 & 0 & \cdots \\ \tilde{a}_0 & \tilde{b}_1 & \tilde{a}_1 & \\ 0 & \tilde{a}_1 & \tilde{b}_2 & \ddots \\ \vdots & & \ddots & \ddots \end{pmatrix},$$

we have

$$UJ = \begin{pmatrix} b_0 & a_0 & 0 & \cdots \\ \lambda_1 a_0 & \lambda_1 b_1 & \lambda_1 a_1 & \\ 0 & \lambda_2 a_1 & \lambda_2 b_2 & \ddots \\ \vdots & & \ddots & \ddots \end{pmatrix}, \qquad \tilde{J}U = \begin{pmatrix} \tilde{b}_0 & \tilde{a}_0\lambda_1 & 0 & \cdots \\ \tilde{a}_0 & \tilde{b}_1\lambda_1 & \tilde{a}_1\lambda_2 & \\ 0 & \tilde{a}_1\lambda_1 & \tilde{b}_2\lambda_2 & \ddots \\ \vdots & & \ddots & \ddots \end{pmatrix}.$$

This yields

$$\begin{aligned}&b_0 = \tilde{b}_0, \ldots, \lambda_k b_k = \tilde{b}_k\lambda_k, \ldots\\ &a_0 = \tilde{a}_0\lambda_1, \lambda_1 a_1 = \tilde{a}_1\lambda_2, \ldots, \lambda_k a_k = \tilde{a}_k\lambda_{k+1}, \ldots.\end{aligned}$$

In view of $a_k > 0$, $\tilde{a}_k > 0$ and $|\lambda_k| = 1$, we get $a_k = \tilde{a}_k$, $\lambda_k = 1$, and $b_k = \tilde{b}_k$. We hence get $J = \tilde{J}$ and $U = I$. □

Now we are ready to show the following.

Theorem 1. *$f(t) \in \mathbf{P}(-1, 1)$ if and only if there is a contractive Jacobi operator J on ℓ^2 or $\mathbf{C}^n$ such that*

$$f(t) = f(0) + f'(0)\langle t(1-tJ)^{-1}\mathbf{e}_0, \mathbf{e}_0\rangle, \tag{10}$$

where $\mathbf{e}_0 = (1, 0, \ldots)$ is a unit vector in ℓ^2 or $\mathbf{C}^n$. Moreover, J is uniquely determined if $f(t)$ is not constant.

Proof. Assume $f(t) \in \mathbf{P}(-1, 1)$. If f is constant, then (10) is valid for every J. Suppose f is non-constant, and represent it as (4). Then $f'(0) = \int_{-1}^{1} d\sigma(\lambda) > 0$. Construct the orthonormal polynomials $\{p_n(t)\}$ associated to σ and then Jacobi operator J with $\{a_n\}, \{b_n\}$ in (5). Notice that $p_0(t)^2 = \frac{1}{f'(0)}$. Two cases occur: $J = J_\infty$ or $J = J_n$. But it is legitimate for us to assume $J = J_\infty$.

Let $U : \ell^2 \to L^2(\sigma)$ be a unitary operator defined by $U\mathbf{e}_n = p_n(t)$. By (3) and (5) we get $UJ = MU$, where $(Mh)(\lambda) = \lambda h(\lambda)$ on $L^2(\sigma)$ (cf. [4]). Since $\|J\| \leqq 1$, we have

$$Ut(1-tJ)^{-1} = t(1-tM)^{-1}U \quad (|t|<1),$$

which yields

$$\langle t(1-tJ)^{-1}\mathbf{e}_0, \mathbf{e}_0\rangle = \langle t(1-tM)^{-1}p_0, p_0\rangle = \frac{1}{f'(0)}\int_{-1}^{1}\frac{t}{1-t\lambda}d\sigma(\lambda).$$

We consequently get (10). By Lemma 3 the right hand side of (10) is operator monotone. It remains to show the uniqueness of J. Assume J and $\tilde{J}$ satisfy (10). Then

$$\langle t(1-tJ)^{-1}\mathbf{e}_0, \mathbf{e}_0\rangle = \langle t(1-t\tilde{J})^{-1}\mathbf{e}_0, \mathbf{e}_0\rangle \quad (|t|<1).$$

This implies

$$\sum_{n=0}^{\infty} t^{n+1}\langle J^n\mathbf{e}_0, \mathbf{e}_0\rangle = \sum_{n=0}^{\infty} t^{n+1}\langle \tilde{J}^n\mathbf{e}_0, \mathbf{e}_0\rangle$$

and hence

$$\langle J^n\mathbf{e}_0, \mathbf{e}_0\rangle = \langle \tilde{J}^n\mathbf{e}_0, \mathbf{e}_0\rangle \quad \textit{for } n = 1, 2, \ldots.$$

By Lemma 4 we get $J = \tilde{J}$. □

The following fact is known (e.g., see p. 86 of [13]), but we have never seen the proof so far. So we give a proof here.

Corollary 1. *A bounded selfadjoint operator T with a cyclic vector is unitarily equivalent to a Jacobi operator.*

Proof. We may assume that $\|T\| \leqq 1$. By Lemma 3 $h(t) := \langle t(1-tT)^{-1}\mathbf{x}_0, \mathbf{x}_0\rangle \in \mathbf{p}(-1,1)$, where $\mathbf{x}_0$ is a cyclic vector for T with $\|\mathbf{x}_0\| = 1$. Since $h(0) = 0$ and $h'(0) = 1$, by Theorem 1, there is a Jacobi operator J such that

$$\langle t(1-tJ)^{-1}\mathbf{e}_0, \mathbf{e}_0\rangle = h(t) = \langle t(1-tT)^{-1}\mathbf{x}_0, \mathbf{x}_0\rangle \quad (|t|<1).$$

This gives

$$\langle J^n\mathbf{e}_0, \mathbf{e}_0\rangle = \langle T^n\mathbf{x}_0, \mathbf{x}_0\rangle \quad (n = 0, 1, 2, \ldots)$$

and hence $\|\sum c_i J^i\mathbf{e}_0\|^2 = \|\sum c_i T^i\mathbf{x}_0\|^2$ for every finite set of c_i. Thus $U : \sum c_i T^i\mathbf{x}_0 \mapsto \sum c_i J^i\mathbf{e}_0$ is unitary and $UT = JU$. □

Theorem 2. *For every open interval I and for every $f(t) \in \mathbf{P}(I)$, there is a sequence of rational functions $f_n(t)$ such that $f_n(t) \in \mathbf{P}(I)$ and*

$$\lim_{n\to\infty} f_n(t) = f(t).$$

Moreover the convergence is uniform on every compact subset of I.

Proof. We first show the case $I = (-1, 1)$. In the proof of Lemma 2 we have seen that $\tilde{J}_n$ converges strongly to J; hence so does $t(1-t\tilde{J}_n)^{-1}$ to $t(1-tJ)^{-1}$ for each $-1 < t < 1$. We therefore need to show $\langle t(1-t\tilde{J}_n)^{-1}\mathbf{e}_0, \mathbf{e}_0\rangle$ is rational. Observe that it is equal to $\langle t(1-tJ_n)^{-1}\mathbf{e}_0, \mathbf{e}_0\rangle$. Denote the zeros of $p_{n+1}(t)$ by $c_0 < c_1 < \cdots < c_n$. By (7) they are eigenvalues of J_n, and

$$\mathbf{x}_i := \frac{1}{\left(\sum_{k=0}^{n}|p_k(c_i)|^2\right)^{1/2}}(p_0, p_1(c_i), \ldots, p_n(c_i))$$

is a unit eigenvector corresponding to c_i. This deduces that

$$\begin{aligned}\langle t(1-tJ_n)^{-1}\mathbf{e}_0, \mathbf{e}_0\rangle &= \sum_{i=0}^{n}\langle t(1-tJ_n)^{-1}\mathbf{e}_0, \mathbf{x}_i\rangle\langle \mathbf{x}_i, \mathbf{e}_0\rangle \\ &= \sum_{i=0}^{n}\frac{t}{1-c_i t}\langle \mathbf{e}_0, \mathbf{x}_i\rangle\langle \mathbf{x}_i, \mathbf{e}_0\rangle = \sum_{i=0}^{n}\frac{t}{1-c_i t}\frac{p_0^2}{\sum_{k=0}^{n}|p_k(c_i)|^2}.\end{aligned}$$

Since this is rational and operator monotone on $(-1, 1)$, the desired result holds in the case of $I = (-1, 1)$. If $I = (-\infty, \infty)$, $f(t)$ is just an affine function as mentioned in the Introduction, so we have only to put $f_n(t) = f(t)$. As we also mentioned in the Introduction we can reduce other cases to the case of $I = (-1, 1)$. So we can obtain the first statement. Since $f(t)$ and $f_n(t)$ are both continuous and non-decreasing, the second statement is a fundamental fact. □

3. Principal inverse and the Löwner kernel

In this section we analyze 'general' orthogonal polynomials $\{p_n\}$ from the perspective of operator monotonicity; here, 'general' means that the support of the measure corresponding to $\{p_n\}$ is not necessarily in a finite interval.

Recall that a kernel function $K(t, s)$ is said to be *positive semi-definite* on $I \times I$ if the associative matrices $(K(t_i, t_j))_{i,j}$ are positive semi-definite, namely

$$\sum_{i,j=1}^{n} K(t_i, t_j) z_i \overline{z_j} \geqq 0 \tag{11}$$

for each n, for all n points $t_i \in I$ and for all n complex numbers z_i. Let $h(t)$ be a bijection from J to I. Then $K(t, s)$ is positive semi-definite on $I \times I$ if and only if $K(h(t), h(s))$ is positive semi-definite on $J \times J$. We invoke the next Schur theorem (e.g., p. 457 of [9]).

If $K_1(t, s)$ and $K_2(t, s)$ are both positive semi-definite kernel functions on $I \times I$, then so is the Schur product $K_1(t, s) \cdot K_2(t, s)$.

$K(t, s)$ is said to be *conditionally (or almost) positive semi-definite* if (11) holds for each n, for all n points $t_i \in I$ and for all n complex numbers z_i such that $\sum_{i=1}^{n} z_i = 0$. If $-K(t, s)$ is conditionally positive semi-definite, then $K(t, s)$ is said to be *conditionally negative semi-definite*. For instance,

$$K(t, s) = t + s + constant$$

is not only conditionally positive semi-definite but also conditionally negative semi-definite. Suppose $K(t, s) \geqq 0$ for every s, t in I. Then $K(t, s)$ is said to be *infinitely divisible* if $K(t, s)^a$ is positive semi-definite for every $a > 0$. The following lemma is known (e.g., p. 152 of [5,6,8]), but for completeness we give a proof, for we often make use of it.

Lemma 5. *Let $K(t, s) > 0$ for $t, s \in I$. If $K(t, s)$ is conditionally negative semi-definite on $I \times I$, then the reciprocal function $\frac{1}{K(t,s)}$ is infinitely divisible there.*

Proof. Take $t_i \in I$ $(i = 1, 2, \ldots, n)$ and put $a_{ij} = K(t_i, t_j)$. Define b_{ij} by

$$b_{ij} = a_{ij} - a_{in} - a_{nj} + a_{nn} \quad (1 \leqq i, j \leqq n).$$

Since $-a_{ij} = -K(t_i, t_j)$ satisfies (11) for $\sum z_i = 0$, the matrix $(-b_{ij})$ is positive semi-definite (see p. 134 of [5]). By the Schur theorem the matrix $(e^{-b_{ij}})$ is positive semi-definite too. Since

$$e^{-a_{ij}} = e^{-a_{in} + \frac{a_{nn}}{2}} e^{-b_{ij}} e^{-a_{nj} + \frac{a_{nn}}{2}},$$

the matrix $\left(\exp(-a_{ij})\right)$ is positive semi-definite as well. The kernel function $\exp(-K(t, s))$ is therefore positive semi-definite. We note that $\exp(-\lambda K(t, s))$ is also positive semi-definite for $\lambda > 0$ since $\lambda K(t, s)$ is conditionally negative semi-definite too. By making use of

$$\Gamma(a) = k^a \int_0^\infty e^{-k\lambda} \lambda^{a-1} d\lambda \quad (a > 0)$$

we get

$$K(t, s)^{-a} = \frac{1}{\Gamma(a)} \int_0^\infty \exp(-\lambda K(t, s)) \lambda^{a-1} d\lambda,$$

which is positive semi-definite. This implies $1/K(t, s)$ is infinitely divisible. □

We remark that the next lemma is not about orthogonal polynomials.

Lemma 6. *For non-decreasing sequence $\{c_n\}$ define a polynomial $h_k(x)$ by $h_k(x) = (x - c_1) \cdots (x - c_k)$ for each k and denote the maximal zero of $h_k'(x)$ by d_k. Then the kernel function*

$$K_n^m(x, y) := \begin{cases} \dfrac{h_m(x) - h_m(y)}{h_n(x) - h_n(y)} & (x \neq y) \\ \dfrac{h_m'(x)}{h_n'(x)} & (x = y) \end{cases} \tag{12}$$

is infinitely divisible on $(d_n, \infty) \times (d_n, \infty)$, provided $1 \leqq m \leqq n$.

Proof. We first show that $K_n^{n-1}(x,y)$ is infinitely divisible for every n. Since $K_2^1(x,y) = \frac{1}{x+y-(c_1+c_2)}$, the matrices associated with this kernel function is the Cauchy matrices. Hence it is an infinitely divisible kernel function on $(d_2,\infty)\times(d_2,\infty)$, where $d_2 = \frac{c_1+c_2}{2}$ (see [2,3]). Assume that $K_k^{k-1}(x,y)$ is infinitely divisible on $(d_k\infty)\times(d_k,\infty)$. We show that $K_{k+1}^k(x,y)$ is also infinitely divisible on $(d_{k+1},\infty)\times(d_{k+1},\infty)$. For $x\neq y$

$$\begin{aligned}\frac{1}{K_{k+1}^k(x,y)} &= \frac{h_{k+1}(x)-h_{k+1}(y)}{h_k(x)-h_k(y)}\\ &= \frac{h_k(x)(x-c_{k+1})-h_k(y)(y-c_{k+1})}{h_k(x)-h_k(y)}\\ &= -c_k-c_{k+1}+(x+y)+\frac{(x-c_k)h_k(y)-(y-c_k)h_k(x)}{h_k(x)-h_k(y)}\\ &= -c_k-c_{k+1}+(x+y)-(y-c_k)\frac{h_{k-1}(x)-h_{k-1}(y)}{h_k(x)-h_k(y)}(x-c_k)\\ &= -c_k-c_{k+1}+(x+y)-(y-c_k)K_k^{k-1}(x,y)(x-c_k).\end{aligned}$$

One can see that the first side equals the last side even for $x=y$. By the assumption, this is a conditionally negative semi-definite kernel function on $(d_k\infty)\times(d_k,\infty)$. Since $K_{k+1}^k(x,y) > 0$ on $(d_{k+1},\infty)\times(d_{k+1},\infty)$, by Lemma 5 $K_{k+1}^k(x,y)$ is infinitely divisible. We next show K_n^m is infinitely divisible. It is evident that $K_n^n(x,y) = 1$ is infinitely divisible. Suppose $m<n$. Then, since

$$K_n^m(x,y) = K_n^{n-1}(x,y)K_{n-1}^{n-2}(x,y)\cdots K_{m+1}^m(x,y),$$

by the Schur theorem, $K_n^m(x,y)$ is infinitely divisible. □

Let $f(t)$ be a real C^1 function on an interval I. Then the kernel function

$$K_f(t,s) := \begin{cases}\dfrac{f(t)-f(s)}{t-s} & (t\neq s)\\ f'(t) & (t=s)\end{cases} \tag{13}$$

is called a *Löwner kernel* of $f(t)$. The following theorem is also due to Löwner [11] (cf. [10,5]):

$f(t)\in\mathbf{P}(I)$ if and only if $K_f(t,s)$ is positive semi-definite on $I\times I$.

Now we go back to orthonormal polynomials $\{p_n\}_{n=0}^\infty$ with positive leading coefficients α_n.

Definition 1. Let d_n be the maximal zero of $p_n'(t)$. Then the restriction of $p_n(t)$ to $[d_n,\infty)$ is increasing. We call its inverse function the *principal inverse* of $p_n(t)$ and write $p_n^{-1}(t)$.

We note that $p_n(d_n)<0$. The following lemma guarantees that the composite $p_{n-1}\circ p_n^{-1}$ is increasing on $[p_n(d_n),\infty)$.

Lemma 7. *$d_{n-1}<d_n$ for $n\geqq 2$.*

Proof. Assume $d_{n-1}\geqq d_n$. Then $p_n'(d_{n-1})\geqq 0$. By substituting d_{n-1} for t in the Christoffel–Darboux formula

$$\sum_{k=0}^{n-1}p_k(t)^2 = \frac{\alpha_{n-1}}{\alpha_{n+1}}(p_n'(t)p_{n-1}(t)-p_{n-1}'(t)p_n(t)),$$

we get $0 < p_n'(d_{n-1})p_{n-1}(d_{n-1})$ since $p_{n-1}'(d_{n-1}) = 0$. But this contradicts $p_{n-1}(d_{n-1}) < 0$, so we obtain the required inequality. □

We are ready to show the main theorem of this section.

Theorem 3. *Let $\{p_n\}_{n=0}^\infty$ be orthonormal polynomials with positive leading coefficients α_n and p_n^{-1} the principal inverse of $p_n(t)$. Then*

$$p_n^{-1}\in\mathbf{P}(p_n(d_n),\infty), \tag{14}$$

where d_n is the maximal zero of $p_n'(t)$. Moreover, $p_n^{-1}(t)$ has a univalent holomorphic extension $p_n^{-1}(z)$ to $\mathbf{C}\setminus(-\infty,p_n(d_n)]$ such that it is a Pick function and satisfies

$$p_n(p_n^{-1}(z)) = z\quad \text{on } \mathbf{C}\setminus(-\infty,p_n(d_n)].$$

Further,

$$p_m\circ p_n^{-1}\in\mathbf{P}(p_n(d_n),\infty)\quad \text{for } m=1,\ldots,n-1. \tag{15}$$

Proof. To see (15) we show that the Löwner kernel $K_{p_m \circ p_n^{-1}}(t,s)$ is positive semi-definite on $(p_n(d_n),\infty)\times(p_n(d_n),\infty)$. Since p_n is increasing on (d_n,∞), it is sufficient to show that

$$K_{p_m \circ p_n^{-1}}(p_n(t),p_n(s))$$

is positive semi-definite on $(d_n,\infty)\times(d_n,\infty)$. We observe that

$$K_{p_m \circ p_n^{-1}}(p_n(t),p_n(s)) = \begin{cases} \dfrac{p_m(t)-p_m(s)}{p_n(t)-p_n(s)} & (t\neq s) \\ \dfrac{p_m'(t)}{p_n'(t)} & (t=s). \end{cases}$$

We first show the case $m=n-1$ by induction for n. For simplicity we put

$$H_k(t,s) := K_{p_{k-1}\circ p_k^{-1}}(p_k(t),p_k(s)).$$

In virtue of

$$p_1(t)-p_1(s)=\alpha_1(t-s), \qquad p_2(t)-p_2(s)=\alpha_2(t-s)(t+s-2d_2),$$

we get

$$H_2(t,s) = \begin{cases} \dfrac{\alpha_1}{\alpha_2}\dfrac{1}{t+s-2d_2} & (t\neq s) \\ \dfrac{\alpha_1}{\alpha_2}\dfrac{1}{2(t-d_2)} & (t=s). \end{cases}$$

The matrices associated with this Kernel function is the Cauchy matrices. Hence it is an infinitely divisible kernel function on $(d_2,\infty)\times(d_2,\infty)$. Assume that $H_k(t,s)$ is an infinitely divisible kernel function on $(d_k,\infty)\times(d_k,\infty)$. Notice that $H_{k+1}(t,s)>0$ for $t,s>d_{k+1}$. By (5)

$$\begin{aligned} \frac{a_k}{H_{k+1}(t,s)} &= \frac{tp_k(t)-sp_k(s)}{p_k(t)-p_k(s)} - b_k - a_{k-1}H_k(t,s) \\ &= (t+s) + \frac{tp_k(s)-sp_k(t)}{p_k(t)-p_k(s)} - b_k - a_{k-1}H_k(t,s). \end{aligned} \tag{16}$$

Let $p_k(t)=\alpha_k(t-c_1)\cdots(t-c_k)$ be the factorization of $p_k(t)$. Then we have

$$\begin{aligned} \frac{tp_k(s)-sp_k(t)}{p_k(t)-p_k(s)} &= (t-c_k)\frac{\frac{p_k(s)}{s-c_k}-\frac{p_k(t)}{t-c_k}}{p_k(t)-p_k(s)}(s-c_k)-c_k \\ &= -(t-c_k)K_k^{k-1}(t,s)(s-c_k)-c_k, \end{aligned}$$

where $K_k^{k-1}(t,s)$ is given in (12). By Lemma 6 the above kernel function is a conditionally negative definite kernel on $(d_k,\infty)\times(d_k,\infty)$. From (16) it follows that $\frac{1}{H_{k+1}(t,s)}$ is conditionally negative definite. $H_{k+1}(t,s)$ is therefore infinitely divisible. Thus we have shown that $H_k(t,s)$ is infinitely divisible on $(d_k,\infty)\times(d_k,\infty)$ for every $k\geqq 2$. Hence it turn out that the Schur product

$$K_{p_m\circ p_n^{-1}}(p_n(t),p_n(s)) = H_m(t,s)H_{m+1}(t,s)\cdots H_n(t,s)$$

is infinitely divisible on $(d_n,\infty)\times(d_n,\infty)$; of course it is positive semi-definite. This indicates (15). Since $p_1(t)=\alpha_1 t+const$, the case of $m=1$ deduces (14). Consequently $p_n^{-1}(z)$ is holomorphic on $\mathbf{C}\setminus(-\infty,p_n(d_n)]$, and it is a Pick function. Since the composite $p_n(p_n^{-1}(z))$ is holomorphic on the domain and coincides with z on the real interval $(p_n(d_n),\infty)$, we get $p_n(p_n^{-1}(z))=z$ on the domain. From this formula it follows that $p_n^{-1}(z)$ is univalent. $\square$

In [15,16], we have shown the following.

Let c_n be the maximal zero of $p_n(t)$ and put $p_{n+}(t)=p_n(t)|_{(c_n,\infty)}$; then

$$p_m\circ p_{n+}^{-1}\in \mathbf{P}(0,\infty) \quad (m=1,\ldots,n-1).$$

At first sight, Theorem 3 seems to be just a slight extension of this result, but the proof is completely different and we cannot extend the domain of p_n^{-1} in Theorem 3 anymore; so we may say that Theorem 3 is an essential extension of it.

Observe that Theorem 3 implies that for $A,B\geqq d_n$

$$p_n(A)\leqq p_n(B) \Rightarrow p_{n-1}(A)\leqq p_{n-1}(B)\Rightarrow\cdots\Rightarrow A\leqq B,$$

because $p_1(A)=\alpha_1 A+\text{constant}$. By using the notation introduced in [17,18] we can express this as follows:

$$t\preceq p_2(t)\preceq\cdots\preceq p_{n-1}(t)\preceq p_n(t) \quad \text{on } [d_n,\infty).$$

References

[1] R. Bhatia, Matrix Analysis, Springer, 1996.
[2] R. Bhatia, Infinitely divisible matrices, Amer. Math. Monthly 113 (2006) 221–235.
[3] R. Bhatia, H. Kosaki, Mean matrices and infinite divisibility, Linear Algebra Appl. 424 (2007) 36–54.
[4] P. Deift, Orthogonal Polynomials and Random Matrices: A Riemann–Hilbert Approach, Amer. Math. Soc., 1998.
[5] W.F. Donoghue, Monotone Matrix Functions and Analytic Continuation, Springer-Verlag, 1974.
[6] C.H. Fitzgerald, On analytic continuation to a Schlicht function, Proc. Amer. Math. Soc. 18 (1967) 788–792.
[7] F. Hansen, G.K. Pedersen, Jensen's inequality for operators and Löwner's theorem, Math. Ann. 258 (1982) 229–241.
[8] R.A. Horn, On boundary values of a Schlicht mapping, Proc. Amer. Math. Soc. 18 (1967) 782–787.
[9] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, 1985.
[10] A. Koranyi, On a theorem of Löwner and its connections with resolvents of selfadjoint transformations, Acta Sci. Math. 17 (1956) 63–70.
[11] K. Löwner, Über monotone matrixfunktionen, Math. Z. 38 (1934) 177–216.
[12] M. Rosenblum, J. Rovnyak, Hardy Classes and Operator Theory, Oxford, 1985.
[13] V. Totik, Orthogonal polynomials, Surv. Approx. Theory 1 (2005) 70–125 (electronic).
[14] M. Uchiyama, The principal inverse of the gamma function, Proc. Amer. Math. Soc. 140 (2012) 1343–1348.
[15] M. Uchiyama, Operator monotone functions which are defined implicitly and operator inequalities, J. Funct. Anal. 175 (2000) 330–347.
[16] M. Uchiyama, Inverse functions of polynomials and orthogonal polynomials as operator monotone functions, Trans. Amer. Math. Soc. 355 (2003) 4111–4123.
[17] M. Uchiyama, A new majorization between functions, polynomials, and operator inequalities, J. Funct. Anal. 231 (1) (2006) 221–244.
[18] M. Uchiyama, A new majorization between functions, polynomials, and operator inequalities II, J. Math. Soc. Japan 60 (1) (2008) 291–310.