J. Math. Anal. Appl. 406 (2013) 243–260

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Summability of formal solutions of linear partial differential equations with divergent initial data

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ARTICLE INFO

Article history:
Received 26 January 2013
Available online 29 April 2013
Submitted by Andrea Cianchi

Keywords:
Linear PDEs with constant coefficients
Formal power series
Moment functions
Moment-PDEs
Gevrey order
Borel summability
Multisummability

ABSTRACT

We study the Cauchy problem for a general homogeneous linear partial differential equation in two complex variables with constant coefficients and with divergent initial data. We state necessary and sufficient conditions for the summability of formal power series solutions in terms of properties of divergent Cauchy data. We consider both the summability in one variable t (with coefficients belonging to some Banach space of Gevrey series with respect to the second variable z) and the summability in two variables (t, z). The results are presented in the general framework of moment-PDEs.

1. Introduction

The problem of summability of formal solutions of linear PDEs was mainly studied under the assumption that the Cauchy data are convergent, see Balser [3], Balser and Loday-Richaud [5], Balser and Miyake [6], Ichinobe [8], Lutz, Miyake and Schäfke [9], Malek [10], Michalik [11–13] and Miyake [15].

The case of more general initial data was investigated only for the complex heat equation (see Balser [1,4]). In [1] Balser considered the case of entire initial data with an appropriate growth condition and he gave some preliminary results for divergent initial data, too. Next, these results were extended in [4], where a characterisation of summable formal power series solutions of the complex heat equation in terms of properties of divergent Cauchy data was given.

The aim of our paper is a generalisation of Balser's results [1,4] to homogeneous linear partial differential equations with constant coefficients.

Namely, we consider the initial value problem for a general linear partial differential equation with constant coefficients in two complex variables (t, z)

$$P(\partial_t, \partial_z)\widehat{u} = 0, \qquad \partial_t^j\widehat{u}(0, z) = \widehat{\varphi}_j(z) \quad (j = 0, \dots, n-1), \tag{1}$$

where $P(\lambda, \zeta)$ is a polynomial in both variables of degree n with respect to λ and the Cauchy data $\widehat{\varphi}_j(z) = \sum_{n=0}^{\infty} \varphi_{jn} z^n \in \mathbb{C}[[z]]$ are formal power series.

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0022-247X/$ – see front matter
http://dx.doi.org/10.1016/j.jmaa.2013.04.062

We study the Gevrey asymptotic properties of formal power series solutions $\widehat{u}$ for a fixed Gevrey order of the initial data. Moreover, we characterise the multisummable formal solutions $\widehat{u}$ of (1) in terms of the Cauchy data.

The results are expressed in the general framework of moment differential equations with the differentiations ∂_t and ∂_z replaced by more general operators of moment differentiations $\partial_{m_1,t}$ and $\partial_{m_2,z}$ respectively (see Definition 12). The general moment differential equations were introduced by Balser and Yoshino [7], who studied the Gevrey order of formal solutions of such equations. A characterisation of the multisummable formal solutions of moment differential equations in terms of analytic continuation properties and growth estimates of the Cauchy data was established in our previous paper [14] under the assumption of convergence of the Cauchy data. In the present paper we continue the study without this assumption. Additionally we consider a wider class of moment functions, which is a group with respect to multiplication, and so the set of moment differential operators contains some integro-differential operators (see Example 3).

We give a meaning to summability of formal solutions $\widehat{u}$ in two variables by two methods. In the first one we treat $\widehat{u}$ as a formal power series in t-variable with the coefficients belonging to some Banach space of Gevrey series (in z-variable). This situation is carried over by the general theory of summability developed by Balser [2]. In the second method we study summability of $\widehat{u}$ in two variables (t, z) using approaches used by Balser [4] and by Sanz [16].

The main idea of the paper is based on the use of appropriate moment Borel transforms $\mathcal{B}_{m_1',t}$ and $\mathcal{B}_{m_2',z}$ (see Definition 5), which transform the formal solution $\widehat{u}$ of the equation $P(\partial_{m_1,t}, \partial_{m_2,z})\widehat{u} = 0$ with the divergent Cauchy data $\widehat{\varphi}_j$ into the analytic solution $v = \mathcal{B}_{m_1',t}\mathcal{B}_{m_2',z}\widehat{u}$ of the equation $P(\partial_{m_1m_1',t}, \partial_{m_2m_2',z})v = 0$ with the convergent Cauchy data $\mathcal{B}_{m_2',z}\widehat{\varphi}_j$. On the other hand we are able to define the summability of $\widehat{u}$ (both in t and in (t, z) variables) in terms of analytic continuation properties of v. In this way, analogously to [14], we reduce the problem of summability of $\widehat{u}$ to the problem of analytic continuation of v.

In the case of summability of $\widehat{u}$ with respect to t-variable, it is sufficient to apply our previous result [14, Theorem 3], which establishes the relation between the analytic continuation properties of v (with respect to t) and the Cauchy data $\mathcal{B}_{m_2',z}\widehat{\varphi}_j$. In the case of summability of $\widehat{u}$ in two variables (t, z) the situation is more complicated, since we have to study the analytic continuation properties of v with respect to both variables. To this end we characterise the analytic continuation properties of v in two variables (t, z) in terms of the Cauchy data.

Finally, in both cases we obtain a characterisation of the multisummable formal solution $\widehat{u}$ of moment differential equations in the terms of the divergent initial data $\widehat{\varphi}_j$.

In the last section we discuss a simple example illustrating the developed theory. Namely, we consider the formal solution $\widehat{u} = \widehat{u}(t, z)$ of the Cauchy problem

$$(\partial_t - \partial_z^q)\widehat{u} = 0, \qquad \widehat{u}(0, z) = \widehat{\varphi}(z).$$

We show the relation between the properties of the Cauchy data $\widehat{\varphi}$, the summability of $\widehat{u}$ in one variable t and in two variables (t, z).

2. Notation

We use the following notation. The complex disc in $\mathbb{C}^n$ with centre at the origin and radius $r > 0$ is denoted by $D_r^n := \{z \in \mathbb{C}^n : |z| < r\}$. To simplify notation, we write D_r instead of D_r^1. If the radius r is not essential, then we denote it briefly by D^n (resp. D).

A sector in a direction $d \in \mathbb{R}$ with an opening $\varepsilon > 0$ in the universal covering space $\widetilde{\mathbb{C} \setminus \{0\}}$ of $\mathbb{C} \setminus \{0\}$ is defined by

$$S_d(\varepsilon) := \{z \in \widetilde{\mathbb{C} \setminus \{0\}} : z = re^{i\theta}, \ d - \varepsilon/2 < \theta < d + \varepsilon/2, \ r > 0\}.$$

Moreover, if the value of opening angle ε is not essential, then we denote it briefly by S_d.

Analogously, by *a disc-sector in a direction $d \in \mathbb{R}$ with an opening $\varepsilon > 0$ and radius $r > 0$* we mean a domain $\widehat{S}_d(\varepsilon; r) := S_d(\varepsilon) \cup D_r$. If the values of ε and r are not essential, we write it as $\widehat{S}_d$ for brevity (i.e. $\widehat{S}_d = S_d \cup D$).

By $\mathcal{O}(G)$ we understand the space of holomorphic functions on a domain $G \subseteq \mathbb{C}^n$. Analogously, the space of analytic functions of the variables $z_1^{1/\kappa_1}, \ldots, z_n^{1/\kappa_n}$ $((\kappa_1, \ldots, \kappa_n) \in \mathbb{N}^n)$ on G is denoted by $\mathcal{O}_{1/\kappa_1,\ldots,1/\kappa_n}(G)$. More generally, if $\mathbb{E}$ denotes a Banach space with a norm $\|\cdot\|_{\mathbb{E}}$, then by $\mathcal{O}(G, \mathbb{E})$ (resp. $\mathcal{O}_{1/\kappa_1,\ldots,1/\kappa_n}(G, \mathbb{E})$) we shall denote the set of all $\mathbb{E}$-valued holomorphic functions (resp. holomorphic functions of the variables $z_1^{1/\kappa_1}, \ldots, z_n^{1/\kappa_n}$) on a domain $G \subseteq \mathbb{C}^n$. For more information about functions with values in Banach spaces we refer the reader to [2, Appendix B]. In the paper, as a Banach space $\mathbb{E}$ we will take the space of complex numbers $\mathbb{C}$ (we abbreviate $\mathcal{O}(G, \mathbb{C})$ to $\mathcal{O}(G)$ and $\mathcal{O}_{1/\kappa_1,\ldots,1/\kappa_n}(G, \mathbb{C})$ to $\mathcal{O}_{1/\kappa_1,\ldots,1/\kappa_n}(G)$) or the space of Gevrey series $G_{s,1/\kappa}(r)$ (see Definition 7).

Definition 1. A function $u \in \mathcal{O}_{1/\kappa}(\widehat{S}_d(\varepsilon; r), \mathbb{E})$ is of *exponential growth of order at most $K \in \mathbb{R}$ as $x \to \infty$ in $\widehat{S}_d(\varepsilon; r)$* if for any $\widetilde{\varepsilon} \in (0, \varepsilon)$ and $\widetilde{r} \in (0, r)$ there exist $A, B < \infty$ such that

$$\|u(x)\|_{\mathbb{E}} < Ae^{B|x|^K} \quad \text{for every } x \in \widehat{S}_d(\widetilde{\varepsilon}; \widetilde{r}).$$

The space of such functions is denoted by $\mathcal{O}_{1/\kappa}^K(\widehat{S}_d(\varepsilon; r), \mathbb{E})$.

Analogously, a function $u \in \mathcal{O}_{1/\kappa_1,1/\kappa_2}(\widehat{S}_{d_1}(\varepsilon_1; r_1) \times \widehat{S}_{d_2}(\varepsilon_2; r_2))$ is of *exponential growth of order at most* $(K_1, K_2) \in \mathbb{R}^2$ *as* $(t, z) \to \infty$ *in* $\widehat{S}_{d_1}(\varepsilon_1; r_1) \times \widehat{S}_{d_2}(\varepsilon_2; r_2)$ if for any $\widetilde{\varepsilon}_i \in (0, \varepsilon_i)$ and any $\widetilde{r}_i \in (0, r_i)$ $(i = 1, 2)$ there exist $A, B_1, B_2 < \infty$ such that

$$|u(t, z)| < Ae^{B_1|t|^{K_1}} e^{B_2|z|^{K_2}} \quad \text{for every } (t, z) \in \widehat{S}_{d_1}(\widetilde{\varepsilon}_1; \widetilde{r}_1) \times \widehat{S}_{d_2}(\widetilde{\varepsilon}_2; \widetilde{r}_2).$$

The space of such functions is denoted by $\mathcal{O}^{K_1,K_2}_{1/\kappa_1,1/\kappa_2}(\widehat{S}_{d_1}(\varepsilon_1; r_1) \times \widehat{S}_{d_2}(\varepsilon_2; r_2))$.

The space of formal power series $\widehat{u}(x) = \sum_{j=0}^{\infty} u_j x^{j/\kappa}$ with $u_j \in \mathbb{E}$ is denoted by $\mathbb{E}[[x^{\frac{1}{\kappa}}]]$. Analogously, the space of formal power series $\widehat{u}(t, z) = \sum_{j,n=0}^{\infty} u_{jn} t^{j/\kappa_1} z^{n/\kappa_2}$ with $u_{jn} \in \mathbb{E}$ is denoted by $\mathbb{E}[[t^{\frac{1}{\kappa_1}}, z^{\frac{1}{\kappa_2}}]]$. We use the "hat" notation $(\widehat{u}, \widehat{v}, \widehat{\varphi}, \widehat{\psi}, \widehat{f})$ to denote the formal power series. If the formal power series $\widehat{u}$ (resp. $\widehat{v}, \widehat{\varphi}, \widehat{\psi}, \widehat{f}$) is convergent, we denote its sum by u (resp. v, φ, ψ, f).

3. Moment functions

In this section we recall the notion of moment methods introduced by Balser [2].

Definition 2 (*See [2, Section 5.5]*)**.** A pair of functions e_m and E_m is said to be *kernel functions of order* k $(k > 1/2)$ if they have the following properties:

1. $e_m \in \mathcal{O}(S_0(\pi/k))$, $e_m(z)/z$ is integrable at the origin, $e_m(x) \in \mathbb{R}_+$ for $x \in \mathbb{R}_+$ and e_m is exponentially flat of order k in $S_0(\pi/k)$ (i.e. $\forall_{\varepsilon>0}\exists_{A,B>0}$ such that $|e_m(z)| \le Ae^{-(|z|/B)^k}$ for $z \in S_0(\pi/k - \varepsilon)$).
2. $E_m \in \mathcal{O}^k(\mathbb{C})$ and $E_m(1/z)/z$ is integrable at the origin in $S_\pi(2\pi - \pi/k)$.
3. The connection between e_m and E_m is given by the *corresponding moment function m of order* $1/k$ as follows. The function m is defined in terms of e_m by
$$m(u) := \int_0^\infty x^{u-1} e_m(x)dx \quad \text{for } \operatorname{Re} u \ge 0 \tag{2}$$
and the kernel function E_m has the power series expansion
$$E_m(z) = \sum_{n=0}^{\infty} \frac{z^n}{m(n)} \quad \text{for } z \in \mathbb{C}. \tag{3}$$

Observe that in case $k \le 1/2$ the set $S_\pi(2\pi - \pi/k)$ is not defined, so the second property in Definition 2 cannot be satisfied. It means that we must define the kernel functions of order $k \le 1/2$ and the corresponding moment functions in another way.

Definition 3 (*See [2, Section 5.6]*)**.** A function e_m is called *a kernel function of order* $k > 0$ if we can find a pair of kernel functions $e_{\widetilde{m}}$ and $E_{\widetilde{m}}$ of order $pk > 1/2$ (for some $p \in \mathbb{N}$) so that

$$e_m(z) = e_{\widetilde{m}}(z^{1/p})/p \quad \text{for } z \in S(0, \pi/k).$$

For a given kernel function e_m of order $k > 0$ we define the *corresponding moment function m of order* $1/k > 0$ by (2) and the *kernel function* E_m *of order* $k > 0$ by (3).

Remark 1. Observe that by Definitions 2 and 3 we have

$$m(u) = \widetilde{m}(pu) \quad \text{and} \quad E_m(z) = \sum_{j=0}^{\infty} \frac{z^j}{m(j)} = \sum_{j=0}^{\infty} \frac{z^j}{\widetilde{m}(jp)}.$$

We extend the notion of moment functions to real orders as follows

Definition 4. We say that m is a *moment function of order* $1/k < 0$ if $1/m$ is a moment function of order $-1/k > 0$.

We say that m is a *moment function of order 0* if there exist moment functions m_1 and m_2 of the same order $1/k > 0$ such that $m = m_1/m_2$.

By Definition 4 and by [2, Theorems 31 and 32] we have

Proposition 1. *Let* m_1, m_2 *be moment functions of orders* $s_1, s_2 \in \mathbb{R}$ *respectively. Then*

(1) $m_1 m_2$ *is a moment function of order* $s_1 + s_2$,
(2) m_1/m_2 *is a moment function of order* $s_1 - s_2$.

Remark 2. By the above proposition we see that the set $\mathcal{M}$ of all moment functions endowed with the multiplication operation has the structure of group $\langle \mathcal{M}, \cdot \rangle$. Moreover, the map ord : $\langle \mathcal{M}, \cdot \rangle \longrightarrow \langle \mathbb{Z}, + \rangle$ defined by ord $(m) := s$ for every moment function m of order s, is a group homomorphism.

Example 1. For any $a \geq 0$, $b \geq 1$ and $k > 0$ we can construct the following examples of kernel functions e_m and E_m of orders $k > 0$ with the corresponding moment function m of order $1/k$ satisfying Definition 2 or 3:

- $e_m(z) = akz^{bk}e^{-z^k}$,
- $m(u) = a\Gamma(b + u/k)$,
- $E_m(z) = \frac{1}{a}\sum_{j=0}^{\infty}\frac{z^j}{\Gamma(b+j/k)}$.

In particular for $a = b = 1$ we get the kernel functions and the corresponding moment function, which are used in the classical theory of k-summability.

- $e_m(z) = kz^k e^{-z^k}$,
- $m(u) = \Gamma(1 + u/k)$,
- $E_m(z) = \sum_{j=0}^{\infty}\frac{z^j}{\Gamma(1+j/k)} =: \mathbf{E}_{1/k}(z)$, where $\mathbf{E}_{1/k}$ is the Mittag-Leffler function of index $1/k$.

Example 2. For any $s \in \mathbb{R}$ we will denote by Γ_s the function

$$\Gamma_s(u) := \begin{cases} \Gamma(1+su) & \text{for } s \geq 0 \\ 1/\Gamma(1-su) & \text{for } s < 0. \end{cases}$$

Observe that by Example 1 and Definition 4, Γ_s is an example of a moment function of order $s \in \mathbb{R}$.

The moment functions Γ_s will be extensively used in the paper, since every moment function m of order s has the same growth as Γ_s. Precisely speaking, we have

Proposition 2 (*See [2, Section 5.5]*)**.** *If m is a moment function of order $s \in \mathbb{R}$ then there exist constants $c, C > 0$ such that*

$$c^n\Gamma_s(n) \leq m(n) \leq C^n\Gamma_s(n) \quad \textit{for every } n \in \mathbb{N}.$$

4. Moment Borel transform, Gevrey order and Borel summability

We use the moment function to define the Gevrey order and the Borel summability. We first introduce

Definition 5. Let $\kappa \in \mathbb{N}$ and m be a moment function. Then the linear operator $\mathcal{B}_{m,x^{1/\kappa}} : \mathbb{E}[[x^{\frac{1}{\kappa}}]] \to \mathbb{E}[[x^{\frac{1}{\kappa}}]]$ defined by

$$\mathcal{B}_{m,x^{1/\kappa}}\left(\sum_{j=0}^{\infty} u_j x^{j/\kappa}\right) := \sum_{j=0}^{\infty}\frac{u_j}{m(j/\kappa)}x^{j/\kappa}$$

is called an *m-moment Borel transform with respect to $x^{1/\kappa}$*.

We define the Gevrey order of formal power series as follows

Definition 6. Let $\kappa \in \mathbb{N}$ and $s \in \mathbb{R}$. Then $\widehat{u} \in \mathbb{E}[[x^{\frac{1}{\kappa}}]]$ is called a *formal power series of Gevrey order s* if there exists a disc $D \subset \mathbb{C}$ with centre at the origin such that $\mathcal{B}_{\Gamma_s,x^{1/\kappa}}\widehat{u} \in \mathcal{O}_{1/\kappa}(D,\mathbb{E})$. The space of formal power series of Gevrey order s is denoted by $\mathbb{E}[[x^{\frac{1}{\kappa}}]]_s$.

Analogously, if $\kappa_1, \kappa_2 \in \mathbb{N}$ and $s_1, s_2 \in \mathbb{R}$ then $\widehat{u} \in \mathbb{E}[[t^{\frac{1}{\kappa_1}}, z^{\frac{1}{\kappa_2}}]]$ is called a *formal power series of Gevrey order* (s_1, s_2) if there exists a disc $D^2 \subset \mathbb{C}^2$ with centre at the origin such that $\mathcal{B}_{\Gamma_{s_1},t^{1/\kappa_1}}\mathcal{B}_{\Gamma_{s_2},z^{1/\kappa_2}}\widehat{u} \in \mathcal{O}_{1/\kappa_1,1/\kappa_2}(D^2,\mathbb{E})$. The space of formal power series of Gevrey order (s_1, s_2) is denoted by $\mathbb{E}[[t^{\frac{1}{\kappa_1}}, z^{\frac{1}{\kappa_2}}]]_{s_1,s_2}$.

Remark 3. By Proposition 2, we may replace Γ_s (resp. Γ_{s_1} and Γ_{s_2}) in Definition 6 by any moment function m of order s (resp. by any moment functions m_1 and m_2 of orders s_1 and s_2).

Remark 4. If $\widehat{u} \in \mathbb{E}[[x^{\frac{1}{\kappa}}]]_s$ and $s \leq 0$ then the formal series $\widehat{u}$ is convergent, so its sum u is well defined. Moreover, $\widehat{u} \in \mathbb{E}[[x^{\frac{1}{\kappa}}]]_0 \Leftrightarrow u \in \mathcal{O}_{1/\kappa}(D,\mathbb{E})$ and $\widehat{u} \in \mathbb{E}[[x^{\frac{1}{\kappa}}]]_s \Leftrightarrow u \in \mathcal{O}_{1/\kappa}^{-1/s}(\mathbb{C},\mathbb{E})$ for $s < 0$.

By Definitions 5 and 6 we obtain

Proposition 3. *For every $\widehat{u} \in \mathbb{E}[[x^{\frac{1}{\kappa}}]]$ the following properties of moment Borel transforms are satisfied:*

- $\mathcal{B}_{m_1,x^{1/\kappa}}\mathcal{B}_{m_2,x^{1/\kappa}}\widehat{u} = \mathcal{B}_{m_1m_2,x^{1/\kappa}}\widehat{u}$ *for every moment functions m_1 and m_2.*
- $\mathcal{B}_{m,x^{1/\kappa}}\mathcal{B}_{1/m,x^{1/\kappa}}\widehat{u} = \mathcal{B}_{1/m,x^{1/\kappa}}\mathcal{B}_{m,x^{1/\kappa}}\widehat{u} = \mathcal{B}_{1,x^{1/\kappa}}\widehat{u} = \widehat{u}$ *for every moment function m.*
- $\widehat{u} \in \mathbb{E}[[x^{\frac{1}{\kappa}}]]_{s_1} \Leftrightarrow \mathcal{B}_{m,x^{1/\kappa}}\widehat{u} \in \mathbb{E}[[x^{\frac{1}{\kappa}}]]_{s_1-s}$ *for every $s, s_1 \in \mathbb{R}$ and for every moment function m of order s.*

As a Banach space $\mathbb{E}$ we will take the space of complex numbers $\mathbb{C}$ or the space of Gevrey series $G_{s,1/\kappa}(r)$ defined below.

Definition 7. Fix $\kappa \in \mathbb{N}$, $r > 0$ and $s \in \mathbb{R}$. By $G_{s,1/\kappa}(r)$ we denote a Banach space of Gevrey series

$$G_{s,1/\kappa}(r) := \{\widehat{\varphi} \in \mathbb{C}[[z^{\frac{1}{\kappa}}]]_s \colon \mathcal{B}_{\Gamma_s,z^{1/\kappa}}\widehat{\varphi} \in \mathcal{O}_{1/\kappa}(D_r) \cap C(\overline{D_r})\}$$

equipped with the norm

$$\|\widehat{\varphi}\|_{G_{s,1/\kappa}(r)} := \max_{|z|\le r} |\mathcal{B}_{\Gamma_s,z^{1/\kappa}}\widehat{\varphi}(z)|.$$

We also set $G_{s,1/\kappa} := \varinjlim_{r>0} G_{s,1/\kappa}(r)$. Analogously, we define $\mathcal{O}_{1/\widetilde{\kappa}}(G, G_{s,1/\kappa}) := \varinjlim_{r>0} \mathcal{O}_{1/\widetilde{\kappa}}(G, G_{s,1/\kappa}(r))$ and $\mathcal{O}^K_{1/\widetilde{\kappa}}(G, G_{s,1/\kappa}) := \varinjlim_{r>0} \mathcal{O}^K_{1/\widetilde{\kappa}}(G, G_{s,1/\kappa}(r))$.

Moreover, we denote by $G_{s_2,1/\kappa}[[t]]_{s_1}$ the space of formal power series $\widehat{u}(t,z) = \sum_{j=0}^{\infty} \widehat{u}_j(z)t^j$ of Gevrey order s_1 with coefficients $\widehat{u}_j(z) \in G_{s_2,1/\kappa}$.

By Definitions 6, 7, Remark 3 and Proposition 3 we conclude

Proposition 4. *For every $\kappa \in \mathbb{N}$, $s, \bar{s} \in \mathbb{R}$ (resp. $s_1, s_2, \bar{s} \in \mathbb{R}$) and for every moment function m of order $\bar{s}$ the following conditions are equivalent:*

- $\widehat{u} \in \mathbb{C}[[x^{\frac{1}{\kappa}}]]_s$ *(resp.* $\widehat{u} \in \mathbb{C}[[t, z^{\frac{1}{\kappa}}]]_{s_1,s_2}$*),*
- $\mathcal{B}_{\Gamma_s,x^{1/\kappa}}\widehat{u} \in \mathcal{O}_{1/\kappa}(D)$ *(resp.* $\mathcal{B}_{\Gamma_{s_1},t}\mathcal{B}_{\Gamma_{s_2},z^{1/\kappa}}\widehat{u} \in \mathcal{O}_{1,1/\kappa}(D^2)$*),*
- *there exists $r > 0$ such that* $\widehat{u} \in G_{s,1/\kappa}(r)$ *(resp.* $\widehat{u} \in G_{s_2,1/\kappa}(r)[[t]]_{s_1}$*),*
- $\widehat{u} \in G_{s,1/\kappa}$ *(resp.* $\widehat{u} \in G_{s_2,1/\kappa}[[t]]_{s_1}$*),*
- $\mathcal{B}_{m,x^{1/\kappa}}\widehat{u} \in \mathbb{C}[[x^{\frac{1}{\kappa}}]]_{s-\bar{s}}$ *(resp.* $\mathcal{B}_{m,z^{1/\kappa}}\widehat{u} \in G_{s_2-\bar{s},1/\kappa}[[t]]_{s_1}$*).*

Now we are ready to define the summability of formal power series in one variable (see Balser [2])

Definition 8. Let $\kappa \in \mathbb{N}$, $K > 0$ and $d \in \mathbb{R}$. Then $\widehat{u} \in \mathbb{E}[[x^{\frac{1}{\kappa}}]]$ is called *K-summable in a direction d* if there exists a disc-sector $\widehat{S}_d$ in a direction d such that $\mathcal{B}_{\Gamma_{1/K},x^{1/\kappa}}\widehat{u} \in \mathcal{O}^K_{1/\kappa}(\widehat{S}_d, \mathbb{E})$.

Remark 5. By Definitions 7 and 8, $\widehat{u} \in G_{s,1/\kappa}[[t]]$ is K-summable in a direction d if and only if $\mathcal{B}_{\Gamma_{1/K},t}\mathcal{B}_{\Gamma_s,z^{1/\kappa}}\widehat{u} \in \mathcal{O}^K_{1,1/\kappa}(\widehat{S}_d \times D)$. Moreover, we may replace Γ_s in the above characterisation by any moment function m of order s.

We can now define the multisummability in a multidirection.

Definition 9. Let $K_1 > \cdots > K_n > 0$. We say that a real vector $(d_1, \dots, d_n) \in \mathbb{R}^n$ is an *admissible multidirection* if

$$|d_j - d_{j-1}| \le \pi(1/K_j - 1/K_{j-1})/2 \quad \text{for } j = 2, \dots, n.$$

Let $\mathbf{K} = (K_1, \dots, K_n) \in \mathbb{R}^n_+$ and let $\mathbf{d} = (d_1, \dots, d_n) \in \mathbb{R}^n$ be an admissible multidirection. We say that a formal power series $\widehat{u} \in \mathbb{E}[[x]]$ is **K**-*multisummable in the multidirection* **d** if $\widehat{u} = \widehat{u}_1 + \cdots + \widehat{u}_n$, where $\widehat{u}_j \in \mathbb{E}[[x]]$ is K_j-summable in the direction d_j for $j = 1, \dots, n$.

Following Sanz [16] we extend the notion of summability to two variables

Definition 10. For $\kappa_1, \kappa_2 \in \mathbb{N}$, $K_1, K_2 > 0$ and $d_1, d_2 \in \mathbb{R}$ the formal power series $\widehat{u} \in \mathbb{C}[[t^{\frac{1}{\kappa_1}}, z^{\frac{1}{\kappa_2}}]]$ is called (K_1, K_2)-*summable in the direction* (d_1, d_2) if there exist disc-sectors $\widehat{S}_{d_1}$ and $\widehat{S}_{d_2}$ such that $\mathcal{B}_{\Gamma_{1/K_1},t^{1/\kappa_1}}\mathcal{B}_{\Gamma_{1/K_2},z^{1/\kappa_2}}\widehat{u} \in \mathcal{O}^{K_1,K_2}_{1/\kappa_1,1/\kappa_2}(\widehat{S}_{d_1} \times \widehat{S}_{d_2})$.

Remark 6. By the general theory of moment summability (see [2, Section 6.5 and Theorem 38]), we may replace $\Gamma_{1/K}$ in Definition 8 (resp. Γ_{1/K_1} and Γ_{1/K_2} in Definition 10) by any moment function m of order $1/K$ (resp. by any moment functions m_1 of order $1/K_1$ and m_2 of order $1/K_2$).

A more general approach to summability in several variables was given by Balser [4]. Namely, he introduced

Definition 11. Let $s_1, s_2 > 0$, $O \subset \{(t_0, z_0) \in (\widetilde{\mathbb{C} \setminus \{0\}})^2 \colon \|(t_0, z_0)\| = 1\}$ be bounded, open and simply connected and let

$$G = \{(t, z) \in (\widetilde{\mathbb{C} \setminus \{0\}})^2 \colon (t, z) = (x^{s_1}t_0, x^{s_2}z_0),\ (t_0, z_0) \in O,\ x > 0\}.$$

Then we say that G is *a (s_1, s_2)-region of infinite radius with an opening O.*

Moreover, for $\kappa_1, \kappa_2 \in \mathbb{N}$ the formal power series

$$\widehat{u}(t, z) = \sum_{j,n=0}^{\infty} u_{jn} t^{j/\kappa_1} z^{n/\kappa_2} \in \mathbb{C}[[t^{\frac{1}{\kappa_1}}, z^{\frac{1}{\kappa_2}}]]$$

is called $(1/s_1, 1/s_2)$-*summable in the direction O* if

$$\mathcal{B}_{(s_1,s_2)}\widehat{u}(t,z) := \sum_{j,n=0}^{\infty} \frac{u_{jn}}{\Gamma(1+s_1 j/\kappa_1 + s_2 n/\kappa_2)} t^{j/\kappa_1} z^{n/\kappa_2}$$

belongs to the space $\mathcal{O}_{1/\kappa_1,1/\kappa_2}(G \cup D^2)$ and for every $O' \Subset O$ there exist $A, B > 0$ such that

$$|\mathcal{B}_{(s_1,s_2)}\widehat{u}(x^{s_1}t_0, x^{s_2}z_0)| \le Ae^{Bx} \quad \text{for every } (t_0,z_0) \in O',\ x > 0.$$

In the paper we will consider only the situation, when G is a polysector $S_{d_1} \times S_{d_2}$ with an opening

$$O = O_{d_1,d_2} := \{(t_0,z_0) \in S_{d_1} \times S_{d_2} : \|(t_0,z_0)\| = 1\}.$$

In this case, immediately by Definition 11, we get

Proposition 5. *Let $s_1, s_2 > 0$, $d_1, d_2 \in \mathbb{R}$ and $\kappa_1, \kappa_2 \in \mathbb{N}$. Then the formal power series $\widehat{u} \in \mathbb{C}[[t^{\frac{1}{\kappa_1}}, z^{\frac{1}{\kappa_2}}]]$ is $(1/s_1, 1/s_2)$-summable in the direction O_{d_1,d_2} if and only if $\mathcal{B}_{(s_1,s_2)}\widehat{u} \in \mathcal{O}^{1/s_1,1/s_2}_{1/\kappa_1,1/\kappa_2}(\widehat{S}_{d_1} \times \widehat{S}_{d_2})$.*

The connection between the Borel type transforms $\mathcal{B}_{\Gamma_{s_1},t}\mathcal{B}_{\Gamma_{s_2},z}$ and $\mathcal{B}_{(s_1,s_2)}$ is given in the next lemma.

Lemma 1. *Let $s_1, s_2 > 0$ and $\widehat{u} \in \mathbb{C}[[t,z]]$. Then the formal power series $\widehat{v}(t,z) := \mathcal{B}_{\Gamma_{s_1},t}\mathcal{B}_{\Gamma_{s_2},z}\widehat{u}(t,z)$ and $\widehat{w}(t,z) := \mathcal{B}_{(s_1,s_2)}\widehat{u}(t,z)$ are connected by the formula*

$$\widehat{w}(t,z) = (1 + s_1 t\partial_t + s_2 z\partial_z)\int_0^1 \widehat{v}(t\varepsilon^{s_1}, z(1-\varepsilon)^{s_2})\, d\varepsilon.$$

Proof. Let $\widehat{u}(t,z) = \sum_{k,n=0}^{\infty} u_{kn}t^k z^n$. Then

$$\widehat{v}(t,z) = \sum_{k,n=0}^{\infty} \frac{u_{kn}t^k z^n}{\Gamma(1+ks_1)\Gamma(1+ns_2)} \quad \text{and} \quad \widehat{w}(t,z) = \sum_{k,n=0}^{\infty} \frac{u_{kn}t^k z^n}{\Gamma(1+ks_1+ns_2)}.$$

Using properties of the beta function

$$\int_0^1 \varepsilon^{ks_1}(1-\varepsilon)^{ns_2}\, d\varepsilon = B(1+ks_1, 1+ns_2) = \frac{\Gamma(1+ks_1)\Gamma(1+ns_2)}{\Gamma(2+ks_1+ns_2)}$$

we conclude that

$$\begin{aligned}\int_0^1 \widehat{v}(t\varepsilon^{s_1}, z(1-\varepsilon)^{s_2})\, d\varepsilon &= \sum_{k,n=0}^{\infty} \frac{u_{kn}t^k z^n}{\Gamma(1+ks_1)\Gamma(1+ns_2)} \int_0^1 \varepsilon^{ks_1}(1-\varepsilon)^{ns_2}\, d\varepsilon \\ &= \sum_{k,n=0}^{\infty} \frac{u_{kn}t^k z^n}{\Gamma(2+ks_1+ns_2)}.\end{aligned}$$

Hence

$$\widehat{w}(t,z) = (1 + s_1 t\partial_t + s_2 z\partial_z)\int_0^1 \widehat{v}(t\varepsilon^{s_1}, z(1-\varepsilon)^{s_2})\, d\varepsilon. \quad \square$$

Remark 7. In Theorem 3 we will show that if $\widehat{u} \in \mathbb{C}[[t, z^{\frac{1}{\kappa}}]]$ is a formal solution of (13) then

$$\mathcal{B}_{\Gamma_{s_1},t}\mathcal{B}_{\Gamma_{s_2},z^{1/\kappa}}\widehat{u} \in \mathcal{O}^{1/s_1,1/s_2}_{1,1/\kappa}(\widehat{S}_{d_1} \times \widehat{S}_{d_2}) \Leftrightarrow \mathcal{B}_{(s_1,s_2)}\widehat{u} \in \mathcal{O}^{1/s_1,1/s_2}_{1,1/\kappa}(\widehat{S}_{d_1} \times \widehat{S}_{d_2}).$$

In other words, for such $\widehat{u}$ we have the equivalence between $(1/s_1, 1/s_2)$-summability in the direction (d_1, d_2) (introduced by Sanz) and $(1/s_1, 1/s_2)$-summability in the direction O_{d_1,d_2} (introduced by Balser). In our opinion it should be possible to extend the general theory of moment summability (see Balser [2, Section 6.5]) to several variables and to show that the above equivalence holds for every formal power series $\widehat{u} \in \mathbb{C}[[t,z]]$.

5. Moment operators

In this section we recall the notion of moment differential operators constructed recently by Balser and Yoshino [7]. We also extend the concept of moment pseudodifferential operators introduced in our previous paper [14].

Definition 12. Let m be a moment function. Then the linear operator $\partial_{m,x}\colon \mathbb{E}[[x]] \to \mathbb{E}[[x]]$ defined by

$$\partial_{m,x}\left(\sum_{j=0}^{\infty}\frac{u_j}{m(j)}x^j\right) := \sum_{j=0}^{\infty}\frac{u_{j+1}}{m(j)}x^j$$

is called the *m-moment differential operator* $\partial_{m,x}$.

More generally, if $\kappa \in \mathbb{N}$ then the linear operator $\partial_{m,x^{1/\kappa}}\colon \mathbb{E}[[x^{\frac{1}{\kappa}}]] \to \mathbb{E}[[x^{\frac{1}{\kappa}}]]$ defined by

$$\partial_{m,x^{1/\kappa}}\left(\sum_{j=0}^{\infty}\frac{u_j}{m(j/\kappa)}x^{j/\kappa}\right) := \sum_{j=0}^{\infty}\frac{u_{j+1}}{m(j/\kappa)}x^{j/\kappa}$$

is called the *m-moment* $1/\kappa$*-fractional differential operator* $\partial_{m,x^{1/\kappa}}$.

Example 3. Below we present some examples of moment differential operators.

- For $m(u) = \Gamma_1(u)$, the operator $\partial_{m,x}$ coincides with the usual differentiation ∂_x.
- For $m(u) = \Gamma_s(u)$ $(s > 0)$, the operator $\partial_{m,x}$ satisfies
  $$(\partial_{m,x}\widehat{u})(x^s) = \partial_x^s(\widehat{u}(x^s)),$$
  where ∂_x^s is the Caputo fractional derivative of order s defined by
  $$\partial_x^s\left(\sum_{j=0}^{\infty}\frac{u_j}{\Gamma_s(j)}x^{sj}\right) := \sum_{j=0}^{\infty}\frac{u_{j+1}}{\Gamma_s(j)}x^{sj}.$$
- For $m(u) \equiv 1$, the corresponding operator $\partial_{m,x}$ satisfies
  $$\partial_{m,x}\widehat{u}(x) = \frac{\widehat{u}(x) - u_0}{x} \quad \text{for every } \widehat{u}(x) = \sum_{j=0}^{\infty}u_jx^j \in \mathbb{E}[[x]].$$
- For $m(u) = \Gamma_{-1}(u)$, the operator $\partial_{m,x}$ satisfies
  $$\partial_{m,x}\widehat{u}(x) = \frac{1}{x}\int_0^x \frac{\widehat{u}(y) - u_0}{y}\,dy \quad \text{for every } \widehat{u}(x) = \sum_{j=0}^{\infty}u_jx^j \in \mathbb{E}[[x]].$$
- For $m(u) = \Gamma_{-s}(u)$ $(s > 0)$, the operator $\partial_{m,x}$ satisfies
  $$(\partial_{m,x}\widehat{u})(x^s) = \frac{1}{x^s}\partial_x^{-s}\frac{\widehat{u}(x^s) - u_0}{x^s} \quad \text{for every } \widehat{u}(x) = \sum_{j=0}^{\infty}u_jx^j \in \mathbb{E}[[x]],$$
  where ∂_x^{-s} is the right-inversion operator to ∂_x^s and is defined by
  $$\partial_x^{-s}\left(\sum_{j=0}^{\infty}\frac{u_j}{\Gamma_s(j)}x^{sj}\right) := \sum_{j=1}^{\infty}\frac{u_{j-1}}{\Gamma_s(j)}x^{sj}.$$

The moment differential operator $\partial_{m,z}$ is well-defined for every $\varphi \in \mathcal{O}(D)$. In addition, we have the following integral representation of $\partial_{m,z}^n\varphi$.

Proposition 6 *(See [14, Proposition 3]). Let $\varphi \in \mathcal{O}(D_r)$ and m be a moment function of order $1/k > 0$. Then for every $|z| < \varepsilon < r$ and $n \in \mathbb{N}$ we have*

$$\partial_{m,z}^n\varphi(z) = \frac{1}{2\pi i}\oint_{|w|=\varepsilon}\varphi(w)\int_0^{\infty(\theta)}\zeta^nE_m(z\zeta)\frac{e_m(w\zeta)}{w\zeta}\,d\zeta\,dw,$$

where $\theta \in (-\arg w - \frac{\pi}{2k}, -\arg w + \frac{\pi}{2k})$.

Using the above formula, we have defined in [14, Definition 8] a moment pseudodifferential operator $\lambda(\partial_{m,z})\colon \mathcal{O}(D) \to \mathcal{O}(D)$ as an operator satisfying

$$\lambda(\partial_{m,z})E_m(\zeta z) := \lambda(\zeta)E_m(\zeta z) \quad \text{for } |\zeta| \geq r_0.$$

Namely, if $\lambda(\zeta)$ is an analytic function for $|\zeta| \geq r_0$ then $\lambda(\partial_{m,z})$ is defined by

$$\lambda(\partial_{m,z})\varphi(z) := \frac{1}{2\pi i}\oint_{|w|=\varepsilon}\varphi(w)\int_{r_0e^{i\theta}}^{\infty(\theta)}\lambda(\zeta)E_m(\zeta z)\frac{e_m(\zeta w)}{\zeta w}\,d\zeta\,dw$$

for every $\varphi \in \mathcal{O}(D_r)$ and $|z| < \varepsilon < r$, where $\theta \in (-\arg w - \frac{\pi}{2k}, -\arg w + \frac{\pi}{2k})$.

We extend this definition to the case where $\lambda(\zeta)$ is an analytic function of the variable $\xi = \zeta^{1/\kappa}$ for $|\zeta| \geq r_0$ (for some $\kappa \in \mathbb{N}$ and $r_0 > 0$). Since $(\partial_{m,z}\varphi)(z^\kappa) = \partial^\kappa_{\widetilde{m},z}(\varphi(z^\kappa))$ for every $\varphi \in \mathcal{O}(D)$, where $\widetilde{m}(u) := m(u/\kappa)$ (see [14, Lemma 3]), the operator $\lambda(\partial_{m,z})$ should satisfy the formula

$$(\lambda(\partial_{m,z})\varphi)(z^\kappa) = \lambda(\partial^\kappa_{\widetilde{m},z})(\varphi(z^\kappa)) \quad \text{for every } \varphi \in \mathcal{O}_{1/\kappa}(D). \tag{4}$$

For this reason we have

Definition 13. Let m be a moment function of order $1/k > 0$ and $\lambda(\zeta)$ be an analytic function of the variable $\xi = \zeta^{1/\kappa}$ for $|\zeta| \geq r_0$ (for some $\kappa \in \mathbb{N}$ and $r_0 > 0$) of polynomial growth at infinity. A *moment pseudodifferential operator* $\lambda(\partial_{m,z}) \colon \mathcal{O}_{1/\kappa}(D) \to \mathcal{O}_{1/\kappa}(D)$ (or, more generally, $\lambda(\partial_{m,z}) \colon \mathbb{E}[[z^{\frac{1}{\kappa}}]]_0 \to \mathbb{E}[[z^{\frac{1}{\kappa}}]]_0$) is defined by

$$\lambda(\partial_{m,z})\varphi(z) := \frac{1}{2\kappa\pi i}\oint^\kappa_{|w|=\varepsilon} \varphi(w) \int_{r_0e^{i\theta}}^{\infty(\theta)} \lambda(\zeta) E_{\widetilde{m}}(\zeta^{1/\kappa} z^{1/\kappa}) \frac{e_m(\zeta w)}{\zeta w}\, d\zeta\, dw \tag{5}$$

for every $\varphi \in \mathcal{O}_{1/\kappa}(D_r)$ and $|z| < \varepsilon < r$, where $\widetilde{m}(u) := m(u/\kappa)$, $E_{\widetilde{m}}(\zeta^{1/\kappa} z^{1/\kappa}) = \sum_{n=0}^\infty \frac{\zeta^{n/\kappa} z^{n/\kappa}}{\widetilde{m}(n)}$, $\theta \in (-\arg w - \frac{\pi}{2k}, -\arg w + \frac{\pi}{2k})$ and $\oint^\kappa_{|w|=\varepsilon}$ means that we integrate κ times along the positively oriented circle of radius ε. Here the integration in the inner integral is taken over a ray $\{re^{i\theta} : r \geq r_0\}$.

Observe that

$$\begin{aligned}
(\lambda(\partial_{m,z})\varphi)(z^\kappa) &= \frac{1}{2\kappa\pi i}\oint^\kappa_{|w|=\varepsilon} \varphi(w) \int_{r_0e^{i\theta}}^{\infty(\theta)} \lambda(\zeta) E_{\widetilde{m}}(\zeta^{1/\kappa} z) \frac{e_m(\zeta w)}{\zeta w}\, d\zeta\, dw \\
&= \frac{1}{2\pi i}\oint_{|w^\kappa|=\varepsilon} \varphi(w^\kappa) \int_{r_0^{1/\kappa}e^{i\theta/\kappa}}^{\infty(\theta/\kappa)} \lambda(\zeta^k) E_{\widetilde{m}}(\zeta z) \frac{e_{\widetilde{m}}(\zeta w)}{\zeta w}\, d\zeta\, dw \\
&= \lambda(\partial^\kappa_{\widetilde{m},z})(\varphi(z^\kappa)),
\end{aligned}$$

so (4) holds for the operators $\lambda(\partial_{m,z})$ defined by (5).

Immediately by the definition, we obtain the following connection between the moment Borel transform and the moment differentiation.

Proposition 7. *Let m and m' be moment functions of positive orders. Then the operators $\mathcal{B}_{m',x}, \partial_{m,x} \colon \mathbb{E}[[x]] \to \mathbb{E}[[x]]$ satisfy the following commutation formulae for every $\widehat{u} \in \mathbb{E}[[x]]$ and for $\overline{m} = mm'$:*

(i) $\mathcal{B}_{m',x}\partial_{m,x}\widehat{u} = \partial_{\overline{m},x}\mathcal{B}_{m',x}\widehat{u}$,
(ii) $\mathcal{B}_{m',x}P(\partial_{m,x})\widehat{u} = P(\partial_{\overline{m},x})\mathcal{B}_{m',x}\widehat{u}$ *for any polynomial P with constant coefficients.*

The same commutation formula holds if we replace $P(\partial_{m,x})$ by $\lambda(\partial_{m,x})$. Namely, we have

Proposition 8. *Let m and m' be moment functions of positive orders and $\lambda(\zeta)$ be an analytic function of the variable $\xi = \zeta^{1/\kappa}$ for $|\zeta| \geq r_0$ (for some $\kappa \in \mathbb{N}$ and $r_0 > 0$) of polynomial growth at infinity. Then the operators $\mathcal{B}_{m',x^{1/\kappa}}, \lambda(\partial_{m,x}) \colon \mathbb{E}[[x^{\frac{1}{\kappa}}]]_0 \to \mathbb{E}[[x^{\frac{1}{\kappa}}]]_0$ satisfy the commutation formula*

$$\mathcal{B}_{m',x^{1/\kappa}}\lambda(\partial_{m,x})\widehat{u} = \lambda(\partial_{\overline{m},x})\mathcal{B}_{m',x^{1/\kappa}}\widehat{u}$$

for every $\widehat{u} \in \mathbb{E}[[x^{\frac{1}{\kappa}}]]_0$ and for $\overline{m} = mm'$.

Proof. Note that, by Proposition 1, $\overline{m} = mm'$ is also a moment function of positive order. Observe that by Definition 13 we have

$$\begin{aligned}
\mathcal{B}_{m',x^{1/\kappa}}\lambda(\partial_{m,x})\widehat{u}(x) &= \frac{1}{2\kappa\pi i}\oint^\kappa_{|w|=\varepsilon} u(w) \int_{r_0e^{i\theta}}^{\infty(\theta)} \lambda(\zeta)\mathcal{B}_{m',x^{1/\kappa}} E_{\widetilde{m}}(\zeta^{1/\kappa} x^{1/\kappa}) \frac{e_m(\zeta w)}{\zeta w}\, d\zeta\, dw \\
&= \frac{1}{2\kappa\pi i}\oint^\kappa_{|w|=\varepsilon} u(w) \int_{r_0e^{i\theta}}^{\infty(\theta)} \lambda(\zeta) E_{\widetilde{\overline{m}}}(\zeta^{1/\kappa} x^{1/\kappa}) \frac{e_m(\zeta w)}{\zeta w}\, d\zeta\, dw \\
&= \lambda(\partial_{\overline{m},x})\frac{1}{2\kappa\pi i}\oint^\kappa_{|w|=\varepsilon} u(w) \int_{r_0e^{i\theta}}^{\infty(\theta)} \mathcal{B}_{m',x^{1/\kappa}} E_{\widetilde{m}}(\zeta^{1/\kappa} x^{1/\kappa}) \frac{e_m(\zeta w)}{\zeta w}\, d\zeta\, dw \\
&= \lambda(\partial_{\overline{m},x})\mathcal{B}_{m',x^{1/\kappa}}\widehat{u}(x),
\end{aligned}$$

where $\widetilde{m}(u) := m(u/\kappa)$ and $\widetilde{\overline{m}}(u) := \overline{m}(u/\kappa) = m(u/\kappa)m'(u/\kappa)$. $\square$

Using Proposition 8 we are able to extend Definition 13 to the formal power series and to the moment functions of real orders.

Definition 14. Let $s \in \mathbb{R}$, m be a moment function of order $\widetilde{s} \in \mathbb{R}$ and $\lambda(\zeta)$ be an analytic function of the variable $\xi = \zeta^{1/\kappa}$ for $|\zeta| \geq r_0$ of polynomial growth at infinity. A *moment pseudodifferential operator* $\lambda(\partial_{m,z})$ for the formal power series $\widehat{\varphi} \in \mathbb{E}[[z^{\frac{1}{\kappa}}]]_s$ is defined by

$$\lambda(\partial_{m,z})\widehat{\varphi}(z) := \mathcal{B}_{\Gamma_{-\bar{s}},z^{1/\kappa}}\lambda(\partial_{\overline{m},z})\mathcal{B}_{\Gamma_{\bar{s}},z^{1/\kappa}}\widehat{\varphi}(z),$$

where $\overline{m} = m\Gamma_{\bar{s}}$, $\bar{s} = \max\{s, 1-\widetilde{s}\}$ and the operator $\lambda(\partial_{\overline{m},z})$ is constructed in Definition 13.

Definition 15 (*[14, Definition 9]*). We define a *pole order* $q \in \mathbb{Q}$ and a *leading term* $\lambda \in \mathbb{C} \setminus \{0\}$ of $\lambda(\zeta)$ as the numbers satisfying the formula $\lim_{\zeta\to\infty} \lambda(\zeta)/\zeta^q = \lambda$. We write it also as $\lambda(\zeta) \sim \lambda\zeta^q$.

At the end of the section we improve the estimate given in [14, Lemma 1] as follows

Lemma 2. *Let $\widehat{\varphi} \in \mathbb{C}[[z^{\frac{1}{\kappa}}]]_s$, $s \leq 0$, m be a moment function of order $1/k > 0$ and $\lambda(\partial_{m,z})$ be a moment pseudodifferential operator with $\lambda(\zeta) \sim \lambda\zeta^q$ and $q \in \mathbb{Q}$. Then there exist $r > 0$ and $A, B < \infty$ such that*

$$\sup_{|z|<r} |\lambda^j(\partial_{m,z})\varphi(z)| \leq |\lambda|^j AB^j \Gamma_{\bar{q}(s+1/k)}(j) \quad \textit{for } j = 0, 1, \ldots,$$

where $\bar{q} := \max\{0, q\}$.

Proof. Repeating the proof of [14, Lemma 1], we may take $r > 0$ and $\varepsilon_r > 0$ such that

$$\sup_{|z|<r} |\lambda^j(\partial_{m,z})\varphi(z)| \leq |\lambda|^j A_1 B_1^j \frac{\Gamma_{\bar{q}/k}(j)}{\varepsilon^{j\bar{q}}} \frac{1}{2\kappa\pi\varepsilon} \oint_{|w|=\varepsilon}^{\kappa} |\varphi(w)|\, d|w|$$

for some $A_1, B_1 < \infty$ and for every $\varepsilon > \varepsilon_r$ such that $D_\varepsilon \Subset D$ and $\varphi \in \mathcal{O}_{1/\kappa}(D)$.

If $s = 0$ then the assertion is given by the estimation

$$\frac{1}{2\kappa\pi\varepsilon} \oint_{|w|=\varepsilon}^{\kappa} |\varphi(w)|\, d|w| \leq A_2.$$

If $s < 0$ then $\varphi \in \mathcal{O}_{1/\kappa}^{-1/s}(\mathbb{C})$. So we estimate

$$\frac{1}{2\kappa\pi\varepsilon} \oint_{|w|=\varepsilon}^{\kappa} |\varphi(w)|\, d|w| \leq A_2 e^{B_2\varepsilon^{-1/s}} \quad \text{for every } \varepsilon > \varepsilon_r.$$

Hence, putting $\varepsilon = (\frac{-sj\bar{q}}{B_2})^{-s}$ and applying the Stirling formula (see [2, Theorem 68]) we conclude that

$$\sup_{|z|<r} |\lambda^j(\partial_{m,z})\varphi(z)| \leq \frac{|\lambda|^j \widetilde{A}\widetilde{B}^j \Gamma_{\bar{q}/k}(j) e^{-sj\bar{q}}}{(-sj\bar{q})^{-sj\bar{q}}} \leq |\lambda|^j AB^j \Gamma_{\bar{q}(s+1/k)}(j). \quad \square$$

6. Formal solutions and Gevrey estimates

In this section we study the formal solutions of the initial value problem for a general linear moment partial differential equation with constant coefficients

$$\begin{cases} P(\partial_{m_1,t}, \partial_{m_2,z})\widehat{u} = 0 \\ \partial_{m_1,t}^j \widehat{u}(0,z) = \widehat{\varphi}_j(z) \in \mathbb{C}[[z]] \quad (j = 0, \ldots, n-1), \end{cases} \tag{6}$$

where m_1, m_2 are moment functions of orders $s_1, s_2 \in \mathbb{R}$ respectively, and

$$P(\lambda, \zeta) = P_0(\zeta)\lambda^n - \sum_{j=1}^{n} P_j(\zeta)\lambda^{n-j} \tag{7}$$

is a general polynomial of two variables, which is of order n with respect to λ.

First, we will show the following

Proposition 9. *Let m_1' and m_2' be moment functions, $\widehat{u} \in \mathbb{C}[[t,z]]$ and $\widehat{v} = \mathcal{B}_{m_1',t}\mathcal{B}_{m_2',z}\widehat{u}$. Then $\widehat{u}$ is a formal solution of (6) if and only if $\widehat{v}$ is a formal solution of*

$$\begin{cases} P(\partial_{\overline{m}_1,t}, \partial_{\overline{m}_2,z})\widehat{v} = 0 \\ \partial_{\overline{m}_1,t}^j \widehat{v}(0,z) = \widehat{\psi}_j(z) := \mathcal{B}_{m_2',z}\widehat{\varphi}_j(z) \in \mathbb{C}[[z]] \quad \textit{for } j = 0, \ldots, n-1, \end{cases} \tag{8}$$

where $\overline{m}_1 := m_1 m_1'$ and $\overline{m}_2 := m_2 m_2'$.

Proof. ($\Longrightarrow$) We assume that $\widehat{u}$ is a formal solution of (6). By Proposition 7 we have

$$\begin{aligned} P(\partial_{\overline{m}_1,t}, \partial_{\overline{m}_2,z})\widehat{v} &= P(\partial_{\overline{m}_1,t}, \partial_{\overline{m}_2,z})\mathcal{B}_{m_1',t}\mathcal{B}_{m_2',z}\widehat{u} \\ &= \mathcal{B}_{m_1',t}\mathcal{B}_{m_2',z}P(\partial_{m_1,t}, \partial_{m_2,z})\widehat{u} = 0 \end{aligned}$$

and

$$\begin{aligned} \partial^j_{\overline{m}_1,t}\widehat{v}(0,z) &= \partial^j_{\overline{m}_1,t}\mathcal{B}_{m_1',t}\mathcal{B}_{m_2',z}\widehat{u}(0,z) = \mathcal{B}_{m_1',t}\mathcal{B}_{m_2',z}\partial^j_{m_1,t}\widehat{u}(0,z) \\ &= \mathcal{B}_{m_2',z}\widehat{\varphi}_j(z) \end{aligned}$$

for $j = 0,\dots,n-1$. So $\widehat{v}$ is a formal solution of (8).

($\Longleftarrow$) Observe that $\widehat{u} = \mathcal{B}_{1/m_1',t}\mathcal{B}_{1/m_2',z}\widehat{v}$ and $\widehat{\varphi}_j = \mathcal{B}_{1/m_2',z}\widehat{\psi}_j$ for $j = 0,\dots,n-1$. Repeating the first part of the proof with $\widehat{u}$ replaced by $\widehat{v}$ and $\widehat{\varphi}_j$ replaced by $\widehat{\psi}_j$, we obtain the assertion. □

If $P_0(\zeta)$ defined by (7) is not a constant, then a formal solution of (6) is not uniquely determined. To avoid this inconvenience we choose some special solution which is already uniquely determined. To this end we factorise the moment differential operator $P(\partial_{m_1,t}, \partial_{m_2,z})$ as follows

$$\begin{aligned} P(\partial_{m_1,t}, \partial_{m_2,z}) &= P_0(\partial_{m_2,z})(\partial_{m_1,t} - \lambda_1(\partial_{m_2,z}))^{n_1}\cdots(\partial_{m_1,t} - \lambda_l(\partial_{m_2,z}))^{n_l} \\ &=: P_0(\partial_{m_2,z})\widetilde{P}(\partial_{m_1,t}, \partial_{m_2,z}), \end{aligned}$$

where $\lambda_1(\zeta),\dots,\lambda_l(\zeta)$ are the roots of the characteristic equation $P(\lambda,\zeta) = 0$ with multiplicity $n_1,\dots,n_l$ $(n_1+\dots+n_l = n)$ respectively.

Since $\lambda_\alpha(\zeta)$ are algebraic functions, we may assume that there exist $\kappa \in \mathbb{N}$ and $r_0 < \infty$ such that $\lambda_\alpha(\zeta)$ are holomorphic functions of the variable $\xi = \zeta^{1/\kappa}$ (for $|\zeta| \geq r_0$) and, moreover, there exist $\lambda_\alpha \in \mathbb{C}\setminus\{0\}$ and $q_\alpha = \mu_\alpha/\nu_\alpha$ (for some relatively prime numbers $\mu_\alpha \in \mathbb{Z}$ and $\nu_\alpha \in \mathbb{N}$) such that $\lambda_\alpha(\zeta) \sim \lambda_\alpha\zeta^{q_\alpha}$ for $\alpha = 1,\dots,l$. Hence the moment pseudodifferential operators $\lambda_\alpha(\partial_{m_2,z})$ are well-defined.

Under the above assumption, by a *normalised formal solution* $\widehat{u}$ of (6) we mean such a solution of (6), which is also a solution of the pseudodifferential equation $\widetilde{P}(\partial_{m_1,t}, \partial_{m_2,z})\widehat{u} = 0$ (see [14, Definition 10]).

Now we are ready to study the Gevrey order of formal solution $\widehat{u}$ of (6), which depends on the orders $s_1, s_2 \in \mathbb{R}$ of the moment functions m_1, m_2 respectively, on the Gevrey order $s \in \mathbb{R}$ of the initial data $\widehat{\varphi}$ and depends on the pole orders $q_\alpha \in \mathbb{Q}$ of the roots $\lambda_\alpha(\zeta)$ $(\alpha = 1,\dots,l)$. We generalise the results for the analytic Cauchy data given in [14, Theorems 1 and 2] as follows

Theorem 1. *Let $s \in \mathbb{R}$ and let $\widehat{u}$ be a normalised formal solution of (6) with $\widehat{\varphi}_j \in \mathbb{C}[[z]]_s$ $(j = 0,\dots,n-1)$ then $\widehat{u} = \sum_{\alpha=1}^{l}\sum_{\beta=1}^{n_\alpha}\widehat{u}_{\alpha\beta}$ with $\widehat{u}_{\alpha\beta}$ being a formal solution of a simple pseudodifferential equation*

$$\begin{cases} (\partial_{m_1,t} - \lambda_\alpha(\partial_{m_2,z}))^\beta\widehat{u}_{\alpha\beta} = 0 \\ \partial^j_{m_1,t}\widehat{u}_{\alpha\beta}(0,z) = 0 \quad (j = 0,\dots,\beta-2) \\ \partial^{\beta-1}_{m_1,t}\widehat{u}_{\alpha\beta}(0,z) = \lambda_\alpha^{\beta-1}(\partial_{m_2,z})\widehat{\varphi}_{\alpha\beta}(z), \end{cases} \tag{9}$$

where $\widehat{\varphi}_{\alpha\beta}(z) := \sum_{j=0}^{n-1} d_{\alpha\beta j}(\partial_{m_2,z})\widehat{\varphi}_j(z) \in \mathbb{C}[[z^{\frac{1}{\kappa}}]]_s$ and $d_{\alpha\beta j}(\zeta)$ are some holomorphic functions of the variable $\xi = \zeta^{1/\kappa}$ and of polynomial growth.

Moreover, if q_α is a pole order of $\lambda_\alpha(\zeta)$ and $\overline{q}_\alpha = \max\{0, q_\alpha\}$, then a formal solution $\widehat{u}_{\alpha\beta}$ is a Gevrey series of order $\overline{q}_\alpha(s_2+s) - s_1$ with respect to t. More precisely, $\widehat{u}_{\alpha\beta} \in \mathbb{C}[[t, z^{\frac{1}{\kappa}}]]_{\overline{q}_\alpha(s_2+s)-s_1,s}$ or, equivalently, $\widehat{u}_{\alpha\beta} \in G_{s,1/\kappa}[[t]]_{\overline{q}_\alpha(s_2+s)-s_1}$.

Proof. For fixed $\bar{s} > \max\{s, -s_2\}$ we define $\widehat{v} := \mathcal{B}_{\Gamma_{\bar{s}},z}\widehat{u}$. By Proposition 9, $\widehat{v}$ is a formal solution of

$$\begin{cases} P(\partial_{m_1,t}, \partial_{\overline{m}_2,z})\widehat{v} = 0 \\ \partial^j_{m_1,t}\widehat{v}(0,z) = \widehat{\psi}_j(z) = \mathcal{B}_{\Gamma_{\bar{s}},z}\widehat{\varphi}_j(z) \in \mathbb{C}[[z]]_{s-\bar{s}} \quad \text{for } j = 0,\dots,n-1, \end{cases}$$

where $\overline{m}_2 := m_2\Gamma_{\bar{s}}$. Since $\overline{m}_2$ is a moment function of order $\bar{s}+s_2 > 0$ and $\widehat{\psi}_j$ are the Gevrey series of order $s - \bar{s} < 0$ for $j = 0,\dots,n-1$, repeating the proof of [14, Theorem 1] we conclude that $\widehat{v} = \sum_{\alpha=1}^{l}\sum_{\beta=1}^{n_\alpha}\widehat{v}_{\alpha\beta}$ with $\widehat{v}_{\alpha\beta}$ being a formal solution of

$$\begin{cases} (\partial_{m_1,t} - \lambda_\alpha(\partial_{\overline{m}_2,z}))^\beta\widehat{v}_{\alpha\beta} = 0 \\ \partial^j_{m_1,t}\widehat{v}_{\alpha\beta}(0,z) = 0 \quad (j = 0,\dots,\beta-2) \\ \partial^{\beta-1}_{m_1,t}\widehat{v}_{\alpha\beta}(0,z) = \lambda_\alpha^{\beta-1}(\partial_{\overline{m}_2,z})\widehat{\psi}_{\alpha\beta}(z), \end{cases}$$

where $\widehat{\psi}_{\alpha\beta}(z) := \sum_{j=0}^{n-1} d_{\alpha\beta j}(\partial_{\overline{m}_2,z})\widehat{\psi}_j(z) \in \mathbb{C}[[z^{\frac{1}{\kappa}}]]_{s-\bar{s}}$ and $d_{\alpha\beta j}(\zeta)$ are some holomorphic functions of the variable $\xi = \zeta^{1/\kappa}$ and of polynomial growth. Hence, by Definition 14, $\widehat{u} = \sum_{\alpha=1}^{l}\sum_{\beta=1}^{n_\alpha}\widehat{u}_{\alpha\beta}$, where $\widehat{u}_{\alpha\beta} = \mathcal{B}_{\Gamma_{-\bar{s}},z^{1/\kappa}}\widehat{v}_{\alpha\beta}$ satisfies (9) with

$$\widehat{\varphi}_{\alpha\beta} = \mathcal{B}_{\Gamma_{-\bar{s}},z^{1/\kappa}}\widehat{\psi}_{\alpha\beta} = \mathcal{B}_{\Gamma_{-\bar{s}},z^{1/\kappa}}\sum_{j=0}^{n-1} d_{\alpha\beta j}(\partial_{\overline{m}_2,z})\widehat{\psi}_j(z) = \sum_{j=0}^{n-1} d_{\alpha\beta j}(\partial_{m_2,z})\widehat{\varphi}_j(z)$$

for $\beta = 1,\dots,n_\alpha$ and $\alpha = 1,\dots,l$.

To find the Gevrey order of $\widehat{v}_{\alpha\beta} = \sum_{j=0}^{\infty} v_{\alpha\beta j}(z)t^j$ with respect to t, observe that by [14, Lemma 2]

$$\widehat{v}_{\alpha,\beta}(t,z) = m_1(0)\sum_{j=\beta-1}^{\infty}\binom{j}{\beta-1}\frac{\lambda^j_{\alpha\beta}(\partial_{\overline{m}_2,z})\psi_{\alpha\beta}(z)}{m_1(j)}t^j.$$

Hence, by Lemma 2, there exist $r > 0$ and $A, B < \infty$ such that

$$\sup_{|z|<r}|v_{\alpha\beta j}(z)| = m_1(0)\binom{j}{\beta-1}\frac{\sup_{|z|<r}|\lambda^j_{\alpha\beta}(\partial_{\overline{m}_2,z})\psi_{\alpha\beta}(z)|}{m_1(j)} \le \widetilde{A}\widetilde{B}^j\frac{\Gamma_{\overline{q}_\alpha(s-\bar{s}+\bar{s}+s_2)}(j)}{\Gamma_{s_1}(j)} \le AB^j\Gamma_{\overline{q}_\alpha(s+s_2)-s_1}(j)$$

for every $j \ge \beta - 1$. It means that $\widehat{v}_{\alpha\beta} \in G_{s-\bar{s},1/\kappa}[[t]]_{\overline{q}_\alpha(s_2+s)-s_1}$. Finally, by Proposition 4 we conclude that $\widehat{u}_{\alpha\beta} = \mathcal{B}_{\Gamma_{-\bar{s}},z^{1/\kappa}}\widehat{v}_{\alpha\beta} \in G_{s,1/\kappa}[[t]]_{\overline{q}_\alpha(s_2+s)-s_1}$ or, equivalently, $\widehat{u}_{\alpha\beta} \in \mathbb{C}[[t, z^{\frac{1}{\kappa}}]]_{\overline{q}_\alpha(s_2+s)-s_1,s}$. □

7. Analytic solutions

In this section we study the analytic continuation properties of the sum of convergent formal power series solutions of

$$\begin{cases}(\partial_{m_1,t} - \lambda(\partial_{m_2,z}))^\beta v = 0\\ \partial^j_{m_1,t}v(0,z) = 0 \quad (j = 0,\dots,\beta-2)\\ \partial^{\beta-1}_{m_1,t}v(0,z) = \lambda^{\beta-1}(\partial_{m_2,z})\varphi(z) \in \mathcal{O}_{1/\kappa}(D),\end{cases} \tag{10}$$

where $\lambda(\zeta)$ is a root of the characteristic equation of (6). It means that $\lambda(\zeta)$ is an analytic function of the variable $\xi = \zeta^{1/\kappa}$ for $|\zeta| \ge r_0$ and $\lambda(\zeta) \sim \lambda\zeta^q$. During this section we assume that m_1 and m_2 are moment functions of orders $1/k_1, 1/k_2 > 0$, respectively.

Repeating the proof of [14, Lemma 4] we get the following representation of solution v of (10)

Lemma 3. *Let v be a solution of (10) and $1/k_1 \ge q/k_2$. Then v belongs to the space $\mathcal{O}_{1,1/\kappa}(D^2)$ and is given by*

$$v(t,z) = \frac{t^{\beta-1}}{(\beta-1)!}\partial_t^{\beta-1}\frac{m_1(0)}{2\kappa\pi i}\oint^{\kappa}_{|w|=\varepsilon}\varphi(w)\int_{r_0e^{i\theta}}^{\infty(\theta)}E_{m_1}(t\lambda(\zeta))E_{\widetilde{m}_2}(\zeta^{1/\kappa}z^{1/\kappa})\frac{e_{m_2}(\zeta w)}{\zeta w}\,d\zeta\,dw,$$

where $\theta \in (-\arg w - \frac{\pi}{2k_2}, -\arg w + \frac{\pi}{2k_2})$ and $\widetilde{m}_2(u) = m_2(u/\kappa)$.

We generalise [14, Lemma 5] as follows

Lemma 4. *Let $\lambda(\zeta) \sim \lambda\zeta^q$ be a root of the characteristic equation of (6) for $q = \mu/\nu$ with relatively prime numbers $\mu, \nu \in \mathbb{N}$, where $\lambda(\zeta)$ is an analytic function of the variable $\xi = \zeta^{1/\kappa}$ for $|\zeta| \ge r_0$ (for some $r_0 > 0$). Moreover, let $1/k_1 = q/k_2$, $K > 0$ and $d \in \mathbb{R}$. We assume that v is a solution of*

$$\begin{cases}(\partial_{m_1,t} - \lambda(\partial_{m_2,z}))^\beta v = 0\\ \partial^j_{m_1,t}v(0,z) = \varphi_j(z) \in \mathcal{O}_{1/\kappa}(D) \quad (j = 0,\dots,\beta-1).\end{cases}$$

If $\varphi_j \in \mathcal{O}^{qK}_{1/\kappa}(\widehat{S}_{(d+\arg\lambda+2k\pi)/q})$ for $k = 0,\dots,q\kappa-1$ and $j = 0,\dots,\beta-1$, then $v \in \mathcal{O}^{K,qK}_{1,1/\kappa}(\widehat{S}_d \times \widehat{S}_{(d+\arg\lambda+2k\pi)/q})$ for $k = 0,\dots,q\kappa-1$. Moreover, if additionally $\varphi_j \in \mathcal{O}(D)$ for $j = 0,\dots,\beta-1$, then $v \in \mathcal{O}^{K,qK}_{1,1/\kappa}(\widehat{S}_{d+2n\pi/\nu} \times \widehat{S}_{(d+\arg\lambda+2k\pi)/q})$ for $k = 0,\dots,q\kappa-1$ and $n = 0,\dots,\nu-1$.

Proof. First, we consider the case $k_1, k_2 > 1/2$. By the principle of superposition of solutions of linear equations, we may assume that v satisfies (10) with $\varphi \in \mathcal{O}^{qK}_{1/\kappa}(\widehat{S}_{(d+\arg\lambda+2k\pi)/q}(\widetilde{\delta};\widetilde{r}))$ for $k = 0,\dots,q\kappa-1$ and for some $\widetilde{\delta}, \widetilde{r} > 0$. Hence, by Lemma 3, the function $v \in \mathcal{O}_{1,1/\kappa}(D^2)$ has the integral representation

$$v(t,z) = \frac{t^{\beta-1}}{(\beta-1)!}\partial_t^{\beta-1}\frac{m_1(0)}{2\kappa\pi i}\oint^{\kappa}_{|w|=\varepsilon}\varphi(w)k(t,z,w)\,dw, \tag{11}$$

where $\varepsilon < \widetilde{r}$ and

$$k(t,z,w) := \int_{r_0 e^{i\theta}}^{\infty(\theta)} E_{m_1}(t\lambda(\zeta)) E_{\widetilde{m}_2}(\zeta^{1/\kappa} z^{1/\kappa}) \frac{e_{m_2}(\zeta w)}{\zeta w}\, d\zeta$$

with $\theta \in (-\arg w - \frac{\pi}{2k_2}, -\arg w + \frac{\pi}{2k_2})$ and $\widetilde{m}_2(u) = m_2(u/\kappa)$. Now we consider the function

$$(t,z) \mapsto k(t,z,w) \quad \text{for every fixed } w \in \mathbb{C}\setminus\{0\}. \tag{12}$$

Observe that by Definition 2 there exist constants A_i and b_i ($i = 1,2,3$) such that $|E_{m_1}(t\lambda(\zeta))| \le A_1 e^{b_1|t|^{k_1}|\zeta|^{k_1 q}}$, $|E_{\widetilde{m}_2}(\zeta^{1/\kappa} z^{1/\kappa})| \le A_2 e^{b_2|\zeta|^{k_2}|z|^{k_2}}$ and $|e_{m_2}(\zeta w)| \le A_3 e^{-b_3|\zeta|^{k_2}|w|^{k_2}}$. Hence, there exist $a, b > 0$ such that for every fixed $w \in \mathbb{C}\setminus\{0\}$ and for every $(t,z) \in \mathbb{C}^2$ satisfying $|t| < a|w|^q$ and $|z| < b|w|$, we have

$$|k(t,z,w)| \le \int_{r_0}^{\infty} \widetilde{A} e^{s^{k_2}(b_1|t|^{k_1} + b_2|z|^{k_2} - b_3|w|^{k_2})}\, ds \le \int_{r_0}^{\infty} \widetilde{A} e^{-\widetilde{b} s^{k_2}|w|^{k_2}}\, ds < \infty$$

with some positive constants $\widetilde{A}, \widetilde{b}$. Hence the function (12) belongs to the space $\mathcal{O}_{1,1/\kappa}(\{(t,z) \in \mathbb{C}^2 : |t| < a|w|^q,\ |z| < b|w|\})$ and the right-hand side of (11) is a well-defined holomorphic function of the variables t and $\zeta = z^{1/\kappa}$ in a complex neighbourhood of the origin.

To show that $v \in \mathcal{O}_{1,1/\kappa}(\widehat{S}_d \times \widehat{S}_{(d+\arg\lambda+2k\pi)/q})$ for $k = 1, \ldots, q\kappa - 1$, we deform the κ-fold circle $|w| = \varepsilon$ in the integral representation (11) of v as in the proof of [14, Lemma 5]. Namely, we split these circles into $2q\kappa$ arcs γ_{2k} and γ_{2k+1} ($k = 0, \ldots, q\kappa - 1$), where γ_{2k} extends between points of argument $(d + \arg\lambda + 2k\pi)/q \pm \widetilde{\delta}/3$ and γ_{2k+1} extends between $(d + \arg\lambda + 2k\pi)/q + \widetilde{\delta}/3$ and $(d + \arg\lambda + 2(k+1)\pi)/q - \widetilde{\delta}/3 \mod 2q\kappa\pi$. Finally, since $\varphi \in \mathcal{O}_{1/\kappa}(S_{(d+\arg\lambda+2k\pi)/q}(\widetilde{\delta}))$, we may deform γ_{2k} into a path γ_{2k}^R along the ray $\arg w = (d + \arg\lambda + 2k\pi)/q - \widetilde{\delta}/3$ to a point with modulus R (which can be chosen arbitrarily large), then along the circle $|w| = R$ to the ray $\arg w = (d + \arg\lambda + 2k\pi)/q + \widetilde{\delta}/3$ and back along this ray to the original circle. So, we have

$$v(t,z) = \frac{t^{\beta-1}}{(\beta-1)!}\partial_t^{\beta-1} v_1(t,z) + \frac{t^{\beta-1}}{(\beta-1)!}\partial_t^{\beta-1} v_2(t,z),$$

where

$$v_1(t,z) := \sum_{k=0}^{q\kappa-1} \frac{m_1(0)}{2\kappa\pi i}\int_{\gamma_{2k+1}} \varphi(w) k(t,z,w)\, dw$$

and

$$v_2(t,z) := \sum_{k=0}^{q\kappa-1} \frac{m_1(0)}{2\kappa\pi i}\int_{\gamma_{2k}^R} \varphi(w) k(t,z,w)\, dw.$$

To study the analytic continuation of v_1, observe that for $\arg t = d$, $\arg z = (d + \arg\lambda + 2k\pi)/q$ ($k = 0, \ldots, q\kappa - 1$), $\arg w \neq (d + \arg\lambda + 2k\pi)/q$ ($k \in \mathbb{Z}$) and for $q = k_2/k_1$, we may choose a direction θ in (12), which satisfies the following conditions

- $$\arg t + 2k\pi + \arg\lambda + q\theta \in \left(\frac{\pi}{2k_1}, 2\pi - \frac{\pi}{2k_1}\right) \quad \text{for some } k \in \mathbb{Z}$$
  (in this case, by Definition 2, we have $|E_{m_1}(t\lambda(\zeta))| \le C|t\lambda(\zeta)|^{-1}$ as $\zeta \to \infty$, $\arg\zeta = \theta$),
- $$\arg z/\kappa + 2l\pi + \theta/\kappa \in \left(\frac{\pi}{2k_2\kappa}, 2\pi - \frac{\pi}{2k_2\kappa}\right) \quad \text{for some } l \in \mathbb{Z}$$
  (in this case, by Definition 2, we have $|E_{\widetilde{m}_2}(\zeta^{1/\kappa} z^{1/\kappa})| \le C'|\zeta z|^{-1/\kappa}$ as $\zeta \to \infty$, $\arg\zeta = \theta$),
- $$\arg w + 2n\pi + \theta \in \left(-\frac{\pi}{2k_2}, \frac{\pi}{2k_2}\right) \quad \text{for some } n \in \mathbb{Z}$$
  (in this case, by Definition 2, there exists $\varepsilon > 0$ such that
  $$\left|\frac{e_{m_2}(\zeta w)}{\zeta w}\right| \le e^{-\varepsilon|\zeta|^{k_2}} \quad \text{as } \zeta \to \infty,\ \arg\zeta = \theta).$$

Hence there exist $\delta > 0$ and $r > 0$ such that the function $v_1 \in \mathcal{O}_{1,1/\kappa}(\widehat{S}_d(\delta; r) \times \widehat{S}_{(d+\arg\lambda+2k\pi)/q}(\delta; r))$ for $k = 0, \ldots, q\kappa-1$. Moreover, there exists $C < \infty$ such that $|k(t, z, w)| < C$ for every $(t, z) \in \widehat{S}_d(\delta; r) \times \widehat{S}_{(d+\arg\lambda+2k\pi)/q}(\delta; r)$ and for every $w \in \bigcup_{k=0}^{q\kappa-1} \gamma_{2k+1}$. Hence

$$|v_1(t, z)| \le \frac{q\kappa}{2\kappa\pi} \max_{k=0,\ldots,q\kappa-1} \int_{\gamma_{2k+1}} |\varphi(w)| C\, d|w| \le \widetilde{C} < \infty$$

and we conclude that v_1 is bounded as $t \to \infty$ and $z \to \infty$.

Now we are ready to study the analytic continuation of v_2, Since the function (12) belongs to the space $\mathcal{O}_{1,1/\kappa}(\{(t, z) \in \mathbb{C}^2 : |t| < a|w|^q, \ |z| < b|w|\})$, one can find $\delta, r > 0$ such that $v_2 \in \mathcal{O}_{1,1/\kappa}(\widehat{S}_d(\delta; r) \times \widehat{S}_{(d+\arg\lambda+2k\pi)/q}(\delta; r))$ for $k = 0, \ldots, q\kappa - 1$ as R tends to infinity. Estimating this integral we obtain

$$|v_2(t, z)| \le \frac{q\kappa}{2\kappa\pi} \max_{k=0,\ldots,q\kappa-1} \int_{\gamma_{2k}^R} |\varphi(w)| C\, d|w| \le ARe^{BR^{qK}} \le \widetilde{A}e^{\widetilde{B}_1|t|^K + \widetilde{B}_2|z|^{qK}},$$

since $|t| \sim |w|^q = R^q$ and $|z| \sim |w|$.

Hence also $v \in \mathcal{O}^{K,qK}_{1,1/\kappa}(\widehat{S}_d \times \widehat{S}_{(d+\arg\lambda+2k\pi)/q})$ for $k = 0, \ldots, q\kappa - 1$.

In the general case $k_1, k_2 > 0$, there exists $p \in \mathbb{N}$ such that $\widetilde{k}_1 := pk_1 > 1/2$ and $\widetilde{k}_2 := pk_2 > 1/2$. By [14, Lemma 3], the function $w(t, z) := v(t^p, z^p)$ is a solution of

$$\begin{cases} (\partial^p_{\widetilde{m}_1,t} - \lambda(\partial^p_{\widetilde{m}_2,z}))^\beta w = 0, \\ \partial^{np}_{\widetilde{m}_1,t} w(0, z) = \varphi_n(z^p) \in \mathcal{O}^{pqK}_{1/\kappa}(\widehat{S}_{(d+\arg\lambda+2k\pi)/(pq)}) \quad \text{for } n = 0, \ldots, \beta - 1 \\ \partial^j_{\widetilde{m}_1,t} w(0, z) = 0 \quad \text{for } j = 1, \ldots, \beta p - 1 \text{ and } p \not| j, \end{cases}$$

where $\widetilde{m}_1(u) := m_1(u/p)$ and $\widetilde{m}_2(u) := m_2(u/p)$ are moment functions of order $1/\widetilde{k}_1$ and $1/\widetilde{k}_2$ respectively.

By Theorem 1 we conclude that $w = w_0 + \cdots + w_{p-1}$ with w_j $(j = 0, \ldots, p - 1)$ satisfying

$$\begin{cases} (\partial_{\widetilde{m}_1,t} - e^{i2j\pi/p}\lambda^{1/p}(\partial^p_{\widetilde{m}_2,z}))^\beta w_j = 0, \\ \partial^n_{\widetilde{m}_1,t} w_j(0, z) = \widetilde{\varphi}_{jn}(z) \in \mathcal{O}^{pqK}_{1/\kappa}(\widehat{S}_{(d+\arg\lambda+2k\pi)/(pq)}) \quad \text{for } n = 0, \ldots, \beta - 1. \end{cases}$$

Applying the first part of the proof to the above equation we see that $w_j(t, z) \in \mathcal{O}^{pK,pqK}_{1,1/\kappa}(\widehat{S}_{(d+2j\pi)/p} \times \widehat{S}_{(d+\arg\lambda+2k\pi)/(pq)})$ for $j = 1, \ldots, p$. It means that $v(t, z) = w(t^{1/p}, z^{1/p}) \in \mathcal{O}^{K,qK}_{1,1/\kappa}(\widehat{S}_d \times \widehat{S}_{(d+\arg\lambda+2k\pi)/q})$ for $k = 0, \ldots, q\kappa - 1$.

To prove the last part of the lemma, observe that if $\varphi_j \in \mathcal{O}^{qK}_{1/\kappa}(\hat{S}_{(d+\arg\lambda+2k\pi)/q})$ and $\varphi_j \in \mathcal{O}(D)$ then also $\varphi_j \in \mathcal{O}^{qK}(\hat{S}_{(d+\arg\lambda+2k\pi)/q})$ and consequently $\varphi_j \in \mathcal{O}^{qK}(\hat{S}_{(d+2n\pi/\nu+\arg\lambda+2k\pi)/q})$ for $n = 0, \ldots, \nu-1$. Hence, replacing d by $d+2n\pi/\nu$ we conclude that $v \in \mathcal{O}^{K,qK}_{1,1/\kappa}(\hat{S}_{d+2n\pi/\nu} \times \widehat{S}_{(d+\arg\lambda+2k\pi)/q})$ for $n = 0, \ldots, \nu - 1$ and $k = 0, \ldots, q\kappa - 1$. □

Now we are ready to generalise [14, Theorem 3] as follows

Theorem 2. *Let $\lambda(\zeta) \sim \lambda\zeta^q$ be a root of the characteristic equation of (6) for $q = \mu/\nu$ with relatively prime numbers $\mu, \nu \in \mathbb{N}$, where $\lambda(\zeta)$ is an analytic function of the variable $\xi = \zeta^{1/\kappa}$ for $|\zeta| \ge r_0$ (for some $r_0 > 0$). Moreover, let us assume that v is a solution of (10), $1/k_1 = q/k_2$, $K > 0$ and $d \in \mathbb{R}$. Then the following conditions are equivalent:*

(a) $\varphi \in \mathcal{O}^{qK}_{1/\kappa}(\widehat{S}_{(d+\arg\lambda+2k\pi)/q})$ *for* $k = 0, \ldots, q\kappa - 1$,
(b) $v \in \mathcal{O}^{K,qK}_{1,1/\kappa}(\widehat{S}_d \times \widehat{S}_{(d+\arg\lambda+2k\pi)/q})$ *for* $k = 0, \ldots, q\kappa - 1$.
(c) $v \in \mathcal{O}^{K}_{1,1/\kappa}(\widehat{S}_d \times D)$,
(d) $\partial^j_{m_2,z^{1/\kappa}} v(t, 0) \in \mathcal{O}^K(\widehat{S}_d)$ *for* $j = 0, \ldots, q\kappa\beta - 1$.

If additionally we assume that $\varphi \in \mathcal{O}(D)$ then the above conditions are also equivalent to

(e) $v \in \mathcal{O}^{K,qK}_{1,1/\kappa}(\widehat{S}_{d+2n\pi/\nu} \times \widehat{S}_{(d+\arg\lambda+2k\pi)/q})$ *for* $n = 0, \ldots, \nu - 1$ *and* $k = 0, \ldots, q\kappa - 1$,
(f) $v \in \mathcal{O}^{K}_{1,1/\kappa}(\widehat{S}_{d+2n\pi/\nu} \times D)$ *for* $n = 0, \ldots, \nu - 1$.

Proof. The implication (a) ⇒ (b) is given immediately by Lemma 4. The implications (b) ⇒ (c) and (c) ⇒ (d) are trivial. To prove the implication (d) ⇒ (a), observe that by [14, Lemma 3] the function $w(t, z) := v(t^{q\kappa}, z^\kappa)$ satisfies

$$(\partial^{q\kappa}_{\widetilde{m}_1,t} - \lambda(\partial^\kappa_{\widetilde{m}_2,z}))^\beta w = 0,$$

where $\widetilde{m}_1(u) := m_1(u/(q\kappa))$ and $\widetilde{m}_2(u) := m_2(u/\kappa)$ are moment functions of orders $1/\widetilde{k}_1 := 1/(k_1q\kappa)$ and $1/\widetilde{k}_2 := 1/(k_2\kappa)$. It means that w is also a solution of the equation

$$(\partial_{\widetilde{m}_1,t} - \widetilde{\lambda}_0(\partial_{\widetilde{m}_2,z}))^\beta \cdots (\partial_{\widetilde{m}_1,t} - \widetilde{\lambda}_{q\kappa-1}(\partial_{\widetilde{m}_2,z}))^\beta w = 0,$$

where

$$\widetilde{\lambda}_j(\zeta) := e^{i2\pi j/(q\kappa)}\lambda^{1/(q\kappa)}(\zeta^\kappa) \quad \text{for } j = 0, \dots, q\kappa - 1.$$

Since $\widetilde{\lambda}_j(\zeta)$ is an analytic function for sufficiently large $|\zeta|$ with a pole order equal to 1 (more precisely $\widetilde{\lambda}_j(\zeta) \sim e^{i2\pi j/(q\kappa)}\lambda^{1/(q\kappa)}\zeta$) and $1/\widetilde{k}_1 = 1/\widetilde{k}_2$, by [14, Lemma 7] and by condition (d), the function w satisfies also

$$\begin{cases} (\partial_{\widetilde{m}_2,z} - \widetilde{\lambda}_0^{-1}(\partial_{\widetilde{m}_1,t}))^\beta \cdots (\partial_{\widetilde{m}_2,z} - \widetilde{\lambda}_{q\kappa-1}^{-1}(\partial_{\widetilde{m}_1,t}))^\beta w = 0, \\ \partial_{\widetilde{m}_2,z}^n w(t,0) = \widetilde{\psi}_n(t) \in \mathcal{O}^{q\kappa K}(\hat{S}_{(d+2\pi k)/(q\kappa)}) \end{cases}$$

for $n = 0, \dots, q\kappa\beta - 1$ and $k = 0, \dots, q\kappa - 1$. Hence, by Theorem 1, $w = w_0 + \cdots + w_{q\kappa-1}$ with w_j ($j = 0, \dots, q\kappa - 1$) satisfying

$$\begin{cases} (\partial_{\widetilde{m}_2,z} - \widetilde{\lambda}_j^{-1}(\partial_{\widetilde{m}_1,t}))^\beta w_j = 0, \\ \partial_{\widetilde{m}_2,z}^n w_j(t,0) = \widetilde{\psi}_{jn}(t) \in \mathcal{O}^{q\kappa K}(\hat{S}_{(d+2\pi k)/(q\kappa)}) \end{cases}$$

for $n = 0, \dots, \beta - 1$ and $k = 0, \dots, q\kappa - 1$. Since $\widetilde{\lambda}_j^{-1}(\tau) \sim e^{-i2\pi j/(q\kappa)}\lambda^{-1/(q\kappa)}\tau$, by Lemma 4 with replaced variables, we conclude that $w_j(t,z) \in \mathcal{O}^{q\kappa K}(D \times \hat{S}_{\theta_{jk}})$, where

$$\theta_{jk} := \frac{d + 2\pi k}{q\kappa} - \arg(e^{-i2\pi j/(q\kappa)}\lambda^{-1/(q\kappa)}) = \frac{d + \arg\lambda + 2\pi(k+j)}{q\kappa}$$

for $k = 0, \dots, q\kappa - 1$. In consequence, also $w(t,z) \in \mathcal{O}^{q\kappa K}(D \times \hat{S}_{(d+\arg\lambda+2\pi k)/(q\kappa)})$ and finally $v(t,z) = w(t^{1/(q\kappa)}, z^{1/\kappa}) \in \mathcal{O}^{qK}_{1/(q\kappa),1/\kappa}(D \times \hat{S}_{(d+\arg\lambda+2\pi k)/q})$. In particular $\varphi(z) \in \mathcal{O}^{qK}_{1/\kappa}(\hat{S}_{(d+\arg\lambda+2\pi k)/q})$ for $k = 0, \dots, q\kappa - 1$, which proves the implication (d) ⇒ (a).

If additionally $\varphi \in \mathcal{O}(D)$ then also $\varphi \in \mathcal{O}^{qK}(\hat{S}_{(d+2n\pi/\nu+\arg\lambda+2k\pi)/q})$ for $n = 0, \dots, \nu - 1$. Hence, replacing d by $d + 2n\pi/\nu$ we conclude by Lemma 4 that $v \in \mathcal{O}^{K,qK}_{1,1/\kappa}(\widehat{S}_{d+2n\pi/\nu} \times \widehat{S}_{(d+\arg\lambda+2k\pi)/q})$ for $n = 0, \dots, \nu - 1$ and $k = 0, \dots, q\kappa - 1$ and the implication (a) ⇒ (e) holds. The last implications (e) ⇒ (f) and (f) ⇒ (c) are obvious. □

By the above theorem we conclude

Corollary 1. *If $K' > 0$, $d' \in \mathbb{R}$, $\varphi \in \mathcal{O}^{K'}(\widehat{S}_{d'})$ and m is a moment function of order 0, then also $\mathcal{B}_{m,z}\varphi \in \mathcal{O}^{K'}(\widehat{S}_{d'})$.*

Proof. Let v be a solution of

$$(\partial_t - \partial_z)v = 0, \qquad v(0,z) = \varphi(z) \in \mathcal{O}^{K'}(\widehat{S}_{d'}).$$

Then $v(t,z) = \varphi(t+z) \in \mathcal{O}^{K'}(\widehat{S}_{d'} \times D)$. Since m is a moment function of order 0, we see that also $\mathcal{B}_{m,z}v \in \mathcal{O}^{K'}(\widehat{S}_{d'} \times D)$. On the other hand, by Proposition 9, $\mathcal{B}_{m,z}v$ is a solution of

$$(\partial_t - \partial_{\Gamma_1 m,z})\mathcal{B}_{m,z}v = 0, \qquad \mathcal{B}_{m,z}v(0,z) = \mathcal{B}_{m,z}\varphi(z) \in \mathcal{O}(D).$$

Hence, applying Theorem 2, we conclude that $\mathcal{B}_{m,z}\varphi \in \mathcal{O}^{K'}(\widehat{S}_{d'})$. □

8. Summable and multisummable solutions

In this section we characterise summable formal solutions $\widehat{u}$ of (9) in terms of the Cauchy data $\widehat{\varphi}$. Next, we also give a similar characterisation of multisummable normalised formal solutions of general equation (6).

Applying Theorem 2 we obtain the following impressive characterisation of summable solutions of simple pseudodifferential equations (9)

Theorem 3. *Let $\lambda(\zeta) \sim \lambda\zeta^q$ be a root of the characteristic equation of (6) for $q = \mu/\nu$ with relatively prime numbers $\mu, \nu \in \mathbb{N}$, where $\lambda(\zeta)$ is an analytic function of the variable $\xi = \zeta^{1/\kappa}$ for $|\zeta| \geq r_0$ (for some $r_0 > 0$). We also assume that m_1, m_2 are moment functions of orders $s_1, s_2 \in \mathbb{R}$ respectively, $d, s \in \mathbb{R}$, $s > -s_2$, $q > \frac{s_1}{s_2+s}$, $K = \left(q(s_2+s) - s_1\right)^{-1}$ and $\widehat{u}$ is a formal solution of*

$$\begin{cases} (\partial_{m_1,t} - \lambda(\partial_{m_2,z}))^\beta \widehat{u} = 0 \\ \partial_{m_1,t}^j \widehat{u}(0,z) = 0 \quad (j = 0, \dots, \beta - 2) \\ \partial_{m_1,t}^{\beta-1}\widehat{u}(0,z) = \lambda^{\beta-1}(\partial_{m_2,z})\widehat{\varphi}(z) \in \mathbb{C}[[z^{\frac{1}{\kappa}}]]_s. \end{cases} \tag{13}$$

Then the following conditions are equivalent:

(a) $\mathcal{B}_{\Gamma_s,z^{1/\kappa}}\widehat{\varphi} \in \mathcal{O}^{qK}_{1/\kappa}(\widehat{S}_{(d+\arg\lambda+2k\pi)/q})$ *for $k = 0, \dots, q\kappa - 1$,*

(b) $\mathcal{B}_{\Gamma_{1/K},t}\mathcal{B}_{\Gamma_s,z^{1/\kappa}}\widehat{u} \in \mathcal{O}^{K}_{1,1/\kappa}(\widehat{S}_d \times D)$,

(c) $\mathcal{B}_{\Gamma_{1/K},t}\mathcal{B}_{\Gamma_s,z^{1/\kappa}}\widehat{u} \in \mathcal{O}^{K,qK}_{1,1/\kappa}(\widehat{S}_d \times \widehat{S}_{(d+\arg\lambda+2k\pi)/q})$ *for* $k = 0, \dots, q\kappa - 1$,
(d) $\mathcal{B}_{\Gamma_{s_1/q-s_2},z^{1/\kappa}}\widehat{\varphi}$ *is* qK*-summable in the directions* $(d + \arg\lambda + 2k\pi)/q$ *for* $k = 0, \dots, q\kappa - 1$,
(e) $\widehat{u}(t, z) \in G_{s,1/\kappa}[[t]]$ *is* K*-summable in direction* d.

Moreover, if additionally $s > 0$ *and* $qs_2 \geq s_1$ *then the above conditions* (a)–(e) *are also equivalent to*

(f) $\widehat{u}(t, z) \in \mathbb{C}[[t, z^{\frac{1}{\kappa}}]]$ *is* $(K, 1/s)$*-summable in the directions* $(d, (d + \arg\lambda + 2k\pi)/q)$ *for* $k = 0, \dots, q\kappa - 1$,
(g) $\widehat{u}(t, z) \in \mathbb{C}[[t, z^{\frac{1}{\kappa}}]]$ *is* $(K, 1/s)$*-summable in the directions* $O_{d,(d+\arg\lambda+2k\pi)/q}$ *for* $k = 0, \dots, q\kappa - 1$.

Remark 8. If we assume additionally that $\varphi \in \mathcal{O}(D)$ then we may replace the direction d by $d + 2n\pi/\nu$ $(n = 0, \dots, \nu - 1)$. Hence conditions (a)–(e) are also equivalent to

(h) $\mathcal{B}_{\Gamma_{1/K},t}\mathcal{B}_{\Gamma_s,z^{1/\kappa}}\widehat{u} \in \mathcal{O}^{K}_{1,1/\kappa}(\widehat{S}_{d+2n\pi/\nu} \times D)$ for $n = 0, \dots, \nu - 1$,
(i) $\mathcal{B}_{\Gamma_{1/K},t}\mathcal{B}_{\Gamma_s,z^{1/\kappa}}\widehat{u} \in \mathcal{O}^{K,qK}_{1,1/\kappa}(\widehat{S}_{d+2n\pi/\nu} \times \widehat{S}_{(d+\arg\lambda+2k\pi)/q})$ for $n = 0, \dots, \nu - 1$ and $k = 0, \dots, q\kappa - 1$,
(j) $\widehat{u}(t, z) \in G_{s,1/\kappa}[[t]]$ is K-summable in the directions $d + 2n\pi/\nu$ for $n = 0, \dots, \nu - 1$,

and conditions (f)–(g) are equivalent to

(k) $\widehat{u}(t, z) \in \mathbb{C}[[t, z^{\frac{1}{\kappa}}]]$ is $(K, 1/s)$-summable in the directions $(d + 2n\pi/\nu, (d + \arg\lambda + 2k\pi)/q)$ for $k = 0, \dots, q\kappa - 1$ and $n = 0, \dots, \nu - 1$.
(l) $\widehat{u}(t, z) \in \mathbb{C}[[t, z^{\frac{1}{\kappa}}]]$ is $(K, 1/s)$-summable in the directions $O_{d+2n\pi/\nu,(d+\arg\lambda+2k\pi)/q}$ for $k = 0, \dots, q\kappa - 1$ and $n = 0, \dots, \nu - 1$.

Proof of Theorem 3. First, observe that by Propositions 7 and 8 the function $v := \mathcal{B}_{\Gamma_{1/K},t}\mathcal{B}_{\Gamma_s,z^{1/\kappa}}\widehat{u}$ satisfies the equation

$$\begin{cases} (\partial_{\overline{m}_1,t} - \lambda(\partial_{\overline{m}_2,z}))^{\beta} v = 0 \\ \partial^j_{\overline{m}_1,t} v(0, z) = 0 \quad (j = 0, \dots, \beta - 2) \\ \partial^{\beta-1}_{\overline{m}_1,t} v(0, z) = \lambda^{\beta-1}(\partial_{\overline{m}_2,z})\mathcal{B}_{\Gamma_s,z^{1/\kappa}}\widehat{\varphi}(z) \in \mathcal{O}_{1/\kappa}(D), \end{cases}$$

where $\overline{m}_1 := m_1\Gamma_{1/K}$ is a moment function of order $1/\overline{k}_1 := s_1 + 1/K = q(s_2 + s) > 0$ and $\overline{m}_2 := m_2\Gamma_s$ is a moment function of order $1/\overline{k}_2 := s_2 + s > 0$. Since $1/\overline{k}_1 = q/\overline{k}_2$, applying Theorem 2 to v we conclude that properties (a)–(c) are equivalent.

Moreover, by Remark 5 we obtain the equivalence (b) $\Leftrightarrow$ (e).

To show the equivalence between (a) and (d), observe that $\mathcal{B}_{\Gamma_{s_1/q-s_2},z^{1/\kappa}}\widehat{\varphi}$ is qK-summable in directions $(d + \arg\lambda + 2k\pi)/q$ for $k = 0, \dots, q\kappa - 1$ if and only if $\mathcal{B}_{\Gamma_{1/qK},z^{1/\kappa}}\mathcal{B}_{\Gamma_{s_1/q-s_2},z^{1/\kappa}}\widehat{\varphi} \in \mathcal{O}^{qK}_{1,1/\kappa}(\widehat{S}_{(d+\arg\lambda+2k\pi)/q})$ for $k = 0, \dots, q\kappa - 1$. By Proposition 3 and Corollary 1, it is equivalent to (a).

Now we assume additionally that $s > 0$ and $qs_2 \geq s_1$. To find the equivalence between (f) and the previous conditions (a)–(e), it is sufficient to show implications (c) $\Rightarrow$ (f) and (f) $\Rightarrow$ (b). To this end observe that $qK \leq 1/s$. Hence if $\mathcal{B}_{\Gamma_{1/K},t}\mathcal{B}_{\Gamma_s,z^{1/\kappa}}\widehat{u} \in \mathcal{O}^{K,qK}_{1,1/\kappa}(\widehat{S}_d \times \widehat{S}_{(d+\arg\lambda+2k\pi)/q})$ then also $\mathcal{B}_{\Gamma_{1/K},t}\mathcal{B}_{\Gamma_s,z^{1/\kappa}}\widehat{u} \in \mathcal{O}^{K,1/s}_{1,1/\kappa}(\widehat{S}_d \times \widehat{S}_{(d+\arg\lambda+2k\pi)/q})$ (for $k = 0, \dots, q\kappa - 1$) and consequently by Definition 10 we conclude (f). On the opposite side, if u satisfies (f) then $\mathcal{B}_{\Gamma_{1/K},t}\mathcal{B}_{\Gamma_s,z^{1/\kappa}}\widehat{u} \in \mathcal{O}^{K,1/s}_{1,1/\kappa}(\widehat{S}_d \times \widehat{S}_{(d+\arg\lambda+2k\pi)/q})$. In particular, $\mathcal{B}_{\Gamma_{1/K},t}\mathcal{B}_{\Gamma_s,z^{1/\kappa}}\widehat{u} \in \mathcal{O}^{K}(\widehat{S}_d \times D)$, which gives (b).

Next we show the equivalence (c) $\Leftrightarrow$ (g). By Proposition 5, $\widehat{u}(t, z) = \sum_{j,n=0}^{\infty} u_{jn}t^j z^{n/\kappa}$ is $(K, 1/s)$-summable in the direction $O_{d,(d+\arg\lambda+2k\pi)/q}$ if and only if

$$\widetilde{v}(t, z) := \sum_{j,n=0}^{\infty} \frac{u_{jn}}{\Gamma(1 + j/K + sn/\kappa)} t^k z^{n/\kappa} \in \mathcal{O}^{K,1/s}_{1,1/\kappa}(\widehat{S}_d \times \widehat{S}_{(d+\arg\lambda+2k\pi)/q}).$$

So, it is sufficient to show

$$v \in \mathcal{O}^{K,qK}_{1,1/\kappa}(\widehat{S}_d \times \widehat{S}_{(d+\arg\lambda+2k\pi)/q}) \Leftrightarrow \widetilde{v} \in \mathcal{O}^{K,1/s}_{1,1/\kappa}(\widehat{S}_d \times \widehat{S}_{(d+\arg\lambda+2k\pi)/q}).$$

By Lemma 1 we get the following connection between $\widetilde{V}(t, z) := \widetilde{v}(t, z^{\kappa})$ and $V(t, z) := v(t, z^{\kappa})$

$$\widetilde{V}(t, z) = (1 + \frac{1}{K}t\partial_t + \frac{s}{\kappa}z\partial_z) \int_0^1 V(t\varepsilon^{1/K}, z(1 - \varepsilon)^{s/\kappa})\, d\varepsilon.$$

By the above formula and by the assumption $Kq \leq 1/s$ we conclude that if $v \in \mathcal{O}^{K,Kq}_{1,1/\kappa}(\widehat{S}_d \times \widehat{S}_{(d+\arg\lambda+2k\pi)/q})$ then $\widetilde{v} \in \mathcal{O}^{K,1/s}_{1,1/\kappa}(\widehat{S}_d \times \widehat{S}_{(d+\arg\lambda+2k\pi)/q})$.

To show the implication on the opposite side, we use the connection between the boundary conditions for v and $\widetilde{v}$. Namely, since

$$\partial^n_{\overline{m}_2,z^{1/\kappa}}\widetilde{v}(t, 0) = \frac{\overline{m}_2(n)}{\overline{m}_2(0)} \sum_{j=0}^{\infty} \frac{u_{jn}}{\Gamma(1 + j/K + sn/\kappa)} t^j$$

and

$$\partial^n_{\overline{m}_2,z^{1/\kappa}} v(t,0) = \frac{\overline{m}_2(n)}{\overline{m}_2(0)} \sum_{j=0}^{\infty} \frac{u_{jn}}{\Gamma(1+j/K)\Gamma(1+sn/\kappa)} t^j,$$

we get

$$\partial^n_{\overline{m}_2,z^{1/\kappa}} v(t,0) = \mathcal{B}_{m'_n,t} \partial^n_{\overline{m}_2,z^{1/\kappa}} \widetilde{v}(t,0),$$

where $m'_n(u) := \frac{\Gamma(1+u/K+sn/\kappa)}{\Gamma(1+u/K)\Gamma(1+sn/\kappa)}$ is a moment function of order 0 for $n = 0, \ldots, q\kappa\beta - 1$. So, since $\partial^n_{\overline{m}_2,z^{1/\kappa}} \widetilde{v}(t,0) \in \mathcal{O}^K(\widehat{S}_d)$, by Corollary 1 we see that also $\partial^n_{\overline{m}_2,z^{1/\kappa}} v(t,0) \in \mathcal{O}^K(\widehat{S}_d)$ for $n = 0, \ldots, q\kappa\beta - 1$. Hence, by Theorem 2 we conclude that $v \in \mathcal{O}^{K,qK}_{1,1/\kappa}(\widehat{S}_d \times \widehat{S}_{(d+\arg\lambda+2k\pi)/q})$. $\square$

Now we return to the general equation (6). For convenience we assume that

$$P(\lambda,\zeta) = P_0(\zeta) \prod_{\alpha=1}^{\widetilde{n}} \prod_{\beta=1}^{l_\alpha} (\lambda - \lambda_{\alpha\beta}(\zeta))^{n_{\alpha\beta}},$$

where $\lambda_{\alpha\beta}(\zeta) \sim \lambda_{\alpha\beta}\zeta^{q_\alpha}$ are the roots of the characteristic equation $P(\lambda,\zeta) = 0$ with pole orders $q_\alpha \in \mathbb{Q}$ and leading terms $\lambda_{\alpha\beta} \in \mathbb{C} \setminus \{0\}$ for $\beta = 1, \ldots, l_\alpha$ and $\alpha = 1, \ldots, \widetilde{n}$.

We also assume that $s, s_1, s_2 \in \mathbb{R}$, $s_1 > 0$, $s + s_2 > 0$ and $\widehat{\varphi}_j \in \mathbb{C}[[z]]_s$ for $j = 0, \ldots, n-1$. Without loss of generality we may assume that there exist exactly N pole orders of the roots of the characteristic equation, which are greater than $\frac{s_1}{s_2+s}$, say $\frac{s_1}{s_2+s} < q_1 < \cdots < q_N < \infty$ and let $K_\alpha > 0$ be defined by $K_\alpha := (q_\alpha(s_2+s) - s_1)^{-1}$ for $\alpha = 1, \ldots, N$.

By Theorem 1, the normalised formal solution $\widehat{u}$ of (6) is given by

$$\widehat{u} = \sum_{\alpha=1}^{\widetilde{n}} \sum_{\beta=1}^{l_\alpha} \sum_{\gamma=1}^{n_{\alpha\beta}} \widehat{u}_{\alpha\beta\gamma} \tag{14}$$

with $\widehat{u}_{\alpha\beta\gamma}$ satisfying

$$\begin{cases} (\partial_{m_1,t} - \lambda_{\alpha\beta}(\partial_{m_2,z}))^\gamma \widehat{u}_{\alpha\beta\gamma} = 0 \\ \partial^j_{m_1,t} \widehat{u}_{\alpha\beta\gamma}(0,z) = 0 \quad \text{for } j = 0, \ldots, \gamma-2 \\ \partial^{\gamma-1}_{m_1,t} \widehat{u}_{\alpha\beta\gamma} = \lambda^{\gamma-1}_{\alpha\beta}(\partial_{m_2,z}) \widehat{\varphi}_{\alpha\beta\gamma}(z), \end{cases}$$

where $\widehat{\varphi}_{\alpha\beta\gamma}(z) = \sum_{j=0}^{n-1} d_{\alpha\beta\gamma j}(\partial_{m_2,z}) \widehat{\varphi}_j(z) \in \mathbb{C}[[z^{\frac{1}{\kappa}}]]_s$ and $d_{\alpha\beta\gamma j}(\zeta)$ are holomorphic functions of the variable $\xi = \zeta^{1/\kappa}$ of polynomial growth at infinity.

Since $q_\alpha \le \frac{s_1}{s_2+s}$ for $\alpha = N+1, \ldots, \widetilde{n}$, by Theorem 1, $\widehat{u}_{\alpha\beta\gamma}$ is convergent for $\gamma = 1, \ldots, n_{\alpha\beta}$, $\beta = 1, \ldots, l_\alpha$ and $\alpha = N+1, \ldots, \widetilde{n}$.

Under the above conditions, immediately by Theorem 3 we get (see also [14, Theorem 5])

Theorem 4. *Let $(d_1, \ldots, d_N) \in \mathbb{R}^N$ be an admissible multidirection with respect to $(K_1, \ldots, K_N)$ and let $q_\alpha = \mu_\alpha/\nu_\alpha$ with relatively prime numbers $\mu_\alpha, \nu_\alpha \in \mathbb{N}$ for $\alpha = 1, \ldots, N$. We assume that*

$$\mathcal{B}_{\Gamma_s,z} \widehat{\varphi}_j(z) \in \mathcal{O}^{q_\alpha K_\alpha}(\widehat{S}_{(d_\alpha+\arg\lambda_{\alpha\beta}+2n_\alpha\pi)/q_\alpha})$$

for every $j = 0, \ldots, n-1$, $n_\alpha = 0, \ldots, \mu_\alpha - 1$, $\beta = 1, \ldots, l_\alpha$ and $\alpha = 1, \ldots, N$. Then the normalised formal solution $\widehat{u} \in G_{s,1/\kappa}[[t]]$ of (6) is $(K_1, \ldots, K_N)$-multisummable in the multidirection $(d_1, \ldots, d_N)$.

In general, the sufficient condition for the multisummability of $\widehat{u}$ given in Theorem 4 is not necessary, since the multisummability of $\widehat{u}$ satisfying (14) does not imply the summability of $\widehat{u}_{\alpha\beta\gamma}$ (see [14, Example 2]). For this reason, following [14], we define a kind of multisummability for which that implication holds.

Definition 16. Let $(d_1, \ldots, d_N)$ be an admissible multidirection with respect to $(K_1, \ldots, K_N)$. We say that $\widehat{u}$ *is $(K_1, \ldots, K_N)$-multisummable in the multidirection $(d_1, \ldots, d_N)$ with respect to the decomposition* (14) if $\widehat{u}_{\alpha\beta\gamma}$ is K_α-summable in the direction d_α (for $\alpha = 1, \ldots, N$) and is convergent (for $\alpha = N+1, \ldots, \widetilde{n}$), where $\beta = 1, \ldots, l_\alpha$ and $\gamma = 1, \ldots, n_{\alpha\beta}$.

Repeating the proof of [14, Theorem 6] with [14, Theorem 4] replaced by Theorem 3, we conclude

Theorem 5. *Let $(d_1, \dots, d_N) \in \mathbb{R}^N$ be an admissible multidirection with respect to $(K_1, \dots, K_N)$ and let $q_\alpha = \mu_\alpha/\nu_\alpha$ with relatively prime numbers $\mu_\alpha, \nu_\alpha \in \mathbb{N}$ for $\alpha = 1, \dots, N$. We assume that $\widehat{u}$ is the normalised formal solution of*

$$\begin{cases} P(\partial_{m_1,t}, \partial_{m_2,z})\widehat{u} = 0 \\ \partial_{m_1,t}^j \widehat{u}(0,z) = 0 \quad (j = 0, \dots, n-2) \\ \partial_{m_1,t}^{n-1}\widehat{u}(0,z) = \widehat{\varphi}(z) \in \mathbb{C}[[z]]_s. \end{cases}$$

Then $\widehat{u} \in G_{s,1/\kappa}[[t]]$ is $(K_1, \dots, K_N)$-multisummable in the multidirection $(d_1, \dots, d_N)$ with respect to the decomposition (14) if and only if

$$\mathcal{B}_{\Gamma_s,z}\widehat{\varphi} \in \mathcal{O}^{q_\alpha K_\alpha}(\widehat{S}_{(d_\alpha + \arg\lambda_{\alpha\beta} + 2n_\alpha\pi)/q_\alpha})$$

for every $n_\alpha = 0, \dots, \mu_\alpha - 1$, $\beta = 1, \dots, l_\alpha$ and $\alpha = 1, \dots, N$.

Remark 9. Analogously, one can also consider the multisummability in two variables using the approaches given by Sanz or Balser. By Theorem 3 we obtain the same characterisation of multisummable solutions in two variables as in Theorems 4 and 5.

9. An example

In this section we give a simple example illustrating the developed theory. For fixed $q \in \mathbb{N}$ and $s \in \mathbb{R}$ we discuss the solution of the equation

$$(\partial_t - \partial_z^q)\widehat{u} = 0, \quad \widehat{u}(0,z) = \widehat{\varphi}(z) \in \mathbb{C}[[z]]_s. \tag{15}$$

Observe that $\widehat{u}$ satisfies equation $(\partial_{m_1,t} - \lambda(\partial_{m_2,z}))\widehat{u} = 0$ with the moment functions $m_1 = m_2 = \Gamma_1$ and $\lambda(\zeta) = \zeta^q$. We have

Corollary 2. *Let $s \in \mathbb{R}$, $q \in \mathbb{N}$ and $\widehat{u}$ be a formal power series solution of (15). Then the following conditions are equivalent:*

(1) $\widehat{u}(0,z) \in \mathbb{C}[[z]]_s$.
(2) $\widehat{u}(t,0) \in \mathbb{C}[[t]]_{q(1+s)-1}$.
(3) $\widehat{u}(t,z) \in \mathbb{C}[[t,z]]_{q(1+s)-1,s}$.

Proof. The implications (3) ⇒ (2) and (3) ⇒ (1) are obvious. The implication (1) ⇒ (3) follows from Theorem 1. So, it is sufficient to show implication (2) ⇒ (3). To this end, observe that $\widehat{u}$ satisfies the equation

$$(\partial_z - \lambda_1(\partial_t)) \cdots (\partial_z - \lambda_q(\partial_t))\widehat{u} = 0, \quad \widehat{u}(t,0) \in \mathbb{C}[[t]]_{q(1+s)-1},$$

where $\lambda_n(\zeta) = e^{i2n\pi/q}\zeta^{1/q}$ for $n = 1, \dots, q$.

Hence, by Theorem 1 with replaced variables t and z, we get $\widehat{u} = \widehat{u}_1 + \cdots + \widehat{u}_q$, where $\widehat{u}_n$ satisfies the equation $(\partial_z - \lambda_n(\partial_t))\widehat{u}_n = 0$ and $\widehat{u}_n \in \mathbb{C}[[t,z]]_{q(1+s)-1,s}$ for $n = 1, \dots, q$. It means that also $\widehat{u} \in \mathbb{C}[[t,z]]_{q(1+s)-1,s}$. □

Assuming $s = 0$ (resp. $s < 0$) in Corollary 2, replacing $\widehat{u}$ and $\widehat{\varphi}$ in (15) by their sums u and φ, and applying Remark 4, we obtain

Corollary 3. *The solution u of (15) is t-analytic in a complex neighbourhood of the origin if and only if $\varphi \in \mathcal{O}(D)$ (for $q = 1$) and $\varphi \in \mathcal{O}^{\frac{q}{q-1}}(\mathbb{C})$ (for $q = 2, 3, \dots$). Furthermore, the solution u of (15) is t-entire of exponential growth of order $k > 0$ if and only if $\varphi \in \mathcal{O}^{\frac{kq}{k(q-1)+1}}(\mathbb{C})$.*

By Theorem 3 we obtain immediately

Proposition 10. *Let $d \in \mathbb{R}$, $\widehat{u}$ be a formal power series solution of (15) and $q(1+s) - 1 > 0$. Then the following conditions are equivalent:*

1. *$\widehat{u} \in G_{s,1}[[t]]$ is $(q(1+s)-1)^{-1}$-summable in direction d.*
2. *$\mathcal{B}_{\Gamma_s,z}\widehat{\varphi} \in \mathcal{O}^{\frac{q}{q(1+s)-1}}(\widehat{S}_{(d+2k\pi)/q})$ (for $k = 0, \dots, q-1$).*
3. *$\mathcal{B}_{\Gamma_{1/q-1},z}\widehat{\varphi}$ is $\frac{q}{q(1+s)-1}$-summable in the directions $(d + 2k\pi)/q$ for $k = 0, \dots, q-1$.*

If additionally $s > 0$ then conditions 1–3 are also equivalent to

4. *$\widehat{u} \in \mathbb{C}[[t,z]]$ is $((q(1+s)-1)^{-1}, s^{-1})$-summable in the directions $(d, (d+2k\pi)/q)$ for $k = 0, \dots, q-1$.*
5. *$\widehat{u} \in \mathbb{C}[[t,z]]$ is $((q(1+s)-1)^{-1}, s^{-1})$-summable in the directions $O_{d,(d+2k\pi)/q}$ for $k = 0, \dots, q-1$.*

References

[1] W. Balser, Divergent solutions of the heat equation: on an article of Lutz, Miyake and Schäfke, Pacific J. Math. 188 (1999) 53–63.
[2] W. Balser, Formal Power Series and Linear Systems of Meromorphic Ordinary Differential Equations, Springer-Verlag, New York, 2000.
[3] W. Balser, Multisummability of formal power series solutions of partial differential equations with constant coefficients, J. Differential Equations 201 (2004) 63–74.

[4] W. Balser, Summability of power series in several variables, with applications to singular perturbation problems and partial differential equations, Ann. Fac. Sci. Toulouse Math. 6 (14) (2005) 593–608.
[5] W. Balser, M. Loday-Richaud, Summability of solutions of the heat equation with inhomogeneous thermal conductivity in two variables, Adv. Dyn. Syst. Appl. 4 (2009) 159–177.
[6] W. Balser, M. Miyake, Summability of formal solutions of certain partial differential equations, Acta Sci. Math. (Szeged) 65 (1999) 543–551.
[7] W. Balser, M. Yoshino, Gevrey order of formal power series solutions of inhomogeneous partial differential equations with constant coefficients, Funkcial. Ekvac. 53 (2010) 411–434.
[8] K. Ichinobe, Integral representation for Borel sum of divergent solution to a certain non-Kovalevski type equation, Publ. Res. Inst. Math. Sci. 39 (2003) 657–693.
[9] D. Lutz, M. Miyake, R. Schäfke, On the Borel summability of divergent solutions of the heat equation, Nagoya Math. J. 154 (1999) 1–29.
[10] S. Malek, On the summability of formal solutions of linear partial differential equations, J. Dyn. Control Syst. 11 (2005) 389–403.
[11] S. Michalik, Summability of divergent solutions of the n-dimensional heat equation, J. Differential Equations 229 (2006) 353–366.
[12] S. Michalik, Summability of formal solutions to the n-dimensional inhomogeneous heat equation, J. Math. Anal. Appl. 347 (2008) 323–332.
[13] S. Michalik, On the multisummability of divergent solutions of linear partial differential equations with constant coefficients, J. Differential Equations 249 (2010) 551–570.
[14] S. Michalik, Analytic solutions of moment partial differential equations with constant coefficients, Funkcial. Ekvac. 56 (2013) 19–50.
[15] M. Miyake, Borel summability of divergent solutions of the Cauchy problem to non-Kovaleskian equations, Partial Differ. Equ. Appl. (1999) 225–239.
[16] J. Sanz, Summability in a direction of formal power series in several variables, Asymptot. Anal. 29 (2002) 115–141.