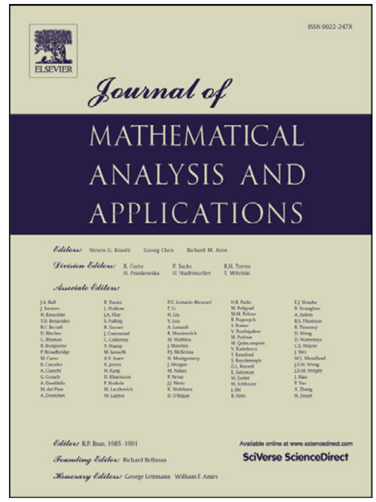

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Spectral Analysis and Exponential Stability of One-Dimensional Wave Equation with Viscoelastic Damping

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Abstract

This paper presents the exponential stability of a one-dimensional wave equation with viscoelastic damping. Using the asymptotic analysis technique, we prove that the spectrum of the system operator consists of two parts: the point and continuous spectrum. The continuous spectrum is a set of N points which are the limits of the eigenvalues of the system, and the point spectrum is a set of three classes of eigenvalues: one is a subset of N isolated simple points, the second is approaching to a vertical line which parallels to the imagine axis, and the third class is distributed around the continuous spectrum. Moreover, the Riesz basis property of the generalized eigenfunctions of the system is verified. Consequently, the spectrum-determined growth condition holds true and the exponential stability of the system is then established.

Keywords: Wave equation, Viscoelastic damping, Asymptotic analysis, Riesz basis, Stability.
2000 MSC: 35B35; 93C20

1. Introduction

It is known that the viscoelastic materials have been widely used in mechanics, chemical engineering, architecture, traffic, information and so on [3, 16]. Many researchers have paid close attention to the dynamic behavior and control of vibration for elastic structures with viscoelasticity in the past several decades. In the early 1990's, the existence and asymptotic stability of a linear hyperbolic integro-differential equation are presented for the Hilbert state space in [4], where an abstract version of the equation of motion for dynamic linear viscoelastic solids is established. The well posedness for damped second order systems with unbounded input operators is considered in [1], and the existence, uniqueness and continuous dependence of solutions in a weak or variational setting are presented. Later on, using a frequency domain method and combining a contradiction argument with the multiplier technique, the exponential stability for a vibrating Euler-Bernoulli beam with Kelvin-Voigt damping distributed locally on any subinterval of the region is studied in [8], and the stability for a vibrating string with local viscoelasticity, that is, one segment of the string is made of viscoelastic material and the other segments are made of elastic material, is discussed in [9]. In [11], the global existence and the asymptotic behaviour of the solution to a non-linear one dimensional wave equation with a viscoelastic boundary condition are analyzed by means of the energy method. Spectral analysis of a wave equation with Kelvin-Voigt damping is

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considered in [5] and it is shown that, with some assumption of the analyticity of the variable coefficients, the continuous spectrum of the system is an interval on the left real axis in [5]. The Riesz basis property of the generalized eigenfunctions of a one-dimensional hyperbolic system, which describes a heat equation incorporating the effect of thermomechanical coupling and the effect of inertia, is studied in [14]. The mathematical equation modelling a vibrating Timoshenko beam, which is made of viscoelastic material of a Kelvin-Voigt type locally in one segment, is deduced and the exponential stability is obtained under certain hypotheses of the smoothness and structural condition of the coefficients of the system in [19]. In [15], a detailed spectral analysis for a heat equation with thermoelastic memory type is presented. The spectrum-determined growth condition and exponential stability are also concluded in [15]. A particular set of functions related to the controllability of the heat equation with memory and finite signal speed, with suitable kernel, is shown to be a Riesz system in [12]. Other studies from different aspects for elastic structures with viscoelasticity can also be found in [2, 13, 17, 18] and the references therein.

In this paper, we are concerned with the following one-dimensional wave equation with viscoelastic damping under the Dirichlet boundary condition: for $0 < x < 1, t > 0$,

$$\begin{cases} w_{tt}(x,t) = a^2 w_{xx}(x,t) - \displaystyle\int_0^t \kappa(t-s) w_{xx}(x,s)ds - cw_t(x,t), \\ w(0,t) = w(1,t) = 0, \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), \end{cases} \tag{1.1}$$

where the kernel is taken for the finite sum of exponential polynomials:

$$\kappa(t) = \sum_{i=1}^{N} a_i e^{-b_i t}, \quad 0 < a_i,\ b_i \in \mathbb{R}, \quad 1 \le N \in \mathbb{N}. \tag{1.2}$$

Moreover, the following assumptions hold true for the coefficients:

$$0 < b_1 < b_2 < \cdots < b_N < c, \qquad a^2 > \sum_{i=1}^{N} \frac{a_i}{b_i}. \tag{1.3}$$

In [6], a different model for a vibrating wave system with Boltzmann integrals is considered. The spectral properties are analyzed and the Riesz basis for the system is verified. The spectrum-determined growth conditions and the exponential stability are also concluded.

In this paper, with the viscous damping forced into system (1.1)-(1.2), the dynamic behavior of the system is investigated. By introducing some new variables for the exponential polynomial kernel, we set up a time-invariant system and prove the existence of solution, the distribution and structure of the spectrum, and the basis property of the generalized eigenfunctions.

The paper is organized as follows. In Section 2, some new variables are introduced to transform the system into a time-invariant one. The detailed spectral analysis of the newly formulated system is presented in Section 3. By the asymptotic analysis technique, it is shown that the eigenvalues have three classes: one is the set of the simple points $\{-b_i,\ i = 1, 2, \cdots, N\}$, the second approaches a line that is parallel to the imaginary axis, the third is located around the continuous spectral points which contains N isolated points of the complex plane. Moreover, the residual spectrum is shown to be empty and the set of continuous spectrum is exactly characterized. Section 4 is devoted to the Riesz basis generation and the exponential stability of the system.

2. System operator setup

Set

$$y_i(x,t) = a_i \int_0^t e^{-b_i(t-s)} w_x(x,s)ds, \quad i = 1, 2, \cdots, N. \tag{2.1}$$

Then y_i satisfies

$$\begin{cases} (y_i)_t(x,t) = a_i w_x(x,t) - b_i y_i(x,t), \\ (y_i)_x(x,t) = a_i \displaystyle\int_0^t e^{-b_i(t-s)} w_{xx}(x,s)ds. \end{cases} \tag{2.2}$$

So we can rewrite (1.1)-(1.2) as

$$\begin{cases} w_{tt}(x,t) - \dfrac{\partial}{\partial x}\left[a^2 w_x(x,t) - \displaystyle\sum_{i=1}^N y_i(x,t)\right] + c w_t(x,t) = 0, \\ (y_i)_t(x,t) = a_i w_x(x,t) - b_i y_i(x,t), \quad i = 1, 2, \cdots, N, \\ w(0,t) = w(1,t) = 0, \qquad t > 0, \\ w(x,0) = w_0(x),\ w_t(x,0) = w_1(x),\ y_i(x,0) = 0, \ \ i = 1, 2, \cdots, N. \end{cases} \tag{2.3}$$

Obviously, the system (2.3) is a time-invariant system. The system energy is given by

$$E(t) = \frac{1}{2}\int_0^1 \left[a^2|w_x(x,t)|^2 + |w_t(x,t)|^2 + \sum_{i=1}^N |y_i(x,t)|^2\right] dx. \tag{2.4}$$

Motivated by the energy function, we consider naturally the system (2.3) in the following Hilbert space $\mathcal{H}$ given by:

$$\begin{cases} \mathcal{H} = H_0^1(0,1) \times (L^2(0,1))^{N+1} \\ H_0^1(0,1) = \{f \in H^1(0,1) \mid f(0) = f(1) = 0\} \end{cases}$$

equipped with the inner product: $\forall\ (w, v, y_1, \cdots, y_N),\ (f, g, h_1, \cdots, h_N) \in \mathcal{H}$,

$$\begin{aligned} &\Big\langle (w, v, y_1, \cdots, y_N), (f, g, h_1, \cdots, h_N)\Big\rangle_{\mathcal{H}} \\ = \ &\int_0^1 a^2 w'(x)\overline{f'(x)}dx + \int_0^1 v(x)\overline{g(x)}dx + \sum_{i=1}^N \int_0^1 y_i(x)\overline{h_i}(x)dx. \end{aligned}$$

Now, define the system operator $\mathcal{A} : D(\mathcal{A})(\subset \mathcal{H}) \to \mathcal{H}$ by

$$\begin{cases} \mathcal{A}\begin{pmatrix} w \\ v \\ y_1 \\ \vdots \\ y_N \end{pmatrix}^\top = \begin{pmatrix} v \\ \left[a^2 w' - \sum_{i=1}^N y_i\right]' - cv \\ a_1 w' - b_1 y_1 \\ \vdots \\ a_N w' - b_N y_N \end{pmatrix}^\top, \\ D(\mathcal{A}) = \left\{ \begin{pmatrix} w \\ v \\ y_1 \\ \vdots \\ y_N \end{pmatrix}^\top \ \middle| \ \begin{array}{c} w, v \in H_0^1(0,1), \\ y_i \in L^2(0,1), i = 1, \cdots, N, \\ \left[a^2 w' - \displaystyle\sum_{i=1}^N y_i\right] \in H^1(0,1). \end{array} \right\}. \end{cases} \tag{2.5}$$

Then (2.3) can be formulated into an abstract evolution equation in $\mathcal{H}$:

$$\frac{d}{dt}Z(t) = \mathcal{A}Z(t), \quad Z(0) = Z_0, \tag{2.6}$$

where $Z(t) = \Big(w(\cdot,t), w_t(\cdot,t), y_1(\cdot,t), \cdots, y_N(\cdot,t)\Big)$ is the state variable and $Z_0 = \Big(w_0(\cdot),\ w_1(\cdot),\ 0,\ \cdots,\ 0\Big)$ is the initial value.

Lemma 2.1. *Let $\mathcal{A}$ be defined by (2.5). Then $0 \in \rho(\mathcal{A})$.*

Proof. Let $\tilde{Z} = (\tilde{w}, \tilde{v}, \tilde{y_1}, \cdots, \tilde{y_N}) \in \mathcal{H}$. Solve $\mathcal{A}Z = \tilde{Z}$ for $Z = (w, v, y_1, \cdots, y_N) \in D(\mathcal{A})$, that is

$$\begin{cases} v(x) = \tilde{w}(x), \\ \left[a^2 w'(x) - \sum_{i=1}^{N} y_i(x)\right]' - cv(x) = \tilde{v}(x), \\ a_i w'(x) - b_i y_i(x) = \tilde{y}_i(x), \quad i = 1, 2, \cdots, N, \\ w(0) = w(1) = 0. \end{cases} \tag{2.7}$$

From (2.7), we have

$$v(x) = \tilde{w}(x), \qquad y_i(x) = \frac{a_i}{b_i}w'(x) - \frac{1}{b_i}\tilde{y}_i(x),\ j = 1, 2, \cdots, N \tag{2.8}$$

and

$$a^2 w'(x) - \sum_{i=1}^{N} y_i(x) = C_1 + \int_0^x \Big[c\tilde{w}(\tau) + \tilde{v}(\tau)\Big]d\tau, \tag{2.9}$$

where C_1 is a constant to be determined. Substituting (2.8) into (2.9) to yield

$$Aw'(x) = C_1 - \sum_{i=1}^{N} \frac{\tilde{y}_i(x)}{b_i} + \int_0^x \Big[c\tilde{w}(\tau) + \tilde{v}(\tau)\Big]d\tau, \tag{2.10}$$

where $A = a^2 - \sum_{i=1}^N \frac{a_i}{b_i}$. Using the boundary condition $w(0) = 0$ gives

$$Aw(x) = C_1 x - \sum_{i=1}^{N} \frac{\int_0^x \tilde{y}_i(\tau)d\tau}{b_i} + \int_0^x (x-\tau)\Big[c\tilde{w}(\tau) + \tilde{v}(\tau)\Big]d\tau, \tag{2.11}$$

by the other boundary condition $w(1) = 0$, it yields

$$C_1 = \sum_{i=1}^{N} \frac{\int_0^1 \tilde{y}_i(\tau)d\tau}{b_i} - \int_0^1 (1-\tau)\Big[c\tilde{w}(\tau) + \tilde{v}(\tau)\Big]d\tau. \tag{2.12}$$

Combining (2.11) and (2.12) to get

$$\begin{aligned} Aw(x) \;\; = \;\; & -\left[\sum_{i=1}^{N} \frac{\int_0^x \tilde{y}_i(\tau)d\tau}{b_i} + \int_0^x \tau\Big[c\tilde{w}(\tau) + \tilde{v}(\tau)\Big]d\tau\right](1-x) \\ & -x\sum_{i=1}^{N} \frac{\int_x^1 \tilde{y}_i(\tau)d\tau}{b_i} - x\int_x^1 (1-\tau)\Big[c\tilde{w}(\tau) + \tilde{v}(\tau)\Big]d\tau. \end{aligned} \tag{2.13}$$

Collecting (2.8),(2.10) and (2.12)-(2.13), we get the unique solution $Z = (w, v, y_1, \cdots, y_N) \in D(\mathcal{A})$ and hence $\mathcal{A}^{-1}$ exists, or $0 \in \rho(\mathcal{A})$. □

3. Spectral analysis of the system

In this section, we analyze the spectrum of $\mathcal{A}$. Firstly, we consider the eigenvalue problem. Suppose $\mathcal{A}Z = \lambda Z$ for $\lambda \in \mathbb{C}$ and $0 \neq Z = (w, v, y_1, \cdots, y_N) \in D(\mathcal{A})$. Then

$$\begin{cases} v(x) = \lambda w(x), \\ \left[a^2 w'(x) - \sum_{i=1}^{N} y_i(x)\right]' - cv(x) = \lambda v(x), \\ a_i w'(x) - b_i y_i(x) = \lambda y_i(x), \quad i = 1, 2, \cdots, N, \\ w(0) = w(1) = 0. \end{cases} \tag{3.1}$$

Proposition 3.1. *Let $\mathcal{A}$ be defined by (2.5). Then $\lambda = -b_i,\ i = 1, 2, \cdots, N$ are eigenvalues of $\mathcal{A}$, which correspond to eigenfunctions $e_{i+2}, i = 1, 2, \cdots, N$ respectively, where e_i is a constant function whose element is the i-th element of the canonical basis of $\mathbb{R}^{N+2}$. Moreover, each of these eigenvalues is algebraically simple.*

Proof. We only give the proof for $\lambda = -b_1$ because other cases can be treated similarly. Let $\lambda = -b_1$ and $Z = (w, v, y_1, \cdots, y_N) \in D(\mathcal{A})$. Solve $(\mathcal{A} + b_1)Z = 0$, that is, for $0 < x < 1$,

$$\begin{cases} v(x) + b_1 w(x) = 0, \\ \left[a^2 w'(x) - \sum_{i=1}^{N} y_i(x)\right]' + (b_1 - c)v(x) = 0, \\ a_1 w'(x) = 0, \\ a_i w'(x) + (b_1 - b_i) y_i(x) = 0, \quad i = 2, \cdots, N, \\ w(0) = w(1) = 0. \end{cases} \tag{3.2}$$

From the first and third equation of (3.2) and boundary condition $w(0) = 0$, we obtain $w = v \equiv 0$. This together with the forth equation of (3.2) yields

$$(b_1 - b_i) y_i(x) = 0, \quad i = 2, \cdots, N.$$

According to the assumption (1.3), we arrive at

$$y_i(x) = 0, \quad i = 2, \cdots, N.$$

By the second equation of (3.2), it has

$$y_1'(x) = 0, \quad 0 < x < 1.$$

Therefore, e_3 is an eigenfunction of $\mathcal{A}$ corresponding to $-b_1$. Further computation of $(b_1 I + \mathcal{A})Z_1 = -e_3$, where $Z_1 = (f, g, h_1, \cdots, h_N) \in D(\mathcal{A})$, it yields

$$\begin{cases} b_1 f(x) + g(x) = 0, \\ \left[a^2 f'(x) - \sum_{i=1}^{N} h_i(x)\right]' + (b_1 - c)g(x) = 0, \\ a_1 f'(x) = -1, \\ (b_1 - b_i) h_i(x) + a_i f'(x) = 0, \quad i = 2, \cdots, N, \\ f(0) = f(1) = 0. \end{cases} \tag{3.3}$$

(3.3) has no solution since f must satisfy

$$f'(x) = -\frac{1}{a_1}, \quad f(0) = f(1) = 0,$$

which is impossible. This shows that $\lambda = -b_1$ is algebraically simple. $\square$

When $\lambda \neq -b_i,\ i = 1, \cdots, N$, it follows from (3.1) that

$$\begin{cases} v(x) = \lambda w(x), \\ y_i(x) = \dfrac{a_i}{\lambda + b_i} w'(x), \quad i = 1, 2, \cdots, N, \end{cases} \tag{3.4}$$

and w satisfies

$$\begin{cases} \left[a^2 - \displaystyle\sum_{i=1}^{N} \frac{a_i}{\lambda + b_i} \right] w''(x) = (\lambda^2 + c\lambda)\ w(x), \\ w(0) = w(1) = 0. \end{cases} \tag{3.5}$$

Lemma 3.2. *Let $\mathcal{A}$ be defined by (2.5) and $\lambda \neq -b_i$, $i = 1, 2, \cdots, N$. Then* $\mathrm{Re}\lambda < 0$ *for each $\lambda \in \sigma_p(\mathcal{A})$, where $\sigma_p(\mathcal{A})$ denotes the point spectrum of $\mathcal{A}$.*

Proof. Firstly, we show that $\mathrm{Re}\lambda \leq 0$. When $\lambda \in \sigma_p(\mathcal{A})$, $\lambda \neq -b_i$, $i = 1, 2, \cdots, N$, the eigenvalue problem becomes (3.5). Multiply the first equation of (3.5) by $\overline{w}$, the conjugate of w, and integrate over $[0, 1]$ with respect to x, to give

$$(\lambda^2 + c\lambda)\|w\|_{L^2}^2 + \left[a^2 - \sum_{i=1}^{N} \frac{a_i}{\lambda + b_i} \right] \|w'\|_{L^2}^2 = 0.$$

Set $\lambda = \sigma + i\tau,\ \sigma, \tau \in \mathbb{R}$. Then

$$\begin{aligned} &(\sigma^2 - \tau^2 + c\sigma)\|w\|_{L^2}^2 + \left[a^2 - \sigma \sum_{i=1}^{N} \frac{a_i}{(\sigma + b_i)^2 + \tau^2} - \sum_{i=1}^{N} \frac{a_i b_i}{(\sigma + b_i)^2 + \tau^2} \right] \|w'\|_{L^2}^2 \\ &+ i\tau \left[(2\sigma + c)\|w\|_{L^2}^2 + \sum_{i=1}^{N} \frac{a_i}{(\sigma + b_i)^2 + \tau^2} \|w'\|_{L^2}^2 \right] = 0. \end{aligned}$$

Let the real and imaginary part of the equation equal to zero to yield

$$\begin{cases} (\sigma^2 - \tau^2 + c\sigma)\|w\|_{L^2}^2 + \left[a^2 - \displaystyle\sum_{i=1}^{N} \frac{a_i(\sigma + b_i)}{(\sigma + b_i)^2 + \tau^2} \right] \|w'\|_{L^2}^2 = 0, \\ \tau(2\sigma + c)\|w\|_{L^2}^2 + \tau \displaystyle\sum_{i=1}^{N} \frac{a_i}{(\sigma + b_i)^2 + \tau^2} \|w'\|_{L^2}^2 = 0. \end{cases} \tag{3.6}$$

If $\tau = 0$, from the first equation of (3.6), it has

$$(\sigma^2 + c\sigma)\|w\|_{L^2}^2 + \left[a^2 - \sum_{i=1}^{N} \frac{a_i}{\sigma + b_i} \right] \|w'\|_{L^2}^2 = 0.$$

We need to prove $\sigma \le 0$. Suppose $\sigma \ge 0$. Since $\sum_{i=1}^N \dfrac{a_i}{\sigma + b_i}$ is not increasing with respect to σ, we have

$$\begin{aligned} \sigma &= -\frac{\sigma^2\|w\|_{L^2}^2 + \left[a^2 - \sum\limits_{i=1}^N \dfrac{a_i}{\sigma+b_i}\right]\|w'\|_{L^2}^2}{c\|w\|_{L^2}^2} \\ &\le -\frac{\sigma^2\|w\|_{L^2}^2 + \left[a^2 - \sum\limits_{i=1}^N \dfrac{a_i}{b_i}\right]\|w'\|_{L^2}^2}{c\|w\|_{L^2}^2}, \end{aligned}$$

which shows $\sigma \le 0$ by (1.3). This is a contradiction.
If $\tau \neq 0$, from the second equation of (3.6), we have

$$(2\sigma + c)\|w\|_{L^2}^2 + \sum_{i=1}^N \frac{a_i}{(\sigma+b_i)^2+\tau^2}\|w'\|_{L^2}^2 = 0.$$

Then

$$2\sigma + c = -\frac{\sum_{i=1}^N \dfrac{a_i}{(\sigma+b_i)^2+\tau^2}\|w'\|_{L^2}^2}{\|w\|_{L^2}^2} \le 0.$$

Hence,

$$\sigma \le -\frac{c}{2}.$$

Secondly, we show that $\mathrm{Re}\lambda \neq 0$. If $\mathrm{Re}\lambda = 0$ for $\lambda \in \sigma_p(\mathcal{A})$ and $\lambda \neq -b_i (i = 1, \cdots, N)$, let $\lambda = i\tau, \tau \in \mathbb{R}(\tau \neq 0)$. From (3.6) it has

$$\begin{cases} -\tau^2\|w\|_{L^2}^2 + \left[a^2 - \sum\limits_{i=1}^N \dfrac{a_i b_i}{b_i^2+\tau^2}\right]\|w'\|_{L^2}^2 = 0, \\ \tau c\|w\|_{L^2}^2 + \tau\sum\limits_{i=1}^N \dfrac{a_i}{b_i^2+\tau^2}\|w'\|_{L^2}^2 = 0. \end{cases} \tag{3.7}$$

From the second equation of (3.7), we have $\tau = 0$ only. This is a contradiction. □

The next lemma is straightforward.

Lemma 3.3. *Let $\mathcal{A}$ be defined by (2.5) and*

$$\Delta = \left\{\lambda \in \mathbb{C} \;\middle|\; a^2 - \sum_{i=1}^N \frac{a_i}{\lambda+b_i} = 0\right\}. \tag{3.8}$$

Then

$$\Delta \not\subset \sigma_p(\mathcal{A}).$$

Lemma 3.4. *Let $\mathcal{A}$ be defined as in (2.5) and Δ is given by (3.8). Then*

$$\Delta = \Big\{\lambda_{ck} \in (-b_k, -b_{k-1}) \subset \mathbb{R},\ k = 1, 2, \cdots, N, b_0 = 0\Big\}. \tag{3.9}$$

Proof. Since $-b_i \notin \Delta, \quad i = 1, 2, \cdots, N$, $p(\lambda) = 0$ is equivalent to $q(\lambda) = 0$, where

$$p(\lambda) = a^2 - \sum_{i=1}^{N} \frac{a_i}{\lambda + b_i}, \quad q(\lambda) = p(\lambda)\Pi_{i=1}^{N}(\lambda + b_i). \tag{3.10}$$

However, $q(\lambda)$ is a N-th order polynomial, and hence there are at most N number of zeros of $p(\lambda)$. Now we find all these zeros. Notice that $p(\lambda)$ is continuous in $(-b_1, \infty) \bigcup \left(\bigcup_{i=1}^{N-1} (-b_{i+1}, -b_i) \right)$, and

$$\lim_{\lambda \to -b_i^-} p(\lambda) = +\infty, \quad \lim_{\lambda \to -b_i^+} p(\lambda) = -\infty, \quad i = 1, 2, \cdots, N,$$

$p(0) > 0$ by (1.3). It follows that there exists a solution to $p(\lambda) = 0$ in $(-b_{i+1}, -b_i)$, $i = 0, 1, \cdots, N-1$, here $b_0 = 0$. □

By Lemma 3.3, the eigenvalue problem (3.5) is equivalent to the following problem:

$$\begin{cases} w''(x) = \dfrac{\lambda^2 + c\lambda}{p(\lambda)} w(x), \\ w(0) = w(1) = 0, \end{cases} \tag{3.11}$$

where $p(\lambda)$ is given by (3.10). Hence

$$w(x) = e^{\sqrt{\frac{\lambda^2 + c\lambda}{p(\lambda)}}\, x} - e^{-\sqrt{\frac{\lambda^2 + c\lambda}{p(\lambda)}}\, x}. \tag{3.12}$$

By the boundary condition $w(1) = 0$, (3.11) has non-trivial solution if and only if

$$e^{\sqrt{\frac{\lambda^2 + c\lambda}{p(\lambda)}}} - e^{-\sqrt{\frac{\lambda^2 + c\lambda}{p(\lambda)}}} = 0, \tag{3.13}$$

that is

$$e^{2\sqrt{\frac{\lambda^2 + c\lambda}{p(\lambda)}}} = 1,$$

which is equivalent to

$$\frac{\lambda^2 + c\lambda}{p(\lambda)} = -n^2\pi^2, \quad n = 1, 2, \cdots. \tag{3.14}$$

Substituting (3.14) into (3.12), we obtain the eigenfunction

$$\left(w(x), \lambda w(x), \frac{a_1}{\lambda + b_1} w'(x), \cdots, \frac{a_N}{\lambda + b_N} w'(x) \right)$$

corresponding to λ, where

$$w(x) = \sin n\pi x, \tag{3.15}$$

for some $n \in \mathbb{N}^+$. When $|\lambda|$ is large enough, since

$$\frac{\lambda^2 + c\lambda}{p(\lambda)} = \frac{\lambda^2 + c\lambda}{a^2 - \sum\limits_{i=1}^{N} \frac{a_i}{\lambda + b_i}} = \frac{\lambda^2 + c\lambda}{a^2} \cdot \frac{1}{1 - \dfrac{1}{a^2} \sum\limits_{i=1}^{N} \dfrac{a_i}{\lambda + b_i}}$$

$$= \frac{1}{a^2}\left[\lambda^2 + \left(c + \frac{1}{a^2}\sum_{i=1}^{N} a_i\right)\lambda + \frac{1}{a^2}\sum_{i=1}^{N} a_i b_i - \frac{c}{a^2}\sum_{i=1}^{N} a_i + \frac{1}{a^4}\left(\sum_{i=1}^{N} a_i\right)^2\right] + \mathcal{O}(|\lambda|^{-1}),$$

we obtain

$$\lambda^2 + \left(c + \frac{1}{a^2}\sum_{i=1}^{N} a_i\right)\lambda + \frac{1}{a^2}\sum_{i=1}^{N} a_i b_i - \frac{c}{a^2}\sum_{i=1}^{N} a_i + \frac{1}{a^4}\left(\sum_{i=1}^{N} a_i\right)^2 + a^2 n^2 \pi^2 + \mathcal{O}(|\lambda|^{-1}) = 0.$$

Thus, the eigenvalues of $\mathcal{A}$ in this case are found to be

$$\lambda_n = -\frac{1}{2}\left(c + \frac{1}{a^2}\sum_{i=1}^{N} a_i\right) \pm ian\pi + \mathcal{O}(n^{-1}), \quad n \to \infty.$$

Obviously, $-\frac{1}{2}\left(c + \frac{1}{a^2}\sum_{i=1}^{N} a_i\right) < 0$.

For any $\lambda_c \in \Delta$, when $\lambda \to \lambda_c$, $\mu = \lambda - \lambda_c \to 0$. Since

$$\begin{aligned} p(\lambda) &= a^2 - \sum_{i=1}^{N} \frac{a_i}{\lambda + b_i} \\ &= a^2 - \sum_{i=1}^{N} \frac{a_i}{\lambda_c + b_i} \frac{1}{1 + \dfrac{\lambda - \lambda_c}{\lambda_c + b_i}} \\ &= \mu \sum_{i=1}^{N} a_i \left[\frac{1}{(\lambda_c + b_i)^2} - \frac{\mu}{(\lambda_c + b_i)^3} + \mathcal{O}(\mu^2)\right], \end{aligned}$$

it has

$$\begin{aligned} \frac{\lambda^2 + c\lambda}{p(\lambda)} &= \frac{\lambda_c^2 + \lambda_c(2\mu + c) + \mu^2 + c\mu}{p(\lambda)} \\ &= \frac{1}{\mu} \frac{\lambda_c^2\left[\left(1 + \dfrac{c}{\lambda_c}\right) + \left(\dfrac{2}{\lambda_c} + \dfrac{c}{\lambda_c^2}\right)\mu + \dfrac{1}{\lambda_c^2}\mu^2\right]}{\sum\limits_{i=1}^{N} \dfrac{a_i}{(\lambda_c + b_i)^2}\left[1 - \dfrac{\sum\limits_{i=1}^{N} \dfrac{a_i}{(\lambda_c + b_i)^3}}{\sum\limits_{i=1}^{N} \dfrac{a_i}{(\lambda_c + b_i)^2}}\mu + \mathcal{O}(\mu^2)\right]} \\ &= \frac{\lambda_c^2\left[\left(1 + \dfrac{c}{\lambda_c}\right) + \left(\dfrac{2}{\lambda_c} + \dfrac{c}{\lambda_c^2}\right)\mu + \dfrac{1}{\lambda_c^2}\mu^2\right]}{\mu \sum\limits_{i=1}^{N} \dfrac{a_i}{(\lambda_c + b_i)^2}} \left[1 + \dfrac{\sum\limits_{i=1}^{N} \dfrac{a_i}{(\lambda_c + b_i)^3}}{\sum\limits_{i=1}^{N} \dfrac{a_i}{(\lambda_c + b_i)^2}}\mu\right] + \mathcal{O}(\mu) \\ &= \frac{1}{\mu}\frac{\lambda_c^2}{\Lambda}\left[\left(1 + \frac{c}{\lambda_c}\right) + \left(\frac{\lambda_c + c}{\lambda_c}\frac{\tilde{\Lambda}}{\Lambda} + \frac{2\lambda_c + c}{\lambda_c^2}\right)\mu\right] + \mathcal{O}(\mu), \end{aligned}$$

where

$$\Lambda = \sum_{i=1}^{N} \frac{a_i}{(\lambda_c + b_i)^2}, \quad \tilde{\Lambda} = \sum_{i=1}^{N} \frac{a_i}{(\lambda_c + b_i)^3}.$$

This together with (3.14) yields

$$\frac{1}{\mu}\frac{\lambda_c^2}{\Lambda}\left[\left(1 + \frac{c}{\lambda_c}\right) + \left(\frac{\lambda_c + c}{\lambda_c}\frac{\Lambda_1}{\Lambda} + \frac{2\lambda_c + c}{\lambda_c^2}\right)\mu\right] + \mathcal{O}(\mu) = -n^2\pi^2, \quad n \to \infty.$$

Thus

$$\mu_n = -\frac{1}{n^2\pi^2}\frac{\lambda_c^2 + c\lambda_c}{\Lambda} + \mathcal{O}(n^{-3}), \quad n \to \infty,$$

or

$$\lambda_n = \lambda_c + \mu_n = \lambda_c - \frac{1}{n^2\pi^2}\frac{\lambda_c^2 + c\lambda_c}{\Lambda} + \mathcal{O}(n^{-3}), \quad n \to \infty.$$

We summarize these results as Proposition 3.5.

Proposition 3.5. *Let $\mathcal{A}$ be defined by (2.5) and λ be an eigenvalue of $\mathcal{A}$, satisfying $\lambda \neq -b_i$, $i = 1, \cdots, N$. Then the eigenfunction corresponding to λ is of the form*

$$\left(w(x), \lambda w(x), \frac{a_1}{\lambda + b_1}w'(x), \cdots, \frac{a_N}{\lambda + b_N}w'(x)\right),$$

where $w(x) = \sin n\pi x$, for some $n \in \mathbb{N}^+$. Moreover,

(i) *For any $1 \leq k \leq N$, there is a sequence of eigenvalues $\{\lambda_{nk}\}$ of $\mathcal{A}$, which have the following asymptotic expressions:*

$$\lambda_{nk} = \lambda_{ck} - \frac{1}{n^2\pi^2}\frac{\lambda_{ck}^2 + c\lambda_{ck}}{\Lambda_k} + \mathcal{O}(n^{-3}), \quad n \to \infty, \tag{3.16}$$

where

$$\Lambda_k = \sum_{i=1}^{N} \frac{a_i}{(\lambda_{ck} + b_i)^2}.$$

The corresponding eigenfunctions

$$\left(w_n(x), \lambda_{nk}w_n(x), \frac{a_1}{\lambda_{nk} + b_1}w_n'(x), \cdots, \frac{a_N}{\lambda_{nk} + b_N}w_n'(x)\right)$$

satisfy

$$w_n(x) = \frac{1}{n\pi}\sin n\pi x. \tag{3.17}$$

(ii) *When $|\lambda| \to \infty$, the eigenvalues $\{\lambda_{n0}, \overline{\lambda_{n0}}\}$ of $\mathcal{A}$ have the following asymptotic expressions:*

$$\lambda_{n0} = -\frac{1}{2}\left(c + \frac{1}{a^2}\sum_{i=1}^{N} a_i\right) + ian\pi + \mathcal{O}(n^{-1}), \quad n \to \infty, \tag{3.18}$$

where $\overline{\lambda_{n0}}$ denotes the complex conjugate of λ_{n0}. In particular,

$$\mathrm{Re}\lambda_{n0} \to -\frac{1}{2}\left(c + \frac{1}{a^2}\sum_{i=1}^{N} a_i\right) < 0, \quad n \to \infty, \tag{3.19}$$

that is to say, $\mathrm{Re}\lambda_{n0} = -\frac{1}{2}\left(c + \frac{1}{a^2}\sum_{i=1}^{N} a_i\right)$ *is the asymptote of the eigenvalues* λ_{n0} *given by (3.18). Furthermore, the corresponding eigenfunctions*

$$\left(w_n, \lambda_{n0} w_n, \frac{a_1}{\lambda_{n0}+b_1} w_n', \cdots, \frac{a_N}{\lambda_{n0}+b_N} w_n'\right)$$

satisfy (3.17).

The following result is a direct consequence of Proposition 3.1 and Proposition 3.5.

Theorem 3.6. *Let* $\mathcal{A}$ *be defined as in (2.5). Then*

(i) $\mathcal{A}$ *has the eigenvalues*

$$\Big\{-b_i,\ i = 1, 2, \cdots, N\Big\} \bigcup \Big\{\lambda_{n1}, \lambda_{n2}, \cdots, \lambda_{nN}, n \in \mathbb{N}^+\Big\} \bigcup \Big\{\lambda_{n0}, \overline{\lambda_{n0}},\ n \in \mathbb{N}^+\Big\}, \tag{3.20}$$

where $\lambda_{nk},\ 1 \le k \le N$ *and* λ_{n0} *have the asymptotic expressions (3.16)and (3.18), respectively.*

(ii) *The eigenfunction corresponding to* $-b_i$ *is* e_{i+2} *for any* $i = 1, 2, \cdots, N$.

(iii) *The eigenfunctions corresponding to* $\lambda_{nk},\ k = 1, 2, \cdots, N$ *are given by*

$$\begin{aligned} W_{nk}(x) &= \left(\frac{1}{n\pi}\sin n\pi x, 0, \frac{a_1}{\lambda_{nk}+b_1}\cos n\pi x, \cdots, \frac{a_N}{\lambda_{nk}+b_N}\cos n\pi x\right) \\ &\quad + \Big(0, \mathcal{O}(n^{-1}), \cdots, \mathcal{O}(n^{-1})\Big), \quad n \to \infty. \end{aligned} \tag{3.21}$$

(iv) *The eigenfunctions corresponding to* λ_{n0} *and* $\overline{\lambda_{n0}}$ *are given by*

$$W_{n0}(x) = \left(\frac{1}{n\pi}\sin n\pi x, ia\sin n\pi x, 0, \cdots, 0\right) + \Big(0, \mathcal{O}(n^{-1}), \cdots, \mathcal{O}(n^{-1})\Big), \tag{3.22}$$

and

$$\overline{W_{n0}}(x) = \left(\frac{1}{n\pi}\sin n\pi x, -ia\sin n\pi x, 0, \cdots, 0\right) + \Big(0, \mathcal{O}(n^{-1}), \cdots, \mathcal{O}(n^{-1})\Big), \tag{3.23}$$

for $n \to \infty$*,respectively.*

In order to investigate the residual and continuous spectrum of $\mathcal{A}$, we need the adjoint operator $\mathcal{A}^*$.

Lemma 3.7. *Let* $\mathcal{A}$ *be defined by (2.5). Then*

$$\begin{cases} \mathcal{A}^* \begin{pmatrix} w \\ v \\ y_1 \\ \vdots \\ y_N \end{pmatrix}^\top = \begin{pmatrix} \frac{1}{a^2}\sum_{i=1}^{N}\int_0^x a_i y_i(\tau)d\tau - v \\ -a^2 w'' - cv \\ v' - b_1 y_1 \\ \vdots \\ v' - b_N y_N \end{pmatrix}^\top, \\ D(\mathcal{A}^*) = \left\{ \begin{pmatrix} w \\ v \\ y_1 \\ \vdots \\ y_N \end{pmatrix}^\top \middle| \begin{array}{c} w, v \in H_0^1(0,1), \\ w'' \in L^2(0,1), \\ \sum_{i=1}^{N}\int_0^x a_i y_i(\tau)d\tau \in H_0^1(0,1), \\ y_i \in L^2(0,1),\ i = 1, \cdots, N. \end{array} \right\} \end{cases} \tag{3.24}$$

Theorem 3.8. *Let $\mathcal{A}$ be defined by (2.5). Then $\sigma_r(\mathcal{A}) = \emptyset$, where $\sigma_r(\mathcal{A})$ denotes the set of residual spectrum of $\mathcal{A}$.*

Proof. Since $\lambda \in \sigma_r(\mathcal{A})$ implies $\overline{\lambda} \in \sigma_p(\mathcal{A}^*)$, the proof will be accomplished if we can show that $\sigma_p(\mathcal{A}) = \sigma_p(\mathcal{A}^*)$. This is because obviously, the eigenvalues of $\mathcal{A}$ are symmetric on the real axis. From (3.24), the eigenvalue problem $\mathcal{A}^* Z = \lambda Z$ for $\lambda \in \mathbb{C}$ and $0 \neq Z = (w, v, y_1, \cdots, y_N) \in D(\mathcal{A}^*)$ reads:

$$\begin{cases} \dfrac{1}{a^2}\displaystyle\sum_{i=1}^{N}\int_0^x a_i y_i(\tau)d\tau - v(x) = \lambda w(x), \\ a^2 w''(x) = -(\lambda + c)v(x), \\ v'(x) - b_i y_i(x) = \lambda y_i(x), \quad i = 1, \cdots, N, \\ v(0) = v(1) = 0. \end{cases} \tag{3.25}$$

Obviously, $\lambda = -b_i, i = 1, 2, \cdots, N$ are the eigenvalues of $\mathcal{A}^*$.
When $\lambda \neq -b_i, i = 1, \cdots, N$, it has

$$y_i(x) = \frac{v'(x)}{\lambda + b_i}, \quad i = 1, 2, \cdots, N.$$

This together with the first equation of (3.25) yields

$$\left[a^2 - \sum_{i=1}^{N}\frac{a_i}{\lambda + b_i}\right] v(x) = -a^2\lambda w(x). \tag{3.26}$$

Then combining (3.26) with the second equation of (3.25) shows

$$\begin{cases} \left[a^2 - \displaystyle\sum_{i=1}^{N}\frac{a_i}{\lambda + b_i}\right] v''(x) = (\lambda^2 + c\lambda)v(x), \\ v(0) = v(1) = 0. \end{cases} \tag{3.27}$$

(3.27) is the same as (3.5) by setting $v = w$. Hence $\mathcal{A}^*$ has the same eigenvalues with $\mathcal{A}$. □

Theorem 3.9. *Let $\mathcal{A}$ be defined as in (2.5) and Δ is given by (3.8). Then*

$$\sigma_c(\mathcal{A}) = \Delta = \Big\{\lambda_{c1}, \lambda_{c2}, \cdots, \lambda_{cN}\Big\}, \tag{3.28}$$

where $\sigma_c(\mathcal{A})$ is the set of the continuous spectrum of $\mathcal{A}$, $\lambda_{ck} \in (-b_k, -b_{k-1})$, $k = 1, 2, \cdots, N$ and $b_0 = 0$.

Proof. Let $\lambda \notin \sigma_p(\mathcal{A})$. For any $\tilde{Z} = (\tilde{w}, \tilde{v}, \tilde{y_1}, \cdots, \tilde{y_N}) \in \mathcal{H}$, since $-b_i \in \sigma_p(\mathcal{A})$, $i = 1, \cdots, N$, solve $(\lambda I - \mathcal{A})Z = \tilde{Z}$ for $Z = (w, v, y_1, \cdots, y_N)$; that is

$$\begin{cases} \lambda w(x) - v(x) = \tilde{w}(x), \\ \lambda v(x) - \left[a^2 w'(x) - \displaystyle\sum_{i=1}^{N} y_i(x)\right]' + cv(x) = \tilde{v}(x), \\ \lambda y_i(x) - \Big[a_i w'(x) - b_i y_i(x)\Big] = \tilde{y_i}(x), \quad i = 1, 2, \cdots, N, \\ w(0) = w(1) = 0, \end{cases} \tag{3.29}$$

to get

$$\begin{cases} v(x) = \lambda w(x) - \tilde{w}(x), \\ y_i(x) = \dfrac{a_i}{\lambda + b_i} w'(x) + \dfrac{1}{\lambda + b_i} \tilde{y}_i(x), \ i = 1, 2, \cdots, N, \end{cases} \tag{3.30}$$

and

$$\begin{cases} \theta(x) = a^2 w'(x) - \displaystyle\sum_{i=1}^{N} y_i(x) = p(\lambda) w'(x) - \tilde{y}(\lambda, x), \\ \theta'(x) = (\lambda^2 + c\lambda) w(x) - \tilde{v}(x) - (\lambda + c)\tilde{w}(x), \end{cases} \tag{3.31}$$

where $p(\lambda)$ is given by (3.10), and

$$\tilde{y}(\lambda, x) = \sum_{i=1}^{N} \frac{\tilde{y}_i(x)}{\lambda + b_i}. \tag{3.32}$$

We claim that $\Delta \subseteq \sigma_c(\mathcal{A})$. In fact, when $\lambda \in \Delta$, it has $p(\lambda) = 0$. By (3.31), we have

$$\begin{cases} w(x) = \dfrac{1}{\lambda^2 + c\lambda} \Big[\tilde{v}(x) + (\lambda + c)\tilde{w}(x) - \tilde{y}'(\lambda, x) \Big], \\ w(0) = w(1) = 0, \end{cases} \tag{3.33}$$

Since $\tilde{w} \in H_0^1(0,1)$, (3.33) means that (3.29) admits a solution if and only if $\tilde{y}(\lambda, x)$ is differentiable, and

$$\tilde{v}(0) - \tilde{y}'(\lambda, 0) = \tilde{v}(1) - \tilde{y}'(\lambda, 1) = 0,$$

which is impossible. Thus $\lambda \notin \rho(\mathcal{A})$, or $\Delta \subseteq \sigma_c(\mathcal{A})$ by Lemma 3.2 and Theorem 3.8.

Now we show that $\sigma_c(\mathcal{A}) \subseteq \Delta$, or equivalently, for any $\lambda \notin \sigma_p(\mathcal{A}) \cup \Delta$, we deduce $\lambda \in \rho(\mathcal{A})$. To do this, assume that $\lambda \notin \sigma_p(\mathcal{A}) \cup \Delta$. By Lemma 2.1, $0 \in \rho(\mathcal{A})$, we need only consider the case of $\lambda \neq 0$. Now, we can rewrite (3.31) as the following first-order system of differential equations:

$$\begin{cases} \dfrac{d}{dx} \begin{bmatrix} w \\ \theta \end{bmatrix} = A(\lambda) \begin{bmatrix} w \\ \theta \end{bmatrix} - \begin{bmatrix} -\dfrac{1}{p(\lambda)} \tilde{y} \\ \tilde{v} + (\lambda + c)\tilde{w} \end{bmatrix}, \\ w(0) = w(1) = 0, \end{cases} \tag{3.34}$$

where

$$A(\lambda) = \begin{bmatrix} 0 & \dfrac{1}{p(\lambda)} \\ \lambda^2 + c\lambda & 0 \end{bmatrix}.$$

Note that

$$e^{A(\lambda)x} = \begin{bmatrix} a_{11}(\lambda, x) & a_{12}(\lambda, x) \\ a_{21}(\lambda, x) & a_{22}(\lambda, x) \end{bmatrix},$$

where

$$\begin{cases} a_{11}(\lambda, x) = \cosh\left(\sqrt{\mu(\lambda)}\ x\right), \ a_{12}(\lambda, x) = \dfrac{1}{p(\lambda)\sqrt{\mu(\lambda)}} \sinh\left(\sqrt{\mu(\lambda)}\ x\right), \\ a_{21}(\lambda, x) = p(\lambda)\sqrt{\mu(\lambda)} \sinh\left(\sqrt{\mu(\lambda)}\ x\right), \ a_{22}(\lambda, x) = \cosh\left(\sqrt{\mu(\lambda)}\ x\right) \end{cases}$$

and

$$\mu(\lambda) = \frac{\lambda^2 + c\lambda}{p(\lambda)}. \tag{3.35}$$

The general solution of (3.34) is given by

$$\begin{aligned}\begin{bmatrix} w(x) \\ \theta(x) \end{bmatrix} &= e^{A(\lambda)x}\begin{bmatrix} w(0) \\ \theta(0) \end{bmatrix} - \int_0^x e^{A(\lambda)(x-\tau)}\begin{bmatrix} -\dfrac{1}{p(\lambda)}\tilde{y}(\lambda,\tau) \\ \tilde{v}(\tau)+(\lambda+c)\tilde{w}(\tau) \end{bmatrix} d\tau \\ &= \begin{bmatrix} a_{12}(\lambda,x)\theta(0) \\ a_{22}(\lambda,x)\theta(0) \end{bmatrix} \\ &\quad + \int_0^x \begin{bmatrix} \dfrac{1}{p(\lambda)}a_{11}(\lambda,x-\tau)\tilde{y}(\lambda,\tau) - a_{12}(\lambda,x-\tau)\Big(\tilde{v}(\tau)+(\lambda+c)\tilde{w}(\tau)\Big) \\ \dfrac{1}{p(\lambda)}a_{21}(\lambda,x-\tau)\tilde{y}(\lambda,\tau) - a_{22}(\lambda,x-\tau)\Big(\tilde{v}(\tau)+(\lambda+c)\tilde{w}(\tau)\Big) \end{bmatrix} d\tau,\end{aligned}$$

that is

$$\begin{cases} w(x) = a_{12}(\lambda,x)\theta(0) + \xi_1(\lambda,x), \\ \theta(x) = a_{22}(\lambda,x)\theta(0) + \xi_2(\lambda,x), \end{cases} \tag{3.36}$$

where

$$\xi_j(\lambda,x) = \int_0^x \left[\frac{\tilde{y}(\lambda,\tau)}{p(\lambda)}a_{j1}(\lambda,x-\tau) - a_{j2}(\lambda,x-\tau)\Big(\tilde{v}(\tau)+(\lambda+c)\tilde{w}(\tau)\Big)\right] d\tau,\ j=1,2.$$

When $\tilde{Z} = (\tilde{w},\tilde{v},\tilde{y_1},\cdots,\tilde{y_N}) = 0$, (3.35) reduces to the eigenvalue problem

$$w(x) = a_{12}(\lambda,x)\theta(0), \quad \theta(x) = a_{22}(\lambda,x)\theta(0).$$

So when $\lambda \in \sigma_p(\mathcal{A})$, $\lambda \neq -b_i, i=1,2,\cdots,N$ if and only if $a_{12}(\lambda,1)=0$, that is

$$a_{12}(\lambda,1) = \frac{1}{\sqrt{p(\lambda)(\lambda^2+c\lambda)}}\sinh\left(\sqrt{\frac{\lambda^2+c\lambda}{p(\lambda)}}\right) = 0,$$

which yields

$$\sinh\left(\sqrt{\frac{\lambda^2+c\lambda}{p(\lambda)}}\right) = 0.$$

This is the characteristic determinant of $\mathcal{A}$, which satisfies (3.14).

Now since $\lambda \notin \sigma_p(\mathcal{A}) \cup \Delta$, by $w(1)=0$, we have

$$\theta(0) = \frac{1}{a_{12}(\lambda,1)}\int_0^1 \left[-\frac{\tilde{y}(\lambda,\tau)}{p(\lambda)}a_{11}(\lambda,1-\tau) + a_{12}(\lambda,1-\tau)\Big(\tilde{v}(\tau)+(\lambda+c)\tilde{w}(\tau)\Big)\right] d\tau. \tag{3.37}$$

So w is uniquely determined by (3.36). By the first equation of (3.31) and the second equation of (3.36), $w' \in L^2(0,1)$. This together with (3.30) shows that $(\lambda I - \mathcal{A})^{-1}$ exists and is bounded. Hence $\lambda \in \rho(\mathcal{A})$. □

4. Riesz basis generation and exponential stability

In this section, we study the Riesz basis property for system (2.3). To do this, we need the following result mentioned in [6] (see also [7]).

Theorem 4.1. *Let A be a densely closed linear operator in a Hilbert space H with isolated eigenvalues $\{\lambda_i\}_1^\infty$ and $\sigma_r(A) = \emptyset$. Let $\{\phi_n\}_1^\infty$ be a Riesz basis for H. Suppose that there are $N_0 \geq 1$ and a sequence of generalized eigenvectors $\{\psi_n\}_{N_0}^\infty$ of A such that*

$$\sum_{n=N_0}^{\infty} \|\psi_n - \phi_n\|_H^2 < \infty. \tag{4.1}$$

Then there exist $M(\geq N_0)$ number of generalized eigenvectors $\{\psi_{n0}\}_1^M$ such that $\{\psi_{n0}\}_1^M \bigcup \{\psi_n\}_{M+1}^\infty$ forms a Riesz basis for H.

Theorem 4.2. *Let $\mathcal{A}$ be defined by (2.5). Then*

(i) *There is a sequence of generalized eigenfunctions of $\mathcal{A}$, which forms a Riesz basis for the state space $\mathcal{H}$.*
(ii) *All eigenvalues with large modulus are algebraically simple.*
(iii) *$\mathcal{A}$ generates a C_0-semigroup $e^{\mathcal{A}t}$ on $\mathcal{H}$.*

Therefore, for the semigroup $e^{\mathcal{A}t}$, the spectrum-determined growth condition holds true: $s(\mathcal{A}) = \omega(\mathcal{A})$, where $s(\mathcal{A}) = \sup\{\mathrm{Re}\lambda \mid \lambda \in \sigma(\mathcal{A})\}$ is the spectral bound of $\mathcal{A}$ and $\omega(\mathcal{A}) = \lim\limits_{t\to\infty} \dfrac{\ln \|e^{\mathcal{A}t}\|}{t}$ is the growth order of $e^{\mathcal{A}t}$.

Proof. By Lemma 3.2, all eigenvalues are located in left half complex plane, the other parts follow directly from (i) and (ii). So we need only to prove (i) and (ii). For any $n \in \mathbb{N}^+$, set

$$U_{n0}(x) = \left(\frac{1}{n\pi}\sin n\pi x, ia\sin n\pi x, 0, \cdots, 0\right), \tag{4.2}$$

$$\begin{cases} \varphi_{n0} = (a\cos n\pi x, ia\sin n\pi x, 0, \cdots, 0) + (0,1,1,\cdots,1)\mathcal{O}(n^{-1}), \\ \varphi_{nk}(x) = \left(a, 0, \dfrac{a_1}{\lambda_{nk}+b_1}, \cdots, \dfrac{a_N}{\lambda_{nk}+b_N}\right)\cos n\pi x + (0,1,\cdots,1)\mathcal{O}(n^{-1}), \\ k = 1,2,\cdots,N. \end{cases} \tag{4.3}$$

Let the reference sequence be given by

$$\begin{cases} \psi_{n0} = (a\cos n\pi x, ia\sin n\pi x, 0, \cdots, 0), \\ \psi_{nk}(x) = \left(0, 0, \dfrac{a_1}{\lambda_{nk}+b_1}, \cdots, \dfrac{a_N}{\lambda_{nk}+b_N}\right)\cos n\pi x, \quad k = 1,2,\cdots,N. \end{cases} \tag{4.4}$$

Since $b_j \neq b_k,\ \lambda_{nj} \neq \lambda_{nk},\ 1 \leq j < k \leq N$, a direct computation shows that

$$\det \begin{bmatrix} \dfrac{a_1}{\lambda_{n1}+b_1} & \dfrac{a_1}{\lambda_{n2}+b_1} & \cdots & \dfrac{a_1}{\lambda_{nN}+b_1} \\ \dfrac{a_2}{\lambda_{n1}+b_2} & \dfrac{a_2}{\lambda_{n2}+b_2} & \cdots & \dfrac{a_2}{\lambda_{nN}+b_2} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{a_N}{\lambda_{n1}+b_N} & \dfrac{a_N}{\lambda_{n2}+b_N} & \cdots & \dfrac{a_N}{\lambda_{nN}+b_N} \end{bmatrix} \neq 0.$$

Hence,

$$\left\{\psi_{n0}, \overline{\psi_{n0}}, \psi_{n1}, \psi_{n2}, \cdots, \psi_{nN}\right\}_1^\infty \tag{4.5}$$

forms a Riesz basis for $\mathcal{H}_1 = \left[L^2(0,1)\right]^{N+2}$. By (4.3), (4.4) and Theorem 3.6, there exists an $N_0 \in \mathbb{N}^+$, such that

$$\begin{aligned}
& \sum_{n=N_0}^{\infty} \left[\|W_{n0} - U_{n0}\|_{\mathcal{H}}^2 + \|\overline{W_{n0}} - \overline{U_{n0}}\|_{\mathcal{H}}^2\right] + \sum_{k=1}^{N} \left\|W_{nk} - \frac{W_{n0} + \overline{W_{n0}}}{2} - \psi_{nk}\right\|_{\mathcal{H}}^2 \\
= \quad & \sum_{n=N_0}^{\infty} \left[\|\varphi_{n0} - \psi_{n0}\|_{\mathcal{H}_1}^2 + \|\overline{\varphi_{n0}} - \overline{\psi_{n0}}\|_{\mathcal{H}_1}^2\right] + \sum_{k=1}^{N} \left\|\varphi_{nk} - \frac{\varphi_{n0} + \overline{\varphi_{n0}}}{2} - \psi_{nk}\right\|_{\mathcal{H}_1}^2 \\
< \quad & \infty.
\end{aligned} \tag{4.6}$$

So by Theorem 4.1, we conclude that the generalized eigenfunctions of $\mathcal{A}$ forms a Riesz basis in $\mathcal{H}$. Next, we need to prove the algebraic simplicity for the eigenvalues of $\mathcal{A}$. From the proof of the second part of Theorem 3.9, the order of each $\lambda \in \sigma_p(\mathcal{A})\backslash\{-b_i,\ i = 1, 2, \cdots, N\}$, as a pole of $R(\lambda, \mathcal{A})$, with sufficiently large modulus is less than or equal to the multiplicity of λ as a zero of the entire function $\sinh(\sqrt{\mu(\lambda)})$, where $\mu(\lambda)$ is given by (3.35). Since λ is geometrically simple and from (3.13) all roots of $\sinh(\sqrt{\mu(\lambda)}) = 0$, which satisfies (3.14), with large moduli are simple, the result then follows from the formula: $m_a \le p \cdot m_g$ (see e.g. [10],p.148), where p denotes the order of the pole of the resolvent operator and m_a, m_g denote the algebraic and geometric multiplicities, respectively. Hence (ii) holds true. $\square$

Now we establish the exponential stability of the system (2.3).

Theorem 4.3. *Let $\mathcal{A}$ be defined by (2.5). Then the spectrum-determined growth condition $\omega(\mathcal{A}) = s(\mathcal{A})$ holds true for the C_0 Semigroup $e^{\mathcal{A}t}$ generated by $\mathcal{A}$. Moreover, the system (2.3) is exponentially stable, i.e., there exist two positive constants M and ω such that the C_0 Semigroup $e^{\mathcal{A}t}$ satisfies*

$$\|e^{\mathcal{A}t}\| \le Me^{-\omega t}, \tag{4.7}$$

for some $M, \omega > 0$.

Proof. The spectrum-determined growth condition follows from Theorem 4.2. By Lemma 3.2, Lemma 3.4, (3.20) and Theorem 3.8, Theorem 3.9, for each $\lambda \in \sigma(\mathcal{A})$, we have $\mathrm{Re}\lambda < 0$. This, together with (3.16), (3.18) and the spectrum-determined growth condition, shows that $e^{\mathcal{A}t}$ is exponentially stable. $\square$

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