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Length-expanding Lipschitz maps on totally regular continua

Vladimír Špitalský

Department of Mathematics, Faculty of Natural Sciences, Matej Bel University, Tajovského 40, 974 01 Banská Bystrica, Slovakia

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ABSTRACT

The tent map is an elementary example of an interval map possessing many interesting properties, such as dense periodicity, exactness, Lipschitzness and a kind of length-expansiveness. It is often used in constructions of dynamical systems on the interval/trees/graphs. The purpose of the present paper is to construct, on totally regular continua (i.e. on topologically rectifiable curves), maps sharing some typical properties with the tent map. These maps will be called length-expanding Lipschitz maps, briefly LEL maps. We show that every totally regular continuum endowed with a suitable metric admits a LEL map. As an application we obtain that every non-degenerate totally regular continuum admits an exactly Devaney chaotic map with finite entropy and the specification property.

1. Introduction

The tent map is the piecewise linear map f on the interval $I = [0, 1]$ given by $f(x) = 2\min\{x, 1 - x\}$. The properties of this map, conjugate to the full logistic map, include Lipschitzness, length-expansiveness (in a sense that it doubles the length of every subinterval J of I not containing $1/2$), exactness, specification, finite positive topological entropy and dense periodicity, just to name a few. This map, together with "generalized" tent maps, i.e. piecewise linear continuous maps $f_k : I \to I$ ($k \geqslant 3$) fixing 0 and mapping linearly every interval $[(i - 1)/k, i/k]$ onto I, are frequently used in dynamics. Usefulness of these maps lies in the fact that on one hand they are very simple (and so we have easy explicit formulae for iterates, periodic points, horseshoes, etc.) and on the other hand they are very "powerful". They are often used in constructions of systems on the interval/trees/graphs with special properties. For example, it is known that to construct a transitive map on the unit interval with the smallest possible topological entropy, one can define $g : I \to I$ in such a way that $1/2$ is a fixed point, g maps linearly $I_0 = [0, 1/2]$ onto $I_1 = [1/2, 1]$ and $g|_{I_1} : I_1 \to I_0$ is "tent-like". Analogously one can define a transitive map with the smallest possible entropy $(1/n)\log 2$ on any n-star S_n ($n \geqslant 3$), see [3]; the map fixes the branch point of S_n, maps cyclically each branch to the next one, all but one linearly and the remaining one in a "tent-like" way.

Unfortunately, when one wants to construct a map with given properties on curves more general than graphs, he/she faces the problem that no direct analogue of the tent map on such curves is known. Take e.g. the ω-star X, which is a very simple dendrite defined as an infinite wedge of arcs. A construction of a transitive finite entropy map on X is much more complicated than on n-stars and, as far as we know, no such construction has been available in literature. The only result in this direction known to us is the theorem of Agronsky and Ceder [1] stating that any finite-dimensional Peano continuum (hence also the ω-star) admits a transitive map; however, the proof does not say anything about the entropy of the map.

The purpose of the present paper is to construct, on continua more general than graphs, a family of maps sharing some typical properties with the tent map. Since the key property of these maps will deal with the *length* (Hausdorff

E-mail address: vladimir.spitalsky@umb.sk.

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one-dimensional measure) of subcontinua and their images, the natural class of spaces to consider is the class of *rectifiable curves*, i.e. continua of finite length. Topologically they coincide with the class of totally regular continua. Recall that a continuum X is *totally regular* if for every point $x \in X$ and every countable set $P \subseteq X$ there is a basis of neighborhoods of x with finite boundary not intersecting P. This notion was introduced in [20], but the class of these continua was studied a long time before, see e.g. [22,8,10,9]. For more details on totally regular continua see Section 2.4.

Before stating the main results of the paper we need to introduce the notion of a length-expanding Lipschitz map. Let X be a non-degenerate totally regular continuum. We say that a family $\mathcal{C}$ of non-degenerate subcontinua of X is *dense* if every nonempty open set in X contains a member of $\mathcal{C}$. Recall that a map $f:(X,d)\to(X',d')$ between metric spaces is *Lipschitz-L* if $d'(f(x),f(y))\leqslant L\cdot d(x,y)$ for every $x,y\in X$. For a metric space (X,d), the Hausdorff one-dimensional measure is denoted by $\mathcal{H}^1_d$.

Definition A. Let $X=(X,d)$, $X'=(X',d')$ be non-degenerate (totally regular) continua of finite length and let $\mathcal{C}$, $\mathcal{C}'$ be dense systems of subcontinua of X, X', respectively. We say that a continuous map $f:X\to X'$ is *length-expanding* with respect to $\mathcal{C}$, $\mathcal{C}'$ if there exists $\varrho>1$ (called *length-expansivity constant* of f) such that, for every $C\in\mathcal{C}$, $f(C)\in\mathcal{C}'$ and

$$\text{if } f(C)\neq X' \quad \text{then } \mathcal{H}^1_{d'}\big(f(C)\big)\geqslant \varrho\cdot\mathcal{H}^1_d(C).$$

Moreover, if f is surjective and Lipschitz-L we say that $f:(X,d,\mathcal{C})\to(X',d',\mathcal{C}')$ is *(ϱ,L)-length-expanding Lipschitz*. Sometimes we briefly say that f is *(ϱ,L)-LEL* or only *LEL*. On the other hand, when we wish to be more precise, we say that f is *$(\mathcal{C},\mathcal{C}',\varrho,L)$-LEL*.

A few comments are necessary. Assume that $f:(X,d,\mathcal{C})\to(X',d',\mathcal{C}')$ is (ϱ,L)-LEL and denote by $\mathcal{C}_X$ and $\mathcal{C}_{X'}$ the systems of *all* non-degenerate subcontinua of X and X', respectively. Obviously, then also $f:(X,d,\mathcal{C})\to(X',d',\mathcal{C}_{X'})$ is (ϱ,L)-LEL. However, one cannot claim that $f:(X,d,\mathcal{C}_X)\to(X',d',\mathcal{C}')$ is (ϱ,L)-LEL. In fact, for some spaces (X,d), (X',d') there is no LEL map $f:(X,d,\mathcal{C}_X)\to(X',d',\mathcal{C}_{X'})$. For instance this is the case when X is the ω-star and $X'=I$. To show this, suppose that there is a (ϱ,L)-LEL map $f:(X,d,\mathcal{C}_X)\to(X',d',\mathcal{C}_{X'})$. Take $k\in\mathbb{N}$ such that $\varrho>L/k$ and find a k-star C in X such that every edge of C is mapped onto the same proper subinterval of X'. Then $\mathcal{H}^1_{d'}(f(C))\leqslant(L/k)\cdot\mathcal{H}^1_d(C)<\varrho\cdot\mathcal{H}^1_d(C)$, a contradiction.

Our first result says that in the special case when $X=X'$ and $\mathcal{C}=\mathcal{C}'$, LEL maps have interesting dynamical properties. (For the definitions of the corresponding notions, see Section 2.)

Proposition B. *Let $f:(X,d,\mathcal{C})\to(X,d,\mathcal{C})$ be a LEL map. Then f is exact and has finite positive entropy. Moreover, if f is the composition $\varphi\circ\psi$ of some maps $\psi:X\to I$ and $\varphi:I\to X$, then f has the specification property and so it is exactly Devaney chaotic.*

The above mentioned tent-like maps $f_k:I\to I$ (where $k\geqslant 3$ and I is endowed with the Euclidean metric d_I) are $(\mathcal{C}_I,\mathcal{C}_I,k/2,k)$-LEL, where $\mathcal{C}_I$ is the system of all non-degenerate closed subintervals of I. Here $k\geqslant 3$ because the classical tent map f_2 is not $(\mathcal{C}_I,\mathcal{C}_I,\varrho,L)$-LEL for any $\varrho>1$ and any L. However, it becomes $(\mathcal{C}_I,\mathcal{C}_I,\varrho,L)$-LEL (for some $\varrho>1$ and L) after a slight change of the metric. One can easily construct examples of LEL maps between arbitrary graphs, even in the form of the composition $\varphi\circ\psi$ as in Proposition B; one can use e.g. the maps from [2, Lemma 3.6]. Further, for a given continuum (X,d) of finite length, one can often find $\mathcal{C}$, $\mathcal{C}'$ and construct LEL-maps $\varphi:(I,d_I,\mathcal{C}_I)\to(X,d,\mathcal{C})$ and $\psi:(X,d,\mathcal{C}')\to(I,d_I,\mathcal{C}_I)$. However, it is not so easy to obtain $\mathcal{C}'\supseteq\mathcal{C}$; this inclusion is desirable since then also the composition $\psi\circ\varphi$ is LEL (see Lemma 9).

Our main results, the proofs of which were inspired by [1] and [6], assert that such LEL maps can always be found provided we allow to change the metric on X (the new metric still being compatible with the topology). Recall that a metric d on X is *convex* if for every $x,y\in X$ there is $z\in X$ such that $d(x,z)=d(z,y)=d(x,y)/2$. For two points $a,b\in X$ of a continuum X, $\mathrm{Cut}_X(a,b)$ denotes the set of points $x\in X$ such that a,b belong to different components of $X\setminus\{x\}$.

Theorem C. *For every non-degenerate totally regular continuum X and every $a,b\in X$ we can find a convex metric $d=d_{X,a,b}$ on X and Lipschitz surjections $\varphi_{X,a,b}:I\to X$, $\psi_{X,a,b}:X\to I$ with the following properties:*

(a) $\mathcal{H}^1_d(X)=1$;
(b) *the system $\mathcal{C}=\mathcal{C}_{X,a,b}=\{\varphi_{X,a,b}(J)\colon J$ is a closed subinterval of $I\}$ is a dense system of subcontinua of X;*
(c) *for every $\varrho>1$ there are a constant L_ϱ (depending only on ϱ) and (ϱ,L_ϱ)-LEL maps*

$$\varphi:(I,d_I,\mathcal{C}_I)\to(X,d,\mathcal{C}) \quad \text{and} \quad \psi:(X,d,\mathcal{C})\to(I,d_I,\mathcal{C}_I)$$

with $\varphi(0)=a$, $\varphi(1)=b$, $\psi(a)=0$ and such that $\varphi=\varphi_{X,a,b}\circ f_k$, $\psi=f_l\circ\psi_{X,a,b}$ for some $k,l\geqslant 3$.

Moreover, if $\mathrm{Cut}_X(a,b)$ is uncountable, d, φ, ψ can be assumed to satisfy

(d) *$d(a,b)>1/2$ and $\psi(b)=1$.*

Theorem D. *Keeping the notation from Theorem C, for every $\varrho > 1$, every non-degenerate totally regular continua X, X' and every points $a, b \in X$, $a', b' \in X'$ there are a constant L_ϱ (depending only on ϱ) and a (ϱ, L_ϱ)-LEL map*

$$f : (X, d_{X,a,b}, \mathcal{C}_{X,a,b}) \to \left(X', d_{X',a',b'}, \mathcal{C}_{X',a',b'}\right)$$

with $f(a) = a'$ and, provided $\mathrm{Cut}_X(a, b)$ is uncountable, $f(b) = b'$. Moreover, f can be chosen to be the composition $\varphi \circ \psi$ of two LEL-maps $\psi : X \to I$ and $\varphi : I \to X'$.

In [1] it was shown that every non-degenerate finite-dimensional Peano continuum admits an exactly Devaney chaotic map and that every finite union of non-degenerate finite-dimensional Peano continua admits a Devaney chaotic map. Theorem D and Proposition B imply the following results which, on one hand, deal with smaller class of spaces, but on the other hand ensure finiteness of the entropy.

Corollary E. *Every non-degenerate totally regular continuum admits an exactly Devaney chaotic map with finite positive entropy and specification.*

Corollary F. *Every finite union of disjoint non-degenerate totally regular continua admits a Devaney chaotic map with finite positive entropy.*

In a subsequent paper we deal with the problem of determining the infima of entropies of transitive/exact/(exactly) Devaney chaotic maps on a given totally regular continuum and we show that under some conditions this infimum is zero. The constructions are heavily based on Theorems C and D. To illustrate usefulness of LEL maps let us sketch here an example which shows how easy is to construct a small entropy transitive system on the ω-star.

Example 1. Let X be the ω-star with the branch point a and edges A_i $(i = 1, 2, \ldots)$; i.e. $X = \bigcup_i A_i$ and $A_i \cap A_j = \{a\}$ for every $i \neq j$. Take arbitrarily large k, put $Y = \bigcup_{i \geqslant k} A_i$ and define a convex metric d on X in such a way that it coincides with $d_{Y,a,a}$ on Y and each of the sets $A_1, \ldots, A_{k-1}$ has length 1. Fix $\varrho > 1$. By Theorem C there are (ϱ, L_ϱ)-maps $f_{k-1} : A_{k-1} \to Y$, $f_k : Y \to A_1$ fixing a. Let $f_i : A_i \to A_{i+1}$ $(i = 1, \ldots, k-2)$ be isometries fixing a. Then it suffices to define $f : X \to X$ by $f|_{A_i} = f_i$ for $i < k$ and $f|_Y = f_k$. The map $f^k|_Y : Y \to Y$ is exact and has dense periodic points by Proposition B; moreover, it is Lipschitz-L_ϱ^2. So f is Devaney chaotic with entropy $h(f) \leqslant (2/k)\log L_\varrho$, where L_ϱ does not depend on k.

The paper is organized as follows. In the next section we recall all the needed definitions and facts. In Section 3 we prove some basic properties of LEL maps. The main part of the paper—Sections 4 and 5—are devoted to the construction of LEL maps from the unit interval onto a given totally regular continuum and vice versa, see Proposition 20. Finally, in Section 6 we prove the main results of the paper, namely Theorems C, D and Corollaries E, F.

2. Preliminaries

Here we briefly recall all the notions and results which will be needed in the rest of the paper. The terminology is taken mainly from [16,19,17,12].

If M is a set, its cardinality is denoted by $\#M$. The cardinality of infinite countable sets is denoted by $\aleph_0$. If M is a singleton set we often identify it with its only point. We write $\mathbb{N}$ for the set of positive integers $\{1, 2, 3, \ldots\}$, $\mathbb{R}$ for the set of reals and I for the unit interval $[0, 1]$. By an interval we mean any nonempty connected subset of $\mathbb{R}$ (possibly degenerate to a point). For intervals J, J' we write $J \leqslant J'$ if $t \leqslant s$ for every $t \in J$, $s \in J'$.

By a *space* we mean any nonempty metric space. A space is called *degenerate* provided it has only one point; otherwise it is called *non-degenerate*. If E is a subset of a space $X = (X, d)$ we denote the closure, the interior and the boundary of E by $\overline{E}$, $\mathrm{int}(E)$ and ∂E, respectively, and we write $d(E)$ for the diameter of E. We say that two sets $E, F \subseteq X$ are *non-overlapping* if they have disjoint interiors. For $x \in X$ and $r > 0$ we denote the closed ball with the center x and radius r by $B(x, r)$. If f is a map defined on X and $\mathcal{C}$ is a system of subsets of X we denote the system $\{f(C)\colon C \in \mathcal{C}\}$ by $f(\mathcal{C})$.

A (*discrete*) *dynamical system* is a pair (X, f) where $X = (X, d)$ is a compact metric space and $f : X \to X$ is a continuous map. For $n \in \mathbb{N}$ we denote the composition $f \circ f \circ \cdots \circ f$ (n-times) by f^n. A point $x \in X$ is a *periodic point* of f if $f^n(x) = x$ for some $n \in \mathbb{N}$. The *topological entropy* of a dynamical system (X, f) is denoted by $h(f)$. We say that (X, f) is (*topologically*) *transitive* if for every nonempty open sets $U, V \subseteq X$ there is $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$. A system (X, f) is (*topologically*) *exact* or *locally eventually onto* if for every nonempty open subset U of X there is $n \in \mathbb{N}$ such that $f^n(U) = X$. Further, (X, f) is *Devaney chaotic* (*exactly Devaney chaotic*) provided X is infinite, f is transitive (exact) and has dense set of periodic points. Finally, a system (X, f) is said to satisfy the *specification property* if for every $\varepsilon > 0$ there is m such that for every $k \geqslant 2$, for every k points $x_1, \ldots, x_k \in X$, for every integers $a_1 \leqslant b_1 < \cdots < a_k \leqslant b_k$ with $a_i - b_{i-1} \geqslant m$ $(i = 2, \ldots, k)$ and for every integer $p \geqslant m + b_k - a_1$, there is a point $x \in X$ with $f^p(x) = x$ such that

$$d\left(f^n(x), f^n(x_i)\right) \leqslant \varepsilon \quad \text{for } a_i \leqslant n \leqslant b_i,\ 1 \leqslant i \leqslant k.$$

2.1. Continua

A *continuum* is a connected compact metric space. A *cut point* (or a *separating point*) of a continuum X is any point $x \in X$ such that $X \setminus \{x\}$ is disconnected. A point x of a continuum X is called a *local separating point* of X if there is a connected neighborhood U of x such that $U \setminus \{x\}$ is not connected. If a, b are points of X then any cut point of X such that a, b belong to different components of $X \setminus \{x\}$ is said to *separate* a, b. The set of all such points is denoted by $\mathrm{Cut}(a,b)$ or $\mathrm{Cut}_X(a,b)$. If $a = b$ then obviously $\mathrm{Cut}(a,b) = \emptyset$.

Let X be a continuum, let $x \in X$ and let m be a cardinal number. We say that the *order* of x is at most m, written $\mathrm{ord}_X(x) \leqslant m$, provided X has a local basis of open neighborhoods of X the boundary of which has cardinality at most m. If m is the least such cardinal we write $\mathrm{ord}_X(x) = m$ with one exception: if $m = \aleph_0$ and x has a basis of neighborhoods with finite boundary, we write $\mathrm{ord}_X(x) = \omega$. If $\mathrm{ord}_X(x) = \omega$ or $\mathrm{ord}_X(x)$ is finite we say that x has *finite order* and we write $\mathrm{ord}_X(x) \leqslant \omega$. The points of order 1 are called *end points*, the points of order 2 are called *ordinary points* and the points of order at least 3 are called *branch points* of X; the sets of all end, ordinary and branch points are denoted by $E(X)$, $O(X)$ and $B(X)$, respectively.

Tightly connected with the order of a point is the following notion, see e.g. [22]. A point x of a continuum X is said to be of *degree* m, written $\deg_X(x) = m$, provided m is the least cardinal such that for every $\varepsilon > 0$ there exists an *uncountable* family of neighborhoods of x with diameters less than ε, each having the boundary of cardinality at most m and such that for any two neighborhoods U, V either $\overline{U} \subseteq V$ or $\overline{V} \subseteq U$. Again if $m = \aleph_0$ and the neighborhoods can be chosen with finite boundary we write $\deg_X(x) = \omega$ instead of $\deg_X(x) = \aleph_0$. We say that x has *finite degree* and write $\deg_X(x) \leqslant \omega$ if the degree of x is either finite or ω. Trivially always $\mathrm{ord}_X(x) \leqslant \deg_X(x)$ but there are examples when $\mathrm{ord}_X(x) < \deg_X(x)$; e.g. if X is the Sierpiński triangle then the order of every point $x \in X$ is at most 4 and the degree is equal to the cardinality of the continuum [22].

Let X be a continuum. A metric d on X is said to be *convex* provided for every distinct $x, y \in X$ there is $z \in X$ such that $d(x,z) = d(z,y) = d(x,y)/2$. By [4, Theorem 8] every locally connected continuum admits a compatible convex metric.

2.2. Graphs

An *arc* A is any homeomorphic image of the unit interval I; the end points of A are the images of the points $0, 1$. A *simple closed curve* is any homeomorphic image of the unit circle $\mathbb{S}^1$.

By a *graph* we mean a continuum which can be written as the union of finitely many arcs which are either disjoint or intersect only at their end points. These arcs are called *edges* and their end points are called *vertices* of the graph. So we allow vertices of order 2 and thus the edges and vertices are not defined uniquely. Notice also that we do not allow simple closed curves to be edges of a graph. By a *subgraph* of a graph G we mean any non-degenerate subcontinuum H of G; so the vertices/edges of H need not be vertices/edges of G.

Let $G = (G,d)$ be a graph with $\mathcal{H}^1_d(G) < \infty$ and let a, b be vertices of G. By a *path* in G from a to b we mean a sequence $\pi = a_0E_1a_1E_2 \dots a_{k-1}E_ka_k$, where a_i $(i = 0, \dots, k)$ are vertices of G such that $a_0 = a$, $a_k = b$ and E_j $(j = 1, \dots, k)$ are edges of G with end points a_{j-1}, a_j; the number k is called the *length* of the path π. A *natural parametrization* of a path $\pi = a_0E_1a_1E_2 \dots a_{k-1}E_ka_k$ is any continuous map $\kappa : J \to G$ defined on a compact interval $J = [s,t] \subseteq \mathbb{R}$ such that $\kappa(s) = a_0$, $\kappa(t) = a_k$ and we can write J as the union $J_1 \cup J_2 \cup \dots \cup J_k$ of non-overlapping closed subintervals such that $J_1 \leqslant J_2 \leqslant \dots \leqslant J_k$ and the restriction of $\kappa|_{J_j} : J_j \to E_j$ is an isometry for every $j = 1, \dots, k$.

2.3. Hausdorff one-dimensional measure and Lipschitz maps

For a Borel subset B of a separable metric space (X,d) the *one-dimensional Hausdorff measure* of B is defined by

$$\mathcal{H}^1_d(B) = \lim_{\delta \to 0} \inf\left\{\sum_{i=1}^{\infty} d(E_i)\colon\ B \subseteq \bigcup_{i=1}^{\infty} E_i,\ d(E_i) \leqslant \delta,\ E_i \subseteq X\right\},$$

with the interpretation $d(\emptyset) = 0$ (here we may assume that the sets E_i are closed). We say that (X,d) has *finite length* if $\mathcal{H}^1_d(X) < \infty$. By e.g. [13, Proposition 4A],

$$\mathcal{H}^1_d(C) \geqslant d(C) \quad \text{whenever } C \text{ is a connected Borel subset of } X.$$

If $A \subseteq X$ is an arc then $\mathcal{H}^1_d(A)$ is equal to the length of A [12, Lemma 3.2]. In the case when (X,d) is the Euclidean real line $\mathbb{R}$ and $J \subset \mathbb{R}$ is an interval, $\mathcal{H}^1_d(J)$ is equal to the length of J and we denote it simply by $|J|$.

If (X,d) is a continuum of finite length endowed with a convex metric d, then it has the so-called *geodesic property* (see e.g. [13, Corollary 4E]): for every distinct $x, y \in X$ there is an arc A with end points x, y such that $d(x,y) = \mathcal{H}^1_d(A)$; any such arc A is called a *geodesic arc* or shortly a *geodesic*. Every subarc of a geodesic is again a geodesic. If x, y are the end points of a geodesic A and $z \in A$ then $d(x,y) = d(x,z) + d(z,y)$.

A map $f:(X,d)\to(Y,\varrho)$ between metric spaces is called *Lipschitz* with a Lipschitz constant $L\geqslant 0$, shortly *Lipschitz-L*, provided $\varrho(f(x),f(x'))\leqslant L\cdot d(x,x')$ for every $x,x'\in X$; the smallest such L is denoted by $\mathrm{Lip}(f)$ and is called the *Lipschitz constant* of f. If $f:X\to Y$ is Lipschitz-L then $\mathcal{H}^1_\varrho(f(B))\leqslant L\cdot\mathcal{H}^1_d(B)$ for every Borel set $B\subset X$ such that $f(B)$ is Borel-measurable [12, p. 10]. We omit the proof of the following lemma.

Lemma 2. *Let $X=(X,d)$ be a non-degenerate (totally regular) continuum of finite length and let $\varphi:I\to X$ be a Lipschitz surjection. Put*

$$\mathcal{C}=\varphi(\mathcal{C}_I)=\{\varphi(J):\ J \text{ is a non-degenerate closed subinterval of } J\}.$$

Then the following hold:

(1) *$\mathcal{C}$ is a dense system of subcontinua of X;*
(2) *for every $\varepsilon>0$ the space X can be covered by some $C_1,\dots,C_k\in\mathcal{C}$ satisfying $\mathcal{H}^1_d(C_i)<\varepsilon$ for $i=1,\dots,k$.*

2.4. Totally regular continua

By e.g. [16,20], a continuum X is called

- a *dendrite* if it is locally connected and contains no simple closed curve;
- a *local dendrite* if it is locally connected and contains at most finitely many simple closed curves;
- *completely regular* if it contains no non-degenerate nowhere dense subcontinuum;
- *totally regular* if for every $x\in X$ and every countable set $P\subseteq X$ there is a basis of neighborhoods of x with finite boundary not intersecting P;
- *regular* if every $x\in X$ has a basis of neighborhoods with finite boundary, i.e. $\mathrm{ord}_X(x)\leqslant\omega$ for every x;
- *hereditarily locally connected* if every subcontinuum of X is locally connected;
- *rational* if every $x\in X$ has a basis of neighborhoods with countable boundary, i.e. $\mathrm{ord}_X(x)\leqslant\aleph_0$ for every x;
- a *curve* if it is one-dimensional.

Notice that (local) dendrites as well as completely regular continua are totally regular and (totally) regular continua are hereditarily locally connected, hence they are locally connected curves. Totally regular continua are also called *continua of finite degree* since they are just those continua X for which every point x has finite degree $\deg_X(x)\leqslant\omega$ [8]. This and other conditions equivalent to total regularity are summarized in the following theorem.

Theorem 3. *For a continuum X the following are equivalent:*

(1) *X is totally regular;*
(2) *X is of finite degree (i.e. $\deg_X(x)\leqslant\omega$ for every x);*
(3) *X has a (convex) metric d such that (X,d) has finite length;*
(4) *X has a (convex) metric d such that (X,d) is a Lipschitz image of the unit interval;*
(5) *X has a (convex) metric d such that for every $x\in X$ and for almost every $r>0$ the boundary of the closed ball $B(x,r)$ is finite;*
(6) *every non-degenerate subcontinuum of X contains uncountably many local separating points;*
(7) *X is locally connected and for every disjoint closed sets $E,F\subseteq X$ there are disjoint perfect sets $N_1,\dots,N_k$ such that every subcontinuum of X intersecting both E and F contains some N_i.*

Proof. The equivalence of (1), (2), (3), (6) and (7) follows from [8,22,10,15,9]. Immediately (4) implies (3) and (5) implies (2). By e.g. [14, Lemma 2A], (3) implies (4). Finally, the fact that (3) implies (5) follows from the following inequality (see e.g. [13, 1A(f)] or [18, Theorem 7.7]) applied to the map $f:X\to\mathbb{R}$, $f(x')=d(x,x')$. The inequality says that if $f:(X,d)\to(Y,\varrho)$ is Lipschitz-1 then

$$\mathcal{H}^1_d(X)\geqslant\int_Y^*\#^*f^{-1}(y)\,d\mathcal{H}^1_\varrho(y)$$

where $\#^*$ denotes the cardinality of a finite set and ∞ for infinite sets and $\int_Y^* h\,d\mu$ is the infimum of integrals $\int_Y g\,d\mu$ as g runs over μ-measurable functions from Y to $[0,\infty]$ such that $g\geqslant h$. Hence if (X,d) has finite length then $f^{-1}(y)$ is finite for $\mathcal{H}^1_\varrho$–almost every $y\in Y$. $\square$

By [9], if d is a metric on a totally regular continuum X with $\mathcal{H}^1_d(X)<\infty$ then there is a (unique) convex metric d^* on X such that $d^*(x,y)\geqslant d(x,y)$ for every $x,y\in X$ and $\mathcal{H}^1_{d^*}(B)=\mathcal{H}^1_d(B)$ for every Borel $B\subseteq X$; it is defined by

$$d^*(x,y)=\inf\{\mathcal{H}^1_d(A):\ A \text{ is an arc from } x \text{ to } y\}.$$

2.5. Monotone inverse limits

An *inverse sequence* is a sequence $(X_n, f_n)_{n\in\mathbb{N}}$ where X_n is a compact metric space and $f_n : X_{n+1} \to X_n$ is a continuous map for every $n \in \mathbb{N}$. The *inverse limit* of an inverse sequence $(X_n, f_n)_{n\in\mathbb{N}}$ is the subspace $X_\infty = \varprojlim(X_n, f_n)$ of the product $\prod_{n=1}^{\infty} X_n$ given by

$$X_\infty = \varprojlim(X_n, f_n) = \left\{ (x_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} X_n \colon\ f_n(x_{n+1}) = x_n \text{ for every } n \in \mathbb{N} \right\}.$$

The maps f_n are called *bonding maps*. For $n \in \mathbb{N}$ the projection from X_∞ onto the n-th coordinate will be denoted by $\pi_n : X_\infty \to X_n$. From now on we will assume that every f_n (and hence every π_n) is surjective.

A fundamental result states that the inverse limit of continua is a continuum [19, Theorem 2.1]. Moreover, if the dimension of every X_n is at most d then also $\dim X_\infty \leqslant d$ [11, Theorem 1.13.4]. Hence the inverse limit of curves is a curve.

The special case important for us is when the bonding maps are monotone. (Recall that a continuous map $f : X \to Y$ is *monotone* if every preimage $f^{-1}(y)$ is connected.) Then also every projection map π_n is monotone [17, Proposition 2.1.13]. The following theorem combines [17, Corollary 2.1.14], [20, Theorem 3.6] and [19, Theorem 10.36].

Theorem 4. *Let $X_\infty = \varprojlim(X_n, f_n)$ be the inverse limit of continua X_n with surjective monotone bonding maps. If every X_n is locally connected* (*totally regular, a dendrite*) *then also X_∞ is locally connected* (*totally regular, a dendrite*)*.*

It is often the case that a continuum X is homeomorphic to the inverse limit of some "simpler" continua X_n. For example every continuum is the inverse limit of compact connected polyhedra [19, Theorem 2.15] and every curve is the inverse limit of graphs [11, Theorem 1.13.2]. Fundamental results for *monotone* inverse limits and locally connected curves are summarized below, see [20, Theorem 2.2], [6, Theorem 3] and e.g. the proof of [19, Theorem 10.32].

Theorem 5. *Every locally connected curve* (*totally regular continuum, dendrite*) *is the monotone inverse limit of regular continua* (*graphs, trees*)*.*

Notice that for non-degenerate totally regular continua and for non-degenerate dendrites the bonding maps f_n in the previous theorem can be chosen such that $f_n^{-1}(x)$ is non-degenerate for exactly one point x.

The following theorem gives us a way to define the so-called *induced map* between inverse limits, see e.g. [17, Theorems 2.1.46–48].

Theorem 6. *Let $(X_n, f_n)_n$, $(X'_n, f'_n)_n$ be inverse sequences and let $g_n : X_n \to X'_n$ $(n \in \mathbb{N})$ be continuous maps such that for every n the left-hand side diagram commutes*:

$$\begin{array}{ccc} X_{n+1} & \xrightarrow{f_n} & X_n \\ {\scriptstyle g_{n+1}}\downarrow & & \downarrow{\scriptstyle g_n} \\ X'_{n+1} & \xrightarrow{f'_n} & X'_n \end{array} \qquad \begin{array}{ccc} X_\infty & \xrightarrow{\pi_n} & X_n \\ {\scriptstyle g_\infty}\downarrow & & \downarrow{\scriptstyle g_n} \\ X'_\infty & \xrightarrow{\pi'_n} & X'_n \end{array}$$

Then there is a unique continuous map $g_\infty = \varprojlim g_n : X_\infty \to X'_\infty$ such that for every n the right-hand side diagram commutes. The map g_∞ is given by

$$g_\infty(x_1, x_2, x_3, \ldots) = \big(g_1(x_1), g_2(x_2), g_3(x_3), \ldots\big).$$

Moreover, if every g_n is surjective (*injective*) *then g_∞ is surjective* (*injective*)*.*

3. Properties of length-expanding Lipschitz maps

Here we briefly state basic properties of the class of LEL maps. We start with the proof of Proposition B stated in the introduction.

Proposition B. *Let $f : (X, d, \mathcal{C}) \to (X, d, \mathcal{C})$ be a LEL map. Then f is exact and has finite positive entropy. Moreover, if f is the composition $\varphi \circ \psi$ of some maps $\psi : X \to I$ and $\varphi : I \to X$, then f has the specification property and so it is exactly Devaney chaotic.*

Proof. Let $f : (X, d, \mathcal{C}) \to (X, d, \mathcal{C})$ be a (ϱ, L)-LEL map. Take any nonempty open subset U of X and fix some $C \in \mathcal{C}$ contained in U. Then $f^n(C) \in \mathcal{C}$ for every n. If $f^n(C) \neq X$ for every n then $\mathcal{H}^1_d(f^n(C)) \geqslant \varrho^n \cdot \mathcal{H}^1_d(C) \to \infty$ for $n \to \infty$, which contradicts the fact that $X = (X, d)$ has finite length. So $f^n(U) \supseteq f^n(C) = X$ for some n, which proves the exactness of f.

Now assume that $f = \varphi \circ \psi$. Since f is exact, then the factor $f' = \psi \circ \varphi : I \to I$ of f is exact. Hence f' has the specification property by [5]. By [7, 21.4] f, also being a factor of f', has the specification property. Finally, by [7, 21.3], f has dense periodic points. □

Recall that d_I denotes the Euclidean metric on I and $\mathcal{C}_I$ is the system of all non-degenerate closed subintervals of I. Note that the following lemma can be substantially generalized, but for our purposes this version is sufficient.

Lemma 7. *Let $k \geqslant 3$ and $f_k : I \to I$ be the piecewise linear map fixing 0 and mapping every $[(i-1)/k, i/k]$ onto I. Then $f_k : (I, d_I, \mathcal{C}_I) \to (I, d_I, \mathcal{C}_I)$ is $(k/2, k)$-LEL.*

Proof. Only length-expansiveness needs a proof. Take any non-degenerate closed subinterval J of I. If there is i such that $J \supseteq [(i-1)/k, i/k]$ then $f(J) = I$. Otherwise there is i such that $J = J_0 \cup J_1$ where $J_0 \subseteq ((i-1)/k, i/k]$ and $J_1 \subseteq [i/k, (i+1)/k)$. Then $|f_k(J)| \geqslant k \cdot \max\{|J_0|, |J_1|\} \geqslant (k/2) \cdot |J|$. □

Lemma 8. *Let $f : (X, d, \mathcal{C}) \to (X', d', \mathcal{C}')$ be a (ϱ, L)-LEL map. Then for every $\mathcal{D}' \supseteq \mathcal{C}'$, $1 < \varrho' \leqslant \varrho$ and $L' \geqslant L$, the map $f : (X, d, \mathcal{C}) \to (X', d', \mathcal{D}')$ is (ϱ', L')-LEL.*

Lemma 9. *Let $f : (X, d, \mathcal{C}) \to (X', d', \mathcal{C}')$ be a (ϱ, L)-LEL map and $f' : (X', d', \mathcal{C}') \to (X'', d'', \mathcal{C}'')$ be a (ϱ', L')-LEL map. Then $f' \circ f : (X, d, \mathcal{C}) \to (X'', d'', \mathcal{C}'')$ is a $(\varrho\varrho', LL')$-LEL map.*

Proof. Put $g = f' \circ f$. Immediately $\mathrm{Lip}(g) \leqslant LL'$. Take any $C \in \mathcal{C}$ and put $C' = f(C) \in \mathcal{C}'$. If $C' = X'$ then, by surjectivity of f', $g(C) = X''$. Otherwise $\mathcal{H}^1_{d'}(C') \geqslant \varrho \mathcal{H}^1_d(C)$ and, if $f'(C') \neq X''$, also $\mathcal{H}^1_{d''}(f'(C')) \geqslant \varrho' \mathcal{H}^1_{d'}(C')$; hence $\mathcal{H}^1_{d''}(g(C)) \geqslant \varrho\varrho' \mathcal{H}^1_d(C)$. □

4. Lipschitz-1 surjections $g : [0, \alpha] \to X$, $h : X \to [0, \beta]$

In this section we show that for a totally regular continuum X there are a compatible convex metric d and two Lipschitz surjections $g : [0, \alpha] \to (X, d)$, $h : (X, d) \to [0, \beta]$ such that

$$\gamma \cdot |J| \leqslant \mathcal{H}^1_d\big(g(J)\big) \leqslant \Gamma \cdot \big|h \circ g(J)\big|$$

for every closed subinterval J of $[0, \alpha]$, where $0 < \gamma < \Gamma$ are constants not depending on J (see Lemma 19).

We start with a simple property of convex metrics on locally connected continua. For a metric space $X = (X, d)$ and a point $a \in X$ put

$$h_a : X \to \mathbb{R}, \qquad h_a(x) = d(a, x) \quad \text{for } x \in X.$$

Lemma 10. *Let $X = (X, d)$ be a locally connected continuum endowed with a convex metric d and let $a \in X$. Then*

$$\big|h_a(A)\big| \geqslant \frac{1}{2} \cdot \mathcal{H}^1_d(A)$$

for any free arc A in X.

Proof. Let y, z be the end points of A. For distinct $u, v \in A$ we will denote by uv the subarc of A with end points u, v. Let α be the length of A and let $\kappa : [0, \alpha] \to A$ be the natural parametrization of A such that $\kappa(0) = y$ and $\kappa(\alpha) = z$. Put $y_t = \kappa(t)$ for $t \in [0, \alpha]$; hence $\mathcal{H}^1_d(y_t y_s) = |s - t|$ for every different $t, s \in [0, \alpha]$.

For every $t \in [0, \alpha]$ such that $y_t \neq a$ take a geodesic arc A_t from a to y_t. Assume first that a is not an interior point of A. Since A is a free arc, every arc (hence also every A_t) from a to a point of A must contain y or z. Take any $t \in [0, \alpha]$. If $y \in A_t$ then $d(a, y_t) = d(a, y) + d(y, y_t) = d(a, y) + t$ since A_t is geodesic. Analogously, if $z \in A_t$ then $d(a, y_t) = d(a, z) + d(z, y_t) = d(a, z) + (\alpha - t)$. So

$$h_a(y_t) = \min\big\{d(a, y) + t, d(a, z) + (\alpha - t)\big\}.$$

Hence immediately $|h_a(A)| \geqslant \alpha/2$.

Now assume that $a = y_s$ for some $s \in (0, \alpha)$; without loss of generality we may assume that $\mathcal{H}^1_d(ay) \leqslant \mathcal{H}^1_d(az)$. Then for every $t \in [0, \alpha]$, $t \neq s$ the geodesic arc A_t is either the subarc ay_t of A or an arc containing both y and z. Hence

$$h_a(y_t) = \min\big\{|t - s|, s + d(y, z) + (\alpha - t)\big\}.$$

So also in this case we easily have $|h_a(A)| \geqslant \alpha/2$. □

4.1. Admissible maps on graphs

Let G be a graph with a metric d and let a, b be (not necessarily distinct) vertices of G. We say that a path $\pi = aE_{j_1}a_1 \dots a_{k-1}E_{j_k}b$ in G (from a to b) is *admissible* provided every edge of G is at least once but at most twice in π; moreover, if a_i is a vertex of G of order 2 then $E_{j_i} \neq E_{j_{i+1}}$ (i.e. π "goes through" the ordinary vertices of G).

A continuous map κ from a compact interval $J = [\alpha, \beta]$ to G is called *fully-admissible* or, more precisely, *fully-admissible for* (G, d) *from* a *to* b, if it is the natural parametrization of some admissible path π from a to b. I.e. $\kappa(\alpha) = a$, $\kappa(\beta) = b$ and there is an admissible path $\pi_\kappa = aE_{j_1}a_1 \dots a_{k-1}E_{j_k}b$ and non-overlapping compact intervals $J_1 \leqslant J_2 \leqslant \dots \leqslant J_k$ such that $J = J_1 \cup \dots \cup J_k$ and every restriction $\kappa|_{J_i} : J_i \to E_{j_i}$ is an isometry.

A map is called *admissible* if it is a restriction of a fully-admissible map onto a compact interval. Notice that any admissible map is finite-to-one and outside of a finite set (the set of points mapped to the vertices of G) is at most two-to-one. Moreover, admissible maps are Lipschitz-1 provided the metric d is convex. The following lemma can be easily proved by induction on the number of edges of G.

Lemma 11. *Let G be a graph and let a, b be vertices of G. Then there is a fully-admissible map $\kappa : [\alpha, \beta] \to G$ for G from a to b.*

Lemma 12. *Let $0 < q < 1$ and let G be a graph. Then there is a convex metric d on G such that for every admissible map $\kappa : J \to G$ and every vertex a of G it holds that*

$$\mathcal{H}^1_d\big(\kappa(J)\big) \geqslant \frac{1}{2} \cdot |J| \quad \text{and} \quad \big|h_a \circ \kappa(J)\big| \geqslant \frac{1-q}{6} \cdot \mathcal{H}^1_d\big(\kappa(J)\big).$$

Moreover, $|h_a(G)| \geqslant \frac{1-q}{2} \cdot \mathcal{H}^1_d(G)$.

Proof. Fix any $0 < q < 1$. Let G be a graph and let $E_0, \dots, E_k$ be the edges of G. Take a convex metric d on G such that $\mathcal{H}^1_d(G) < \infty$ and

$$\mathcal{H}^1_d(E_i) \leqslant q \cdot \mathcal{H}^1_d(E_{i-1}) \quad \text{for every } i \geqslant 1. \tag{4.1}$$

Such a metric can be constructed as follows: We may assume that G is a subset of $\mathbb{R}^3$ endowed with the Euclidean metric and that the (Euclidean) lengths of edges of G are finite and exponentially decreasing with quotient q. Then it suffices to take the convex metric on G generated by the Euclidean one.

Let a be a vertex of G and let $\kappa : J \to G$ be an admissible map for (G, d); put $Y = \kappa(J)$. Let $\pi = a_0E_{i_1}a_1E_{i_2} \dots a_{k-1}E_{i_k}a_k$ be the admissible path given by a fully-admissible extension of κ. Since π is admissible, we immediately have $\mathcal{H}^1_d(Y) \geqslant (1/2) \cdot |J|$.

Now we show the lower bound for the length of $h_a(Y)$. Realize that there are at most two indices j such that

$$Y \cap E_j \text{ is non-degenerate and } Y \nsupseteq E_j \tag{4.2}$$

(indeed, for any such j the edge E_j must contain the κ-image of an end point of J in its interior). For simplicity we will assume that there are exactly two j's satisfying (4.2)—we denote them by j_1, j_2—and that there is an index j such that $E_j \subseteq Y$; the other cases can be described analogously. Let j_0 be the smallest index j such that $E_j \subseteq Y$. Then using (4.1) we have

$$\begin{aligned}\mathcal{H}^1_d(Y) &= \sum_j \mathcal{H}^1_d(E_j \cap Y) \leqslant \mathcal{H}^1_d(E_{j_1} \cap Y) + \mathcal{H}^1_d(E_{j_2} \cap Y) + \sum_{j \geqslant j_0} \mathcal{H}^1_d(E_j) \\ &\leqslant \mathcal{H}^1_d(E_{j_1} \cap Y) + \mathcal{H}^1_d(E_{j_2} \cap Y) + \mathcal{H}^1_d(E_{j_0})/(1-q).\end{aligned} \tag{4.3}$$

On the other hand, Lemma 10 gives

$$\begin{aligned}\big|h_a(Y)\big| &\geqslant \max\big\{\big|h_a(E_{j_1} \cap Y)\big|, \big|h_a(E_{j_2} \cap Y)\big|, \big|h_a(E_{j_0})\big|\big\} \\ &\geqslant \frac{1}{2} \cdot \max\big\{\mathcal{H}^1_d(E_{j_1} \cap Y), \mathcal{H}^1_d(E_{j_2} \cap Y), \mathcal{H}^1_d(E_{j_0})\big\}.\end{aligned}$$

The simple fact that

$$\max_{i=1,\dots,p} c_i \leqslant \sum_{i=1}^p c_i \leqslant p \cdot \max_{i=1,\dots,p} c_i \quad \text{for any non-negative } c_1, \dots, c_p,$$

applied to (4.3) immediately implies $|h_a(Y)| \geqslant \mathcal{H}^1_d(Y) \cdot (1-q)/6$.

The final assertion of the lemma follows from the facts that $\mathcal{H}^1_d(G) \leqslant \mathcal{H}^1_d(E_0)/(1-q)$ and $|h_a(G)| \geqslant |h_a(E_0)| \geqslant \mathcal{H}^1_d(E_0)/2$ by Lemma 10. $\square$

4.2. Construction of d, g, h

Now we embark on the construction of a convex metric d on X and Lipschitz surjections $g : [0, \alpha] \to X$, $h : X \to [0, \beta]$ for a given totally regular continuum X (see Lemma 19).

Let $0 < q < 1$, let X be a non-degenerate totally regular continuum and let a, b be two points of X. By [6] there is an inverse sequence $(X_n, f_n)_{n\in\mathbb{N}}$ of graphs X_n with monotone surjective bonding maps $f_n : X_{n+1} \to X_n$ such that X is (homeomorphic to) the inverse limit

$$\varprojlim(X_n, f_n).$$

Without loss of generality we may assume that for every integer $n \geqslant 1$ the following hold:

- there is $\tilde{x}_n \in X_n$ such that $\tilde{X}_{n+1} = f_n^{-1}(\tilde{x}_n)$ is a non-degenerate subgraph of X_{n+1};
- $f_n^{-1}(x)$ is a singleton for every $x \neq \tilde{x}_n$;
- $\tilde{x}_n$ is a vertex of X_n;
- every vertex of $\tilde{X}_{n+1}$ is a vertex of X_{n+1}; moreover, every point of the boundary (in X_{n+1}) of $\tilde{X}_{n+1}$ is a vertex of both $\tilde{X}_{n+1}$ and X_{n+1}; so an edge of $\tilde{X}_{n+1}$ is also an edge of X_{n+1};
- the f_n-preimage of every vertex $x \neq \tilde{x}_n$ of X_n is a vertex of X_{n+1}; so the f_n-image of any edge in X_{n+1} which is not an edge of $\tilde{X}_{n+1}$ is a free arc contained in an edge of X_n.

Let $\pi_n : X \to X_n$ $(n \in \mathbb{N})$ be the natural projections; put $a_n = \pi_n(a)$, $b_n = \pi_n(b)$. We may assume that a_n, b_n are vertices of X_n and, if $a \neq b$, $a_1 \neq b_1$ (otherwise we remove finitely many of the first X_n's). Then $a_n \neq b_n$ for every n provided $a \neq b$.

Let d_1 be a convex metric on X_1 obtained using Lemma 12 such that $\mathcal{H}^1_{d_1}(X_1) = 1 - q$ and let $g_1 : I_1 \to X_1$ be a fully-admissible map for (X_1, d_1) from a_1 to b_1. Assume that $n \geqslant 2$ and that for every $1 \leqslant m \leqslant n-1$ we have defined a metric d_m on X_m, maps $g_m : I_m \to X_m$ and, provided $m \geqslant 2$, a map $\varrho_{m-1} : I_m \to I_{m-1}$. Put

$$\mu_{n-1} = \min\bigl\{\mathcal{H}^1_{d_{n-1}}(E):\ E \text{ is an edge of } X_{n-1}\bigr\}.$$

Let $\tilde{d}_n$ be a convex metric on $\tilde{X}_n$ obtained from Lemma 12 such that

$$\mathcal{H}^1_{\tilde{d}_n}(\tilde{X}_n) < \frac{q \cdot \mu_{n-1}}{2p} \quad \text{where } p = \#g_{n-1}^{-1}(\tilde{x}_{n-1}). \tag{4.4}$$

Denote by d_n the only convex metric on X_n such that for every edge E of X_n and every two points $x, y \in E$ the following holds:

$$d_n(x, y) = \begin{cases} \tilde{d}_n(x, y) & \text{if } E \subseteq \tilde{X}_n; \\ d_{n-1}(f_{n-1}(x), f_{n-1}(y)) & \text{otherwise.} \end{cases}$$

Let $s_1 < s_2 < \cdots < s_p$ be the points of I_{n-1} mapped by g_{n-1} to $\tilde{x}_{n-1}$. Write I_{n-1} as the union $J'_0 \cup J'_1 \cup \cdots \cup J'_p$ of non-overlapping compact subintervals such that $J'_0 \leqslant s_1 \leqslant J'_1 \leqslant s_2 \leqslant \ldots \leqslant s_p \leqslant J'_p$ (here J'_0, J'_p can be degenerate). For every $i = 1, \ldots, p$ let K'_i be an interval and $\kappa_i : K'_i \to (\tilde{X}_n, \tilde{d}_n)$ be a fully-admissible map (see Lemma 11); the images of end points of K'_i will be fixed later. Now let $I_n = [0, \alpha_n]$ be a compact interval of length $\alpha_n = |I_{n-1}| + \sum_{i=1}^p |K'_i|$ and define $g_n : I_n \to X_n$ by "concatenating" the maps

$$g_{n-1}|_{J'_0},\ \kappa_1,\ g_{n-1}|_{J'_1},\ \kappa_2,\ \ldots, \kappa_p,\ g_{n-1}|_{J'_p}.$$

I.e. we write I_n as the union of non-overlapping compact intervals

$$I_n = J_0 \cup K_1 \cup J_1 \cup K_2 \cup \cdots \cup K_p \cup J_p$$

such that $J_0 \leqslant K_1 \leqslant \cdots \leqslant K_p \leqslant J_p$ and $|J_i| = |J'_i|$, $|K_j| = |K'_j|$ for every i, j; then we define g_n such that

$$g_n|_{J_i} \approx g_{n-1}|_{J'_i} \quad \text{and} \quad g_n|_{K_j} \approx \kappa_j \quad \text{for every } i, j.$$

(Here we write $f \approx g$ for maps f, g defined on real intervals J, K if there is a constant s_0 such that $J = K + s_0$ and $f(s + s_0) = g(s)$ for every $s \in K$.) By an "appropriate" specification of κ_i-images of the end points of K'_i we obtain that g_n is continuous and that $g_n(0) = a_n$, $g_n(\alpha_n) = b_n$. Notice that

$$g_n : I_n \to (X_n, d_n) \quad \text{is a natural parametrization of some (not necessarily admissible) path in } X_n \text{ from } a_n \text{ to } b_n.$$

Let $\varrho_{n-1} : I_n \to I_{n-1}$ be the piecewise linear continuous surjection with slopes 0 and 1 which collapses every K_i into a point. For $1 \leqslant k < n$ denote the composition $\varrho_k \circ \varrho_{k+1} \circ \cdots \circ \varrho_{n-1}$ by $\varrho_{n,k} : I_n \to I_k$; for convenience put $\varrho_{n,n} = \mathrm{id}_{I_n}$. Analogously define $f_{n,k} : X_n \to X_k$ for $1 \leqslant k \leqslant n$. Notice that the following diagram commutes for every $1 \leqslant k \leqslant n$:

$$\begin{tikzcd} X_n \arrow[r, "f_{n,k}"] & X_k \\ I_n \arrow[u, "g_n"] \arrow[r, "\varrho_{n,k}"] & I_k \arrow[u, "g_k"'] \end{tikzcd} \tag{4.5}$$

After finishing the induction we obtain the metrics d_n on X_n and the maps $g_n : I_n \to X_n$. As in [6] define

$$d(x, y) = \sup_{n\in\mathbb{N}} d_n(x_n, y_n) \quad \text{for } x = (x_n)_n,\ y = (y_n)_n \in X.$$

(By Lemma 13, d is a convex metric on X.) Define also

$$I_\infty = \varprojlim(I_n, \varrho_n).$$

The corresponding projection map from I_∞ onto I_n ($n \in \mathbb{N}$) will be denoted by π'_n. It is easy to see that the map

$$\eta : I_\infty \to [0, \alpha], \qquad (s_n)_{n\in\mathbb{N}} \mapsto t = \lim_{n\to\infty} s_n = \sup_{n\in\mathbb{N}} s_n, \quad \text{where } \alpha = \lim_{n\to\infty} \alpha_n,$$

defines a homeomorphism of I_∞ onto the interval $[0, \alpha]$, which is even isometry if we use the following metric d' on I_∞ (see Lemma 15):

$$d'(s, t) = \sup_n |s_n - t_n| \quad \text{for } s = (s_n)_n,\ t = (t_n)_n \in I_\infty.$$

Since the diagrams in (4.5) commute, the surjective maps $g_n : I_n \to X_n$ induce the continuous surjective map $g = \varprojlim\{g_n\} : I_\infty \to X$ between $I_\infty = \varprojlim(I_n, \varrho_n)$ and $X = \varprojlim(X_n, f_n)$ such that the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "\pi_n"] & X_n \\ I_\infty \arrow[u, "g"] \arrow[r, "\pi'_n"] & I_n \arrow[u, "g_n"'] \end{tikzcd}$$

(see Theorem 6); the map g is given by

$$g(s_1, s_2, s_3 \ldots) = \big(g_1(s_1), g_2(s_2), g_3(s_3), \ldots\big).$$

Finally define $h_n : X_n \to \mathbb{R}$, $h : X \to \mathbb{R}$ by

$$h_n(x_n) = d_n(x_n, a_n) \quad \text{for } x_n \in X_n, \qquad h(x) = d(x, a) \quad \text{for } x \in X.$$

4.3. Properties of d, g, h

The following lemmas summarize properties of the constructed metrics and maps. Due to space limitations we just outline the proofs; the full proofs can be found in [21].

Lemma 13. *The map d is a convex metric on X compatible with the topology of X.*

Lemma 14. *Let $B \subseteq X$ be a closed set and let $B_n = \pi_n(B) \subseteq X_n$ for every $n \in \mathbb{N}$. Then*

$$\mathcal{H}^1_d(B) = \sup_{n\in\mathbb{N}} \mathcal{H}^1_{d_n}(B_n) = \lim_{n\to\infty} \mathcal{H}^1_{d_n}(B_n).$$

Moreover, $\mathcal{H}^1_d(X) \leqslant 1$.

Lemma 15. *Put $\alpha = \sup_n \alpha_n = \lim_n \alpha_n$ (recall that α_n is the length of $I_n = [0, \alpha_n]$). Then $\alpha < \infty$ and the following hold:*

(a) *d' is a metric on I_∞ compatible with the topology;*
(b) *for every $s = (s_n)_n, s' = (s'_n)_n \in I_\infty$ we have $d'(s, s') = \lim_{n\to\infty} |s_n - s'_n|$;*
(c) *the projection maps $\pi'_n : (I_\infty, d') \to I_n$ ($n \in \mathbb{N}$) are Lipschitz-1;*
(d) *the map $\eta : (I_\infty, d') \to [0, \alpha]$ sending $s = (s_n)_{n\in\mathbb{N}}$ to $\eta(s) = \lim_{n\to\infty} s_n = \sup_{n\in\mathbb{N}} s_n$ is an isometry;*
(e) *for every subcontinuum J of I_∞ it holds that*

$$\mathcal{H}^1_{d'}(J) = \lim_{n\to\infty} |J_n| = \sup_{n\in\mathbb{N}} |J_n|, \quad \textit{where } J_n = \pi'_n(J) \textit{ for } n \in \mathbb{N}.$$

Lemma 13 is a straightforward consequence of the definition of the metrics d_n. The proof of Lemma 14 requires a fine relationship between the diameter of subcontinua Y of X and the diameters of the projections $\pi_n(Y) \subseteq X_n$, see [21, Lemma 14]. Lemma 15 is trivial.

Lemma 16. *The map $g_n : I_n \to (X_n, d_n)$ is a Lipschitz-1 surjection. Moreover, for every compact interval $J \subseteq I_n$ and for $Y = g_n(J)$, $L = h_n(Y)$ it holds that*

$$\mathcal{H}^1_{d_n}(Y) \geqslant \frac{1-q}{2} \cdot |J| \quad \text{and} \quad |L| \geqslant \frac{1-4q}{12} \cdot \mathcal{H}^1_{d_n}(Y).$$

The proof of Lemma 16 is divided into three cases, see [21, Lemmas 17–19]. If $Y = g_n(J)$ contains at most one vertex of X_n, we easily obtain that $\mathcal{H}^1_{d_n}(Y) \geqslant (1/2) \cdot |J|$ and $|L| \geqslant (1/4) \cdot \mathcal{H}^1_{d_n}(Y)$; in fact, it suffices to use Lemma 10 and the fact that g_n restricted to any edge of X_n is an isometry. If Y contains at least two vertices of X_n and $Y' = f_{n-1}(Y)$ contains at most one vertex of X_{n-1}, we write Y as the non-overlapping union of (at most) two free arcs and a subgraph $\tilde{Y}$ of $\tilde{X}_n$; then we employ the property of $\tilde{d}_n$ from Lemma 12 to get $\mathcal{H}^1_{d_n}(Y) \geqslant (1/2) \cdot |J|$ and $|L| \geqslant ((1-q)/12) \cdot \mathcal{H}^1_{d_n}(Y)$. In the third case, denote by m the smallest integer such that $Y_m = f_{n,m}(Y)$ contains at least two vertices of X_m. Then the lemma follows by (4.4) and the second case applied to Y_m instead of Y.

Lemma 17. *The map $g : (I_\infty, d') \to (X, d)$ is a Lipschitz-1 surjection.*

Lemma 18. *Let J be a subcontinuum of I_∞ and let $Y = g(J)$, $L = h(Y)$. Then*

$$\mathcal{H}^1_{d}(Y) \geqslant \frac{1-q}{2} \cdot \mathcal{H}^1_{d'}(J) \quad \text{and} \quad |L| \geqslant \frac{1-4q}{12} \cdot \mathcal{H}^1_{d}(Y).$$

Lemma 17 is immediate and Lemma 18 easily follows from Lemmas 14–16. The following is the main result of Section 4.

Lemma 19. *There are constants $0 < \gamma < \Gamma$ such that for any $\delta > 0$ the following hold: For any non-degenerate totally regular continuum X and any two points a, b of X there are a compatible convex metric d on X and maps $g : [0, \alpha] \to X$, $h : X \to [0, \beta]$ with the following properties:*

(a) *$g(0) = a$, $g(\alpha) = b$ and $h(a) = 0$;*
(b) *g, h are Lipschitz-1 surjections;*
(c) *$\gamma \cdot |J| \leqslant \mathcal{H}^1_d(g(J)) \leqslant \Gamma \cdot |h \circ g(J)|$ for every closed subinterval J of $[0, \alpha]$;*
(d) *$\mathcal{H}^1_d(X) \in [1-\delta, 1]$, $\mathcal{H}^1_d(X) \leqslant \alpha \leqslant 2 \cdot \mathcal{H}^1_d(X)$ and $(1/2 - \delta) \cdot \mathcal{H}^1_d(X) \leqslant \beta \leqslant \mathcal{H}^1_d(X)$.*

Moreover, if $\mathrm{Cut}(a, b)$ is uncountable then a metric d and maps g, h can be chosen such that also:

(e) *$h(b) = \beta$;*
(f) *$d(a, b) > (1-\delta) \cdot \mathcal{H}^1_d(X)$.*

To prove this result, we fix $\gamma < 1/2$, $\Gamma > 24$, $\delta \leqslant 1/2$ and choose $q < \delta/2$ such that $(1-q)/2 \geqslant \gamma$ and $(1-4q)/12 \geqslant 2/\Gamma$. Let the metric d and maps g, h be those constructed in Section 4.2. Then the first part of the lemma follows from Lemmas 17 and 18. The proof of the second part requires an easy fact on cardinalities of cut points and their π_n-projections [21, Lemma 23] and a small refinement of the construction: we need the arc $[\pi_1(a), \pi_1(b)]$ to be the longest edge of X_1 and we define h by $h(x) = \lambda(d(x, a))$, where $\lambda(s) = s$ for $s \leqslant d(a, b)$ and $\lambda(s) = 2d(a, b) - s$ for $d(a, b) < s \leqslant \beta = \max_x d(a, x)$. For the details see [21, Lemma 24].

5. Length-expanding Lipschitz maps from/to the interval

The following proposition provides the key tool for constructing LEL maps. Basically it is just a reformulation of Lemma 19.

Proposition 20. *There are constants $0 < \gamma < \Gamma$ and $L > 1$ such that the following hold: For every non-degenerate totally regular continuum X and every two points $a, b \in X$ there are a compatible convex metric d on X and maps $\varphi : I \to X$, $\psi : X \to I$ with the following properties:*

(a) *$\varphi(0) = a$, $\varphi(1) = b$ and $\psi(a) = 0$;*
(b) *φ, ψ are Lipschitz-L surjections;*
(c) *$\gamma \cdot |J| \leqslant \mathcal{H}^1_d(\varphi(J)) \leqslant \Gamma \cdot |\psi \circ \varphi(J)|$ for every closed subinterval J of I;*
(d) *$\mathcal{H}^1_d(X) = 1$.*

Moreover, if Cut(a,b) *is uncountable then for any* $\delta > 0$ *a metric* d *and maps* φ, ψ *can be chosen such that it also holds*:

(e) $\psi(b) = 1$;
(f) $d(a,b) > 1-\delta$.

Proof. Take any $L > 2$ and let $0 < \gamma < \Gamma$ be constants from Lemma 19. Fix a non-degenerate totally regular continuum X, a pair $a,b \in X$ and a positive real δ; we may assume that $2/(1-2\delta) < L$. We give the proof only in the case when Cut(a,b) is uncountable; the other case can be described analogously.

Let $\tilde{d}$ be a convex metric on X and $g : [0,\alpha] \to X$, $h : X \to [0,\beta]$ be maps satisfying (a)–(f) from Lemma 19. Now define $d : X \times X \to \mathbb{R}$, $\varphi : I \to X$ and $\psi : X \to I$ by

$$d(x,y) = \frac{1}{c}\cdot\tilde{d}(x,y), \qquad \varphi(t) = g(\alpha t) \quad \text{and} \quad \psi(x) = \frac{1}{\beta}\cdot h(x),$$

where $c = \mathcal{H}^1_{\tilde{d}}(X)$. Then (a) and (d)–(f) are immediately satisfied. Since

$$\mathrm{Lip}_d(\varphi) = \frac{\alpha}{c}\cdot\mathrm{Lip}_{\tilde{d}}(g) \leqslant 2 < L \quad \text{and} \quad \mathrm{Lip}_d(\psi) = \frac{c}{\beta}\cdot\mathrm{Lip}_{\tilde{d}}(h) \leqslant \frac{2}{1-2\delta} < L,$$

also (b) is fulfilled. The property (c) follows from

$$\mathcal{H}^1_d\big(\varphi(J)\big) \geqslant \frac{\gamma\alpha}{c}\cdot|J| \quad \text{and} \quad \Gamma\cdot\big|\psi\circ\varphi(J)\big| \geqslant \frac{c}{\beta}\cdot\mathcal{H}^1_d\big(\varphi(J)\big)$$

and from $\alpha \geqslant c \geqslant \beta$. □

Corollary 21. *Every non-degenerate totally regular continuum* X*, endowed with a suitable convex metric* d *and a dense system* $\mathcal{C}$ *of subcontinua of* X*, admits LEL-maps* $\tilde{\varphi} : (I, d_I, \mathcal{C}_I) \to (X, d, \mathcal{C})$ *and* $\tilde{\psi} : (X, d, \mathcal{C}) \to (I, d_I, \mathcal{C}_I)$.

Proof. Fix arbitrary $a,b \in X$; let d,φ,ψ be as in Proposition 20. Put $\mathcal{C} = \varphi(\mathcal{C}_I)$; this is a dense system by Lemma 2. Let f_k be the map from Lemma 7, where $k \geqslant 3$ is such that $\varrho = \gamma k/2 > 1$. Then the map $\tilde{\varphi} = \varphi\circ f_k : I \to X$ is (ϱ, kL)-LEL. Analogously, if $k' \geqslant 3$ is such that $\varrho' = k'/(2\Gamma) > 1$ then $\tilde{\psi} = f_{k'}\circ\psi : X \to I$ is $(\varrho', k'L)$-LEL. □

Notice that to fulfill only the conditions (a)–(d) from Proposition 20 we can find d, φ, ψ such that

$$\psi(x) = c\cdot d(a,x) \quad \text{for every } x \in X,$$

where c is a constant. One can also see that any constants $0 < \gamma < \frac{1}{2}$, $\Gamma > 24$ and $L > 2$ are suitable in Proposition 20. Derivation of the "best" values for γ, Γ and L is out of the scope of this paper. However, we can at least say that L and the ratio Γ/γ cannot be arbitrarily close to 1. In fact, if X is the 3-star then easy arguments show that we must have $\Gamma/\gamma \geqslant 3$. Further, if $X = (X,d)$ is a simple closed curve of length 1 then, for any $\psi : X \to I$ from Proposition 20, we can write X as the union $A \cup B$ of two non-overlapping arcs such that $\psi(A) = \psi(B) = I$; so $L \geqslant \mathrm{Lip}(\psi) \geqslant 2$.

The following example shows that in the second part of Proposition 20 one cannot replace the assumption Cut(a,b) is uncountable by Cut(a,b) is infinite.

Example 22. Take an integer $p \geqslant 3$, put $a = (-1,0)$, $b = (1,0)$, $a_0 = (0,0)$, $a_k = (1-2^{-k}, 0)$, $a_{-k} = -a_k$ ($k \in \mathbb{N}$) and define a continuum $X_p \subseteq \mathbb{R}^2$ by

$$X_p = \bigcup_{k\in\mathbb{Z}} G_k \cup \{a,b\}$$

where every G_k ($k \in \mathbb{Z}$) is a graph with exactly two vertices a_{k-1}, a_k, these vertices have order p (in G_k) and $G_k \cap G_l$ is empty for $l > k+1$ and is equal to $\{a_k\}$ for $l = k+1$; see Fig. 1 for $p = 3$. In this case a,b are end points of X_p (so Cut(a,b) is infinite), but neither (e) nor (f) can be fulfilled for small δ.

To show this realize that $\mathcal{H}^1_d(X_p) \geqslant p\cdot d(a,b)$ for any convex metric d on X_p; indeed, X_p is the union of p arcs with ends a,b (so the length of any of them is greater than or equal to $d(a,b)$) and with countable intersections. So immediately (f) is not true for $\delta < 1-\frac{1}{p}$. Moreover, since ψ is Lipschitz-L and $\psi(a) = 0$ we have that

$$\psi(b) = \psi(b) - \psi(a) \leqslant L\cdot d(a,b) \leqslant L\cdot\frac{\mathcal{H}^1_d(X_p)}{p} = \frac{L}{p}$$

which is smaller than 1 for $p > L$. So also (e) is not true. Notice that any metric d satisfying (a)–(d) must be such that the diameter of X_p is approximately p-times larger than the distance of a,b; so for some k the shortest edge of G_k must be "very small" when compared to the longest one.

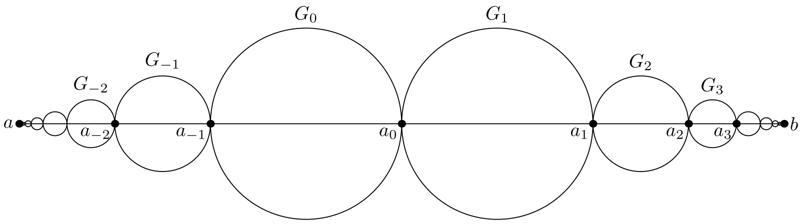

Fig. 1. The continuum X_3.

Remark 23. If we replace the metric d from Proposition 20 by $d' = c \cdot d$ (where $c > 0$ is a constant), the Lipschitz constants of φ, ψ change to $\mathrm{Lip}_{d'}(\varphi) = c \cdot \mathrm{Lip}_d(\varphi)$, $\mathrm{Lip}_{d'}(\psi) = (1/c) \cdot \mathrm{Lip}_d(\psi)$. So instead of the conditions (d) and (f) we can have

(d′) $\mathcal{H}^1_d(X) < 1/(1-\delta)$;
(f′) $d(a, b) = 1$.

6. Proofs of the main results

Now we are ready to prove the main results of the paper stated in the introduction. For convenience we repeat the statements of them.

Theorem C. *For every non-degenerate totally regular continuum X and every $a, b \in X$ we can find a convex metric $d = d_{X,a,b}$ on X and Lipschitz surjections $\varphi_{X,a,b} : I \to X$, $\psi_{X,a,b} : X \to I$ with the following properties*:

(a) $\mathcal{H}^1_d(X) = 1$;
(b) *the system $\mathcal{C} = \mathcal{C}_{X,a,b} = \{\varphi_{X,a,b}(J)\colon\ J$ is a closed subinterval of $I\}$ is a dense system of subcontinua of X*;
(c) *for every $\varrho > 1$ there are a constant L_ϱ (depending only on ϱ) and (ϱ, L_ϱ)-LEL maps*

$$\varphi : (I, d_I, \mathcal{C}_I) \to (X, d, \mathcal{C}) \quad \text{and} \quad \psi : (X, d, \mathcal{C}) \to (I, d_I, \mathcal{C}_I)$$

with $\varphi(0) = a$, $\varphi(1) = b$, $\psi(a) = 0$ and such that $\varphi = \varphi_{X,a,b} \circ f_k$, $\psi = f_l \circ \psi_{X,a,b}$ for some $k, l \geqslant 3$.

Moreover, if $\mathrm{Cut}_X(a, b)$ is uncountable, d, φ, ψ can be assumed to satisfy

(d) *$d(a, b) > 1/2$ and $\psi(b) = 1$.*

Proof. Let γ, Γ and L be constants from Proposition 20. Let X be a non-degenerate totally regular continuum and a, b be two points of X. Put $\delta = 1/2$ and fix a metric $d = d_{X,a,b}$ on X and maps $\varphi_{X,a,b} : I \to X$, $\psi_{X,a,b} : X \to I$ satisfying (a)–(d) (or (a)–(f) if $\mathrm{Cut}_X(a, b)$ is uncountable) from Proposition 20. Recall that $\mathcal{H}^1_d(X) = 1$ and, provided $\mathrm{Cut}_X(a, b)$ is uncountable, $d(a, b) > 1/2$. By Lemma 2, $\mathcal{C}_{X,a,b} = \varphi_{X,a,b}(\mathcal{C}_I)$ is a dense system of subcontinua of X.

Let $k, l \geqslant 3$ be the smallest odd integers such that $\gamma k/2 \geqslant \varrho$ and $l/(2\Gamma) \geqslant \varrho$. Put $L_\varrho = 2L\varrho \cdot (1 + \max\{\Gamma, 1/\gamma\}) > 1$. As in the proof of Corollary 21, the maps $\varphi = \varphi_{X,a,b} \circ f_k : I \to X$ and $\psi = f_l \circ \psi_{X,a,b} : X \to I$ are (ϱ, L_ϱ)-LEL. Since k, l are odd we have $\varphi(0) = \varphi_{X,a,b}(0) = a$, $\varphi(1) = \varphi_{X,a,b}(1) = b$, $\psi(a) = \psi_{X,a,b}(a) = 0$ and, provided $\mathrm{Cut}_X(a, b)$ is uncountable, $\psi(b) = \psi_{X,a,b}(b) = 1$. $\square$

Theorem D. *Keeping the notation from Theorem C, for every $\varrho > 1$, every non-degenerate totally regular continua X, X' and every points $a, b \in X$, $a', b' \in X'$ there are a constant L_ϱ (depending only on ϱ) and a (ϱ, L_ϱ)-LEL map*

$$f : (X, d_{X,a,b}, \mathcal{C}_{X,a,b}) \to \left(X', d_{X',a',b'}, \mathcal{C}_{X',a',b'}\right)$$

with $f(a) = a'$ and, provided $\mathrm{Cut}_X(a, b)$ is uncountable, $f(b) = b'$. Moreover, f can be chosen to be the composition $\varphi \circ \psi$ of two LEL-maps $\psi : X \to I$ and $\varphi : I \to X'$.

Proof. The theorem follows from Theorem C and Lemma 9. $\square$

Corollary E. *Every non-degenerate totally regular continuum admits an exactly Devaney chaotic map with finite positive entropy and specification.*

Proof. This immediately follows from Theorem D and Proposition B. $\square$

Corollary F. *Every finite union of disjoint non-degenerate totally regular continua admits a Devaney chaotic map with finite positive entropy.*

Proof. Let $X = \bigsqcup_{i=1}^{k} X_i$, where X_i's are non-degenerate totally regular continua. Fix $\varrho > 1$, $a_i \in X_i$ $(i = 1, \dots, k)$ and put $d_i = d_{X_i,a_i,a_i}$, $\mathcal{C}_i = \mathcal{C}_{X_i,a_i,a_i}$. Let $f_i : X_i \to X_{i+1}$ $(i = 1, \dots, k-1)$ and $f_k : X_k \to X_1$ be LEL maps from Theorem D. Finally, let d be the metric on X such that $d(x, y) = d_i(x, y)$ for any $x, y \in X_i$ $(i = 1, \dots, k)$ and $d(x, y) = 2$ for $x \in X_i$, $y \in X_j$ $(i \neq j)$. Since $d_i(X_i) \leqslant 1$, the metric d is compatible with the topology of X.

Define $f : X \to X$ by $f|_{X_i} = f_i$ for $i = 1, \dots, k$. For every i the restriction $f^k|_{X_i} : X_i \to X_i$ is LEL, hence it is exactly Devaney chaotic with positive finite entropy and specification by Proposition B. Since f permutes $X_1, \dots, X_k$, the assertion follows. $\square$

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