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Favard's theorem of piecewise continuous almost periodic functions and its application ☆

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ABSTRACT

In this paper, by constructing Bochner–Fejér polynomials for piecewise continuous almost periodic functions (PCAP, for short), the authors establish Favard's theorem of PCAP functions, which illustrates when the primitive function of PCAP function is a PCAP function. As its application, combining coincidence degree theory, we consider the existence of PCAP solution of impulsive single population model with hereditary effects. To our best knowledge, it is the first time when coincidence degree theory is used to study the existence of PCAP solution of impulsive differential equation.

1. Introduction

The theory of almost periodic functions (AP, for short) [2,3] was mainly created by the Danish mathematician H. Bohr during 1925–1926. Almost periodic functions, with a superior spatial structure, are a generalization of periodic functions. The development of almost periodic functions concentrates on two directions, that is, the broader study of almost periodic functions and the application of almost periodic type functions in equations (see [1,7,9,19,24] etc.). Thereinto, the broader study of almost periodic functions includes bringing up some new function types, such as asymptotically almost periodic functions, pseudo almost periodic functions [24], Stepanov almost periodic functions [20], piecewise continuous almost periodic functions [11] (PCAP, for short). More function types and related topics can be found in [5,6,13,21,23] and the references therein.

PCAP function was considered for the first time by Halanay and Wexler [11] in connection with the determination of a piecewise continuous almost periodic solution for impulsive system. According to [11], a piecewise continuous function f with first kind discontinuities at the points of a fixed sequence $\{t_k\}$ is called a piecewise continuous almost periodic function if:

(1) The sequence $\{t_k\}$ is such that the derived sequence $\{t_i^j = t_{i+j} - t_i\}$, $j = 0, \pm 1, \pm 2, \ldots$ is equipotentially almost periodic;
(2) $\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon) > 0$ such that if the points t_1 and t_2 belong to the same interval of continuity and $|t_1 - t_2| < \delta$, then $|f(t_1) - f(t_2)| < \varepsilon$;
(3) For any $\varepsilon > 0$, there exists a relatively dense set $\Gamma(f, \varepsilon)$ such that if $\tau \in \Gamma(f, \varepsilon)$, then $|f(t+\tau) - f(t)| < \varepsilon$ for all $t \in \mathbb{R}$ which satisfy the condition $|t - t_i| > \varepsilon$, $i = 0, \pm 1, \pm 2, \ldots$.

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Many people make intensive study on the property of PCAP functions. Samoilenko and Trofimchukt [16,17] studied the boundedness of PCAP functions, the limit of uniformly convergent PCAP function sequences, rational arithmetic of PCAP functions, and so on. The relationship between PCAP functions and its discontinuous points was illuminated in [15]. Stamov [18] showed some equivalent definitions for PCAP functions. Liu [14] investigated the Fourier expansion of PCAP functions, and obtained the uniqueness theorem for PCAP functions.

In this paper, we will search for some conditions under which the primitive function of PCAP function is a PCAP function. To our best knowledge, there is no published paper considering similar property for PCAP functions. In fact, this paper is partly inspired by Favard's theorem of AP functions [13]. Favard's theorem of AP functions shows that if the Fourier exponents of almost periodic function aren't dense in zero, then the primitive function of AP function is an AP function. The main result (Theorem 2.3) obtained in this paper shows that if the Fourier exponents of PCAP function are bounded and they aren't dense in zero, the Fourier coefficients are absolutely convergent, then the primitive function of PCAP function is an AP function. Because Theorem 2.3 and Favard's theorem of AP functions solve the problem of the same nature, in this sense, we might as well call the main result (Theorem 2.3) of this paper: Favard's theorem of PCAP functions.

As an application of Favard's theorem of PCAP functions, we consider impulsive single population model with hereditary effects:

$$\begin{cases} N'(t) = N(t)\big[a(t) - b(t)N(t) - d(t)N\big(t - \tau(t)\big)\big], & t \neq t_k, \\ N\big(t_k^+\big) = (1 + c_k)N(t_k). \end{cases}$$

By means of Favard's theorem of PCAP functions and coincidence degree theory, the existence of strictly positive PCAP solution is obtained. Coincidence degree theory has been widely used to prove the existence of periodic solution of differential equation, regardless of the equation is impulsive or not (see e.g. [4,8]). For almost periodic cases, there are rarely papers applying coincidence degree theory to investigate the existence of almost periodic solution of differential equation without impulse except [22]. In this paper, we firstly establish Favard's theorem of PCAP functions, then apply coincidence degree theory and Favard's theorem of PCAP functions to study the existence of PCAP solution of impulsive single population model with hereditary effects. To our best knowledge, it is the first time applying coincidence degree theory to study the existence of PCAP solution of impulsive differential equation.

The paper is organized as follows: after introducing some preliminaries in the next section, we present Favard's theorem of PCAP functions. Then, in Section 3, we firstly show how to use Favard's theorem of PCAP functions and coincidence degree theory to study the existence of PCAP solution of impulsive single population model with hereditary effects, and then, two examples are provided to show the results obtained in this section.

2. Favard's theorem of piecewise continuous almost periodic functions

In this section, we will introduce some preliminaries firstly, and then prove Favard's theorem of PCAP functions.

Suppose $f \in \mathrm{PCAP}$, the discontinuous points of f are denoted by t_k, $\inf_k t_k^1 = \theta > 0$. It follows from [14] that the limit mean of f $(m(f) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T+a}^{T+a} f(t)\,dt)$ exists uniformly with respect to $a \in \mathbb{R}$. Furthermore, the limit is independent of a. Let

$$a(\lambda, f) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(t)e^{-i\lambda t}\,dt.$$

The set $\Lambda_f = \{\lambda \in \mathbb{R}\colon\ a(\lambda, f) \neq 0\}$ is called the frequency set of f. Members of Λ_f are called the Fourier exponents of f and $a(\lambda, f)$ are called the Fourier coefficients of f. Liu [14] has proved that there exists at most a countable set of λ for which $a(\lambda, f) \neq 0$. Let $\Lambda_f = \{\lambda_k\}$ and $A_k = a(\lambda_k, f)$, hence, there exists a Fourier series associated with f

$$f(t) \sim \sum_{k=1}^{\infty} A_k e^{i\lambda_k t}.$$

Furthermore, the uniqueness theorem holds for PCAP functions, that is, two distinct piecewise continuous almost periodic functions have distinct Fourier series.

Next, we consider $\mathbb{R}$ as a $\mathbb{Q}$ vectorial space. The linear independence of countable set in $\mathbb{R}$, and the basis for the set can be defined similarly. A lemma of interest is the following

Lemma 2.1. *(See [5].) Any countable set of real numbers admits a basis, whose elements belong to the given set.*

Similar to the almost periodic case [13], we will define the Bochner–Fejér polynomials for piecewise continuous almost periodic function f.

From above, we know that with f one can associate a Fourier series and the frequency set of f is countable. According to Lemma 2.1, we choose $\beta_1, \beta_2, \beta_3, \ldots,$ which belong to the frequency set of f, as a basis of frequency set of f. Let r, m, $n_1, n_2, \ldots, n_r$ be positive integers. Define Fejér synthetic kernel for f

$$K^{(m)}_{(n_1,n_2,\dots,n_r,\,\beta_1,\beta_2,\dots,\beta_r)}(t)=K_{n_1}\left(\frac{\beta_1 t}{m!}\right)K_{n_2}\left(\frac{\beta_2 t}{m!}\right)\cdots K_{n_r}\left(\frac{\beta_r t}{m!}\right),$$

where $K_{n_i}(\frac{\beta_i t}{m!})$, $i=1,2,\dots,r$ are Fejér kernels of f which have the form

$$K_n(\beta t)=\frac{\sin^2\frac{n\beta}{2}t}{n\sin^2\frac{\beta}{2}t}=\sum_{\nu=-n}^{n}\left(1-\frac{|\nu|}{n}\right)e^{-i\nu\beta t}.$$

For convenience, let $B=(n_1,n_2,\dots,n_r,\ \beta_1,\beta_2,\dots,\beta_r)$.

Obviously, the Fejér synthetic kernels of f satisfy the following two properties: (1) it is nonnegative; (2) $m(K^{(m)}_B)=1$. In fact,

$$\begin{aligned}K^{(m)}_B(t)&=\sum_{|\nu_i|\leqslant n_i,\ i=1,2,\dots,r}\left(1-\frac{|\nu_1|}{n_1}\right)\left(1-\frac{|\nu_2|}{n_2}\right)\cdots\left(1-\frac{|\nu_r|}{n_r}\right)e^{-i(\frac{\nu_1}{m!}\beta_1+\frac{\nu_2}{m!}\beta_2+\frac{\nu_r}{m!}\beta_r)t}\\&=\sum_{|\nu_i|\leqslant n_i,\ i=1,2,\dots,r}k_{n_1,n_2,\dots,n_r,\,\beta_1,\beta_2,\dots,\beta_r}e^{-i(\frac{\nu_1}{m!}\beta_1+\frac{\nu_2}{m!}\beta_2+\frac{\nu_r}{m!}\beta_r)t}.\end{aligned}$$

By virtue of the linear independence of $\beta_1,\beta_2,\dots,\beta_r$, we obtain $m(K^{(m)}_B)=1$.

Now, we give the Bochner–Fejér polynomials of PCAP function f. Since Bochner–Fejér polynomials only associate with the Fourier expansion, the Bochner–Fejér polynomials of PCAP functions have the same form as the Bochner–Fejér polynomials of AP functions. Specifically, define

$$\begin{aligned}P^{(m)}_B(x)&=m\bigl\{f(x+t)K^{(m)}_B(t)\bigr\}\\&=\lim_{T\to\infty}\int_{-T}^{T}f(x+t)K^{(m)}_B(t)\,dt\\&=\sum_{|\nu_i|\leqslant n_i,\ i=1,2,\dots,r}k_{n_1,\dots,n_r,\,\beta_1,\dots,\beta_r}a\left(\frac{\nu_1}{m!}\beta_1+\frac{\nu_2}{m!}\beta_2+\frac{\nu_r}{m!}\beta_r,\ f\right)e^{i(\frac{\nu_1}{m!}\beta_1+\frac{\nu_2}{m!}\beta_2+\frac{\nu_r}{m!}\beta_r)x}.\end{aligned}$$

$P^{(m)}_B(x)$ are called the Bochner–Fejér polynomials of PCAP function f. According to the properties of the Fejér synthetic kernel of f, we obtain

$$\bigl|P^{(m)}_B(x)\bigr|\leqslant m\bigl\{\bigl|f(x+t)\bigr|K^{(m)}_B(t)\bigr\}\leqslant\sup_x\bigl|f(x)\bigr|,\tag{2.1}$$

$$\bigl|P^{(m)}_B(x+h)-P^{(m)}_B(x)\bigr|\leqslant m\bigl\{\bigl|f(x+h+t)-f(x+t)\bigr|K^{(m)}_B(t)\bigr\}\leqslant\sup_x\bigl|f(x+h)-f(x)\bigr|.\tag{2.2}$$

Since r, m, $n_1,n_2,\dots,n_r$ appearing in $P^{(m)}_B(x)$ are countable, when r, m, $n_1,n_2,\dots,n_r$ change, the Bochner–Fejér polynomials of f form a function sequence. For convenience, we denote this sequence by $\{P_n(x)\}$. Concerning $\{P_n(x)\}$, we have the following lemma:

Lemma 2.2. *Suppose $f\in PCAP$, there exists $\alpha_1>0$ such that for any $\lambda\in\Lambda_f$, $\alpha_1>|\lambda|$ and $\Sigma_{i=1}^{\infty}|a(\lambda_i,f)|<+\infty$, then there exist the subsequences of Bochner–Fejér polynomials $\{P_n(x)\}$ of f, which converge to f uniformly.*

Proof. Since $f\in$ PCAP, for any $\frac{\inf_k t^1_k}{2}>\varepsilon>0$, $\exists\delta(\varepsilon)$, $\varepsilon>\delta>0$ such that if the points t_1 and t_2 belong to the same interval of continuity and $|t_1-t_2|<\delta$, then

$$\bigl|f(t_1)-f(t_2)\bigr|<\varepsilon,$$

besides, there exists an $l>0$ with the property that any interval of length l contains a τ such that

$$\bigl|f(t+\tau)-f(t)\bigr|<\varepsilon,\quad\forall t\in\mathbb{R},\ |t-t_i|>\varepsilon,\ i=0,\pm1,\pm2,\dots.$$

Inequality (2.1) implies that $\{P_n(x)\}$ is uniformly bounded. Let $x_1,x_2,\dots$ be a countable and dense set on $\mathbb{R}$, using the diagonal argument, we can choose a subsequence of $\{P_n(x)\}$ (we still use $\{P_n(x)\}$ to denote the subsequence) such that $\{P_n(x)\}$ is convergent in x_i, $i=1,2,\dots$. The interval $[0,l]$ (l is mentioned above) can be covered by finite small intervals of length δ. In each small interval, we choose a number from $\{x_i\}$ and denote those numbers by $x_1,x_2,\dots,x_k$. Since $\{P_n(x_i)\}_n$ is a Cauchy sequence, for any $\varepsilon>0$ there exists $N=N(\varepsilon)$ such that for any $n>N$, $p>0$

$$\bigl|P_{n+p}(x_i)-P_n(x_i)\bigr|<\varepsilon,\quad i=1,2,\dots,k.\tag{2.3}$$

In fact, inequality (2.3) holds for any $x\in\mathbb{R}$. The proof can be divided into three steps.

Firstly, for δ mentioned above, set

$$f_\delta(t)=\frac{1}{\delta}\int_t^{t+\delta} f(u)\,du.$$

The uniform continuity of f_δ can be derived from the boundedness of f. Since $f\in\mathrm{PCAP}$, f is a Stepanov almost periodic function. It follows from [13] that f_δ is an almost periodic function. By means of approximation theorem of AP functions, there exists a trigonometric polynomial

$$S(t)=\sum_{k=1}^{n(\varepsilon)} b_{k,\varepsilon}a(\lambda_k,f_\delta)e^{i\lambda_k t}$$

such that

$$\bigl|S(t)-f_\delta(t)\bigr|\leqslant\varepsilon,\quad \forall t\in\mathbb{R},$$

where $0\leqslant b_{k,\varepsilon}\leqslant 1$. Besides, the following equalities hold: $\Lambda_f=\Lambda_{f_\delta}$, $a(0,f)=a(0,f_\delta)$, and for any $\lambda\neq 0$

$$\begin{aligned}
a(\lambda,f_\delta)&=\lim_{T\to\infty}\frac{1}{T\delta}\int_0^T\int_t^{t+\delta} f(u)e^{-i\lambda t}\,du\,dt\\
&=\lim_{T\to\infty}\frac{1}{T\delta}\left(\int_0^\delta\int_0^u f(u)e^{-i\lambda t}\,dt\,du+\int_\delta^T\int_{u-\delta}^u f(u)e^{-i\lambda t}\,dt\,du+\int_T^{T+\delta}\int_{u-\delta}^T f(u)e^{-i\lambda t}\,dt\,du\right)\\
&=\lim_{T\to\infty}\frac{-1}{i\lambda\delta T}\left(\int_0^T f(u)e^{-i\lambda u}\,du-\int_0^\delta f(u)\,du-\int_\delta^{T+\delta} f(u)e^{-i\lambda(u-\delta)}\,du+\int_T^{T+\delta} f(u)e^{-i\lambda T}\,du\right)\\
&=\frac{-1}{i\lambda\delta}\lim_{T\to\infty}\frac{1}{T}\left(\int_0^T f(u)e^{-i\lambda u}\,du-e^{i\lambda\delta}\int_\delta^{T+\delta} e^{-i\lambda u}f(u)\,du\right)\\
&=\frac{e^{i\lambda\delta}-1}{i\lambda\delta}\lim_{T\to\infty}\frac{1}{T}\int_0^T f(u)e^{-i\lambda u}\,du\\
&=\frac{e^{i\lambda\delta}-1}{i\lambda\delta}a(\lambda,f).
\end{aligned}$$

Secondly, since $\Gamma(f,\varepsilon)\cap\Gamma(S,\varepsilon)$ is relatively dense in $\mathbb{R}$, i.e., there exists $\tau\in\Gamma(f,\varepsilon)\cap\Gamma(S,\varepsilon)$ in any interval of length l (without loss of generality, we suppose l here is the same as above) such that

$$\bigl|S(t)-S(t+\tau)\bigr|<\varepsilon,\quad\forall t\in\mathbb{R},\quad\text{and}\quad \bigl|f(t+\tau)-f(t)\bigr|<\varepsilon,\quad \forall t\in\mathbb{R},\ |t-t_i|>\varepsilon.$$

For any x, taking $\tau\in\Gamma(f,\varepsilon)\cap\Gamma(S,\varepsilon)$ and $\tau\in[-x,-x+l]$, we have

$$\begin{aligned}
\bigl|P_n(x)-P_{n+p}(x)\bigr|\leqslant{}&\bigl|P_n(x)-P_n(x+\tau)\bigr|+\bigl|P_n(x+\tau)-P_n(x_i)\bigr|+\bigl|P_n(x_i)-P_{n+p}(x_i)\bigr|\\
&+\bigl|P_{n+p}(x_i)-P_{n+p}(x+\tau)\bigr|+\bigl|P_{n+p}(x)-P_{n+p}(x+\tau)\bigr|.
\end{aligned}\tag{2.4}$$

Notice that $\forall\lambda\in\Lambda_f$, $|\lambda|<\alpha_1$, $\Sigma_{i=1}^{\infty}|a(\lambda_i,f)|<+\infty$ and for any $x,y\in\mathbb{R}$,

$$\bigl|e^{ix}-e^{iy}\bigr|\leqslant|x-y|.$$

It follows from the expression of Bochner–Fejér polynomials of PCAP function f that

$$\begin{aligned}
\bigl|P_n(x+\tau)-P_n(x_i)\bigr|&\leqslant\sum_k\bigl|a(\lambda_k,f)\bigr|\bigl|e^{i\lambda_k(x+\tau)}-e^{i\lambda_k x_i}\bigr|\\
&\leqslant\sum_k\bigl|a(\lambda_k,f)\lambda_k\delta\bigr|\leqslant\sum_k\bigl|a(\lambda_k,f)\bigr|\alpha_1\varepsilon.
\end{aligned}\tag{2.5}$$

Similarly, we can show that

$$\left|P_{n+p}(x+\tau)-P_{n+p}(x_i)\right| \leqslant \sum_k \left|a(\lambda_k, f)\right|\alpha_1 \varepsilon. \tag{2.6}$$

At last, we firstly estimate the first item on the right-hand side of (2.4). The last item on the right-hand side of (2.4) can be obtained similarly. Inequality (2.2) implies that

$$\begin{aligned}\left|P_n(x)-P_n(x+\tau)\right| &\leqslant \sup_t \left|f(t)-f(t+\tau)\right| \\ &\leqslant \sup_{|t-t_k|\leqslant\varepsilon}\left|f(t)-f(t+\tau)\right| + \sup_{|t-t_k|>\varepsilon}\left|f(t)-f(t+\tau)\right| \\ &\leqslant \sup_{0\leqslant t-t_k\leqslant\varepsilon}\left|f(t)-f(t+\tau)\right| + \sup_{-\varepsilon\leqslant t-t_k\leqslant 0}\left|f(t)-f(t+\tau)\right| + \varepsilon.\end{aligned}$$

Now, we estimate $\sup_{0\leqslant t-t_k\leqslant\varepsilon}|f(t)-f(t+\tau)|$. Since $0\leqslant t-t_k\leqslant\varepsilon$, then

$$\left|f(t)-f_\delta(t)\right| \leqslant \frac{1}{\delta}\int_t^{t+\delta}\left|f(t)-f(u)\right|du \leqslant \varepsilon.$$

If $-\varepsilon\leqslant t+\tau-t_k\leqslant 0$, then the points $t+\tau$ and u, $u\in(t+\tau-\delta, t+\tau)$, belong to the same interval of continuity of f, hence, $|f(t+\tau)-f_\delta(t+\tau-\delta)|\leqslant\varepsilon$. As for S, we have

$$\begin{aligned}\left|S(t+\tau)-S(t+\tau-\delta)\right| &\leqslant \sum_{k=1}^{n(\epsilon)}\left|a(\lambda_k, f_\delta)\right|\left|e^{i\lambda_k(t+\tau)}-e^{i\lambda_k(t+\tau-\delta)}\right| \\ &\leqslant \sum_k\left|\frac{e^{i\lambda_k\delta}-1}{\lambda_k\delta}a(\lambda_k, f)\lambda_k\delta\right| \\ &\leqslant \sum_k\left|\lambda_k\delta a(\lambda_k, f)\right| \\ &\leqslant \alpha_1\sum_k\left|a(\lambda_k, f)\right|\varepsilon.\end{aligned}$$

Hence, for any $0\leqslant t-t_k\leqslant\varepsilon$

$$\begin{aligned}\left|f(t)-f(t+\tau)\right| &\leqslant \left|f(t)-f_\delta(t)\right|+\left|f_\delta(t)-S(t)\right|+\left|S(t)-S(t+\tau)\right|+\left|S(t+\tau)-S(t+\tau-\delta)\right| \\ &\quad+\left|S(t+\tau-\delta)-f_\delta(t+\tau-\delta)\right|+\left|f_\delta(t+\tau-\delta)-f(t+\tau)\right| \\ &\leqslant \left(5+\alpha_1\sum_k\left|a(\lambda_k, f)\right|\right)\varepsilon.\end{aligned}$$

If $t_k\leqslant t+\tau\leqslant t_{k+1}-\varepsilon$, then the points $t+\tau$ and u, $u\in(t+\tau, t+\tau+\delta)$, belong to the same interval of continuity of f, hence, $|f(t+\tau)-f_\delta(t+\tau)|\leqslant\varepsilon$. Now,

$$\begin{aligned}\left|f(t)-f(t+\tau)\right| &\leqslant \left|f(t)-f_\delta(t)\right|+\left|f_\delta(t)-S(t)\right|+\left|S(t)-S(t+\tau)\right| \\ &\quad+\left|S(t+\tau)-f_\delta(t+\tau)\right|+\left|f_\delta(t+\tau)-f(t+\tau)\right| \\ &\leqslant \left(5+\alpha_1\sum_k\left|a(\lambda_k, f)\right|\right)\varepsilon.\end{aligned}$$

To sum up,

$$\sup_{0\leqslant t-t_k\leqslant\varepsilon}\left|f(t)-f(t+\tau)\right| \leqslant \left(5+\alpha_1\sum_k\left|a(\lambda_k, f)\right|\right)\varepsilon.$$

For $\sup_{-\varepsilon\leqslant t-t_k\leqslant 0}|f(t)-f(t+\tau)|$, by a similar argument as above, we can obtain

$$\sup_{-\varepsilon\leqslant t-t_k\leqslant 0}\left|f(t)-f(t+\tau)\right| \leqslant \left(5+\alpha_1\sum_k\left|a(\lambda_k, f)\right|\right)\varepsilon.$$

Hence, for any $x \in \mathbb{R}$

$$\left|P_n(x) - P_n(x+\tau)\right| \leqslant \left(11 + 2\alpha_1 \sum_k \left|a(\lambda_k, f)\right|\right)\varepsilon. \tag{2.7}$$

Similarly, we have

$$\left|P_{n+p}(x) - P_{n+p}(x+\tau)\right| \leqslant \left(11 + 2\alpha_1 \sum_k \left|a(\lambda_k, f)\right|\right)\varepsilon, \quad \forall x \in \mathbb{R}. \tag{2.8}$$

Combining inequalities (2.3)–(2.8), we know that for any $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that for any $n > N$, $p > 0$

$$\left|P_{n+p}(x) - P_n(x)\right| < \left(23 + 6\alpha_1 \sum_k \left|a(\lambda_k, f)\right|\right)\varepsilon, \quad \forall x \in \mathbb{R}.$$

Consequently, $\{P_n(x)\}$ is a Cauchy sequence. We denote the limit of $\{P_n(x)\}$ by $\psi(x)$. Since $k_{n_1,\dots,n_r,\beta_1,\dots,\beta_r}$ appearing in the Bochner–Fejér polynomials of f tend to 1 as $n_1, \dots, n_r$ tend to infinity, it is obvious that the Fourier series of ψ is the same as the Fourier series of f, it follows from the uniqueness theorem of PCAP function that $\psi = f$. Therefore, there exist the subsequences of Bochner–Fejér polynomials $\{P_n(x)\}$ of f, which converge to f uniformly. The proof is complete. $\square$

Now, we are in the position to give the main theorem (Theorem 2.3) in this paper. Because Theorem 2.3 shows that under some conditions, the primitive function of PCAP function is an AP function, and is similar to Favard's theorem of AP functions, it might as well be called Favard's theorem of PCAP functions.

Theorem 2.3 *(Favard's theorem of PCAP functions). Suppose $f \in PCAP$, there exist $\alpha_1 > \alpha > 0$ such that $\forall \lambda \in \Lambda_f$, $\alpha_1 > |\lambda| > \alpha$, $\Sigma_{i=1}^{\infty} |a(\lambda_i, f)| < +\infty$, then the primitive function of f is an almost periodic function.*

Proof. The proof of this theorem is similar to the proof of Favard's theorem of almost periodic functions. We give a simple description here. The details can be found in Ref. [13].

Set

$$\phi(x) = \begin{cases} \frac{1}{i\alpha^2}x, & 0 \leqslant x \leqslant \alpha, \\ \frac{1}{ix}, & x > \alpha, \end{cases} \qquad -\phi(x) = \phi(-x).$$

From [13], we know that $\phi(x)$ is square-integrable on the interval $(-\infty, +\infty)$. The Fourier transform formula of $\phi(x)$ is

$$\psi(u) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \phi(x) e^{-ixu}\, dx = \frac{-1}{\pi\alpha^2} \int_0^{\alpha} x \sin xu\, dx - \frac{1}{\pi} \int_{\alpha}^{\infty} \frac{\sin xu}{x}\, dx.$$

ψ is an odd and bounded function, continuous on the intervals $(-\infty, 0]$ and $[0, +\infty)$. Furthermore, $\psi \in L(-\infty, \infty)$. According to the Fourier inversion formula, we get

$$\phi(x) = \int_{-\infty}^{+\infty} \psi(u) e^{ixu}\, du. \tag{2.9}$$

Considering the function

$$F(x) = \int_{-\infty}^{+\infty} f(x+u)\psi(u)\, du,$$

for any $\tau \in \mathbb{R}$, we have

$$\sup_x \left|F(x+\tau) - F(x)\right| \leqslant \sup_t \left|f(t+\tau) - f(t)\right| \int_{-\infty}^{+\infty} \left|\psi(u)\right| du.$$

Similar to the proof of Lemma 2.2, we can obtain that $F(x)$ is an almost periodic function. Using (2.9) and the expression of $\phi(x)$, we get the Fourier series associated with F

$$F(x) \sim \sum \frac{a(\lambda_k, f)}{i\lambda_k} e^{i\lambda_k x}.$$

We use $P_B^{(m)}(x,f)$ and $P_B^{(m)}(x,F)$ to denote the Bochner–Fejér polynomials for f and F, respectively. Obviously, $\frac{dP_B^{(m)}(x,F)}{dx} = P_B^{(m)}(x,f)$. Hence

$$P_B^{(m)}(x,F) - P_B^{(m)}(0,F) = \int_0^x P_B^{(m)}(t,f)\,dt.$$

Taking the limit on both sides of the equality, combining Lemma 2.2, we obtain

$$F(x) - F(0) = \int_0^x f(t)\,dt,$$

hence, the almost periodic function F is the primitive function of f. The proof is complete. □

Inequality (2.1) implies that if the Fourier coefficients of f don't change sign, then its Fourier coefficients are absolutely convergent. Consequently, we have the following lemma:

Lemma 2.4. *Suppose $f \in PCAP$, there exist $\alpha_1 > \alpha > 0$ such that $\forall \lambda \in \Lambda_f$, $\alpha_1 > |\lambda| > \alpha$, and the Fourier coefficients of f don't change sign, then the primitive function of f is an almost periodic function.*

3. The application of Favard's theorem of piecewise continuous almost periodic functions

In this section, by means of Favard's theorem of PCAP functions and coincidence degree theory, we investigate the existence of strictly positive PCAP solution of impulsive single population model with hereditary effects:

$$\begin{cases} N'(t) = N(t)\big[a(t) - b(t)N(t) - d(t)N\big(t-\tau(t)\big)\big], & t \neq t_k, \\ N\big(t_k^+\big) = (1+c_k)N(t_k), \end{cases} \tag{3.1}$$

where $a(\cdot)$, $b(\cdot)$, $d(\cdot)$, $\tau(\cdot)$ are positive almost periodic functions, $m(a) > 0$, $\{c_k\}$ is an almost periodic sequence, $\{t_k\}$ is an equipotentially almost periodic sequence. We consider functions $\prod_{0<t_k<t}(1+c_k)$ and $\prod_{0<t_k<t-\tau(t)}(1+c_k)$, namely,

$$\prod_{0<t_k<t}(1+c_k) = \begin{cases} 1, & t \in (-\infty, t_1], \\ (1+c_1)(1+c_2)\cdots(1+c_k), & t \in (t_k, t_{k+1}],\ k = 1,2,3,\ldots, \end{cases}$$

$$\prod_{0<t_k<t-\tau(t)}(1+c_k) = \begin{cases} 1, & t-\tau(t) \in (-\infty, t_1], \\ (1+c_1)(1+c_2)\cdots(1+c_k), & t-\tau(t) \in (t_k, t_{k+1}],\ k = 1,2,3,\ldots. \end{cases}$$

We suppose the following condition (H) is satisfied, that is

(H) $\prod_{0<t_k<t}(1+c_k)$, $\prod_{0<t_k<t-\tau(t)}(1+c_k)$ are positive piecewise continuous almost periodic functions, and

$$\inf_{t\in\mathbb{R}} \prod_{0<t_k<t} \big|(1+c_k)\big| > 0.$$

Remark. There exist a great number of functions satisfying the assumption (H). For instance, we suppose $\{c_k\}$ is a periodic sequence with period K (K is a positive integer), $(1+c_1)(1+c_2)\cdots(1+c_K) = 1$, $1+c_i > 0$, $\forall i \in \mathbb{Z}$.

(1) Let $\{t_k\}$ be an arbitrary equipotentially almost periodic sequence, $\tau(t) = \tau > 0$, then $\prod_{0<t_k<t}(1+c_k)$ is a positive PCAP function with discontinuous points t_k, and $\inf_{t\in\mathbb{R}} \prod_{0<t_k<t}|(1+c_k)| > 0$. $\prod_{0<t_k<t-\tau(t)}(1+c_k)$ is also a positive PCAP function with discontinuous points $t_k + \tau$.

(2) Let $\{t_k\} = \{1,2,3,\ldots\}$,

$$\tau(t) = \begin{cases} t - 2n, & t \in [2n, 2n+1), \\ -t + 2n + 2, & t \in [2n+1, 2n+2),\ n = 0, \pm 1, \pm 2, \ldots. \end{cases}$$

$\tau(t)$ is a positive periodic function. $\prod_{0<t_k<t}(1+c_k)$ is a positive PCAP function with discontinuous points t_k, and $\inf_{t\in\mathbb{R}} \prod_{0<t_k<t}|(1+c_k)| > 0$. $\prod_{0<t_k<t-\tau(t)}(1+c_k)$ is also a positive PCAP function with discontinuous points $\frac{3}{2}, 3, \frac{7}{2}, 5, \frac{11}{2}, 7, \frac{15}{2}, \ldots$, which is an equipotentially almost periodic sequence.

Before studying the existence of strictly positive PCAP solution of system (3.1), we firstly introduce a few concepts and the continuous theorem, which are summarized in [10].

Let X and Z be real Banach spaces, $L : dom\,L \subset X \to Z$ be a linear mapping, $N : X \to Z$ be a continuous mapping. L is called a Fredholm mapping of index zero if $dim\,Ker\,L = codim\,Im\,L < \infty$ and $Im\,L$ is close in Z. If L is a Fredholm mapping of index zero, there are continuous projects $P : X \to X$, $Q : Z \to Z$ such that $Im\,P = Ker\,L$, $Im\,L = Ker\,Q = Im(I - Q)$. It follows that $L|dom\,L \cap Ker\,P : (I - P)X \to Im\,L$ is invertible. We denote the inverse of that map by K_p. If Ω is an open subset of X, the mapping N will be called L-compact on Ω if $QN(\bar{\Omega})$ is bounded and $K_p(I - Q)N : \bar{\Omega} \to X$ is compact. Since $Im\,Q$ is isomorphic to $Ker\,L$, there exists isomorphism $J : Im\,Q \to Ker\,L$. The following continuous theorem holds:

Lemma 3.1 *(Continuous theorem). Let $\Omega \subset X$ be an open bounded set, let L be a Fredholm mapping of index zero and N be L-compact on $\bar{\Omega}$. Assume*

(1) *For each $\lambda \in (0, 1)$, every solution x of $Lx = \lambda Nx$ is such that $x \notin \partial\Omega$;*
(2) *For each $x \in Ker\,L \cap \partial\Omega$, $QNx \neq 0$;*
(3) *$deg(JQN, Ker\,L \cap \Omega, 0) \neq 0$.*

Then $Lx = Nx$ has at least one solution in $dom\,L \cap \bar{\Omega}$.

Consider the following equation

$$y'(t) = y(t)\big[a(t) - B(t)y(t) - D(t)y\big(t - \tau(t)\big)\big], \tag{3.2}$$

where $B(t) = \prod_{0<t_k<t}(1 + c_k)b(t)$, $D(t) = \prod_{0<t_k<t-\tau(t)}(1 + c_k)d(t)$. The solutions of Eqs. (3.1) and (3.2) satisfy the following relationships:

Lemma 3.2. *Suppose* (H) *is satisfied, the following results hold*:

(1) *If $N(\cdot) \in PCAP$ is a solution of Eq. (3.1), then $y(\cdot) = \prod_{0<t_k<t}(1 + c_k)^{-1}N(\cdot)$ is an AP solution of Eq. (3.2);*
(2) *If $y(\cdot) \in AP$ is a solution of Eq. (3.2), then $N(\cdot) = \prod_{0<t_k<t}(1 + c_k)y(\cdot)$ is a PCAP solution of Eq. (3.1).*

From [15] we know that if $\prod_{0<t_k<t}(1 + c_k)$ is a positive piecewise continuous almost periodic function and $\inf_{t\in\mathbb{R}} \prod_{0<t_k<t} |(1 + c_k)| > 0$, then $\prod_{0<t_k<t}(1 + c_k)^{-1}$ is a positive piecewise continuous almost periodic function. Besides, the scalar product of two piecewise continuous almost periodic functions which have a common sequence of points of discontinuity is also a piecewise continuous almost periodic function. Similar to [12], the proof of Lemma 3.2 can be obtained easily, we omit it here.

Let $y(t) = e^{x(t)}$, Eq. (3.2) is translated to

$$x'(t) = a(t) - B(t)e^{x(t)} - D(t)e^{x(t-\tau(t))}. \tag{3.3}$$

Obviously, if Eq. (3.3) has an almost periodic solution, then Eq. (3.2) has a strictly positive almost periodic solution. It follows from Lemma 3.2 that Eq. (3.1) has a strictly positive piecewise continuous almost periodic solution. Due to this, we concentrate on solving the existence of almost periodic solution of Eq. (3.3).

Lemma 3.3. *If* (H) *is satisfied, $m(B) + m(D) \neq 0$, then Eq.* (3.3) *has at least one almost periodic solution.*

Proof. We divide the proof into four steps.

Step 1. Let

$$X_1 = \big\{x(\cdot) \in \mathrm{AP}\colon\ mod(x) \subset mod(F_1),\ \forall\lambda \in \Lambda_x,\ \alpha_1 > |\lambda| > \alpha\big\} \cup \{0\},$$

$$Z_1 = \Bigg\{z(\cdot) \in \text{PCAP with discontinuous points } \{t_k\},\ mod(z) \subset mod(F_1),$$

$$\forall\lambda \in \Lambda_z,\ \alpha_1 > |\lambda| > \alpha,\ \sum_{i=1}^{\infty}\big|a(\lambda_i, z)\big| < +\infty\Bigg\} \cup \{0\},$$

$$Z_2 = X_2 = \big\{x(\cdot),\ x(\cdot) = h \in \mathbb{R}\big\},$$

where α and α_1 are given positive constants, and

$$F_1(t) = a(t) - b(t)e^{\varphi(0)} - d(t)e^{\varphi(-\tau(t))}, \quad \varphi(\cdot) \in C\left(\left[-\sup_{t\in\mathbb{R}} \tau(t), 0\right]\right).$$

Define $X = X_1 \oplus X_2$, $Z = Z_1 \oplus Z_2$ with the norm $\|\phi\| = \sup_{t\in\mathbb{R}} |\phi(t)|$, $\phi \in X$ or Z. Similar to [22], we know that X is a Banach space. In fact, Z is also a Banach space. If $\{z_n\} \subset Z$ and z_n converges to z uniformly, it follows from the property of PCAP functions that $z \in$ PCAP with discontinuous points $\{t_k\}$. According to [22], it is easy to show that $mod(z) \subset mod(F_1)$, $\forall \lambda \in \Lambda_z$, $\alpha_1 > |\lambda| > \alpha$. Besides, we assert that $\sum_{i=1}^{\infty} |a(\lambda_i, z)| < +\infty$. If not, $\forall M > 0$, $\exists k > 0$ such that $\sum_{i=1}^{k} |a(\lambda_i, z)| > M$. Since $\{z_n\}$ is a Cauchy sequence, for $1/2^k$, $\exists K$ such that for any $m, n > K$,

$$\left|a(\lambda, z_m) - a(\lambda, z_n)\right| < \frac{1}{2^k}, \quad \forall \lambda \in \mathbb{R}.$$

Since z_n converges to z uniformly,

$$\begin{aligned}
&\text{for } \lambda_1 \in \Lambda_z \text{ and } \frac{1}{2},\ \exists n_1 > K,\ \text{s.t.,} \quad \left|a(\lambda_1, z)\right| < \left|a(\lambda_1, z_{n_1})\right| + \frac{1}{2},\\
&\text{for } \lambda_2 \in \Lambda_z \text{ and } \frac{1}{4},\ \exists n_2 > K,\ \text{s.t.,} \quad \left|a(\lambda_2, z)\right| < \left|a(\lambda_2, z_{n_2})\right| + \frac{1}{4},\\
&\quad \vdots\\
&\text{for } \lambda_k \in \Lambda_z \text{ and } \frac{1}{2^k},\ \exists n_k > K,\ \text{s.t.,} \quad \left|a(\lambda_k, z)\right| < \left|a(\lambda_k, z_{n_k})\right| + \frac{1}{2^k}.
\end{aligned}$$

Since $n_1, n_2, \ldots, n_k > K$, then

$$\left|a(\lambda_i, z_{n_i})\right| < \left|a(\lambda_i, z_{n_1})\right| + \frac{1}{2^k}, \quad i = 1, 2, \ldots, k.$$

Hence,

$$\begin{aligned}
M &< \sum_{i=1}^{k} \left|a(\lambda_i, z)\right| < \sum_{i=1}^{k} \left|a(\lambda_i, z_{n_i})\right| + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^k}\\
&< \sum_{i=1}^{k} \left|a(\lambda_i, z_{n_i})\right| + 1 < \sum_{i=1}^{k} \left|a(\lambda_i, z_{n_1})\right| + 1 + \frac{k}{2^k} < \sum_{i=1}^{\infty} \left|a(\lambda_i, z_{n_1})\right| + \frac{3}{2}.
\end{aligned}$$

The arbitrariness of M leads to a contradiction, so the assertion holds. Therefore, Z is a Banach space.

Step 2. Let

$$L : X \to Z, \quad Lx = \frac{dx}{dt}.$$

We prove that L is a Fredholm mapping of index zero.

Obviously, $Ker\,L = X_2$. We claim that $Im\,L = Z_1$. Firstly, for any $\varphi \in Im\,L \subset Z$, $\varphi = \varphi_1 + \varphi_2$, $\varphi_1 \in Z_1$, $\varphi_2 \in Z_2$. Since $\int_0^t \varphi(s)\,ds \in$ AP, $\int_0^t \varphi_1(s)\,ds \in$ AP, then $\varphi_2 = 0$. Thus, $Im\,L \subset Z_1$. Secondly, for any $\varphi \in Z_1$ (without loss of generality, we suppose $\varphi \neq 0$), $\int_0^t \varphi(s)\,ds \in$ AP, since

$$\Lambda_{\int_0^t \varphi(s)\,ds - m(\int_0^t \varphi(s)\,ds)} = \Lambda_\varphi,$$

$\int_0^t \varphi(s)\,ds - m(\int_0^t \varphi(s)\,ds) \in X$ is the primitive function of φ. Hence, $\varphi \in Im\,L$, $Z_1 \subset Im\,L$. To sum up, $Z_1 = Im\,L$. Besides, one can easily show that $Im\,L$ is closed in Z and $dim\,Ker\,L = 1 = codim\,Im\,L$. Therefore, L is a Fredholm mapping of index zero.

Step 3. We define some operators:

$$\begin{aligned}
&N : X \to Z, \quad Nx(t) = a(t) - B(t)e^{x(t)} - D(t)e^{x(t-\tau(t))},\\
&P : X \to X, \quad Px = m(x), \qquad Q : Z \to Z, \quad Qz = m(z).
\end{aligned}$$

Firstly, it is easy to show that P and Q are continuous projectors such that

$$Im\,P = Ker\,L, \qquad Im\,L = Im(I - Q) = Ker\,Q,$$

where I is an identity mapping. Thus, $L|dom\,L \cap Ker\,P : (I - P)X \to Im\,L$ is invertible. We denote the inverse of that map by K_p. $K_p : Im\,L \to Ker\,P \cap dom\,L$ has the form

$$K_p z(t) = \int_0^t z(s)\,ds - m\left(\int_0^t z(s)\,ds\right),$$

then,

$$QNx(t) = m\big(a(t) - B(t)e^{x(t)} - D(t)e^{x(t-\tau(t))}\big),$$
$$K_p(I-Q)Nx(t) = f\big(x(t)\big) - Qf\big(x(t)\big),$$

where,

$$f\big(x(t)\big) = \int_0^t \big(Nx(s) - QNx(s)\big)\,ds.$$

Obviously, QN and $(I-Q)N$ are continuous. In fact, K_p is also continuous. For any $1 > \varepsilon > 0$, let l_1 be the inclusion interval of $\Gamma(F_1, \varepsilon)$. Suppose $z \in Z_1 = Im\,L$, then $\int_0^t z(s)\,ds \in \mathrm{AP}$. Since

$$\Lambda_{\int_0^t z(s)\,ds} = \Lambda_{\int_0^t z(s)\,ds - m(\int_0^t z(s)\,ds)} \cup \{0\} = \Lambda_z \cup \{0\}, \quad mod(z) \subset mod(F_1),$$

then $mod(\int_0^t z(s)\,ds) \subset mod(F_1)$. Therefore, there exists δ, $0 < \delta < \varepsilon$ such that $\Gamma(F_1,\delta) \subset \Gamma(\int_0^t z(s)\,ds, \varepsilon)$. Let l be the inclusion interval of $\Gamma(F_1,\delta)$, then $l \geqslant l_1$. For any $t \notin [0,l]$, there exists $\xi \in \Gamma(F_1,\delta) \subset \Gamma(\int_0^t z(s)\,ds, \varepsilon)$ such that $t+\xi \in [0,l]$,

$$\begin{aligned}
\sup_{t\in R}\left|\int_0^t z(s)\,ds\right| &\leqslant \sup_{t\in[0,l]}\left|\int_0^t z(s)\,ds\right| + \sup_{t\notin[0,l]}\left|\int_0^t z(s)\,ds\right| \\
&\leqslant \int_0^l \big|z(s)\big|\,ds + \sup_{t\notin[0,l]}\left|\int_0^t z(s)\,ds - \int_0^{t+\xi} z(s)\,ds\right| + \sup_{t\notin[0,l]}\left|\int_0^{t+\xi} z(s)\,ds\right| \\
&\leqslant 2\int_0^l \big|z(s)\big|\,ds + \sup_{t\notin[0,l]}\left|\int_0^t z(s)\,ds - \int_0^{t+\xi} z(s)\,ds\right| \\
&\leqslant 2\int_0^l \big|z(s)\big|\,ds + \varepsilon.
\end{aligned}$$

Thus, we can conclude that K_p is continuous, and consequently, $K_p(I-Q)N$ is also continuous. In addition, we can easily obtain that $K_p(I-Q)N$ is uniformly bounded in $\bar{\Omega}$, $QN(\bar{\Omega})$ is bounded and $K_p(I-Q)N$ is equicontinuous in $\bar{\Omega}$. By using the Arzela–Ascoli theorem, we can immediately conclude that $\overline{K_p(I-Q)N\bar{\Omega}}$ is compact, thus N is L-compact on $\bar{\Omega}$.

Step 4. Let the isomorphism $J : Im\,Q \to Ker\,L$ be an identity mapping. We search an appropriate bounded open subset Ω for the application of Lemma 3.1. Corresponding to operator equation $Lx = \lambda Nx$, $\lambda \in (0,1)$, we have

$$x'(t) = \lambda\big(a(t) - B(t)e^{x(t)} - D(t)e^{x(t-\tau(t))}\big). \tag{3.4}$$

If $x \in X$ is a solution of system (3.4), taking the limit mean for system (3.4), we obtain $m(a(t)) = m(B(t)e^{x(t)} + D(t)e^{x(t-\tau(t))})$. Since $\frac{m(a)}{m(B)+m(D)} > 0$, then $\sup_{t\in\mathbb{R}} x(t) \geqslant \ln\frac{m(a)}{m(B)+m(D)} \geqslant \inf_{t\in\mathbb{R}} x(t)$. Hence there exists at least one $t^* \in \mathbb{R}$ such that

$$\big|x(t^*)\big| \leqslant \left|\ln\frac{m(a)}{m(B)+m(D)}\right| + 1. \tag{3.5}$$

Since $Lx \in Z_1$, by a similar argument as in Step 3, we can derive that

$$\|x\| = \sup_{t\in\mathbb{R}}\big|x(t)\big| \leqslant \big|x(t^*)\big| + \sup_{t\in\mathbb{R}}\left|\int_{t^*}^t x'(s)\,ds\right| \leqslant x(t^*) + 2\int_{t^*}^{t^*+l}\big|x'(s)\big|\,ds + 1. \tag{3.6}$$

Taking $\bar{\tau} \in [l, 2l] \cap \Gamma(F_1,\delta) \subset \Gamma(\int_0^t x'(s)\,ds, \varepsilon)$, we have

$$\begin{aligned}\int_{t^*}^{t^*+l} \left|x'(s)\right| ds &\leqslant \int_{t^*}^{t^*+\bar\tau} \left|x'(s)\right| ds \leqslant \lambda \int_{t^*}^{t^*+\bar\tau} \left|a(s)\right| ds + \lambda \int_{t^*}^{t^*+\bar\tau} \left|B(s)e^{x(s)} + D(s)e^{x(s-\tau(s))}\right| ds \\ &\leqslant 2\lambda \int_{t^*}^{t^*+\bar\tau} \left|a(s)\right| ds - \lambda \int_{t^*}^{t^*+\bar\tau} x'(s)\, ds \leqslant 2 \int_{t^*}^{t^*+2l} \left|a(s)\right| ds + 1.\end{aligned} \tag{3.7}$$

Combining (3.5)–(3.7), we obtain

$$\|x\| \leqslant \left|\ln \frac{m(a)}{m(B)+m(D)}\right| + 4 + 4 \int_{t^*}^{t^*+l} \left|x'(s)\right| ds. \tag{3.8}$$

Take $\Omega = \{x \in X,\ \|x\| \leqslant 4|\ln \frac{m(a)}{m(B)+m(D)}| + 4 + 4\int_{t^*}^{t^*+l} |x'(s)|\, ds\}$, then it is clear that Ω verifies all the requirements in Lemma 3.1, hence system (3.3) has at least one almost periodic solution. The proof is complete. □

Remark. If $f \in AP$ and $\forall\lambda \in \Lambda_f$, $|\lambda| > \alpha > 0$, then the primitive function of f is an AP function. It doesn't hold for PCAP functions. Theorem 2.3 implies that if $f \in PCAP$, $\forall\lambda \in \Lambda_f$, $\alpha_1 > |\lambda| > \alpha > 0$, $\sum_{i=1}^{\infty} |a(\lambda_i, f)| < +\infty$, then the primitive function of f is an AP function. That is the reason why in Lemma 3.3 we take Z_1 like that.

Combining Lemmas 3.2 and 3.3, for Eq. (3.1), we have the result

Theorem 3.4. *If* (H) *is satisfied,* $m(B) + m(D) \neq 0$, *then Eq.* (3.1) *has at least one strictly positive piecewise continuous almost periodic solution.*

In the following, we present two examples to demonstrate the result obtained in this section.

Example 1. Consider the following impulsive single population model with hereditary

$$\begin{cases} N'(t) = N(t)\left[(3 - \cos\sqrt{3}t) - (2 - \cos 2\pi t)N(t) - (2 + \sin 2\pi t)N(t-1)\right], & t \neq 1, 2, 3, \ldots, \\ N(2k^+) = 2N(2k), \quad N\left((2k+1)^+\right) = \dfrac{1}{2}N(2k+1), & k = 0, 1, 2, \ldots. \end{cases} \tag{3.9}$$

Obviously, $a(\cdot)$, $b(\cdot)$, $d(\cdot)$, $\tau(\cdot)$ are positive almost periodic functions, $m(a) = 3 > 0$, $\{c_k\}$ is an almost periodic sequence, $\{t_k\}$ is an equipotentially almost periodic sequence. The assumption (H) holds. Besides, $m(B) + M(D) = \frac{15}{4} > 0$. From Theorem 3.4, we know that system (3.9) has at least one strictly positive piecewise continuous almost periodic solution.

Example 2. If $c_k = 0$, $\prod_{0<t_k<t}(1 + c_k)$, $\prod_{0<t_k<t-\tau(t)}(1 + c_k)$ are constant functions, then the assumption (H) holds. System (3.1) can be rewritten as the following system [22]

$$N'(t) = N(t)\left[a(t) - b(t)N(t) - d(t)N\left(t - \tau(t)\right)\right].$$

Similar result as in Theorem 3.4 is given in [22].

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