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Local dimensions of measures on infinitely generated self-affine sets

Eino Rossi [1]

Department of Mathematics and Statistics, P.O. Box 35 (MaD), FI-40014, University of Jyväskylä, Finland

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ABSTRACT

We show the existence of the local dimension of an invariant probability measure on an infinitely generated self-affine set, for almost all translations. This implies that an ergodic probability measure is exactly dimensional. Furthermore the local dimension equals the minimum of the local Lyapunov dimension and the dimension of the space.

1. Introduction

The *upper* and *lower local dimensions* of a locally finite Borel measure μ, denoted by $\overline{\dim}_{\text{loc}}(\mu, x)$ and $\underline{\dim}_{\text{loc}}(\mu, x)$ respectively, are the lim sup and lim inf of the ratio

$$\frac{\log \mu(B(x,r))}{\log r},$$

as $r \to 0$. When they agree, we say that the *local dimension*, denoted by $\dim_{\text{loc}}(\mu, x)$, exists and equals to this common value. If the local dimension is constant almost everywhere, we say that μ is *exactly dimensional*. The local dimension does not only give information about the geometry of the measure, but also about the support of the measure. For example, if the upper local dimension of μ is smaller than t for all $x \in A$, then the packing dimension of A is at most t, see e.g. [6, Proposition 2.3(d)].

Our main interest is to study the local dimensions of the canonical projection $\pi\mu$ of an invariant Borel probability measure μ onto a self-affine set. In 2009, Feng and Hu [10] showed that the local dimension of $\pi\mu$ exists almost everywhere if the underlying iterated function system, IFS for short, is conformal. They also showed that the local dimension exists if the mappings of the IFS satisfy $f_i(x) = A_i x + a_i$

E-mail address: eino.rossi@jyu.fi.

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and the matrices A_i commute. When μ is ergodic, these results give that μ is exactly dimensional. The general affine case remained open. In 2011, Falconer and Miao [8] calculated the local dimension in a specific affine case. They showed that $\pi\mu$ is exactly dimensional for Lebesgue almost all translation vectors $a \in \mathbb{R}^{d\kappa}$, where κ is the number of mappings in the IFS, assuming that μ is a Bernoulli measure and that $\sup_i \|A_i\| < \frac{1}{2}$, see [8, Theorem 6.1]. By taking a minor change in the proof of [11, Theorem 4.3] we can have the same result for any ergodic measure. This was noted in a very restrictive case by Feng and Barral in [9, Theorem 2.6] and giving the general proof was one of our motivations at the beginning of this work. In [1], the published version of [9], it is mentioned that this generalization is also known by the authors of [11]. However the proof is not written out. All the works mentioned here assume that the IFS is finitely generated.

Our main result, Theorem 1.2, generalizes the results mentioned above. We show that even in the infinitely generated case, the local dimension of an invariant Borel measure exists, assuming again that $\sup_i \|A_i\| < \frac{1}{2}$. As a corollary we get that ergodic measures are exactly dimensional. We also remark how to obtain estimates for the local dimensions that hold for all translations, with only assuming that the mappings A_i are contractive. Finally, we make remarks on the connections of our results to the dimensions of the limit set.

Let us now introduce some notation. Let I be a finite or countable set. We define $I^* = \bigcup_{n=1}^{\infty} I^n$. If I is finite, we say that $I^{\mathbb{N}}$ is *finitely generated* and otherwise $I^{\mathbb{N}}$ is *infinitely generated*. When $\mathtt{i} \in I^*$, we denote by $\mathtt{ij}$ the symbol obtained by juxtaposing $\mathtt{i}$ and $\mathtt{j}$. Furthermore, for $\mathtt{i} \in I^*$, we set $[\mathtt{i}] = \{\mathtt{ij}\colon \mathtt{j} \in I^{\mathbb{N}}\}$ and call this set a *cylinder* of $\mathtt{i}$. When $\mathtt{i} = (i_1, i_2, \dots)$ we denote $\mathtt{i}|_n = (i_1, \dots, i_n)$. On the symbol space $I^{\mathbb{N}}$ we consider the *left shift* σ, defined by $\sigma(i_1, i_2, i_3, \dots) = (i_2, i_3, \dots)$ and study Borel measures that are *invariant* with respect to this shift, that is $\mu(B) = \mu(\sigma^{-1}B)$ for all Borel sets B. An invariant measure is called *ergodic*, if for all Borel sets B with $B = \sigma^{-1}B$, we have $\mu(B) = 0$ or $\mu(B) = 1$. We denote the set of invariant and ergodic Borel probability measures on $I^{\mathbb{N}}$ by $\mathcal{M}_\sigma(I^{\mathbb{N}})$ and $\mathcal{E}_\sigma(I^{\mathbb{N}})$ respectively. Throughout the paper, μ denotes a Borel probability measure. By $\pi\mu$, we mean the push-forward measure $\mu \circ \pi^{-1}$.

For each $i \in I$, we fix an invertible $d \times d$ matrix A_i and a translation vector $a_i \in \mathbf{Q}$, where $\mathbf{Q} = [-\frac{1}{2}, \frac{1}{2}]^d$. Due to Kolmogorov extension theorem $\mathbf{Q}^{\mathbb{N}}$ supports a natural probability measure $\boldsymbol{m} = (\mathcal{L}^d|_{\mathbf{Q}})^{\mathbb{N}}$. We assume that $\sup_{i \in I} \|A_i\| = \bar{\alpha} < 1$ and consider the IFS $\{f_i\}_{i \in I}$, where $f_i(x) = A_i x + a_i$, and the *canonical projection* $\pi_{\boldsymbol{a}} : I^{\mathbb{N}} \to \mathbb{R}^d$ defined by $\{\pi_{\boldsymbol{a}}(\mathtt{i})\} = \bigcap_{n \in \mathbb{N}} f_{\mathtt{i}|_n}(B(0,R))$, where $f_{\mathtt{i}|_n} = f_{i_1} \circ \cdots \circ f_{i_n}$ and R is so large that $f_i(B(0,R)) \subset B(0,R)$ for all $i \in I$. We call $F_{\boldsymbol{a}} = \bigcup_{\mathtt{i} \in I^{\mathbb{N}}} \pi_{\boldsymbol{a}}(\mathtt{i})$ the *limit set* of this IFS. It is not restrictive to assume that each a_i is in the cube $\mathbf{Q}$, since this is just a matter of scaling the limit set. This only excludes the case where $\sup_i |a_i| = \infty$.

The *singular values* $\|A_{\mathtt{i}|_n}\| = \alpha_1(\mathtt{i}|_n) \geqslant \cdots \geqslant \alpha_d(\mathtt{i}|_n) > 0$ of $A_{\mathtt{i}|_n} = A_{i_1} \cdots A_{i_n}$ are the lengths of the principal semiaxis of the ellipsoid $A_{\mathtt{i}|_n}(B(0,1))$. For $0 \leqslant s < d$, the *singular value function* is defined as

$$\phi^s(\mathtt{i}|_n) = \alpha_1(\mathtt{i}|_n) \cdots \alpha_k(\mathtt{i}|_n) \alpha_{k+1}(\mathtt{i}|_n)^{s-k},$$

where k is the integer part of s. When $s \geqslant d$, we set $\phi^s(\mathtt{i}|_n) = (\alpha_1(\mathtt{i}|_n) \cdots \alpha_d(\mathtt{i}|_n))^{s/d}$.

For convenience, fix partitions $\mathcal{P}_n = \{[\mathtt{i}]\}_{\mathtt{i} \in I^n}$, and set $H_\mu(\mathcal{P}_n) = -\sum_{\mathtt{i} \in I^n} \mu[\mathtt{i}] \log \mu[\mathtt{i}]$. *Entropy* and *energy* of $\mu \in \mathcal{M}_\sigma(I^{\mathbb{N}})$, defined by

$$h_\mu = -\lim_{n \to \infty} \frac{1}{n} H(\mathcal{P}_n) \quad \text{and} \quad \Lambda_\mu(s) = \lim_{n \to \infty} \frac{1}{n} \int_{I^{\mathbb{N}}} \log \phi^s(\mathtt{i}|_n)\, d\mu$$

respectively, are the basic tools in the study of ergodic measures in the field of iterated function systems. In order to work with invariant measures, we need to localize these concepts. By theorems [18, Theorem 7 in Section 2] and [22, Theorem 10.1], for $\mu \in \mathcal{M}_\sigma(I^{\mathbb{N}})$, there exist $L^1(\mu)$ functions $h_\mu(\mathtt{i})$ and $\Lambda_\mu(s, \mathtt{i})$ so that

$$h_\mu(\mathtt{i}) = -\lim_{n\to\infty}\frac{1}{n}\log\mu[\mathtt{i}|_n] \quad\text{and}\quad \Lambda_\mu(s,\mathtt{i}) = \lim_{n\to\infty}\frac{1}{n}\log\phi^s(\mathtt{i}|_n), \tag{1.1}$$

for μ almost all $\mathtt{i}\in I^{\mathbb{N}}$ and

$$\int_{I^{\mathbb{N}}} h_\mu(\mathtt{i})\,d\mu = h_\mu \quad\text{and}\quad \int_{I^{\mathbb{N}}}\Lambda_\mu(s,\mathtt{i})\,d\mu = \Lambda_\mu(s). \tag{1.2}$$

Furthermore, for $\mu\in\mathcal{E}_\sigma(I^{\mathbb{N}})$, we have $h_\mu(\mathtt{i}) = h_\mu$ and $\Lambda_\mu(s,\mathtt{i}) = \Lambda_\mu(s)$ for μ almost all $\mathtt{i}\in I^{\mathbb{N}}$. We call $h_\mu(\mathtt{i})$ the *local entropy* of μ at $\mathtt{i}$ and $\Lambda_\mu(s,\mathtt{i})$ the *local energy* of μ at $\mathtt{i}$. In order to use [18, Theorem 7 in section 2], we need to assume that $H(\mathcal{P}_n)<\infty$ at some level n. Since $h_\mu = \lim_{n\to\infty}\frac{1}{n}H(\mathcal{P}_n) = \inf_{n\in\mathbb{N}}\frac{1}{n}H(\mathcal{P}_n)$ we can equivalently assume that $h_\mu<\infty$.

We define the *measure-theoretical pressure* function $P_\mu(\cdot,\mathtt{i}):[0,\infty]\to\mathbb{R}$ by

$$P_\mu(s,\mathtt{i}) = -\lim_{n\to\infty}\frac{1}{n}\log\frac{\mu[\mathtt{i}|_n]}{\phi^s(\mathtt{i}|_n)},$$

when $\mathtt{i}$ is so that both equations in (1.1) hold. If $h_\mu<\infty$ the limit exists for μ almost all $\mathtt{i}\in I^{\mathbb{N}}$. When $h_\mu(\mathtt{i})<\infty$ or $\Lambda_\mu(s,\mathtt{i})>-\infty$, then $P_\mu(s,\mathtt{i})$ is just $h_\mu(\mathtt{i})+\Lambda_\mu(s,\mathtt{i})$.

It is not yet said, that there exists $\mathtt{i}\in I^{\mathbb{N}}$, so that $\lim_{n\to\infty}\frac{1}{n}\log\phi^s(\mathtt{i}|_n)$ exists for all s. Fortunately, this happens for μ almost all $\mathtt{i}\in I^{\mathbb{N}}$. By repetitive use of the second equation in (1.1), we get that for μ almost all $\mathtt{i}\in I^{\mathbb{N}}$, the limit $\lim_{n\to\infty}\frac{1}{n}\log\alpha_l(\mathtt{i}|_n)$ exists for all $1\leqslant l\leqslant d$. We call these values the *Lyapunov exponents* of μ at $\mathtt{i}$ and denote them by $\lambda_l(\mu,\mathtt{i})$. For $s<d$, it now easily follows that

$$\Lambda_\mu(s,\mathtt{i}) = \lambda_1(\mu,\mathtt{i})+\cdots+\lambda_k(\mu,\mathtt{i})+(s-k)\lambda_{k+1}(\mu,\mathtt{i}), \tag{1.3}$$

where k is the integer part of s, with the interpretation that $0\cdot(-\infty)=0$. If $s\geqslant d$, we get $\Lambda_\mu(s,\mathtt{i}) = \frac{s}{d}(\lambda_1(\mu,\mathtt{i})+\cdots+\lambda_d(\mu,\mathtt{i}))$. From this we see that $\Lambda_\mu(\cdot,\mathtt{i})$ is strictly decreasing function with $\Lambda_\mu(0,\mathtt{i})=0$. Also we see that $\Lambda_\mu(\cdot,\mathtt{i})$ has at most one point of discontinuity and at this point it is continuous from left. The point of discontinuity equals to $\min\{k:\ \lambda_{k+1}(\mu,\mathtt{i})=-\infty\}$.

With the assumption $h_\mu<\infty$, we have that for μ almost all $\mathtt{i}\in I^{\mathbb{N}}$, the equations in (1.1) hold for all s. Also, the first equation in (1.2) gives that $h_\mu(\mathtt{i})<\infty$ for μ almost all $\mathtt{i}\in I^{\mathbb{N}}$. In this light, we give the following definition.

Definition 1.1. Let $\mu\in\mathcal{M}_\sigma(I^{\mathbb{N}})$ and $h_\mu<\infty$. When $\mathtt{i}$ is so that $h_\mu(\mathtt{i})<\infty$ and both equations in (1.1) hold, the *local Lyapunov dimension* of μ at $\mathtt{i}$, denoted by $\dim_{LY}(\mu,\mathtt{i})$, is defined to be the infimum of the numbers s, for which $P_\mu(s,\mathtt{i})<0$.

We remark that, for ergodic μ, the above functions $h_\mu(\mathtt{i})$, $\lambda_l(\mu,\mathtt{i})$, $\Lambda_\mu(s,\mathtt{i})$, $P_\mu(s,\mathtt{i})$ and $\dim_{LY}(\mu,\mathtt{i})$ are constants for μ almost all $\mathtt{i}$. In such case, we use the notations h_μ, $\lambda_l(\mu)$, $\Lambda_\mu(s)$, $P_\mu(s)$ and $\dim_{LY}(\mu)$ to emphasize the independence of $\mathtt{i}$. We are now ready to state our main result.

Theorem 1.2. *Assume that $\mu\in\mathcal{M}_\sigma(I^{\mathbb{N}})$, $h_\mu<\infty$, $\sup_{i\in I}\|A_i\|<\frac{1}{2}$ and that there exists $s\in[0,\infty)$ so that $0>P_\mu(s,\mathtt{i})>-\infty$. Then $\dim_{\mathrm{loc}}(\pi_{\boldsymbol{a}}\mu,\pi_{\boldsymbol{a}}(\mathtt{i}))=\min\{d,\dim_{LY}(\mu,\mathtt{i})\}$ for μ almost all $\mathtt{i}\in I^{\mathbb{N}}$ and $\boldsymbol{m}$ almost all $\boldsymbol{a}\in\mathbf{Q}^{\mathbb{N}}$.*

We only need the assumption $0>P_\mu(s,\mathtt{i})>-\infty$ in the proof of the upper bound to ensure that $\lambda_{k+1}(\mu,\mathtt{i})>-\infty$, where k is the integer part of $\dim_{LY}(\mu,\mathtt{i})$.

2. Local dimensions of invariant measures

In this section we prove Theorem 1.2. The proof is divided into upper and lower estimates, namely to Theorems 2.1 and 2.2. We remark that Theorem 2.1 was proven in [11, Proposition 4.4] for an ergodic measure on a finitely generated affine IFS.

Theorem 2.1. *Assume that $\mu \in \mathcal{M}_\sigma(I^{\mathbb{N}})$, $h_\mu(\mathtt{i}) < \infty$ and $\sup_{i\in I}\alpha_1(i) < \frac{1}{2}$. Then we have that $\underline{\dim}_{\mathrm{loc}}(\pi_{\boldsymbol{a}}\mu, \pi_{\boldsymbol{a}}(\mathtt{i})) \geqslant \min\{d, \dim_{LY}(\mu,\mathtt{i})\}$ for μ almost all $\mathtt{i}\in I^{\mathbb{N}}$ and $\boldsymbol{m}$ almost all $\boldsymbol{a}\in\mathbf{Q}^{\mathbb{N}}$.*

Proof. Assume first that $\dim_{LY}(\mu,\mathtt{i}) \leqslant d$. For arbitrary $\varepsilon > 0$, we choose $\gamma(\mathtt{i}) = \dim_{LY}(\mu,\mathtt{i}) - 2\varepsilon$ and $\theta(\mathtt{i}) = \dim_{LY}(\mu,\mathtt{i}) - \varepsilon$. Since $P_\mu(\cdot,\mathtt{i})$ is strictly decreasing, we find $\varepsilon' > 0$, so that $\Lambda_\mu(\theta(\mathtt{i}),\mathtt{i}) \geqslant -h_\mu(\mathtt{i}) + 2\varepsilon'$. By Egoroff's theorem, for each $\delta > 0$ there is a measurable set $H_\delta \subset I^{\mathbb{N}}$ and integer N_δ, such that $\mu(I^{\mathbb{N}}\setminus H_\delta) < \delta$ and

$$\frac{1}{n}\log\mu[\mathtt{i}|_n] \leqslant -h_\mu(\mathtt{i}) + \varepsilon' \leqslant \Lambda_\mu\big(\theta(\mathtt{i}),\mathtt{i}\big) - \varepsilon' \leqslant \frac{1}{n}\log\phi^{\theta(\mathtt{i})}(\mathtt{i}|_n)$$

for all $n \geqslant N_\delta$ and $\mathtt{i}\in H_\delta$. Therefore we find a constant $c' > 0$, independent of $\mathtt{i}$, so that

$$\mu[\mathtt{i}|_n] \leqslant c'\phi^{\theta(\mathtt{i})}(\mathtt{i}|_n) \tag{2.1}$$

for all $n\in\mathbb{N}$ and $\mathtt{i}\in H_\delta$. Next we consider the integral

$$\int_{\mathbf{Q}^{\mathbb{N}}} \frac{d\boldsymbol{m}(\boldsymbol{a})}{|\pi_{\boldsymbol{a}}(\mathtt{i}) - \pi_{\boldsymbol{a}}(\mathtt{j})|^{\gamma(\mathtt{i})}} = \int_{\mathbf{Q}^{\mathbb{N}}}\int_{\mathbf{Q}} \frac{d\mathcal{L}^d(a_1)}{|\pi_{\boldsymbol{a}}(\mathtt{i}) - \pi_{\boldsymbol{a}}(\mathtt{j})|^{\gamma(\mathtt{i})}}\, d\boldsymbol{m}(\boldsymbol{a}'),$$

where $\boldsymbol{a} = (a_1,\boldsymbol{a}')\in\mathbf{Q}^{\mathbb{N}}$. We can make the change of variable in the inner integral as in [3, Lemma 3.1]. By using this lemma with Fubini's theorem, and then inequality (2.1) and the properties of the singular value function, we get

$$\begin{aligned}
\int_{\mathbf{Q}^{\mathbb{N}}}\int_{H_\delta}\int_{I^{\mathbb{N}}} \frac{d\mu(\mathtt{j})\,d\mu(\mathtt{i})\,d\boldsymbol{m}(\boldsymbol{a})}{|\pi_{\boldsymbol{a}}(\mathtt{i}) - \pi_{\boldsymbol{a}}(\mathtt{j})|^{\gamma(\mathtt{i})}} &\leqslant c\int_{H_\delta}\int_{I^{\mathbb{N}}} \big(\phi^{\gamma(\mathtt{i})}(\mathtt{i}\wedge\mathtt{j})\big)^{-1}\,d\mu(\mathtt{j})\,d\mu(\mathtt{i}) \\
&\leqslant c\int_{H_\delta}\sum_{n=1}^{\infty}\big(\phi^{\gamma(\mathtt{i})}(\mathtt{i}|_n)\big)^{-1}\mu[\mathtt{i}|_n]\,d\mu(\mathtt{i}) \\
&\leqslant cc'\int_{H_\delta}\sum_{n=1}^{\infty}\big(\phi^{\gamma(\mathtt{i})}(\mathtt{i}|_n)\big)^{-1}\phi^{\theta(\mathtt{i})}(\mathtt{i}|_n)\,d\mu(\mathtt{i}) \\
&\leqslant cc'\int_{H_\delta}\sum_{n=1}^{\infty}2^{n(\gamma(\mathtt{i})-\theta(\mathtt{i}))}\,d\mu(\mathtt{i}) \\
&\leqslant cc'\sum_{n=0}^{\infty}2^{-n\varepsilon}\int_{H_\delta}d\mu(\mathtt{i}) < \infty,
\end{aligned}$$

where $\mathtt{i}\wedge\mathtt{j} = \mathtt{i}|_{\min\{k-1:\ i_k\neq j_k\}}$ and c is the constant from [3, Lemma 3.1], independent of $\mathtt{i}$ and $\mathtt{j}$. Originally, the bound of the norms of the linear maps in [3, Lemma 3.1] is $\frac{1}{3}$, but by [21, Proposition 3.1], $\frac{1}{2}$ suffices. Now we have that

$$\int_{H_\delta}\int_{I^{\mathbb{N}}} \frac{d\mu(\mathtt{j})\,d\mu(\mathtt{i})}{|\pi_{\boldsymbol{a}}(\mathtt{i})-\pi_{\boldsymbol{a}}(\mathtt{j})|^{\gamma(\mathtt{i})}} < \infty \tag{2.2}$$

for $\boldsymbol{m}$ almost all $\boldsymbol{a} \in \mathbf{Q}^{\mathbb{N}}$. Next we fix $\boldsymbol{a}$ so that (2.2) holds. We deduce that the integral $\int_{I^{\mathbb{N}}} |\pi_{\boldsymbol{a}}(\mathtt{i}) - \pi_{\boldsymbol{a}}(\mathtt{j})|^{-\gamma(\mathtt{i})}\,d\mu(\mathtt{j})$ is finite for μ almost all $\mathtt{i} \in H_\delta$ and so we find constants $M(\mathtt{i})$ for μ almost every $\mathtt{i} \in H_\delta$, so that $\int_{I^{\mathbb{N}}} |\pi_{\boldsymbol{a}}(\mathtt{i}) - \pi_{\boldsymbol{a}}(\mathtt{j})|^{-\gamma(\mathtt{i})}\,d\mu(\mathtt{j}) < M(\mathtt{i})$. This implies that $\pi_{\boldsymbol{a}}\mu(B(\pi_{\boldsymbol{a}}(\mathtt{i}), r)) \leqslant r^{\gamma(\mathtt{i})}M(\mathtt{i})$ for all $r > 0$ and for μ almost all $\mathtt{i} \in H_\delta$.

We have obtained that $\underline{\dim}_{\mathrm{loc}}(\pi_{\boldsymbol{a}}\mu, \pi_{\boldsymbol{a}}(\mathtt{i})) \geqslant \gamma(\mathtt{i})$ for μ almost all $\mathtt{i} \in H_\delta$ and $\boldsymbol{m}$ almost all $\boldsymbol{a} \in \mathbf{Q}^{\mathbb{N}}$. Since δ was arbitrary, this also holds for μ almost all $\mathtt{i} \in I^{\mathbb{N}}$.

If $\dim_{LY}(\mu, \mathtt{i}) > d$, then we get the proof by choosing $\theta(\mathtt{i}) = d$ and $\gamma(\mathtt{i}) = d - \varepsilon$. $\square$

Theorem 2.2. *If $\mu \in \mathcal{M}_\sigma(I^{\mathbb{N}})$, $h_\mu < \infty$ and $\Lambda_\mu(s, \mathtt{i}) > -\infty$ for some $s > \dim_{LY}(\mu, \mathtt{i})$, then $\overline{\dim}_{\mathrm{loc}}(\pi_{\boldsymbol{a}}\mu, \pi_{\boldsymbol{a}}(\mathtt{i})) \leqslant \min\{d, \dim_{LY}(\mu, \mathtt{i})\}$ for μ almost all $\mathtt{i} \in I^{\mathbb{N}}$ and for all $\boldsymbol{a} \in \mathbf{Q}^{\mathbb{N}}$.*

Proof. As mentioned in the Introduction, we follow the lines of the proof of [11, Theorem 4.3].

We may assume that $\dim_{LY}(\mu, \mathtt{i}) < d$. Fix an integer k, so that $k \leqslant \dim_{LY}(\mu, \mathtt{i}) < k + 1$. We have that $\pi_a[\mathtt{i}|_n] \in f_{\mathtt{i}|_n}(B(0, R))$ for some $R \in \mathbb{N}$. The ellipsoid $f_{\mathtt{i}|_n}(B(0, R))$ can be covered by a rectangular box, call it $B(\mathtt{i}|_n)$, with side-lengths $2R\alpha_1(\mathtt{i}|_n), \ldots, 2R\alpha_d(\mathtt{i}|_n)$. We can cover $B(\mathtt{i}|_n)$ with $N(\mathtt{i}|_n)$ non-overlapping "half-open" boxes with side-lengths

$$\alpha_{k+1}(\mathtt{i}|_n), \ldots, \alpha_{k+1}(\mathtt{i}|_n), \alpha_{k+2}(\mathtt{i}|_n), \ldots, \alpha_d(\mathtt{i}|_n),$$

where $N(\mathtt{i}|_n) \leqslant (2R)^d \alpha_1(\mathtt{i}|_n) \cdots \alpha_k(\mathtt{i}|_n)\alpha_{k+1}(\mathtt{i}|_n)^{-k}$. Let $P_n(\mathtt{i})$ be the box that contains $\pi_{\boldsymbol{a}}(\mathtt{i})$, and let $Q_n(\mathtt{i}) := [\mathtt{i}|_n] \cap \pi_{\boldsymbol{a}}^{-1}(P_n(\mathtt{i}))$. In other words, $Q_n(\mathtt{i})$ is the part of the cylinder $[\mathtt{i}|_n]$ that gets projected into $P_n(\mathtt{i})$. For fixed j we define

$$A_n^j := \left\{\mathtt{i} \in I^{\mathbb{N}}\colon\ \mu\big(Q_n(\mathtt{i})\big) \geqslant 2^{-n/j} \frac{\mu[\mathtt{i}|_n]}{N(\mathtt{i}|_n)}\right\}$$

for all $n \in \mathbb{N}$. Now we have

$$\mu\big(I^{\mathbb{N}} \setminus A_n^j\big) = \mu\Bigg(\bigcup_{\mathtt{i} \in I^{\mathbb{N}}} Q_n(\mathtt{i}) \setminus A_n^j\Bigg) = \sum_{Q_n(\mathtt{i}) \not\subset A_n^j} \mu\big(Q_n(\mathtt{i})\big) \leqslant \sum_{\mathtt{i} \in I^n} N(\mathtt{i}) 2^{-n/j} \frac{\mu[\mathtt{i}]}{N(\mathtt{i})} = 2^{-n/j}.$$

Thus for the set $A^j := \bigcup_{N \in \mathbb{N}} \bigcap_{n=N}^{\infty} A_n^j$ we have

$$\mu\big(A^j\big) = \lim_{N\to\infty} \mu\Bigg(\bigcap_{n=N}^{\infty} A_n^j\Bigg) = 1 - \lim_{N\to\infty} \mu\Bigg(\bigcup_{n=N}^{\infty} \big(I^{\mathbb{N}} \setminus A_n^j\big)\Bigg) \geqslant 1 - \lim_{N\to\infty} \sum_{n=N}^{\infty} 2^{-n/j} = 1.$$

By definition, for all $\mathtt{i} \in A^j$, we find $M(\mathtt{i}) \in \mathbb{N}$ such that the inequality

$$\mu\big(Q_n(\mathtt{i})\big) \geqslant 2^{-n/j} \frac{\mu[\mathtt{i}|_n]}{N(\mathtt{i}|_n)} \tag{2.3}$$

holds for all $n \geqslant M(\mathtt{i})$.

We assumed that $\Lambda_\mu(s, \mathtt{i}) > -\infty$ for some $s > \dim_{LY}(\mu, \mathtt{i})$. This implies that $\lambda_l(\mu) > -\infty$ for all $1 \leqslant l \leqslant k + 1$. For $\gamma(\mathtt{i}) > \dim_{LY}(\mu, \mathtt{i})$, we have

$$\begin{aligned}\lim_{n\to\infty}\frac{\log\mu[\mathtt{i}|_n]-\log N(\mathtt{i}|_n)}{\log\alpha_{k+1}(\mathtt{i}|_{n-1})}&=\lim_{n\to\infty}\frac{\frac{1}{n}(\log\mu[\mathtt{i}|_n]-\sum_{l=1}^{k}\log\alpha_l(\mathtt{i}|_n)-\log\alpha_{k+1}(\mathtt{i}|_n)^{-k})}{\frac{n-1}{n}\frac{1}{n-1}\log\alpha_{k+1}(\mathtt{i}|_{n-1})}\\&\leqslant\frac{h_\mu(\mathtt{i})+\lambda_1(\mu,\mathtt{i})+\cdots+\lambda_k(\mu,\mathtt{i})}{-\lambda_{k+1}(\mu,\mathtt{i})}+k<\gamma(\mathtt{i})\end{aligned}\tag{2.4}$$

for μ almost all $\mathtt{i}\in I^{\mathbb{N}}$. The first inequality follows by the definition of $N(\mathtt{i}|_n)$ and the fact that $h_\mu(\mathtt{i})$ and $\lambda_l(\mu,\mathtt{i})$ are finite for $1\leqslant l\leqslant k+1$ and $0>\lambda_{k+1}(\mu,\mathtt{i})$. The second inequality follows by (1.3), since $P_\mu(\gamma(\mathtt{i}),\mathtt{i})<0$. In the calculation, we have omitted the constant $(2R)^d$ from $N(\mathtt{i}|_n)$, since it has no effect on the result.

Let r_l be any sequence of positive numbers converging to zero. For each l, we find an integer n_l, so that $\sqrt{d}\alpha_{k+1}(\mathtt{i}|_{n_l})\leqslant r_l<\sqrt{d}\alpha_{k+1}(\mathtt{i}|_{n_l-1})$. To avoid complicated notation, we only write n instead of n_l. We have $\mathtt{i}\in Q_n(\mathtt{i})$ and $\pi_{\boldsymbol{a}}Q_n(\mathtt{i})\subset P_n(\mathtt{i})$ and the greatest side-length of $P_n(\mathtt{i})$ is $\alpha_{k+1}(\mathtt{i}|_n)$. Therefore we have $\pi_{\boldsymbol{a}}Q_n(\mathtt{i})\subset B(\pi_{\boldsymbol{a}}(\mathtt{i}),\sqrt{d}\alpha_{k+1}(\mathtt{i}|_n))$. Using this and (2.3) and (2.4), we get that

$$\begin{aligned}\limsup_{l\to\infty}\frac{\log\pi_{\boldsymbol{a}}\mu B(\pi_{\boldsymbol{a}}(\mathtt{i}),r_l)}{\log r_l}&\leqslant\limsup_{n\to\infty}\frac{\log\pi_{\boldsymbol{a}}\mu B(\pi_{\boldsymbol{a}}(\mathtt{i}),\sqrt{d}\alpha_{k+1}(\mathtt{i}|_n))}{\log\sqrt{d}\alpha_{k+1}(\mathtt{i}|_{n-1})}\\&\leqslant\limsup_{n\to\infty}\frac{\log\mu Q_n(\mathtt{i})}{\log\alpha_{k+1}(\mathtt{i}|_{n-1})}\\&\leqslant\limsup_{n\to\infty}\left(\frac{\log 2^{-\frac{n}{j}}}{\log\alpha_{k+1}(\mathtt{i}|_{n-1})}+\frac{\log\mu[\mathtt{i}|_n]-\log N(\mathtt{i}|_n)}{\log\alpha_{k+1}(\mathtt{i}|_{n-1})}\right)\\&\leqslant j^{-1}\frac{\log 2}{-\log\bar{\alpha}}+\gamma(\mathtt{i}),\end{aligned}$$

where $\bar{\alpha}=\sup_{i\in I}\alpha_1(i)<1$. Since j and the sequence r_l were arbitrary and $\mu(A^j)=1$ for all $j\in\mathbb{N}$, we have obtained $\overline{\dim}_{\mathrm{loc}}(\pi_{\boldsymbol{a}}\mu,\pi_{\boldsymbol{a}}(\mathtt{i}))\leqslant\gamma(\mathtt{i})$ for μ almost all $\mathtt{i}\in I^{\mathbb{N}}$. $\square$

Remark 2.3. It is natural to ask, what can be said of the local dimensions, when one only assumes $\sup_{i\in I}\alpha_1(i)\leqslant\bar{\alpha}<1$, and what results can be obtained for all translations $\boldsymbol{a}$. Observe that Theorem 2.2 already applies to this case. By using an essentially identical proof as the proof of [10, Theorem 2.6], one can get the following estimates.

Assume that $\mu\in\mathcal{M}_\sigma(I^{\mathbb{N}})$, $h_\mu<\infty$ and $\log\alpha_d(\mathtt{i}|_1)\in L^1(\mu)$. Then we have for μ almost all $\mathtt{i}\in I^{\mathbb{N}}$ and for all $\boldsymbol{a}\in\mathbf{Q}^{\mathbb{N}}$ that

$$\frac{h_\mu^\pi(\mathtt{i})}{-E_\mu(\log\alpha_d(\mathtt{i}|_1)\mid\mathcal{I})}\leqslant\underline{\dim}_{\mathrm{loc}}\big(\pi\mu,\pi(\mathtt{i})\big)\leqslant\overline{\dim}_{\mathrm{loc}}\big(\pi\mu,\pi(\mathtt{i})\big)\leqslant\frac{h_\mu^\pi(\mathtt{i})}{-E_\mu(\log\alpha_1(\mathtt{i}|_1)\mid\mathcal{I})},\tag{2.5}$$

where $h_\mu^\pi(\mathtt{i})$ is the *local projection entropy* defined as in [10, Definition 2.1], m is so that $H(\mathcal{P}_m)<\infty$, and $\mathcal{I}$ is the σ-algebra of σ invariant sets. For the definition of the conditional expectation E_μ, see [18]. If the index set I is finite then (2.5) is strictly included in [10, Theorem 2.6].

The assumptions $h_\mu<\infty$ and $\log\alpha_d(\mathtt{i}|_1)\in L^1(\mu)$ are needed in the ergodic theorems that are used in the proof and the number m can be chosen to be the least integer for which $H(\mathcal{P}_m)<\infty$. In the finitely generated case these assumptions are of course satisfied and $m=1$. We also note that the proof of [10, Theorem 6.2], which is a more general version of [10, Theorem 2.6], deals with a direct product of two IFS and the conditional measures used there are not needed to obtain (2.5).

In most cases, the upper bound in Eq. (2.5) is not as good as the result of Theorem 2.2. However, in the exceptional case, where Theorem 2.1 does not hold, the upper estimate in (2.5) might give a better estimate since $h_\mu^\pi\leqslant h_\mu$, see [10, Proposition 4.1].

3. Pressure function and dimensions of the limit set

In order to determine the Hausdorff dimension of the limit set $F_{\boldsymbol{a}}$, one often considers the *pressure* function defined by

$$P(s) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\mathtt{i}\in I^n} \phi^s(\mathtt{i}).$$

In the finitely generated setting it is known that if $\max_{i\in I} \|A_i\| < \frac{1}{2}$, then the Hausdorff dimension of $F_{\boldsymbol{a}}$ equals to the zero of the pressure for $\mathcal{L}^{d\kappa}$ almost all $\boldsymbol{a} \in \mathbb{R}^{d\kappa}$, where κ denotes cardinality of the index set I, see [3]. In [13, Theorem B], Käenmäki and Reeve generalize this result for an infinitely generated affine IFS, with the extra assumption of quasi-multiplicativity, see [13, (2.1)] for the definition. Since their results on the Hausdorff dimension of the limit set are closely related to our results on measures, we give some notes on this paper.

The pressure function satisfies $P(s) \geqslant P_\mu(s)$ for all $\mu \in \mathcal{M}_\sigma(I^{\mathbb{N}})$ and all $s \in [0,\infty)$, see [13, Lemma 2.2]. Furthermore, if the singular value function is quasi-multiplicative and $s > s_\infty = \inf\{s\colon\ P(s) < \infty\}$, then there exists an ergodic measure μ_s, called the Gibbs measure, satisfying $P(s) = P_{\mu_s}(s)$, $h_{\mu_s} < \infty$ and $\Lambda_{\mu_s}(s) > -\infty$, [13, Theorems 3.5 and 3.6 and Lemma 4.2]. In Example 4.2 we show that $P(s)$ can be nonzero everywhere. In this case, any ergodic measure μ with $h_\mu < \infty$ satisfies $\dim_{LY}(\mu) < s_\infty$. This follows since $P(s_\infty) \geqslant P_\mu(s_\infty)$ and P_μ is continuous from left. The next theorem gives a necessary and sufficient condition for the existence of the zero of the pressure function under the quasi-multiplicativity assumption.

Lemma 3.1. *Suppose that the singular value function $\phi^s(\mathtt{i})$ is quasi-multiplicative for all $0 \leqslant s \leqslant d$. Then $P(s)$ is continuous and strictly decreasing on the interval $[s_\infty,\infty)$. Furthermore if $P(s_\infty) \geqslant 0$, then there exists a unique s satisfying $P(s) = 0$.*

Käenmäki and Vilppolainen have proved a similar result, [15, Lemma 2.1], and we will make use of that proof. Their lemma deals with a finitely generated IFS, but some parts of the proof apply directly to the infinitely generated case.

Proof of Lemma 3.1. It is easy to see that $P(s)$ is decreasing and thus it is finite for all $s > s_\infty$. As in [15, Lemma 2.1], we deduce that for any $s > s_\infty$, we have

$$P(s) - P(s+\delta) \geqslant -\delta \log \sup_{i\in I} \alpha_1(\mathtt{i}),$$

which gives that $P(s)$ is strictly decreasing for $s > s_\infty$ and that $\lim_{s\to\infty} P(s) = -\infty$. Now we only need to show the continuity. By inspecting the proof of [15, Lemma 2.1], we get that $P(s)$ is convex on intervals $[m, m+1]$. Since $P(s)$ is also decreasing, we get that $P(s)$ is left-continuous for all $s > s_\infty$. Since $\phi^s(\mathtt{i})$ is quasi-multiplicative, $P(s)$ can be approximated pointwise by continuous functions from below, namely by the pressures of finite sub-systems, see [13, Proposition 3.2]. Again, using the fact that $P(s)$ is decreasing, we get right-continuity. Especially, $P(s)$ is right-continuous at s_∞. Note also that quasi-multiplicativity was only used to get the right-continuity. □

The lower local dimension of the Gibbs measure is also estimated in [13, Theorem 4.1]. By Lemma 3.1 and Theorem 1.2 we get the following corollary. Due to Lemma 3.1 the assumption of the existence of s_0 in [13, Theorem 4.1] can be relaxed to $P(s_\infty) \geqslant 0$.

Corollary 3.2. *Suppose that the singular value function $\phi^s(\mathtt{i})$ is quasi-multiplicative for all $0 \leqslant s \leqslant d$ and $P(s_\infty) > 0$. Then there exists s_0 so that $P(s_0) = 0$ and the Gibbs measure μ_{s_0} satisfies $\Lambda_{\mu_{s_0}}(s_0) > -\infty$. If in addition $\Lambda_{\mu_{s_0}}(s_0 + \delta) > -\infty$ for some $\delta > 0$, then*

$$\dim_{\mathrm{loc}}\bigl(\pi_{\boldsymbol{a}}\mu_{s_0}, \pi_{\boldsymbol{a}}(\mathtt{i})\bigr) = s_0,$$

for μ almost all $\mathtt{i} \in I^{\mathbb{N}}$ and m almost all $\boldsymbol{a} \in \mathbf{Q}^{\mathbb{N}}$.

Proof. The existence of s_0 is clear by Lemma 3.1. The assumption $\Lambda_{\mu_{s_0}}(s_0+\delta) > -\infty$ is only needed when s is an integer, to ensure that we may use Theorem 1.2. The result follows since $s_0 = \dim_{LY}(\mu, \mathtt{i})$. □

By [13, Theorem B], $\dim_{\mathrm{H}} F_{\boldsymbol{a}} = \sup\{\dim_{\mathrm{H}} \pi_{\boldsymbol{a}}(J^{\mathbb{N}})\colon J \subset I \text{ is finite}\}$ for $\boldsymbol{m}$ almost all $\boldsymbol{a}$. We do not know whether a similar approximation holds for $\dim_{\mathrm{P}}$ and $\overline{\dim}_{\mathrm{B}}$. Recalling [6, Theorem 10.1], one could use Corollary 3.2 and hope for results on packing dimension of the limit set. The problem is that we only know the local dimension of μ_{s_0} for almost all $\mathtt{i}$ and not for all $\mathtt{i}$. Mauldin and Urbański have given an example of an infinitely generated self similar set F satisfying the open set condition, for which $\dim_{\mathrm{H}} F < \dim_{\mathrm{P}} F$, see [16, Example 5.2]. On the other hand, for all finite subsystems it holds that $\dim_{\mathrm{H}} \pi_{\boldsymbol{a}}(J^{\mathbb{N}}) = \dim_{\mathrm{P}} \pi_{\boldsymbol{a}}(J^{\mathbb{N}})$, see [4]. Therefore the dimension approximation property does not hold for this, or similar examples. Note also that $\overline{\dim}_{\mathrm{B}} F_{\boldsymbol{a}} = \dim_{\mathrm{P}} F_{\boldsymbol{a}}$ for infinitely generated self-affine sets $F_{\boldsymbol{a}}$ by [16, Theorem 3.1]. The following theorem gives an estimate for the relation between Hausdorff and packing dimensions of infinitely generated self affine sets. For $x \in F_{\boldsymbol{a}}$, we set the notation $L_n(x) = \{f_{\mathtt{i}}(x)\colon \mathtt{i} \in I^n\}$.

Theorem 3.3. *Let $\{f_i\}_{i \in I}$, be an infinitely generated affine IFS. Then we have that*

$$\sup_{\substack{x \in F_{\boldsymbol{a}} \\ n \in \mathbb{N}}} \bigl\{\dim_{\mathrm{H}} F_{\boldsymbol{a}}, \overline{\dim}_{\mathrm{B}} L_n(x)\bigr\} \leqslant \dim_{\mathrm{P}} F_{\boldsymbol{a}} \leqslant \sup_{\substack{x \in F_{\boldsymbol{a}} \\ n \in \mathbb{N}}} \bigl\{s_0, \overline{\dim}_{\mathrm{B}} L_n(x)\bigr\},$$

where $s_0 = \inf\{s\colon \lim_{n\to\infty} \frac{1}{n} \log \sum_{\mathtt{i} \in I^n} \alpha_1(\mathtt{i})^s = 0\}$.

Proof. We have $\dim_{\mathrm{P}} F_{\boldsymbol{a}} = \overline{\dim}_{\mathrm{B}} F_{\boldsymbol{a}}$ by [16, Theorem 3.1] and so the first inequality is trivial. The proof of the last inequality is essentially the same as the proof of [17, Lemma 2.8], since $\|f_{\mathtt{i}}'\| = \alpha_1(\mathtt{i})$. □

Note that if $s_0 \leqslant 1$, then $s_0 = \inf\{s\colon P(s) < 0\} = \dim_{\mathrm{H}} F_{\boldsymbol{a}} = \dim_{\mathrm{P}} F_{\boldsymbol{a}}$ for $\boldsymbol{m}$ almost all $\boldsymbol{a} \in \mathbf{Q}^{\mathbb{N}}$ by [13, Theorem B].

4. Examples and final remarks

Here we give some examples on the entropies and pressures of measures. In Example 4.1 we show that the measure-theoretical pressure function can be non-zero everywhere and in Example 4.2 we show that the pressure function can be non-zero everywhere, as mentioned earlier. In the examples, we make use of *Bernoulli measures*: Fix reals $0 \leqslant p_i \leqslant 1$ so that $\sum_{i=1}^{\infty} p_i = 1$. The unique measure satisfying $\mu[\mathtt{i}|_n] = p_{i_1} p_{i_2} \cdots p_{i_n}$ is called a Bernoulli measure. It is well known that Bernoulli measures are ergodic. It is also easy to see that the entropy of a Bernoulli measure can be infinite.

Example 4.1. ($P_\mu(s) \neq 0$ everywhere.) Let μ be a Bernoulli measure with $\mu[i] = c(i+1)^{-2}$, where $c = (\frac{\pi^2}{6} - 1)^{-1}$. Let

$$A_i = \begin{bmatrix} 2\mu[i] & 0 \\ 0 & c4^{-i} \end{bmatrix}.$$

We can now calculate

$$h_\mu = -\sum_{i=2}^{\infty} ci^{-2}\log ci^{-2} = \log c + 2c\sum_{i=2}^{\infty} i^{-2}\log i < \infty$$

and thus μ is a probability measure with finite entropy. Also, by induction we see that $\lambda_1(\mu) = \sum_{i=1}^{\infty}\mu[i]\log 2\mu[i] = \log 2 - h_\mu$ and

$$\lambda_2(\mu) = c\sum_{i=1}^{\infty}(i+1)^{-2}\log c4^{-i} = -\infty.$$

Thus μ is a Bernoulli measure with finite entropy and $P_\mu(t) \geqslant \log 2$ for all $t \leqslant 1$ and $P_\mu(t) = -\infty$ for all $t > 1$. Since $\sup_{\mathtt{i}\in I}\alpha_1(i) = 2\mu[1] = \frac{2c}{4} < \frac{1}{4}$, Theorem 2.1 gives that $\underline{\dim}_{\mathrm{loc}}(\pi_{\boldsymbol{a}}\mu, \pi_{\boldsymbol{a}}(\mathtt{i})) \geqslant 1$ for μ almost all $\mathtt{i} \in I^{\mathbb{N}}$ and m almost all $\boldsymbol{a} \in \mathbf{Q}^{\mathbb{N}}$.

Example 4.2. ($P(s) \neq 0$ everywhere.) Let $c_i = i^{-\frac{1}{2}}$, $d_i = i^{-1}$ and $A_i = \left[\begin{smallmatrix} c_i & 0 \\ 0 & d_i \end{smallmatrix}\right]$ for all $i \in \mathbb{N}$. Now $A_{\mathtt{i}} = \left[\begin{smallmatrix} c_{\mathtt{i}} & 0 \\ 0 & d_{\mathtt{i}} \end{smallmatrix}\right]$ for all $\mathtt{i} \in I^n$, where $c_{\mathtt{i}} = c_{i_1}\cdots c_{i_n}$ and $d_{\mathtt{i}} = d_{i_1}\cdots d_{i_n}$. Therefore, for all $t = 1+s$, we have

$$\phi^t(\mathtt{i}) = \frac{1}{i_1^{\frac{1}{2}}}\cdots\frac{1}{i_n^{\frac{1}{2}}}\cdot\frac{1}{i_1^s}\cdots\frac{1}{i_n^s},$$

which implies

$$\sum_{\mathtt{i}\in I^n}\phi^t(\mathtt{i}) = \left(\sum_{\mathtt{i}\in\mathbb{N}}\frac{1}{i^{\frac{1}{2}+s}}\right)^n \quad \text{and} \quad P(t) = \log\sum_{\mathtt{i}\in\mathbb{N}}\frac{1}{i^{\frac{1}{2}+s}}.$$

Choose $I = \{\lfloor i(\log i)^2\rfloor\colon\ i \geqslant n_0\}$. Now we have that

$$P\left(\frac{3}{2}\right) = \log\sum_{\mathtt{i}\in I}\frac{1}{i^{\frac{1}{2}+\frac{1}{2}}} = \log\sum_{i=n_0}^{\infty}\frac{1}{\lfloor i(\log i)^2\rfloor} < 0$$

for n_0 large enough. For all $t < \frac{3}{2}$ we get $P(t) = \infty$, since $\log i \leqslant i^\delta$ for large i when $\delta > 0$.

We end with final remarks on the assumptions and results of this paper.

Remarks 4.3.

(1) Considering the proof of Theorem 2.2, suppose that $\lambda_{k+1}(\mu, \mathtt{i}) = -\infty$. We face difficulties at (2.4) since we are to calculate the limit

$$\lim_{n\to\infty}\frac{\log\alpha_{k+1}(\mathtt{i}|_n)}{\log\alpha_{k+1}(\mathtt{i}|_{n-1})}.$$

Since $\alpha_{k+1}(i) \to 0$ as $i \to \infty$, there are sequences for which the above limit is infinite. If one has extra information about the support of the measure then the set of these sequences can be studied. For example, if $\mu(I^{\mathbb{N}}\setminus\bigcup J^{\mathbb{N}}) = 0$, where the union is over all finite sets $J \subset I$, then we find constants $c(\mathtt{i})$ for almost all $\mathtt{i}$ so that $\alpha_{k+1}(\mathtt{i}|_n) \geqslant c(\mathtt{i})\alpha_{k+1}(\mathtt{i}|_{n-1})$ and the set of the exceptional sequences is of measure zero. Unfortunately these measures are rather trivial. This can be seen from [14, Lemma 2.3]. Note that one can always use the sequence $\alpha_{k+1}(\mathtt{i}|_n)$ to obtain the estimate $\underline{\dim}_{\mathrm{loc}}(\pi_{\boldsymbol{a}}\mu, \pi_{\boldsymbol{a}}(\mathtt{i})) \leqslant \dim_{LY}(\mu, \mathtt{i})$.

(2) Since we assumed that the limit set F is bounded it is reasonable to also assume that $\alpha_d(i) \to 0$ as $i \to \infty$. Therefore we could have $-\sum_{i\in I} \mu[i] \log \alpha_d(\mathtt{i}|_1) = \infty$ and so the assumption $\log \alpha_d(\mathtt{i}|_1) \in L^1$ in Eq. (2.5) is necessary.

(3) Considering the finitely generated case, suppose that $\#I = \kappa$ and that s_0 is the zero of the pressure function. Käenmäki proved the existence of an equilibrium measure μ_{s_0} in [12]. For this measure, $\dim_{LY}(\mu_{s_0})$ equals to s_0. By Theorem 1.2, we get that $\dim_{\mathrm{H}}(F) \geqslant s_0$ for Lebesgue almost all $\boldsymbol{a} \in \mathbb{R}^{d\kappa}$. This shows that we cannot remove the assumption $\sup_{i\in I} \|A_i\| < \frac{1}{2}$ from Theorem 1.2. For examples where $\dim_{\mathrm{H}}(F) < s_0$, see [2,19,20]. Also there are examples showing that for particular $\boldsymbol{a}$, Theorem 1.2 cannot hold, see e.g. [5, Example 9.11]. The size of the set of these exceptional translations has been studied by Falconer and Miao in [7].

(4) Supposing that $h^\pi_\mu(\mathtt{i}) < \infty$, we may slightly modify the definition of the Lyapunov dimension, namely by setting

$$\dim^\pi_{LY}(\mu, \mathtt{i}) = \inf\{s\colon\ h^\pi_\mu(\mathtt{i}) - \Lambda_\mu(s, \mathtt{i}) < 0\}.$$

Perhaps we could have $\dim_{\mathrm{loc}}(\pi_{\boldsymbol{a}}\mu, \pi_{\boldsymbol{a}}(\mathtt{i})) = \min\{d, \dim^\pi_{LY}(\mu, \mathtt{i})\}$ for μ almost all $\mathtt{i} \in I^{\mathbb{N}}$ and all $\boldsymbol{a} \in \mathbf{Q}^{\mathbb{N}}$, when $\mu \in \mathcal{E}_\sigma(I^{\mathbb{N}})$, $h^\pi_\mu < \infty$ and $\sup_{i\in I} \|A_i\| < 1$.

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