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On multistability of equations with a distributed delay, monotone production and the Allee effect

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ABSTRACT

We consider the delay population dynamics model $dN/dt = f(N_\tau(t)) - d(N(t))$ with an increasing fecundity function f and any mortality function d where the Allee effect or multistability can be observed. The study includes non-autonomous equations with a distributed delay. Attractivity of certain positive solutions, persistence, extinction of solutions with initial values below a threshold, boundedness and oscillation are discussed.

1. Introduction

The scalar differential equation

$$\frac{dN}{dt} = f\big(N(t-\tau)\big) - d\big(N(t)\big) \tag{1.1}$$

with a constant delay τ has been extensively studied in literature, in particular, in the case when there is only one positive equilibrium point K where $f(K) = d(K)$, and $f(x) > d(x)$ for $0 < x < K$, while $f(x) < d(x)$ for $x > K$. Such equations occur in population dynamics, where K is interpreted as a carrying capacity of the environment.

In our paper [6] we investigated equations with delays in the production term, while mortality is defined by the present population size only, and there is a positive equilibrium K such that $f(x) > d(x)$ for $0 < x < K$, $f(x) < d(x)$ for $x > K$. The delay was assumed to be of the general form

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$$\frac{dN}{dt} = \int_{h(t)}^{t} f\big(N(s)\big)\, d_s R(t,s) - d\big(N(t)\big), \tag{1.2}$$

including both concentrated and distributed delays, and it was demonstrated that, once $f(x)$ is monotone increasing and $\lim_{t\to\infty} h(t) = \infty$, all positive solutions tend to the positive equilibrium. Moreover, if the initial function is less (greater) than the equilibrium, so is the solution.

The proofs in [6] used some ideas previously developed in [10,11]. Logistic-type equations with concentrated delays in the production part were considered in [3,4], they illustrated the monotone behaviour and inspired the study of more general models in [6,11].

In this note, we explore the situation when the equation has at least two positive equilibrium points $m < K$ and $f(x) < d(x)$ for $0 < x < m$; this can correspond to the case of the Allee effect in population dynamics when some minimal initial population size is required to survive, see, for example, [2,9,14,15]. Also, other multistability cases with several positive equilibrium points will be considered.

Multistability of systems of delay equations describing real-world phenomena has attracted a lot of attention, see, for example, the recent papers [12,17,19,24]. The simplest example of a model with the Allee effect is the equation

$$\frac{dx}{dt} = rx(a-x)(x-b), \quad a > b > 0, \tag{1.3}$$

where a is the carrying capacity of the environment, b is the minimal size providing population survival: for lower population sizes, extinction is due to possible problems with finding mates, group defence, or social functioning.

Eq. (1.3) can also be rewritten in the form

$$\frac{dx}{dt} = f(x) - d(x), \tag{1.4}$$

where $f(x) = r(a+b)x^2$, $d(x) = rx(x^2+ab)$, f is a monotone increasing function, $f(x) > 0$, $d(x) > 0$, $x > 0$, $f(0) = d(0) = 0$, and

$$f(x) < d(x), \quad x \in (0,b) \cup (a,\infty); \qquad f(x) > d(x), \quad x \in (b,a).$$

We can apply the same functions to consider a similar equation with a delay in the production term

$$\dot{x} = f\big(x\big(h(t)\big)\big) - d(x), \tag{1.5}$$

or with the functions f and d

$$f(x) = \frac{r(a+b)x^2}{1+x^n}, \qquad d(x) = \frac{r(x^3+abx)}{1+x^n}, \quad 0 \leqslant n \leqslant 2$$

corresponding to the Mackey–Glass equation.

The paper is organized as follows. Section 2 discusses persistence and boundedness of solutions for models with a distributed delay. An existence and uniqueness result is also presented but its proof is postponed to Appendix A. Section 3 includes the main results of the present paper: multistability, attractivity of some of the positive equilibrium points and possible convergence to zero of solutions with initial values below a certain threshold. In Section 4 non-oscillation about all positive equilibrium points is studied and justified in some sense. Section 5 involves examples of general models investigated in the present paper, some discussion of the results and relevant open problems.

2. Persistence of solutions

We consider the equation with a distributed delay

$$x'(t) = r(t)\left[\int_{h(t)}^{t} f\big(x(s)\big)\, d_s R(t,s) - d\big(x(t)\big)\right], \quad t \geqslant 0, \tag{2.1}$$

where $\int_{h(t)}^{t} d_s R(t,s) \equiv 1$ and $r : \mathbb{R}^+ \to \mathbb{R}^+$, $\mathbb{R}^+ = [0,\infty)$, with the initial condition

$$x(t) = \varphi(t), \quad t \leqslant 0. \tag{2.2}$$

Definition 2.1. An absolutely continuous in $\mathbb{R}^+$ function $x : \mathbb{R} \to \mathbb{R}$ is called *a solution of problem* (2.1), (2.2) if it satisfies Eq. (2.1) for almost all $t \in \mathbb{R}^+$ and conditions (2.2) for $t \leqslant 0$.

The initial condition should be chosen in a certain class of functions which would guarantee the existence of the integral in the right hand side of (2.1) almost everywhere in t. In particular, for integro-differential equations where $R(t,\cdot)$ is absolutely continuous, φ can be any Lebesgue measurable essentially bounded function, for equations with several concentrated delays where $R(t,\cdot)$ is a step function, the initial function φ should be a Borel measurable bounded function. For any distribution $R(t,s)$, the integral exists if φ is bounded and continuous (f is continuous for all the models considered). Besides, as is widely assumed in population dynamics models, $\varphi(t)$ is nonnegative and the value at the initial point is positive.

Consider (2.1), (2.2) under some of the following assumptions:

(a1) $f : \mathbb{R}^+ \to \mathbb{R}^+$ and $d : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous functions, f is monotone increasing in its domain, $f(x) > 0$ and $d(x) > 0$ for any $x > 0$;
(a2) $h : \mathbb{R}^+ \to \mathbb{R}$ is a Lebesgue measurable function, $h(t) \leqslant t$, $\lim_{t\to\infty} h(t) = \infty$;
(a3) r is a Lebesgue measurable essentially bounded on $\mathbb{R}^+$ function, $r(t) \geqslant 0$ for any $t \geqslant 0$, $\int_0^\infty r(s)\, ds = \infty$;
(a4) $R(t,\cdot)$ is a left continuous non-decreasing function for any t, $R(\cdot,s)$ is locally integrable for any s, $R(t,s) = 0$, $s \leqslant h(t)$, $R(t,t^+) = 1$; here $u(t^+)$ is the right-side limit of function u at the point t;
(a5) $\varphi : (-\infty,0] \to \mathbb{R}$ is a continuous bounded function, $\varphi(t) \geqslant 0$, $\varphi(0) > 0$;
(a6) f and d satisfy local Lipschitz conditions: for any interval $[a,b] \subset \mathbb{R}^+$ there exist positive numbers $\alpha_{[a,b]}$ and $\beta_{[a,b]}$ such that $|f(x) - f(y)| \leqslant \alpha_{[a,b]}|x - y|$ and $|d(x) - d(y)| \leqslant \beta_{[a,b]}|x - y|$ for any $x, y \in [a,b]$.

For some distributed delays we can relax (a5): φ can be a Borel measurable bounded function for variable concentrated delays and a Lebesgue measurable locally integrable function for an integral equation with a locally essentially bounded Lebesgue measurable kernel and a bounded initial memory. Also, a solution will exist even if (a6) is omitted. However, if (a1)–(a6) are satisfied then the solution of (2.1), (2.2) exists and is unique, for any delay distributions.

The proof of the existence and uniqueness theorem is similar to the proofs in [5,6] and thus will be presented in Appendix A.

Theorem 2.2. *Suppose* (a1)–(a6) *hold and there exists* $K > 0$ *such that* $f(x) < d(x)$ *for* $x > K$. *Then* (2.1), (2.2) *has a unique global solution which is positive for* $t \geqslant 0$.

Example 2.3. The assumption that $f(x) < d(x)$ in Theorem 2.2 for any $x > K$ is necessary. For instance, the solution of the initial value problem

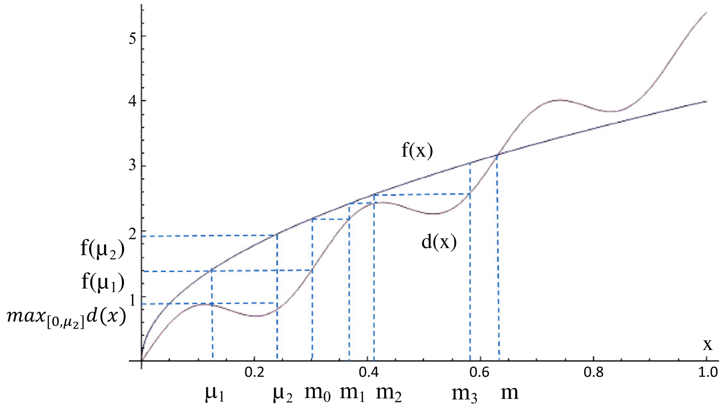

Fig. 1. Illustration of lower solution bounds in the proof of Theorem 2.6.

$$\frac{dx}{dt} = 2x^2(t) - x^2(t), \qquad x(0) = \frac{1}{3} \tag{2.3}$$

is $x(t) = (3-t)^{-1}$ and thus cannot be extended for $t \geqslant 3$.

Remark 2.4. In (a1) it is assumed that the functions f, d map the open interval $(0,\infty)$ to itself, however the proof of Theorem 2.2 is readily adapted to the case when $f : \mathbb{R} \to (0,\infty)$ and $d : \mathbb{R} \to (0,\infty)$.

Definition 2.5. The solution $x(t)$ of (2.1), (2.2) is *persistent* if there exists $A > 0$ such that $\liminf_{t\to\infty} x(t) \geqslant A$. Solutions of (2.1) are *uniformly persistent* if there exists $A > 0$ such that $\liminf_{t\to\infty} x(t) \geqslant A$ for any initial function in (2.2) satisfying (a5).

Let us prove persistence of Eq. (2.1) with a positive initial function, under certain conditions.

Theorem 2.6. *Suppose* (a1)–(a6) *hold and there exists $m > 0$ such that $f(x) > d(x)$ for $0 < x < m$. Then solutions of* (2.1) *are uniformly persistent*; *moreover,* $\liminf_{t\to\infty} x(t) \geqslant m$.

Proof. By the assumption of the theorem and Theorem 2.2 the solution $x(t)$ of (2.1), (2.2) is positive. By (a2), there exists $t_0 \geqslant 0$ such that $h(t) > 0$ for any $t > t_0$. Since $x(t) > 0$ for $t \in [0,t_0]$, and it is continuous, its lower bound on $[0,t_0]$ is also positive

$$m_0 := \inf_{t\in[0,t_0]} x(t) > 0. \tag{2.4}$$

Without loss of generality we can assume that $m_0 < m$: if $x(t) \geqslant m_0 \geqslant m$ on any $[0,t_0]$, this implies the statement of the theorem.

Since $d(x) < f(x)$ for $m_0 < x < m$ and f is increasing, there is

$$m_1 := \inf\{x \mid d(x) \geqslant f(m_0)\} > m_0.$$

First, let us prove that $x(t) \geqslant m_0$ for any $t \geqslant 0$. By (a1) and (2.4), there exists $\mu_1 = f^{-1}(\sup_{x\in[0,m_0]} d(x)) < m_0$, thus $d(x) < f(\mu_1)$ for $x \in [0,m_0]$, see Fig. 1.

Assume the contrary that for some t we have $x(t) = \mu_2 < m_0$; without loss of generality we can take $\mu_2 > \mu_1$, then $d(x) < f(y)$ for any $x \in [0,m_0]$, $y \in [\mu_2,\infty)$. Let $t^* = \inf\{t > 0\colon\ x(t) = \mu_2\}$ and $\hat{t} = \sup\{0 < t < t^*\colon\ x(t) = m_0\}$. Then on $[\hat{t},t^*]$ the function $x(t)$ is changing from m_0 to a smaller value of μ_2, thus $d(x(t)) \leqslant f(\mu_1)$ and $f(x(t)) \geqslant f(\mu_2) > f(\mu_1)$, while its derivative on (t,t^*) is positive:

$$x'(t) = r(t)\left[\int_{h(t)}^{t} f\big(x(s)\big)\,d_sR(t,s) - d\big(x(t)\big)\right] > r(t)\big[f(\mu_1) - d\big(x(t)\big)\big] \geqslant 0,$$

which leads to a contradiction. Thus, $x(t) \geqslant m_0$ for any $t \geqslant 0$.

Next, let us justify that there is a point $t_1 > t_0$ such that $x(t) \geqslant m_1$ for $t \geqslant t_1$. Obviously $d(m_1) = f(m_0)$. Let t_* be such that $h(t) \geqslant t_0$ for $t \geqslant t_*$. Due to monotonicity of f and the fact that $d(m_1) = f(m_0)$, for any $x < m_1$ we have for any $t \geqslant t_*$ and $x(t) < m_1$

$$x'(t) = r(t)\left[\int_{h(t)}^{t} f\big(x(s)\big)\,d_sR(t,s) - d\big(x(t)\big)\right] \geqslant r(t)\big[f(m_0) - d\big(x(t)\big)\big] > 0, \tag{2.5}$$

i.e. the solution increases as long as it is less than m_1. There is a possibility that $x(t) \geqslant m_1$ for any $t \geqslant t_*$; then m_1 becomes a new lower bound. Let $x(t_*) < m_1$. The solution $x(t)$ can either tend to a number $q \leqslant m_1$ or can be equal to m_1 at a certain point t^*. Let us demonstrate that the former case is impossible. Assuming $\lim_{t\to\infty} x(t) = q \leqslant m_1$ and defining an arbitrary $q_1 \in (m_0, q)$, we obtain $x(t) > q_1$ for $t \geqslant \bar{t}$; moreover, $h(t) \geqslant \bar{t}$ for $t \geqslant \hat{t}$. Then $x'(t) \geqslant r(t)[f(q_1) - d(m_1)] > 0$ for $t \geqslant \hat{t}$ yields that $\lim_{t\to\infty} x(t) = \infty$, as r satisfies (a3). This immediately implies $x(t) \geqslant m_1$ for t large enough.

Further, once the solution satisfies $x(t) \geqslant m_1$, it cannot become less than m_1 since the derivative is nonnegative for any $x \leqslant m_1$ due to $d(m_1) \leqslant f(x)$, $x \geqslant m_0$. Thus, we obtain a new eventual lower solution bound m_1: $x(t) \geqslant m_1$ for $t > t_1$.

We are now in the position to verify the induction step. Denote

$$m_k := \inf\big\{x \;\big|\; d(x) \geqslant f(m_{k-1})\big\} > m_0, \quad k \in \mathbb{N}. \tag{2.6}$$

Similar to the transition from the inequality $x(t) > m_0$, $t > t_0$ to $x(t) > m_1$, $t > t_1$, we can prove that as long as $m_k < m$, there is $t_k > t_{k-1}$ such that $x(t) > m_k$ for $t > t_k$.

We have an increasing sequence

$$m_0 < m_1 < \cdots < m_k < \cdots, \quad f^{-1}\big(d(m_1)\big) = m_0, \; \ldots, \; f^{-1}\big(d(m_{k+1})\big) = m_k, \tag{2.7}$$

where $x(t) > m_k$, $t > t_k$. The monotone increasing sequence $\{m_k\}$ has a positive limit $\mu = \lim_{k\to\infty} m_k$. If this limit is finite then (2.7) implies $f(\mu) = d(\mu)$, by continuity of f and d. As $f(x) > d(x)$ for $x < m$, this implies $\mu \geqslant m$, or $\liminf_{t\to\infty} x(t) \geqslant m$. The same is valid in the case $\mu = \infty$, which concludes the proof. □

Theorem 2.7. *If* (a1)–(a6) *hold and* $f(x) > d(x)$ *for any* $x > 0$, *then* $\lim_{t\to\infty} x(t) = \infty$.

Proof. As in the proof of Theorem 2.6, define $t_0 \geqslant 0$ such that $h(t) > 0$ for any $t > t_0$ and denote $m_0 := \inf_{t\in[0,t_0]} x(t) > 0$. Using the same argument, it is possible to construct a sequence $\{m_k\}$ for which (2.7) holds, and $x(t) > m_k$, $t > t_k$. The sequence $\{m_k\}$ is monotone increasing and due to the assumption $f(x) > d(x)$ for any $x > 0$ tends to infinity, so $\lim_{t\to\infty} x(t) = \infty$. □

3. Multistability and equations with the Allee effect

Further, let us proceed to models with multiple positive equilibrium points. We start with the case of two positive fixed points.

Theorem 3.1. *Suppose* (a1)–(a6) *hold,* $f(x) > d(x)$ *for* $m < x < K$ *and* $f(x) < d(x)$ *for* $0 < x < m$ *and* $x > K$.

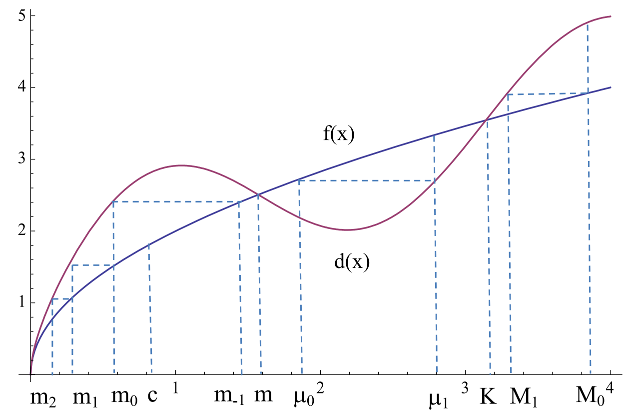

Fig. 2. Illustration of solution bounds in the proof of Theorem 3.1.

If $x(t)$ is a positive solution of (2.1), (2.2) with $\sup_{t\leqslant 0}\varphi(t) < m$ *then* $\lim_{t\to\infty} x(t) = 0$. *If $x(t)$ is a solution of (2.1), (2.2) with* $\inf_{t\leqslant 0}\varphi(t) > m$ *then* $\lim_{t\to\infty} x(t) = K$.

Proof. Let $x(t)$ be a positive solution of (2.1), (2.2).

First, consider the case when $m_0 = \sup_{t\leqslant 0}\varphi(t) < m$. Let us prove that $x(t) \leqslant m_0$ for any $t \geqslant 0$. By the assumption of the theorem, the value $m_{-1} = f^{-1}(\inf_{x\in[m_0,m]} d(x))$ is in the interval $(m_0, m]$ since $d(x) \geqslant f(m_0)$ for $x \in [m_0, m]$, d is continuous, $f(m) = d(m)$, and f is monotone increasing.

Assume the contrary that for some $t > 0$ we have $x(t) = c > m_0$; without loss of generality we can take $c < m_{-1}$. Then $d(x) > f(y)$ for any $x \in [m_0, c]$ and $y \in (0, c)$.

Let $t^* = \inf\{t > 0\colon\ x(t) = c\}$ and $\hat{t} = \sup\{0 < t < t^*\colon\ x(t) = m_0\}$. Then on $[\hat{t}, t^*]$ the function $x(t)$ is changing from m_0 to a greater value of c. However, by the choice of m_{-1} and c we have $d(x(t)) \geqslant f(m_{-1})$ and $f(x(t)) \leqslant f(c) < f(m_{-1})$, so the derivative of $x(t)$ on $[\hat{t}, t^*]$ is non-positive

$$x'(t) = r(t)\left[\int_{h(t)}^{t} f\big(x(s)\big)\, d_s R(t,s) - d\big(x(t)\big)\right] \leqslant r(t)\big[f(m_{-1}) - d\big(x(t)\big)\big] \leqslant 0,$$

which leads to a contradiction. Thus, $x(t) \leqslant m_0$ for any $t \geqslant 0$.

Next, denote

$$m_1 := \inf\big\{x \leqslant m_0\colon\ d(x) > f(m_0)\big\}, \tag{3.1}$$

here $m_1 < m_0$ since $d(m_1) = f(m_0)$ and $f(x) < d(x)$ for $x \in (0, m)$, see Fig. 2. Moreover, $f(x(t)) < d(m_1)$ for any $x \in (m_1, m_0]$, since $x(t) \leqslant m_0$ for any t, as demonstrated above, thus

$$x'(t) = r(t)\left[\int_{h(t)}^{t} f\big(x(s)\big)\, d_s R(t,s) - d\big(x(t)\big)\right] < r(t)\big[f(m_0) - d\big(x(t)\big)\big] \leqslant 0$$

as long as $x(t) \geqslant m_1$. There may be two options: either $x(t) \geqslant m_1$ for any t, while the solution is monotone decreasing, or $x(t_0) = m_1$ for some $t_0 > 0$. Let us prove that the former option is impossible.

A monotone decreasing solution should have a limit $\lim_{t\to\infty} x(t) = b \geqslant m_1$. Then $d(b) > f(m_0)$ and for any $\varepsilon \in (0, 0.5(d(b) - f(m_0)))$ and t large enough, $x'(t) < f(m_0) - d(b) + \varepsilon < 0.5(f(m_0) - d(b)) < 0$, which leads to the equality $\lim_{t\to\infty} x(t) = -\infty$ contradictory to the positivity of a solution. Thus $x(t_0) = m_1$ for some $t_0 > 0$. Since $x'(t) \leqslant 0$ for any $x(t) \geqslant m_1$, we have $x(t) \leqslant m_1$ for any $t \geqslant t_0$. By (a2), there exists $t_1 > t_0$ such that $h(t) \geqslant t_0$ for $t \geqslant t_1$. Thus, for $t \geqslant t_1$, we have $x(t) \leqslant m_1$, $x(h(t)) \leqslant m_1$, which becomes a new solution estimate, similar to m_0.

Continuing this process, we define the decreasing positive sequence of upper solution limits

$$m_k := \inf\{x \geqslant m_0\colon\ d(x) > f(m_{k-1})\} < m_{k-1}, \quad k \in \mathbb{N} \tag{3.2}$$

which has a nonnegative limit and a sequence of points $0 < t_0 < t_1 < \cdots$ such that

$$x(t) \leqslant m_k, \quad t \geqslant t_k. \tag{3.3}$$

Due to the equalities $d(m_k) = f(m_{k-1})$ this limit is a point $x^* < m$ where $d(x^*) = f(x^*)$. By the conditions of the theorem $x^* = 0$, so (3.3) implies $\lim_{t\to\infty} x(t) = 0$.

Next, let $\mu_0 := \inf_{t\leqslant 0} \varphi(t) > m$, $M_0 := \sup_{t\leqslant 0} \varphi(t)$, here M_0 is finite by (a5). Without loss of generality, we can assume the upper bound $M_0 > K$.

Since f and d are continuous, f is increasing, $f(\mu_0) > d(\mu_0)$ and $f(M_0) < d(M_0)$, there exist μ_1 and M_1 such that

$$\mu_1 = \sup\{x \geqslant \mu_0\colon\ d(x) < f(\mu_0)\}, \quad K > \mu_1 > \mu_0, \tag{3.4}$$

$$M_1 = \inf\{x \leqslant M_0\colon\ d(x) > f(M_0)\}, \quad K < M_1 < M_0, \tag{3.5}$$

see Fig. 2.

The derivative $x'(t)$ becomes positive if the solution first changes from μ_0 to some μ^*, $\mu_0 < \mu^* < \mu_1$, which leads to a contradiction, so $x(t) \geqslant \mu_0$ for any t, as in the proof of Theorem 2.6. Similarly, $x(t) \leqslant M_0$ for any t, and we have the estimate

$$\mu_0 \leqslant x(t) \leqslant M_0, \quad t \geqslant 0. \tag{3.6}$$

Also,

$$d(\mu_1) = f(\mu_0), \qquad d(M_1) = f(M_0). \tag{3.7}$$

Let t_* be such that $h(t) \geqslant t_0$ for $t \geqslant t_*$. Due to monotonicity of f and (3.6), for any $x(t) < \mu_1$, where $t \geqslant t_*$, we have

$$x'(t) = r(t)\left[\int_{h(t)}^{t} f\bigl(x(s)\bigr)\, d_s R(t,s) - d\bigl(x(t)\bigr)\right] \geqslant r(t)\bigl[f(\mu_0) - d\bigl(x(t)\bigr)\bigr] > 0,$$

i.e. the solution increases as long as it is less than μ_1. There is a possibility that $x(t) > \mu_1$ for any $t \geqslant t_*$; then μ_1 becomes a new lower bound. Let $x(t_*) < \mu_1$. This solution can either tend to some number $q \leqslant \mu_1$ or can be equal to μ_1 at a certain point t^*. Let us demonstrate that the former case is impossible. Assuming $\lim_{t\to\infty} x(t) = q \leqslant \mu_1$ and defining an arbitrary $q_1 \in (\mu, q)$, we obtain $x(t) > q_1$ for $t \geqslant t_3$; moreover, $h(t) \geqslant t_3$ for $t \geqslant t_4$. Then $x'(t) \geqslant r(t)[f(q_1) - d(\mu_1)] > 0$ for $t \geqslant t_4$, as f is increasing and thus $\lim_{t\to\infty} x(t) = \infty$, since r satisfies (a3). This contradiction with (3.6) implies $x(t) = \mu_1$ for some $t > t_*$. Next, once the solution satisfies $x(t) \geqslant \mu_1$, it cannot become less than μ_1 since the derivative is nonnegative for any $x \leqslant \mu_1$ due to $d(\mu_1) \leqslant f(x)$, $x \geqslant \mu_0$. Thus, we obtain a new eventual lower solution bound μ_1: $x(t) \geqslant \mu_1$, $t > t^*$.

Similarly, we justify that there exists a point $\bar{t}$ such that $x(t) \leqslant M_1$ for $t \geqslant \bar{t}$. Let us choose $t_1 = \max\{t^*, \bar{t}\}$. Then

$$\mu_1 \leqslant x(t) \leqslant M_1, \quad \mu_0 < \mu_1 < K < M_1 < M_0,\ t \geqslant t_1. \tag{3.8}$$

Further, constructing monotone sequences of lower and upper bounds

$$\mu_{k+1} := \sup\{x \geqslant \mu_k\colon\ d(x) < f(\mu_k)\}, \quad K > \mu_{k+1} > \mu_k > \cdots > \mu_1 > \mu_0,\ k \in \mathbb{N}, \tag{3.9}$$

$$M_{k+1} := \inf\{x \leqslant M_k\colon\ d(x) > f(M_k)\}, \quad K < M_{k+1} < M_k < \cdots < M_1 < M_0,\ k \in \mathbb{N}, \tag{3.10}$$

where

$$\begin{gathered} \mu_k \leqslant x(t) \leqslant M_k, \quad t \geqslant t_k, \\ d(\mu_{k+1}) = f(\mu_k), \qquad d(M_{k+1}) = f(M_k), \end{gathered} \tag{3.11}$$

we deduce that both sequences have positive limits. By (3.11), these limits should satisfy $f(x) = d(x)$ and lie in the interval (m, ∞); the only possible point is K. Thus $\liminf_{t\to\infty} x(t) = \limsup_{t\to\infty} x(t) = K$, and K attracts any solution $x(t)$, which completes the proof. □

Remark 3.2. Similarly to existence and uniqueness Theorem 2.2, the stability result of Theorem 3.1 is easily extended to the case when $f : \mathbb{R} \to (0,\infty)$, $d : \mathbb{R} \to (0,\infty)$ instead of $f, d : (0,\infty) \to (0,\infty)$, all other conditions of Theorem 3.1 being satisfied.

Theorem 3.3. *Suppose* (a1)–(a6) *hold, and there exist numbers M and K, $M > K > 0$ such that $f(x) > d(x)$ for $0 < x < K$ and $x > M$ and $f(x) < d(x)$ for $K < x < M$.*

If $x(t)$ is a positive solution of (2.1), (2.2) with $\sup_{t\leqslant 0} \varphi(t) \in (0, M)$ then $\lim_{t\to\infty} x(t) = K$. If $x(t)$ is a positive solution of (2.1), (2.2) with $\inf_{t\leqslant 0} \varphi(t) > M$ then $\lim_{t\to\infty} x(t) = \infty$.

Proof. The proof of the case $\sup_{t\leqslant 0} \varphi(t) \in (0, M)$ coincides with the proof of the main stability result in [6]. The case $\inf_{t\leqslant 0} \varphi(t) > M$ is considered similarly to the proof of Theorem 2.7. □

Remark 3.4. The conditions of Theorem 3.3 can describe the case when for the population or cell density initiated at the levels below a certain critical value, the process converges to some positive equilibrium. As long as the solution exceeds this critical level, the uncontrolled growth begins: this can correspond to either a population outbreak or spread of malignant tumour cells.

Similarly to the proof of Theorem 3.1, we can prove the result for the case when there are more than two positive equilibrium points $0 = K_0 < K_1 < K_2 < \cdots < K_n$ where $f(K_j) = d(K_j)$, $j = 1, \dots, n$, and the zero point may be an attractor as in Theorem 3.1, or a repeller, as well as for the cases when $f(x) - d(x)$ is either always positive or always negative for $x > 0$.

Theorem 3.5. *Let* (a1)–(a6) *hold and $f(K_j) = d(K_j)$, $j = 0, 1, \dots, n$.*

If in addition $f(x) < d(x)$ for $K_{2j} < x < K_{2j+1}$ and $f(x) > d(x)$ for $K_{2j+1} < x < K_{2j+2}$, $j = 0, 1, \dots,$ then all solutions of (2.1), (2.2) with the initial function $\varphi(t) \in (K_0, K_1)$ tend to zero, all the solutions with the initial function $\varphi(t) \in (K_{2j-1}, K_{2j+1})$ tend to K_{2j}, $j = 1, 2, \dots$.

If $f(x) > d(x)$ for $K_{2j} < x < K_{2j+1}$ and $f(x) < d(x)$ for $K_{2j+1} < x < K_{2j+2}$, $j = 0, 1, \dots,$ then all solutions of (2.1), (2.2) with the initial function $\varphi(t) \in (K_{2j-2}, K_{2j})$ tend to K_{2j-1} for any $j = 1, 2, \dots$.

Theorem 3.6. *Let* (a1)–(a6) *hold and $f(K) = d(K)$, $K > 0$. If $f(x) < d(x)$ for $x > 0$, $x \neq K$, then K is semi-stable, all solutions of (2.1), (2.2) with the initial function $\varphi(t) \in (K, \infty)$ tend to the equilibrium K while all the solutions with the initial function $\varphi(t) \in (0, K)$ tend to zero. If $f(x) > d(x)$ for $x > 0$, $x \neq K$, then all solutions of (2.1), (2.2) with the initial function $\varphi(t) \in (K, \infty)$ tend to infinity while all the solutions with the initial function $\varphi(t) \in (0, K)$ tend to K.*

Theorem 3.7. *Let* (a1)–(a6) *hold. If* $d(x) > f(x)$ *for any* $x > 0$ *then any solution of* (2.1), (2.2) *tends to zero. If* $d(x) < f(x)$ *for any* $x > 0$ *then any solution of* (2.1), (2.2) *tends to infinity.*

Example 3.8. By Theorem 3.1, all solutions of the equation

$$\dot{x}(t) = 3x^2(t-\tau) - \big(x^3(t) + 2x(t)\big)$$

with the initial function exceeding $m = 1$ converge to $K = 2$. If the initial function is less than $m = 1$, the solution converges to zero as $t \to \infty$.

By Theorem 3.3, any solution of the equation

$$\dot{x}(t) = x^3(t-\tau) + 2x(t-\tau) - 3x^2(t)$$

with the initial function in $(0, 2)$ tends to $K = 1$, all solutions with the initial function greater than 2 tend to infinity.

By Theorem 3.6, all positive solutions of the equation

$$\dot{x}(t) = 2x^2(t-\tau) - \big(x^3(t) + x(t)\big)$$

with $\varphi(t) \in (1, \infty)$ tend to 1, but with $\varphi(t) \in (0, 1)$ tend to 0.

All positive solutions of the equation

$$\dot{x}(t) = x^3(t-\tau) + x(t-\tau) - 2x^2$$

with $\varphi(t) \in (1, \infty)$ tend to infinity, but with $\varphi(t) \in (0, 1)$ tend to 1.

By Theorem 3.7, all positive solutions of the equation

$$\dot{x}(t) = 2x^2(t-\tau) - \big(x^3(t) + 2x(t)\big)$$

tend to zero, while of the equation

$$\dot{x}(t) = x^3(t-\tau) + 2x(t-\tau) - 2x^2(t)$$

tend to infinity.

Next, let us illustrate that the results of the present section are applicable to equations with a distributed and, generally, unbounded delays.

Example 3.9. The equation with a pantograph-type unbounded delay

$$\dot{x}(t) = \frac{1}{(1-\alpha)t} \int_{\alpha t}^{t} \big(x(s) + \sin\big(x(t)\big)\big)\, ds - x(t), \quad t \geqslant 1,\ 0 < \alpha < 1$$

by Theorem 3.1 is multistable: if $\varphi(t) \in (2\pi(n-1), 2\pi n)$, then $x(t) \to \pi(2n-1)$, $n \in \mathbb{N}$, while the equilibrium points $2\pi(n-1)$ are unstable, including the zero equilibrium. The equation

$$\dot{x}(t) = \frac{1}{(1-\alpha)t} \int_{\alpha t}^{t} \big(x(s) - \sin\big(x(t)\big)\big)\, ds - x(t), \quad t \geqslant 1,\ 0 < \alpha < 1,$$

has the Allee effect: all solutions with $0 \leqslant \varphi(t) < \pi$ tend to zero while if $\varphi(t) \in (\pi(2n-1), \pi(2n+1))$, the solution converges to $2\pi n$, $n \in \mathbb{N}$, and the equilibrium points $\pi(2n-1)$ are unstable, $n \in \mathbb{N}$. It is easy to check that the functions $x \pm \sin x$ are monotone increasing.

Using Theorem 3.3, we obtain that all solutions of the equation

$$\dot{x}(t) = \int_{t-1}^{t} \big(x^3(s) + 2x(s)\big)\, ds - 3x^2(t), \quad t \geqslant 0$$

with $\varphi(t) \in (0, 2)$ tend to $K = 1$, all solutions with the initial function greater than 2 tend to infinity.

Similarly, applying Theorem 3.6, we get that all positive solutions of the equation

$$\dot{x}(t) = \frac{1}{\tau} \int_{t-\tau}^{t} 2x^2(s)\, ds - \big(x^3(t) + x(t)\big)$$

with $\varphi(t) \in (1, \infty)$ tend to 1, but with $\varphi(t) \in (0, 1)$ tend to zero. Any positive solution of the equation

$$\dot{x}(t) = \frac{1}{\tau} \int_{t-\tau}^{t} \big(x^3(s) + x(s)\big)\, ds - 2x^2(t)$$

with $\varphi(t) \in (1, \infty)$ tends to infinity, while with $\varphi(t) \in (0, 1)$ tends to 1.

By Theorem 3.7, all positive solutions of the equation

$$\dot{x}(t) = \frac{1}{2 - \cos t} \int_{t-2+\cos t}^{t} \big(2x(s) + \sin\big(x(t)\big)\big)\, ds - x(t), \quad t \geqslant 0$$

tend to zero, while of the equation

$$\dot{x}(t) = \frac{1}{2 - \cos t} \int_{t-2+\cos t}^{t} \big(x(s) - \sin\big(x(t)\big)\big)\, ds - 2x(t), \quad t \geqslant 0$$

tend to infinity.

4. Oscillation

In this section, we discuss non-oscillation properties of (2.1).

Definition 4.1. A solution $x(t)$ of (2.1), (2.2) is *non-oscillatory about the set* K_j, $j = 1, \dots, n$, if there exists $t_0 > 0$ such that either $x(t) > K_j$ or $x(t) < K_j$ is satisfied for all $t \geqslant t_0$. Otherwise, $x(t)$ *oscillates about* K_j. Eq. (2.1) is *non-oscillatory about the set* K_j, $j = 1, \dots, n$, if for any K_j there exists an initial function such that the solution of (2.1), (2.2) is non-oscillatory about K_j.

Definition 4.2. An oscillating solution of (2.1) is called *slowly oscillating about* K if for any $t_0 > 0$ there exist three points $t_3 > t_2 > t_1 > t_0$ such that $h(t) \geqslant t_1$ for any $t \geqslant t_2$, and $x(t) - K$ preserves its sign on $[t_1, t_3)$ and vanishes at the point t_3:

$$\big(x(s) - K\big)\big(x(t) - K\big) > 0, \quad s, t \in [t_1, t_3), \qquad x(t_3) = K.$$

Otherwise, the solution is *rapidly oscillating*.

Remark 4.3. The relation of existence of slowly oscillating solutions and oscillation is not very simple even in the case of one equilibrium point (or oscillation about zero). Generally, nonoscillation does not imply nonexistence of slowly oscillating about this unique equilibrium solutions. However, a dichotomy is observed for a wide class of delay equations: either there are no slowly oscillating solutions, or all solutions oscillate, see [1] for some results of this type.

If for all K_j an equation has no slowly oscillating about K_j solutions, then it is nonoscillatory: nonexistence of slowly oscillating about any of $K_j > 0$ solutions guarantees that any solution with a positive initial function in an interval not containing equilibrium points is non-oscillatory about K_j. This certainly does not exclude the existence of rapidly oscillating about K_j solutions, where oscillation is a result of the oscillatory nature of the initial function.

Theorem 4.4. *Suppose* (a1)–(a6) *hold and* $0 = K_0 < K_1 < \cdots < K_n$ *be equilibrium points where* $f(K_j) = d(K_j)$. *Then* (2.1) *is non-oscillatory about the set* K_j, $j = 1, \dots, n$. *Moreover, for any* K_j, $j = 1, \dots, n$, *Eq.* (2.1) *has no slowly oscillating about* K_j *solutions. If* $\varphi(t) \in (K_j, K_{j+1})$ *then the solution* $x(t)$ *of* (2.1), (2.2) *for any* $t \geqslant 0$ *satisfies* $x(t) \in (K_j, K_{j+1})$.

Proof. By Theorem 2.2, any solution $x(t)$ of (2.1), (2.2) is positive. Choose t_0 such that $h(t_0) > 0$ for $t \geqslant t_0$ and t_1, t_2, $t_2 > t_1 > t_0$ such that $h(t) \geqslant t_1$ for any $t \geqslant t_2$.

Let $K_j < x(t) < K_{j+1}$ for $t \in [t_1, t_3)$, where $t_3 > t_2$, and $x(t_3) = K_{j+1}$. First we assume that $f(x) > d(x)$ for $x \in (K_j, K_{j+1})$. Let us compare the solution $x(t)$ of (2.1), (2.2) and $y(t)$ of the initial-value problem for the ordinary differential equation

$$y'(t) = r(t)\big[f(K_{j+1}) - d\big(y(t)\big)\big] = r(t)\big[d(K_{j+1}) - d\big(y(t)\big)\big], \qquad y(t_2) = x(t_2). \tag{4.1}$$

On the one hand, since $y(t) \equiv K_{j+1}$ is a solution of the equation involved in (4.1) and, by (a6), problem (4.1) has a unique increasing solution, $y(t) < K_{j+1}$ for any $t \geqslant t_2$. On the other hand, on $[t_2, t_3]$ the solution $x(t)$ of (2.1), (2.2) satisfies

$$\begin{aligned} x'(t) &= r(t)\left[\int_{h(t)}^{t} f\big(x(s)\big)\, d_s R(t,s) - d\big(x(t)\big)\right] \\ &\leqslant r(t)\left[\int_{h(t)}^{t} f(K_{j+1})\, d_s R(t,s) - d\big(x(t)\big)\right] \\ &= r(t)\big[d(K_{j+1}) - d\big(x(t)\big)\big], \qquad x(t_2) = y(t_2). \end{aligned}$$

Moreover, on $[t_2, t_3)$ the strict inequality is valid, so

$$x(t_3) \leqslant y(t_3) < K_{j+1},$$

which contradicts the assumption $x(t_3) = K_{j+1}$, thus $x(t)$ is nonoscillatory about K_{j+1}.

Due to monotonicity of f the solution $x(t)$ is also nonoscillatory about K_j.

Similarly, assuming $f(x) < d(x)$ for $x \in (K_j, K_{j+1})$, we compare the solution with a decreasing solution of the initial value problem

$$y'(t) = r(t)\big[f(K_j) - d\big(y(t)\big)\big] = r(t)\big[d(K_j) - d\big(y(t)\big)\big], \qquad y(t_2) = x(t_2), \tag{4.2}$$

this solution tends to K_j and exceeds K_j for any $t \geqslant t_2$. The argument similar to the previous case completes the proof. $\square$

Remark 4.5. First, let us note that all positive solutions considered in Examples 3.8 and 3.9 are nonoscillatory about the set of their equilibrium points, moreover, any solution is nonoscillatory, once the initial function is positive and all its values belong to an open interval which does not include any of the equilibrium points. It is easy to check that this is no longer true if $f(x)$ is non-monotone, there are solutions slowly oscillating about the equilibrium K, see the details for the Nicholson blowflies

$$\dot{N}(t) = -\delta N(t) + PN\big(h(t)\big)e^{-N(h(t))}, \quad P > \delta e$$

and the Mackey–Glass equations

$$\dot{x}(t) = -r(t)\left[\gamma x(t) - \beta(t)\frac{x(h(t))}{1 + x^n(h(t))}\right], \quad n > 1$$

in the review papers [7] and [8], respectively.

5. Applications, summary and discussion

As an application of the results of the present paper, population dynamics models with the Allee effect can be designed in such a way that m is the minimal survival level and K is another positive equilibrium, $0 < m < K$. For instance, Eq. (2.1) with

$$f(x) = \frac{(m+K)x^2}{1 + x^n}, \qquad d(x) = \frac{x^3 + mKx}{1 + x^n}, \quad 0 \leqslant n \leqslant 2$$

or, more generally, with

$$f(x) = \frac{(m+K)x^2}{g(x)}, \qquad d(x) = \frac{x^3 + mKx}{g(x)}, \quad g(x) > 0,\ x \geqslant 0,$$

where $\frac{x^2}{g(x)}$ is a monotone increasing function, can be considered.

The idea to study systems with a monotone production function was first proposed in [22] and implemented in detail for autonomous equations, see also [18,20,21,16]. As an example, the equation

$$x'(t) = -x(t) + h\big(x(t-\tau)\big)$$

was considered in [22] under the assumption that h is strictly monotone increasing and $h(x) < \lambda x$ for some $\lambda \in (0,1)$ and all x large enough. According to [22, Proposition 4.2], there is an open and dense set of initial conditions corresponding to solutions which converge to an equilibrium. If there is only one equilibrium, then all solutions converge to it; if there are only two equilibrium points, then all solutions converge to one of these.

Compared to this model, we consider a more general type of the mortality function (non-necessarily monotone), which leads to more options (semi-stable equilibrium points are possible), and also a more general delay (distributed or concentrated, or both).

The results of the present paper illustrate the fact that asymptotic properties of Eq. (2.1) are very similar to the properties of the same equation without delay. This effect for equations with monotone nonlinearities was first noticed in the papers [21,23]. We have obtained similar results for rather general models with an arbitrary number of positive equilibrium points.

The global attractivity results can be extended to the case when $f : \mathbb{R} \to (0,\infty)$ is monotone increasing and is continuous together with $d : \mathbb{R} \to (0,\infty)$, see Remarks 2.4 and 3.2. This allows to study the model

$$x'(t) = r(t)x(t)\left[\int_{h(t)}^{t} f\big(x(s)\big)\, d_s R(t,s) - d\big(x(t)\big)\right], \tag{5.1}$$

where in addition (a2)–(a6) are valid. In fact, after the substitution

$$x(t) = e^{y(t)} \tag{5.2}$$

we have the equation

$$y'(t) = r(t)\left[\int_{h(t)}^{t} f\big(e^{y(s)}\big)\, d_s R(t,s) - d\big(e^{y(t)}\big)\right], \tag{5.3}$$

where $f_1(y) = f(e^y)$ is still monotone increasing, and Eq. (5.3) has the equilibrium $\ln K$ for any positive equilibrium K of (5.1), so the results of Theorems 3.1, 3.3, 3.5, and 3.6 can be applied.

Let us outline some open problems.

1. What is an asymptotic behaviour of solutions without the assumption that values of an initial function belong to the interval between two successive equilibrium points?
2. Consider the vector equation

$$x'(t) = r(t)\left[\int_{h(t)}^{t} f\big(x(s)\big)\, d_s R(t,s) - d\big(x(t)\big)\right], \quad t \geqslant 0, \tag{5.4}$$

with initial condition (2.2), where the vector functions

$$x = \{x_1, \dots, x_n\}, \qquad f = \{f_1, \dots, f_n\},$$
$$d = \{d_1, \dots, d_n\}, \qquad \varphi = \{\varphi_1, \dots, \varphi_n\},$$

the matrix-function $R(t,s) = \{R_{ij},\ i,j = 1,\dots,n\}$ and r, f_j, d_j, R_{ij}, h, φ_j satisfy conditions (a1)–(a6) including monotonicity properties of functions f_j, $j = 1,\dots,n$.
Generalize and extend the results obtained for scalar equation (2.1) in [6] and in the present paper to vector equation (5.4).
3. Consider the differential inequalities

$$y'(t) \leqslant r(t)\left[\int_{h(t)}^{t} f\big(y(s)\big)\, d_s R(t,s) - d\big(y(t)\big)\right], \quad t \geqslant 0, \tag{5.5}$$

$$z'(t) \geqslant r(t)\left[\int_{h(t)}^{t} f\big(z(s)\big)\, d_s R(t,s) - d\big(z(t)\big)\right], \quad t \geqslant 0, \tag{5.6}$$

with initial conditions (2.2). Assume that conditions (a1)–(a6) hold for inequalities (5.5) and (5.6).
Prove or disprove:
If x, y, z are solutions of (2.1), (5.5) and (5.6) such that $x(t) = y(t) = z(t)$ for $t \leqslant 0$, then $y(t) \leqslant x(t) \leqslant z(t)$, $t \geqslant 0$.

Appendix A

Denote by $\mathbf{L}^2([t_0,t_1])$ the space of Lebesgue measurable functions $x(t)$ such that $Q=\int_{t_0}^{t_1}(x(t))^2\,dt<\infty$, with the usual norm $\|x\|_{\mathbf{L}^2([t_0,t_1])}=\sqrt{Q}$, by $\mathbf{C}([t_0,t_1])$ the space of continuous on $[t_0,t_1]$ functions with the sup-norm, by $\mathbf{L}^\infty([t_0,t_1])$ the space of Lebesgue measurable essentially bounded on $[t_0,t_1]$ functions with the essential supremum norm.

The following result from the book of Corduneanu [13, Theorem 4.5, p. 95] will be applied. We recall that an operator N is *causal* (or *Volterra*) if for any two functions x and y and each t the fact that $x(s)=y(s)$, $s\leqslant t$, implies $(Nx)(s)=(Ny)(s)$, $s\leqslant t$.

Lemma A.1. *(See [13].) Consider the equation*

$$y'(t)=(\mathcal{L}y)(t)+(\mathcal{N}y)(t),\quad t\in[t_0,t_1],\tag{A.1}$$

where $\mathcal{L}:\mathbf{C}([t_0,t_1])\to\mathbf{L}^2([t_0,t_1])$ is a linear bounded causal operator, $N:\mathbf{C}([t_0,t_1])\to\mathbf{L}^2([t_0,t_1])$ is a nonlinear causal operator which satisfies

$$\|\mathcal{N}x-\mathcal{N}y\|_{\mathbf{L}^2([t_0,t_1])}\leqslant\lambda\|x-y\|_{\mathbf{C}([t_0,t_1])}\tag{A.2}$$

for λ sufficiently small. Then there exists a unique absolutely continuous on $[t_0,t_1]$ solution of (A.1), with the initial function being equal to zero for $t<t_0$.

Proof of Theorem 2.2. To reduce (2.1) to the equation with the zero initial function, for any $t_0\geqslant 0$ we can present the integral as a sum of two integrals

$$x'(t)=-r(t)d\big(x(t)\big)+r(t)\int_{t_0}^{t}f\big(x(s)\big)\,d_sR(t,s)+r(t)\int_{h(t)}^{t}f\big(\varphi(s)\big)\,d_sR(t,s),\tag{A.3}$$

where $x(t)=0$, $t<t_0$, $\varphi(t)=0$, $t\geqslant t_0$.

Let $\alpha=\alpha_{[t_0,t_1]}$ and $\beta=\beta_{[t_0,t_1]}$ defined in (a6) be the appropriate Lipschitz constants. Then in (A.1)

$$(\mathcal{L}x)(t)=0,\qquad(\mathcal{N}x)(t)=r(t)\int_{t_0}^{t}f\big(x(s)\big)\,d_sR(t,s)-r(t)d\big(x(t)\big)+F(t),$$

where

$$F(t)=r(t)\int_{h(t)}^{t}f\big(\varphi(s)\big)\,d_sR(t,s),\qquad\varphi(t)=0,\quad t\geqslant t_0.$$

By (a5) and (a6), for any $\lambda>0$ there is t_1, such that

$$\begin{aligned}&\|\mathcal{N}x-\mathcal{N}y\|_{\mathbf{L}^2([t_0,t_1])}\\&\quad\leqslant\left\|r(t)\int_{t_0}^{t}\big[f\big(x(s)\big)-f\big(y(s)\big)\big]\,d_sR(t,s)\right\|_{\mathbf{L}^2([t_0,t_1])}+\big\|r(t)\big[d\big(x(t)\big)-d\big(y(t)\big)\big]\big\|_{\mathbf{L}^2([t_0,t_1])}\\&\quad\leqslant\|r\|_{\mathbf{L}^\infty([t_0,t_1])}\left[\left\|\int_{t_0}^{t}\big|f\big(x(s)\big)-f\big(y(s)\big)\big|\,d_sR(t,s)\right\|_{\mathbf{L}^2([t_0,t_1])}+\big\|d\big(x(t)\big)-d\big(y(t)\big)\big\|_{\mathbf{L}^2([t_0,t_1])}\right]\end{aligned}$$

$$\leqslant \|r\|_{\mathbf{L}^\infty([t_0,t_1])}\left[\alpha\|x-y\|_{\mathbf{C}([t_0,t_1])}\left\|\int_{t_0}^{t} d_sR(t,s)\right\|_{\mathbf{L}^2([t_0,t_1])}+\beta\|x-y\|_{\mathbf{C}([t_0,t_1])}\sqrt{|t_1-t_0|}\right]$$
$$\leqslant (\alpha+\beta)\|r\|_{\mathbf{L}^\infty([t_0,\infty))}\big\|x(s)-y(s)\big\|_{\mathbf{C}([t_0,t_1])}\sqrt{|t_1-t_0|}\leqslant \lambda\|x-y\|_{\mathbf{C}([t_0,t_1])}$$

for $t_1-t_0 \leqslant \lambda^2/((\alpha+\beta)\operatorname{ess\,sup}_{t\geqslant 0}|r(t)|)^2$, where λ can be chosen small enough. By Lemma A.1, this implies existence and uniqueness of a local solution for (2.1).

This solution is either global or there exists t_2 such that either

$$\liminf_{t\to t_2} x(t) = -\infty \tag{A.4}$$

or

$$\limsup_{t\to t_2} x(t) = \infty. \tag{A.5}$$

The initial value is positive, so as far as $x(t) > 0$, the solution is not less than the solution of the initial value problem

$$x'(t) + r(t)d\big(x(t)\big) = 0, \qquad x(t_0) = x_0 > 0. \tag{A.6}$$

Assuming that the solution of (A.6) becomes negative, letting t_1 be the smallest number exceeding t_0, where $x(t_1) = 0$, and taking into account that $d(0) = 0$, we obtain that there are two solutions (the solution of (A.6) and $x(t) \equiv 0$) through the point $(t_1, 0)$ which contradicts the uniqueness theorem for ordinary differential equations. Thus all solutions of (2.1), (2.2) are positive, which excludes the possibility of (A.4).

Next, we demonstrate that (A.5) also cannot be satisfied. Let us fix some $\varepsilon > 0$ and denote $M = \max\{K + \varepsilon, \sup_{s\leqslant 0}\varphi(s) + \varepsilon\}$. Since $f(x) < d(x)$ for $x > K$ by the assumption of the theorem and f is increasing, we have

$$M_1 = \sup\big\{x > K \bigm| d(x) \leqslant f(M)\big\} < M.$$

Assume that $x(t) > M$ for some $t > 0$, then the points $t^* = \inf\{t > 0 \mid x(t) \geqslant M\}$ and $\hat{t} = \sup\{t < t^* \mid x(t) \geqslant M_1\}$ exist, $x(\hat{t}) = M_1$, $x(t^*) = M$ and $M_1 \leqslant x(t) \leqslant M$ for $t \in [\hat{t}, t^*]$. However, for $t \in [\hat{t}, t^*]$, due to monotonicity of f we have

$$x'(t) = r(t)\left[\int_{h(t)}^{t} f\big(x(s)\big)\,d_sR(t,s) - d\big(x(t)\big)\right] \leqslant r(t)\big[f(M) - d(x)\big] \leqslant r(t)\big(f(M) - d(M_1)\big) = 0.$$

Non-positivity of the derivative of x on $[\hat{t}, t^*]$ together with $x(\hat{t}) = M_1 < x(t^*) = M$ lead to a contradiction. Thus, (A.5) is impossible, which concludes the proof. □

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