



Generalizations of generating functions for higher continuous hypergeometric orthogonal polynomials in the Askey scheme



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ARTICLE INFO

Article history:

Received 24 June 2014
Available online 9 February 2015
Submitted by S. Tikhonov

Keywords:

Orthogonal polynomials
Generating functions
Connection coefficients
Generalized hypergeometric functions
Eigenfunction expansions
Definite integrals

ABSTRACT

We use connection relations and series rearrangement to generalize generating functions for several higher continuous orthogonal polynomials in the Askey scheme, namely the Wilson, continuous dual Hahn, continuous Hahn, and Meixner–Pollaczek polynomials. We also determine corresponding definite integrals using the orthogonality relations for these polynomials.

Published by Elsevier Inc.

1. Introduction

In Cohl (2013) [3] (see (2.1) therein), we developed a series rearrangement technique which produced a generalization of the generating function for Gegenbauer polynomials. We have since demonstrated that this technique is valid for a larger class of orthogonal polynomials. For instance, in Cohl (2013) [2], we applied this same technique to Jacobi polynomials and in Cohl, MacKenzie and Volkmer (2013) [4], we extended this technique to many generating functions for Jacobi, Gegenbauer, Laguerre, and Wilson polynomials.

The series rearrangement technique starts by combining a connection relation with a generating function. This results in a series with multiple sums. The order of summations is then rearranged to produce a generalized generating function. This technique is especially productive when using connection relations with one free parameter. In this case, the connection relation is usually a product of Pochhammer symbols and

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the resulting generalized generating function has coefficients given in terms of generalized hypergeometric functions.

In this paper, we continue this procedure by generalizing generating functions for the remaining hypergeometric orthogonal polynomials in the Askey scheme [6, Chapter 9] with continuous orthogonality relations. We have also computed definite integrals corresponding to our generalized generating function expansions using continuous orthogonality relations. The orthogonal polynomials that we treat in this paper are the Wilson, continuous dual Hahn, continuous Hahn, and Meixner–Pollaczek polynomials. The generalized generating functions we produce through series rearrangement usually arise using connection relations with one free parameter. While connection relations with one free parameter are preferred for their simplicity, relations with more free parameters were considered when necessary.

Hypergeometric orthogonal polynomials with more than one free parameter, such as the Wilson polynomials, have connection relations with more than one free parameter. These connection relations are in general given by single or multiple summation expressions. For the Wilson polynomials, the connection relation with four free parameters is given as a double hypergeometric series. The fact that the four free parameter connection coefficient for Wilson polynomials is given by a double sum was known to Askey and Wilson as far back as in 1985 (see [5, p. 444]). When our series rearrangement technique is applied to cases with more than one free parameter, the resulting coefficients of the generalized generating function are rarely given in terms of a generalized hypergeometric series. The more general problem of generalized generating functions with more than one free parameter requires the theory of multiple hypergeometric series and is not treated in this paper. However, in certain cases when applying the series rearrangement technique to generating functions using connection relations with one free parameter, the generating function remains unchanged. In these cases, we have found that the introduction of a second free parameter can sometimes yield generalized generating functions whose coefficients are given in terms of generalized hypergeometric series (see for instance, Section 3 below and [2, Theorem 1]).

An interesting question regarding our generalizations is, “What is the origin of specific hypergeometric orthogonal polynomial generating functions?” There only exist two known non-equivalent generating functions for the Wilson polynomials, with the Wilson polynomials being at the top of the Askey scheme. Unlike the orthogonal polynomials in the Askey scheme which arise through a limiting procedure from the Wilson polynomials, most known generating functions for these polynomials do not arrive by this same limiting procedure from the two known non-equivalent generating functions for Wilson polynomials. All of the generating functions treated in this paper for the continuous Hahn, continuous dual Hahn and Meixner–Pollaczek polynomials, as well as most of those generating functions treated in our previous papers, do not arrive by this same limiting procedure from the Wilson polynomial generating functions. Therefore, the generalized generating functions for non-Wilson polynomials we present in this paper are interesting by themselves.

Here, we provide a brief introduction into the symbols and special functions used in this paper. We denote the real and complex numbers by \mathbb{R} and \mathbb{C} , respectively. Similarly, the sets $\mathbb{N} = 1, 2, 3, \dots$ and $\mathbb{Z} = 0, \pm 1, \pm 2, \dots$ denote the natural numbers and the integers. We also use the notation $\mathbb{N}_0 = \{0, 1, 2, \dots\} = \mathbb{N} \cup \{0\}$. If $a_1, a_2, a_3, \dots \in \mathbb{C}$, and $i, j \in \mathbb{Z}$ such that $j < i$, then $\sum_{n=i}^j a_n = 0$, and $\prod_{n=i}^j a_n = 1$. Let $z \in \mathbb{C}$, $n \in \mathbb{N}_0$. Let $a_1, \dots, a_p \in \mathbb{C}$, and $b_1, \dots, b_q \in \mathbb{C} \setminus -\mathbb{N}_0$. The generalized hypergeometric function ${}_pF_q$ is defined as [7, Chapter 16]

$${}_pF_q \left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}; z \right) := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}. \quad (1.1)$$

If $p \leq q$ then ${}_pF_q$ is defined for all $z \in \mathbb{C}$. If $p = q + 1$ then ${}_pF_q$ is defined in the unit disk $|z| < 1$, and can be continued analytically to $\mathbb{C} \setminus [1, \infty)$. The generalized hypergeometric function is used in the definitions

of hypergeometric orthogonal polynomials and for the coefficients of our generalized generating functions. When $a \in \mathbb{C} \setminus -\mathbb{N}_0$, and $\nu \in \mathbb{C}$, we define the Pochhammer symbol (rising factorial) as

$$(a)_\nu := \frac{\Gamma(a + \nu)}{\Gamma(a)},$$

where $\Gamma : \mathbb{C} \setminus -\mathbb{N}_0 \rightarrow \mathbb{C}$ is the gamma function (see Olver et al. (2010) [7, Chapter 5]). For $n \in \mathbb{N}_0$, $a \in \mathbb{C}$, one has

$$(a)_n = (a)(a + 1) \cdots (a + n - 1) = \prod_{i=0}^{n-1} (a + i), \tag{1.2}$$

which is a polynomial in a . Note that for some $a_k \in -\mathbb{N}_0$, $1 \leq k \leq p$, the generalized hypergeometric series (1.1) is terminating and represents a polynomial in z . This is the case for the generalized hypergeometric orthogonal polynomials treated in this paper, so we are primarily interested in the definition of the Pochhammer symbol given by (1.2).

2. Wilson polynomials

Koekoek et al. (2010) [6, (9.1.1)] define the Wilson polynomials $W_n(x^2; a, b, c, d)$ by

$$W_n(x^2; a, b, c, d) := (a + b)_n (a + c)_n (a + d)_n {}_4F_3 \left(\begin{matrix} -n, n + a + b + c + d - 1, a + ix, a - ix \\ a + b, a + c, a + d \end{matrix} ; 1 \right),$$

where the parameters a, b, c, d are positive, or complex with positive real parts occurring in conjugate pairs. It is known that $W_n(x^2; a, b, c, d)$ is a symmetric polynomial in the parameters a, b, c, d . The Wilson polynomials occupy the highest echelon of the Askey scheme, and using limit relations and special parameter values it is possible to obtain many other hypergeometric orthogonal polynomials – see for example Chapter 9.1 of [6]. Sánchez-Ruiz and Dehesa (1999) [8, equation just below (15)] and others previously (see for instance Askey and Wilson (1985) [1]) have given a connection relation for the Wilson polynomials with one free parameter:

$$\begin{aligned} W_n(x^2; a, b, c, d) &= \sum_{k=0}^n \frac{n!}{k!(n-k)!} W_k(x^2; a, b, c, h) \\ &\times \frac{(n + a + b + c + d - 1)_k (d - h)_{n-k} (k + a + b)_{n-k} (k + a + c)_{n-k} (k + b + c)_{n-k}}{(k + a + b + c + h - 1)_k (2k + a + b + c + h)_{n-k}}. \end{aligned} \tag{2.1}$$

This is a special case of a more general identity given by Sánchez-Ruiz and Dehesa which gives the connection coefficients for the Wilson polynomials with three free parameters:

$$\begin{aligned} W_n(x^2; a, b, c, d) &= \sum_{k=0}^n \binom{n}{k} \frac{(n + a + b + c + d - 1)_k (k + a + b)_{n-k} (k + a + c)_{n-k} (k + a + d)_{n-k}}{(k + a + f + g + h - 1)_k} \\ &\times {}_5F_4 \left(\begin{matrix} k - n, k + n + a + b + c + d - 1, k + a + f, k + a + g, k + a + h \\ 2k + a + f + g + h, k + a + b, k + a + c, k + a + d \end{matrix} ; 1 \right) \\ &\times W_k(x^2; a, f, g, h). \end{aligned} \tag{2.2}$$

This identity, combined with limit relations, is useful for deriving connection coefficients for hypergeometric orthogonal polynomials lower down in the Askey scheme.

In the process of generalizing generating functions for these lower hypergeometric orthogonal polynomials, it may be necessary to rearrange the terms in a double or triple sum. In order to do so, we show the absolute convergence of that double sum. Therefore, we develop upper bounds for the quantities of interest. We rely on several useful bounds for Pochhammer symbols and factorials given by [4, (48)–(53)]. Let $j \in \mathbb{N}$, $k, n \in \mathbb{N}_0$, $\Re u > 0$ and $v \in \mathbb{C}$. Then

$$|(u)_j| \geq (\Re u)(j - 1)!, \tag{2.3}$$

$$\frac{|(v)_n|}{n!} \leq (1 + n)^{|v|}, \tag{2.4}$$

$$|(k + v)_{n-k}| \leq (1 + n)^{|v|} \frac{n!}{k!} \quad (k \leq n), \tag{2.5}$$

$$\left| \frac{(k + v)_n}{(k + u)_n} \right| \leq \max\{(\Re u)^{-1}, 1\}(1 + n)^{1+|v|}. \tag{2.6}$$

Theorem 2.1. *Let $\rho \in \mathbb{C}$, $|\rho| < 1$, $x \in (0, \infty)$, and a, b, c, d, h complex parameters with positive real parts, non-real parameters occurring in conjugate pairs among a, b, c, d and a, b, c, h . Then*

$$\begin{aligned} & {}_2F_1 \left(\begin{matrix} a + ix, c + ix \\ a + c \end{matrix} ; \rho \right) {}_2F_1 \left(\begin{matrix} b - ix, d - ix \\ b + d \end{matrix} ; \rho \right) = \sum_{k=0}^{\infty} \frac{(k + a + b + c + d - 1)_k}{(k + a + b + c + h - 1)_k (a + c)_k (b + d)_k k!} \\ & \times {}_4F_3 \left(\begin{matrix} d - h, 2k + a + b + c + d - 1, k + a + b, k + b + c \\ k + a + b + c + d - 1, 2k + a + b + c + h, k + b + d \end{matrix} ; \rho \right) \rho^k W_k(x^2; a, b, c, h). \end{aligned} \tag{2.7}$$

Proof. A generating function for the Wilson polynomials is given by [6, (9.1.13)], namely:

$${}_2F_1 \left(\begin{matrix} a + ix, c + ix \\ a + c \end{matrix} ; \rho \right) {}_2F_1 \left(\begin{matrix} b - ix, d - ix \\ b + d \end{matrix} ; \rho \right) = \sum_{n=0}^{\infty} \frac{W_n(x^2; a, b, c, d) \rho^n}{(a + c)_n (b + d)_n n!}. \tag{2.8}$$

Substituting (2.1) into (2.8) gives the double sum

$${}_2F_1 \left(\begin{matrix} a + ix, c + ix \\ a + c \end{matrix} ; \rho \right) {}_2F_1 \left(\begin{matrix} b - ix, d - ix \\ b + d \end{matrix} ; \rho \right) = \sum_{n=0}^{\infty} c_n \sum_{k=0}^n a_{n,k} W_k(x^2; a, b, c, h), \tag{2.9}$$

where

$$c_n = \frac{\rho^n}{(a + c)_n (b + d)_n n!},$$

and $a_{n,k}$ are the coefficients satisfying

$$W_n(x^2; a, b, c, d) = \sum_{k=0}^n a_{n,k} W_k(x^2; a, b, c, h). \tag{2.10}$$

In order to justify reversing the order of summation, we show that

$$\sum_{n=0}^{\infty} |c_n| \sum_{k=0}^n |a_{n,k}| |W_k(x^2; a, b, c, h)| < \infty.$$

Using (2.3), we see that

$$|c_n| \leq K_1 \frac{|\rho|^n (1+n)^2}{(n!)^3}, \tag{2.11}$$

where $K_1 = \max \{1, (\Re(a+c)\Re(b+d))^{-1}\}$. It follows from [4, (47) and (60)] that

$$\sum_{k=0}^n |a_{n,k}| |W_k(x^2; a, b, c, h)| \leq K_2 (n!)^3 (1+n)^{\sigma_2}, \tag{2.12}$$

where K_2 and σ_2 are positive constants independent of n . Combining (2.11) and (2.12), we see that

$$\sum_{n=0}^{\infty} |c_n| \sum_{k=0}^n |a_{n,k}| |W_k(x^2; a, b, c, h)| \leq K_1 K_2 \sum_{n=0}^{\infty} |\rho|^n (1+n)^{\sigma_2+2} < \infty$$

since $|\rho| < 1$. Reversing the summation, shifting the n summation index by k , and simplifying completes the proof. \square

Theorem 2.2. *Let $\rho \in \mathbb{C}$, $|\rho| < 1$, $x \in (0, \infty)$, and a, b, c, d, h complex parameters with positive real parts, non-real parameters occurring in conjugate pairs among a, b, c, d and a, b, c, h . Then*

$$\begin{aligned} & (1-\rho)^{1-a-b-c-d} {}_4F_3 \left(\begin{matrix} \frac{1}{2}(a+b+c+d-1), \frac{1}{2}(a+b+c+d), a+ix, a-ix \\ a+b, a+c, a+d \end{matrix} ; -\frac{4\rho}{(1-\rho)^2} \right) \\ &= \sum_{k=0}^{\infty} \frac{(k+a+b+c+d-1)_k (a+b+c+d-1)_k}{(k+a+b+c+h-1)_k (a+b)_k (a+c)_k (a+d)_k k!} \\ & \times {}_3F_2 \left(\begin{matrix} 2k+a+b+c+d-1, d-h, k+b+c \\ 2k+a+b+c+h, a+d+k \end{matrix} ; \rho \right) \rho^k W_k(x^2; a, b, c, h). \end{aligned} \tag{2.13}$$

Proof. Another generating function for the Wilson polynomials is given by [6, (9.1.15)]

$$\begin{aligned} & (1-\rho)^{1-a-b-c-d} {}_4F_3 \left(\begin{matrix} \frac{1}{2}(a+b+c+d-1), \frac{1}{2}(a+b+c+d), a+ix, a-ix \\ a+b, a+c, a+d \end{matrix} ; -\frac{4\rho}{(1-\rho)^2} \right) \\ &= \sum_{n=0}^{\infty} \frac{(a+b+c+d-1)_n}{(a+b)_n (a+c)_n (a+d)_n n!} W_n(x^2; a, b, c, d) \rho^n. \end{aligned} \tag{2.14}$$

It should be noted that $\rho \mapsto -\frac{4\rho}{(1-\rho)^2}$ maps the unit disk $|\rho| < 1$ bijectively onto the cut plane $\mathbb{C} \setminus [1, \infty)$, so the left-hand side of (2.14) is well-defined and analytic for $|\rho| < 1$.

Substituting (2.1) into (2.14) gives the double sum

$$\begin{aligned} & (1-\rho)^{1-a-b-c-d} {}_4F_3 \left(\begin{matrix} \frac{1}{2}(a+b+c+d-1), \frac{1}{2}(a+b+c+d), a+ix, a-ix \\ a+b, a+c, a+d \end{matrix} ; -\frac{4\rho}{(1-\rho)^2} \right) \\ &= \sum_{n=0}^{\infty} c_n \sum_{k=0}^n a_{n,k} W_k(x^2; a, b, c, h), \end{aligned} \tag{2.15}$$

where

$$c_n = \frac{(a + b + c + d - 1)_n \rho^n}{(a + b)_n (a + c)_n (a + d)_n n!},$$

and $a_{n,k}$ are the connection coefficients satisfying (2.10). We wish to show

$$\sum_{n=0}^{\infty} |c_n| \sum_{k=0}^n |a_{n,k}| |W_k(x^2; a, b, c, h)| < \infty. \tag{2.16}$$

By (2.4), we determine that

$$\left| \frac{(a + b + c + d - 1)_n}{n!} \right| \leq (1 + n)^{|a+b+c+d-1|},$$

and thus, by (2.3),

$$|c_n| \leq K_1 \frac{(1 + n)^{\sigma_1}}{(n!)^3}, \tag{2.17}$$

where $\sigma_1 = |a + b + c + d - 1| + 3$ and $K_1 = \max \{1, (\Re(a + b)\Re(a + c)\Re(a + d))^{-1}\}$.

Combining (2.12) and (2.17), we see that

$$\sum_{n=0}^{\infty} |c_n| \sum_{k=0}^n |a_{n,k}| |W_k(x^2; a, b, c, h)| \leq K_1 K_2 \sum_{n=0}^{\infty} (1 + n)^{\sigma_1 + \sigma_2} |\rho|^n < \infty$$

since $|\rho| < 1$. Swapping the sums, shifting the inner index, and simplifying gives the desired result. \square

3. Continuous dual Hahn polynomials

The continuous dual Hahn polynomials are defined by [6, (9.3.1)]

$$S_n(x^2; a, b, c) := (a + b)_n (a + c)_n {}_3F_2 \left(\begin{matrix} -n, a + ix, a - ix \\ a + b, a + c \end{matrix}; 1 \right),$$

where $a, b, c > 0$, except for possibly a pair of complex conjugates with positive real parts. It is known that $S_n(x^2; a, b, c)$ is a symmetric polynomial in the parameters a, b, c .

In order to generalize generating functions of the continuous dual Hahn polynomials, it is necessary to derive the connection coefficients for the continuous dual Hahn polynomials with two free parameters.

Lemma 3.1. *Let $x \in (0, \infty)$, and $a, b, c, f, g \in \mathbb{C}$ with positive real parts and non-real values appearing in conjugate pairs among a, b, c and a, f, g . Then*

$$\begin{aligned} S_n(x^2; a, b, c) &= \sum_{k=0}^n \binom{n}{k} (k + a + b)_{n-k} (k + a + c)_{n-k} \\ &\quad \times {}_3F_2 \left(\begin{matrix} k - n, k + a + f, k + a + g \\ k + a + b, k + a + c \end{matrix}; 1 \right) S_k(x^2; a, f, g). \end{aligned} \tag{3.1}$$

Proof. Letting $h \mapsto d$ in (2.2) gives

$$W_n(x^2; a, b, c, d) = \sum_{k=0}^n \binom{n}{k} \frac{(n + a + b + c + d - 1)_k (k + a + b)_{n-k} (k + a + c)_{n-k} (k + a + d)_{n-k}}{(k + a + f + g + d - 1)_k}$$

$$\begin{aligned} & \times {}_4F_3 \left(\begin{matrix} k - n, k + n + a + b + c + d - 1, k + a + f, k + a + g \\ 2k + a + f + g + d, k + a + b, k + a + c \end{matrix} ; 1 \right) \\ & \times W_k(x^2; a, f, g, d). \end{aligned} \tag{3.2}$$

The limit relation between the Wilson and continuous dual Hahn polynomials is given by [6, Section 9.3, Limit Relations]

$$\lim_{d \rightarrow \infty} \frac{W_n(x^2; a, b, c, d)}{(a + d)_n} = S_n(x^2; a, b, c).$$

We apply this limit relation to reduce the Wilson connection coefficients to those for the continuous dual Hahn polynomials. Dividing both sides of (3.2) by $(a + d)_n$, multiplying the right-hand side by $(a + d)_k / (a + d)_k$, and taking the limit as $d \rightarrow \infty$ gives the desired result. □

For the other two generating functions, we use a connection relation for the continuous dual Hahn polynomials with one free parameter. Let $a, b, c, d > 0$ except for possibly a pair of complex conjugates with positive real parts among a, b, c and a, b, d . Then

$$S_n(x^2; a, b, c) = \sum_{k=0}^n \binom{n}{k} (k + a + b)_{n-k} (c - d)_{n-k} S_k(x^2; a, b, d). \tag{3.3}$$

This relation follows by letting $f \mapsto b, g \mapsto d$ in (3.1), and applying the Chu–Vandermonde identity (see [7, (15.4.24)]) to the resulting hypergeometric function.

We also need the following bound on the continuous dual Hahn polynomials.

Lemma 3.2. *Let $x > 0$ and $a, b, c \in \mathbb{C}$ with positive real parts and non-real values occurring in conjugate pairs. Then, for $n \in \mathbb{N}_0$,*

$$|S_n(x^2; a, b, c)| \leq K(n!)^2(1 + n)^\sigma, \tag{3.4}$$

where K and σ are constants independent of n (and x).

Proof. The generating function [6, (9.3.12)]

$$(1 - \rho)^{-c+ix} {}_2F_1 \left(\begin{matrix} a + ix, b + ix \\ a + b \end{matrix} ; \rho \right) = \sum_{n=0}^\infty \frac{S_n(x^2; a, b, c)}{(a + b)_n n!} \rho^n \tag{3.5}$$

leads to the representation

$$S_n(x^2; a, b, c) = (a + b)_n n! \sum_{k=0}^n \frac{(a + ix)_k (b + ix)_k (c - ix)_{n-k}}{(a + b)_k k! (n - k)!}.$$

Straightforward estimation using (2.3) and (2.4) gives (3.4). □

Lemma 3.3. *Let $b, c, d, f \in \mathbb{C}$ with $\Re b > 0, \Re c > 0$. Then, for all $\rho \in \mathbb{C}$ with $|\rho| < 1$,*

$$(1 - \rho)^{d-c} {}_2F_1 \left(\begin{matrix} b - f, d \\ b \end{matrix} ; \rho \right) = \sum_{m=0}^\infty \frac{\rho^m}{m!} (c)_m {}_3F_2 \left(\begin{matrix} -m, d, f \\ b, c \end{matrix} ; 1 \right). \tag{3.6}$$

Proof. On the right-hand inside of (3.6), we substitute

$${}_3F_2 \left(\begin{matrix} -m, d, f \\ b, c \end{matrix}; 1 \right) = \sum_{\ell=0}^m (-1)^\ell \frac{m!}{(m-\ell)! \ell!} \frac{(d)_\ell (f)_\ell}{(b)_\ell (c)_\ell},$$

and reverse sums. The reversal of sums is justified provided $|\rho| < \frac{1}{2}$. Then we obtain

$$\sum_{m=0}^{\infty} \frac{\rho^m}{m!} (c)_m {}_3F_2 \left(\begin{matrix} -m, d, f \\ b, c \end{matrix}; 1 \right) = (1-\rho)^{-c} {}_2F_1 \left(\begin{matrix} d, f \\ b \end{matrix}; \frac{\rho}{\rho-1} \right).$$

Now [7, 15.8.1] yields (3.6) for $|\rho| < \frac{1}{2}$. Since the left-hand side of (3.6) is an analytic function in the unit disk $|\rho| < 1$ and the right-hand side is its Maclaurin expansion, we see that (3.6) is valid for $|\rho| < 1$. □

Theorem 3.4. Let $\rho \in \mathbb{C}$ with $|\rho| < 1$, $x \in (0, \infty)$ and $a, b, c, d, f > 0$ except for possibly pairs of complex conjugates with positive real parts among a, b, c and a, d, f . Then

$$(1-\rho)^{-d+ix} {}_2F_1 \left(\begin{matrix} a+ix, b+ix \\ a+b \end{matrix}; \rho \right) = \sum_{k=0}^{\infty} \frac{S_k(x^2; a, d, f) \rho^k}{(a+b)_k k!} {}_2F_1 \left(\begin{matrix} b-f, k+a+d \\ k+a+b \end{matrix}; \rho \right). \tag{3.7}$$

Proof. Substituting (3.1) for $S_n(x^2; a, b, c)$ in the generating function (3.5) yields

$$(1-\rho)^{-c+ix} {}_2F_1 \left(\begin{matrix} a+ix, b+ix \\ a+b \end{matrix}; \rho \right) = \sum_{n=0}^{\infty} c_n \sum_{k=0}^n a_{n,k} b_{n,k} S_k(x^2; a, d, f), \tag{3.8}$$

where

$$\begin{aligned} c_n &= \frac{\rho^n}{(a+b)_n n!}, \\ a_{n,k} &= \binom{n}{k} (k+a+b)_{n-k} (k+a+c)_{n-k}, \\ b_{n,k} &= {}_3F_2 \left(\begin{matrix} k-n, k+a+d, k+a+f \\ k+a+b, k+a+c \end{matrix}; 1 \right). \end{aligned}$$

We wish to reverse the order of summation so we show that

$$\sum_{n=0}^{\infty} |c_n| \sum_{k=0}^n |a_{n,k}| |b_{n,k}| |S_k(x^2; a, d, f)| < \infty. \tag{3.9}$$

Using (2.3) we find

$$|c_n| \leq K_1 \frac{(1+n)|\rho|^n}{(n!)^2}, \tag{3.10}$$

where $K_1 = \max\{1, (\Re(a+b))^{-1}\}$. Using (2.5) we determine

$$|a_{n,k}| \leq \binom{n}{k} (1+n)^{|a+b|+|a+c|} \frac{(n!)^2}{(k!)^2}.$$

Combining this with (3.4) yields

$$|a_{n,k}| |S_k(x^2; a, d, f)| \leq K_2 \binom{n}{k} (1+n)^{\sigma_2} (n!)^2, \tag{3.11}$$

where $\sigma_2 = |a + b| + |a + c| + \sigma$, $K_2 = K$ with σ and K defined as in (3.4).

By (2.6)

$$\begin{aligned} |b_{n,k}| &\leq \sum_{s=0}^{n-k} \binom{n-k}{s} \left| \frac{(k+a+d)_s (k+a+f)_s}{(k+a+b)_s (k+a+c)_s} \right| \\ &\leq \sum_{s=0}^{n-k} \binom{n-k}{s} K_3 (1+s)^{\sigma_3} \leq K_3 2^n (1+n)^{\sigma_3}, \end{aligned} \tag{3.12}$$

where $K_3 = \max\{\Re a^{-2}, 1\}$, $\sigma_3 = 2 + |a + d| + |a + f|$.

Combining (3.10), (3.11), (3.12), we find

$$\begin{aligned} \sum_{n=0}^{\infty} |c_n| \sum_{k=0}^n |a_{n,k}| |b_{n,k}| |S_k| &\leq K_1 K_2 K_3 \sum_{n=0}^{\infty} (1+n)^{1+\sigma_2+\sigma_3} |\rho|^n 2^n \sum_{k=0}^n \binom{n}{k} \\ &= K_1 K_2 K_3 \sum_{n=0}^{\infty} (1+n)^{1+\sigma_2+\sigma_3} |\rho|^n 4^n. \end{aligned}$$

Therefore, condition (3.9) is satisfied provided that $|\rho| < \frac{1}{4}$. When we reverse sums in (3.8), we obtain

$$\begin{aligned} (1-\rho)^{-c+ix} {}_2F_1 \left(\begin{matrix} a+ix, b+ix \\ a+b \end{matrix}; \rho \right) \\ = \sum_{k=0}^{\infty} \frac{\rho^k}{k!(a+b)_k} S_k(x^2; a, d, f) \sum_{m=0}^{\infty} \frac{\rho^m}{m!} (k+a+c)_m {}_3F_2 \left(\begin{matrix} -m, k+a+d, k+a+f \\ k+a+b, k+a+c \end{matrix}; 1 \right). \end{aligned}$$

We now use Lemma 3.3 and obtain (3.7) for $|\rho| < \frac{1}{4}$. Using (2.6) one can show that the right-hand side of (3.7) converges locally uniformly for $|\rho| < 1$, so by analytic continuation, (3.7) holds for all $\rho \in \mathbb{C}$ with $|\rho| < 1$. □

Theorem 3.5. *Let $\rho \in \mathbb{C}$, $x \in (0, \infty)$, and $a, b, c, d > 0$ except for possibly a pair of complex conjugates with positive real parts among a, b, c and a, b, d . Then*

$$e^\rho {}_2F_2 \left(\begin{matrix} a+ix, a-ix \\ a+b, a+c \end{matrix}; -\rho \right) = \sum_{k=0}^{\infty} \frac{\rho^k S_k(x^2; a, b, d)}{(a+b)_k (a+c)_k k!} {}_1F_1 \left(\begin{matrix} c-d \\ k+a+c \end{matrix}; \rho \right). \tag{3.13}$$

Proof. Another generating function for the continuous dual Hahn polynomials is given by [6, (9.3.15)]

$$e^\rho {}_2F_2 \left(\begin{matrix} a+ix, a-ix \\ a+b, a+c \end{matrix}; -\rho \right) = \sum_{n=0}^{\infty} \frac{S_n(x^2; a, b, c)}{(a+b)_n (a+c)_n n!} \rho^n. \tag{3.14}$$

We substitute the term $S_n(x^2; a, b, c)$ using (3.3), which gives

$$e^\rho {}_2F_2 \left(\begin{matrix} a+ix, a-ix \\ a+b, a+c \end{matrix}; -\rho \right) = \sum_{n=0}^{\infty} c_n \sum_{k=0}^n a_{n,k} S_k(x^2; a, b, d), \tag{3.15}$$

where

$$c_n = \frac{\rho^n}{(a+b)_n(a+c)_n n!},$$

$$a_{n,k} = \binom{n}{k} (k+a+b)_{n-k} (c-d)_{n-k}. \tag{3.16}$$

We wish to reverse the order of summation, so we show

$$\sum_{n=0}^{\infty} |c_n| \sum_{k=0}^n |a_{n,k}| |S_k(x^2; a, b, d)| < \infty.$$

Using (2.3), we determine

$$|c_n| \leq K_1 \frac{(1+n)^{\sigma_1} |\rho|^n}{(n!)^3}, \tag{3.17}$$

where $\sigma_1 = 2$, $K_1 = \max\{1, (\Re a)^{-2}\}$. Using (2.4) and (2.5), we determine

$$|a_{n,k}| \leq \frac{(n!)^2}{(k!)^2} (1+n)^{\sigma_2}, \tag{3.18}$$

where $\sigma_2 = |c-d| + |a+b|$. Combining this with (3.4), we see that

$$\sum_{k=0}^n |a_{n,k} S_k(x^2; a, b, d)| \leq \sum_{k=0}^n K(n!)^2 (1+n)^{\sigma_2+\sigma} = K(1+n)^{\sigma_2+\sigma+1} (n!)^2. \tag{3.19}$$

Combining (3.17) and (3.19), we see that

$$\sum_{n=0}^{\infty} |c_n| \sum_{k=0}^n |a_{n,k} S_k(x^2; a, b, d)| \leq K_1 K \sum_{n=0}^{\infty} \frac{(1+n)^{\sigma_1+\sigma_2+\sigma+1} |\rho|^n}{n!} < \infty$$

for any $\rho \in \mathbb{C}$. Reversing the order of summation in (3.15), shifting the n index by k , and simplifying gives the desired result. \square

Theorem 3.6. *Let $\rho \in \mathbb{C}$ with $|\rho| < 1$, $x \in (0, \infty)$, $\gamma \in \mathbb{C}$ and $a, b, c, d > 0$ except for possibly a pair of complex conjugates with positive real parts among a, b, c and a, b, d . Then*

$$(1-\rho)^{-\gamma} {}_3F_2 \left(\begin{matrix} \gamma, a+ix, a-ix \\ a+b, a+c \end{matrix}; \frac{\rho}{\rho-1} \right) = \sum_{n=0}^{\infty} \frac{(\gamma)_k \rho^k}{(a+b)_k (a+c)_k k!}$$

$$\times {}_2F_1 \left(\begin{matrix} -d, \gamma+k \\ k+a+c \end{matrix}; \rho \right) S_k(x^2; a, b, d). \tag{3.20}$$

Proof. Yet another generating function for the continuous dual Hahn polynomials [6, (9.3.16)] is

$$(1-\rho)^{-\gamma} {}_3F_2 \left(\begin{matrix} \gamma, a+ix, a-ix \\ a+b, a+c \end{matrix}; \frac{\rho}{\rho-1} \right) = \sum_{n=0}^{\infty} \frac{(\gamma)_n S_n(x^2; a, b, c)}{(a+b)_n (a+c)_n n!} \rho^n. \tag{3.21}$$

Substituting (3.3) into (3.21) yields the double sum

$$(1 - \rho)^{-\gamma} {}_3F_2 \left(\begin{matrix} \gamma, a + ix, a - ix \\ a + b, a + c \end{matrix}; \frac{\rho}{\rho - 1} \right) = \sum_{n=0}^{\infty} c_n \sum_{k=0}^n a_{n,k} S_k(x^2; a, b, d), \tag{3.22}$$

where

$$c_n = \frac{(\gamma)_n \rho^n}{(a + b)_n (a + c)_n n!},$$

and $a_{n,k}$ is defined by (3.16). By (2.3), (2.4), we obtain

$$|c_n| \leq K_1 \frac{(1 + n)^{\sigma_1} |\rho|^n}{(n!)^2}, \tag{3.23}$$

where $\sigma_1 = 2 + |\gamma|$ and $K_1 = \max \{1, (\Re a)^{-2}\}$.

We recall the bound on $a_{n,k} S_k(x^2; a, b, d)$ given by (3.19). Using this, we have shown

$$\sum_{n=0}^{\infty} |c_n| \sum_{k=0}^n |a_{n,k}| |S_k(x^2; a, b, d)| \leq K_1 K \sum_{n=0}^{\infty} (1 + n)^{\sigma_1 + \sigma_2 + \sigma + 2} |\rho|^n < \infty,$$

since $|\rho| < 1$. Reversing the summation, shifting the n index by k , and simplifying completes the proof. \square

4. Continuous Hahn polynomials

The continuous Hahn polynomial with two pairs of conjugate parameter are defined by [6, (9.4.1)]

$$p_n(x; a, b, \bar{a}, \bar{b}) := i^n \frac{(2\Re a)_n (a + \bar{b})_n}{n!} {}_3F_2 \left(\begin{matrix} -n, n + 2\Re a + 2\Re b - 1, a + ix \\ 2\Re a, a + \bar{b} \end{matrix}; 1 \right),$$

where a and b are complex numbers with positive real parts. The continuous Hahn polynomials are symmetric in a, b . The limit relation between the continuous Hahn and Wilson polynomials is given by [6, Section 9.4, Limit Relations]

$$\lim_{t \rightarrow \infty} \frac{W_n((x + t)^2; a - it, b - it, \bar{a} + it, \bar{b} + it)}{(-2t)^n n!} = p_n(x; a, b, \bar{a}, \bar{b}). \tag{4.1}$$

In the following results for continuous Hahn polynomials, we assume the restriction $\Im a = \Im b = \Im c$. This is because in the general case, one cannot transform the ${}_3F_2$ in the connection relation with one free parameter (see (4.3) below) to a simple product of gamma functions. The case $\Im a = \Im b = \Im c$ is the most general complex solution to the problem of obtaining a generalized generating function for continuous Hahn polynomials using a connection relation with one free parameter. It is interesting to note however that the case $\Im a = \Im b = \Im c$ can be reduced to the case $\Im a = \Im b = \Im c = 0$, because the change of $a, b, c > 0$ to $a + ih, b + ih, c + ih$ only leads to a shift in the x -variables of polynomials involved.

Theorem 4.1. *Let $a, b, c \in \mathbb{C}$ such that $\Re a > 0, \Re b > 0, \Re c > 0$ and $\Im a = \Im b = \Im c$. Then*

$$p_n(x; a, b, \bar{a}, \bar{b}) = \sum_{k=0}^n a_{n,k} p_k(x; a, c, \bar{a}, \bar{c}), \tag{4.2}$$

where $a_{n,k} = 0$ if $n - k$ is odd and, if $n = k + 2p$, $p \in \mathbb{N}_0$,

$$a_{n,k} = \frac{(-1)^p (n + 2\Re a + 2\Re b - 1)_k (k + 2\Re a)_{n-k} (k + a + \bar{b})_{n-k} (b - c)_p}{4^p p! (k + 2\Re a + 2\Re c - 1)_k (k + a + \bar{c} + \frac{1}{2})_p (a + \bar{b} + k)_p}.$$

Proof. Let $a \mapsto a - it$, $b \mapsto b - it$, $c \mapsto \bar{a} + it$, $d \mapsto \bar{b} + it$, $f \mapsto c - it$, $h \mapsto \bar{c} + it$, and $x \mapsto x + t$ in (2.2). Divide both sides by $(-2t)^n n!$, multiply the right-hand side by $(-2t)^k k! / (-2t)^k k!$, and take the limit as $t \rightarrow \infty$. This yields connection coefficients for the continuous Hahn polynomials with one free parameter

$$p_n(x; a, b, \bar{a}, \bar{b}) = \sum_{k=0}^n \frac{(n + 2\Re a + 2\Re b - 1)_k (k + 2\Re a)_{n-k} (k + a + \bar{b})_{n-k}}{(n - k)! (k + 2\Re a + 2\Re c - 1)_k} i^{n-k} p_k(x; a, c, \bar{a}, \bar{c}) \times {}_3F_2 \left(\begin{matrix} k - n, k + n + 2\Re a + 2\Re b - 1, k + a + \bar{c} \\ 2k + 2\Re a + 2\Re c, k + a + \bar{b} \end{matrix}; 1 \right). \tag{4.3}$$

We further reduce this using Whipple’s sum [7, (16.4.7)]

$${}_3F_2 \left(\begin{matrix} a', b', c' \\ \frac{1}{2}(a' + b' + 1), 2c' \end{matrix}; 1 \right) = \frac{\sqrt{\pi} \Gamma(c' + \frac{1}{2}) \Gamma(\frac{1}{2}(a' + b' + 1)) \Gamma(c' + \frac{1}{2}(1 - a' - b'))}{\Gamma(\frac{1}{2}(a' + 1)) \Gamma(\frac{1}{2}(b' + 1)) \Gamma(c' + \frac{1}{2}(1 - a')) \Gamma(c' + \frac{1}{2}(1 - b'))}. \tag{4.4}$$

To apply (4.4) in (4.3), we assume that $\Im a = \Im b = \Im c$. Then the hypergeometric series may be rewritten as

$$\begin{aligned} & {}_3F_2 \left(\begin{matrix} k - n, k + n + 2\Re a + 2\Re b - 1, k + a + \bar{c} \\ 2(\Re a + \Re c + k), a + \bar{b} + k \end{matrix}; 1 \right) \\ &= {}_3F_2 \left(\begin{matrix} k - n, k + n + 2\Re a + 2\Re b - 1, k + \Re a + \Re c \\ 2(\Re a + \Re c + k), \Re a + \Re b + k \end{matrix}; 1 \right). \end{aligned}$$

Setting $a' = k - n$, $b' = k + n + 2\Re a + 2\Re b - 1$, and $c' = k + \Re a + \Re c$, we apply (4.4) and determine

$$\begin{aligned} & {}_3F_2 \left(\begin{matrix} k - n, k + n + 2\Re a + 2\Re b - 1, k + a + \bar{c} \\ 2(\Re a + \Re c + k), k + a + \bar{b} \end{matrix}; 1 \right) = \frac{\sqrt{\pi} \Gamma(k + \Re a + \Re c + \frac{1}{2})}{\Gamma(\frac{1}{2}(k - n + 1)) \Gamma(\frac{1}{2}(k + n + 2\Re a + 2\Re b))} \\ & \times \frac{\Gamma(\Re a + \Re b + k) \Gamma(\Re c - \Re b + 1)}{\Gamma(\frac{1}{2}(k + n + 2\Re a + 2\Re c + 1)) \Gamma(\frac{1}{2}(k - n + 2 + 2\Re c - 2\Re b))}. \end{aligned} \tag{4.5}$$

It follows from (4.3), (4.5) that $a_{n,k} = 0$ if $n - k$ is odd. If $n = k + 2p$ with $p \in \mathbb{N}_0$, then

$${}_3F_2 \left(\begin{matrix} k - n, k + n + 2\Re a + 2\Re b - 1, k + a + \bar{c} \\ 2(\Re a + \Re c + k), k + a + \bar{b} \end{matrix}; 1 \right) = \frac{(b - c)_p (2p)!}{(k + a + \bar{c} + \frac{1}{2})_p (a + \bar{b} + k)_p p! 4^p}. \tag{4.6}$$

Substituting (4.6) into (4.3) with simplification completes the proof. \square

We derive a bound for the continuous Hahn polynomials.

Lemma 4.2. *Let $a, b \in \mathbb{C}$ with positive real part, and $x \in \mathbb{R}$. Then, for all $n \in \mathbb{N}_0$,*

$$|p_n(x; a, b, \bar{a}, \bar{b})| \leq n!(1 + n)^\sigma, \tag{4.7}$$

where σ is a constant independent of n .

Proof. Wilson (1991) [9, (2.2)] showed that

$$W_n(x^2; a, b, c, d) = n! \sum_{k=0}^n u_k(ix) u_{n-k}(-ix) \frac{2ix - n + 2k}{2ix}, \tag{4.8}$$

where

$$u_k(x) := \frac{(a+x)_k (b+x)_k (c+x)_k (d+x)_k}{k!(1+2x)_k}.$$

Applying the limit relation (4.1) to (4.8), we find

$$p_n(x; a, b, \bar{a}, \bar{b}) = i^n \sum_{k=0}^n (-1)^k \frac{(a+ix)_k (b+ix)_k (\bar{a}-ix)_{n-k} (\bar{b}-ix)_{n-k}}{k!(n-k)!}. \tag{4.9}$$

Using (2.4), we obtain

$$\begin{aligned} |p_n(x; a, b, \bar{a}, \bar{b})| &\leq \sum_{k=0}^n (1+k)^\sigma k!(1+n-k)^\sigma (n-k)! \\ &\leq (1+n)^{2\sigma} \sum_{k=0}^n k!(n-k)! \leq (1+n)^{2\sigma+1} n!, \end{aligned}$$

where $\sigma = |a+ix| + |b+ix|$. \square

Theorem 4.3. Let $\rho \in \mathbb{C}$, $x \in \mathbb{R}$, and $a, b, c \in \mathbb{C}$ such that $\Re a > 0$, $\Re b > 0$, $\Re c > 0$ and $\Im a = \Im b = \Im c$. Then

$$\begin{aligned} {}_1F_1 \left(\begin{matrix} a+ix \\ 2\Re a \end{matrix} ; -i\rho \right) {}_1F_1 \left(\begin{matrix} \bar{b}-ix \\ 2\Re b \end{matrix} ; i\rho \right) &= \sum_{k=0}^{\infty} \frac{(k+2\Re a+2\Re b-1)_k}{(2\Re a)_k (2\Re b)_k (k+2\Re a+2\Re c-1)_k} \rho^k p_k(x, a, c, \bar{a}, \bar{c}) \\ &\times {}_4F_5 \left(\begin{matrix} \frac{1}{2}(\Re a + \Re b + k), \frac{1}{2}(\Re a + \Re b + k + 1), \Re a + \Re b + k - \frac{1}{2}, \Re b - \Re c \\ \Re a + \Re b + \frac{k-1}{2}, \Re a + \Re b + \frac{k}{2}, \Re b + \frac{k}{2}, \Re b + \frac{k+1}{2}, \Re a + \Re c + k + \frac{1}{2} \end{matrix} ; \frac{-\rho^2}{4} \right). \end{aligned}$$

Proof. A generating function for the continuous Hahn polynomials is given by [6, (9.4.11)]

$${}_1F_1 \left(\begin{matrix} a+ix \\ 2\Re a \end{matrix} ; -i\rho \right) {}_1F_1 \left(\begin{matrix} \bar{b}-ix \\ 2\Re b \end{matrix} ; i\rho \right) = \sum_{n=0}^{\infty} \frac{p_n(x; a, b, \bar{a}, \bar{b})}{(2\Re a)_n (2\Re b)_n} \rho^n. \tag{4.10}$$

Substituting (4.2) in (4.10) gives the double sum

$${}_1F_1 \left(\begin{matrix} a+ix \\ 2\Re a \end{matrix} ; -i\rho \right) {}_1F_1 \left(\begin{matrix} \bar{b}-ix \\ 2\Re b \end{matrix} ; i\rho \right) = \sum_{n=0}^{\infty} c_n \sum_{k=0}^n a_{n,k} p_k(x; a, c, \bar{a}, \bar{c}),$$

where

$$c_n = \frac{\rho^n}{(2\Re a)_n (2\Re b)_n},$$

and the $a_{n,k}$ are the coefficients satisfying (4.2).

We wish to reverse the order of summation, so we need to show that

$$\sum_{n=0}^{\infty} |c_n| \sum_{k=0}^n |a_{n,k}| |p_k(x; a, c, \bar{a}, \bar{c})| < \infty.$$

Using (2.3), we determine

$$|c_n| \leq K_1 \frac{(1+n)^2 |\rho|^n}{(n!)^2},$$

where $K_1 = \max\{1, (4\Re a \Re b)^{-1}\}$. Using the bounds (2.3), (2.4), (2.5), (2.6), we estimate

$$|a_{n,k}| \leq K_2 (1+n)^{\sigma_2} \frac{n! k! 4^k}{(2k)!}, \tag{4.11}$$

where we used that $\binom{2m}{m} \leq 4^m$ with $m = p + k$. By these estimates and (4.7),

$$\begin{aligned} \sum_{n=0}^{\infty} |c_n| \sum_{k=0}^n |a_{n,k} p_k| &\leq K_1 K_2 \sum_{n=0}^{\infty} (1+n)^{\sigma_2+2} \frac{|\rho|^n}{n!} \sum_{k=0}^n \frac{4^k (k!)^2}{(2k)!} \\ &\leq K_3 \sum_{n=0}^{\infty} (1+n)^{\sigma_2+7/2} \frac{|\rho|^n}{n!} < \infty \end{aligned}$$

for any $\rho \in \mathbb{C}$, where we used that the sequence $4^{-k} \binom{2k}{k} \sqrt{k}$ converges to $\pi^{-1/2}$.

Therefore, we are justified to reverse summation in the double sum, and we obtain

$$\sum_{n=0}^{\infty} c_n \sum_{k=0}^n a_{n,k} p_k(x; a, c, \bar{a}, \bar{c}) = \sum_{k=0}^{\infty} p_k(x; a, c, \bar{a}, \bar{c}) \sum_{m=0}^{\infty} c_{m+k} a_{m+k,k}.$$

Since $a_{m+k,k} = 0$ for odd m , we may set $m = 2p$. Then using $a_{2p+k,k}$ as given by Theorem 4.1, we obtain the desired result after some simplification. \square

Theorem 4.4. Let $\rho \in \mathbb{C}$, $|\rho| < 1$, $x \in \mathbb{R}$, and $a, b, c \in \mathbb{C}$ such that $\Re a > 0$, $\Re b > 0$, $\Re c > 0$ and $\Im a = \Im b = \Im c$. Then

$$\begin{aligned} &(1-\rho)^{1-2\Re a-2\Re b} {}_3F_2 \left(\begin{matrix} \Re a + \Re b - \frac{1}{2}, \Re a + \Re b, a + ix \\ 2\Re a, a + \bar{b} \end{matrix}; -\frac{4\rho}{(1-\rho)^2} \right) \\ &= \sum_{k=0}^{\infty} \frac{(2\Re a + 2\Re b - 1)_{2k}}{(2\Re a)_k (\Re a + \Re b)_k (2\Re a + 2\Re c + k - 1)_k} {}_2F_1 \left(\begin{matrix} \Re a + \Re b + k - \frac{1}{2}, \Re b - \Re c \\ \Re a + \Re c + k + \frac{1}{2} \end{matrix}; \rho^2 \right) \\ &\quad \times (-i\rho)^k p_k(x; a, c, \bar{a}, \bar{c}). \end{aligned} \tag{4.12}$$

Proof. Another generating function for the continuous Hahn polynomials is given by [6, (9.4.13)]

$$\begin{aligned} &(1-\rho)^{1-2\Re a-2\Re b} {}_3F_2 \left(\begin{matrix} \Re a + \Re b - \frac{1}{2}, \Re a + \Re b, a + ix \\ 2\Re a, a + \bar{b} \end{matrix}; -\frac{4\rho}{(1-\rho)^2} \right) \\ &= \sum_{n=0}^{\infty} \frac{(2\Re a + 2\Re b - 1)_n}{(2\Re a)_n (a + \bar{b})_n i^n} p_n(x; a, b, \bar{a}, \bar{b}) \rho^n. \end{aligned} \tag{4.13}$$

Note that the left-hand side of (4.13) is an analytic function of ρ in the unit disk $|\rho| < 1$, and (4.13) is valid for $|\rho| < 1$. We substitute $p_n(x; a, b, \bar{a}, \bar{b})$ in the generating function (4.13) using (4.2). This gives the double sum

$$(1 - \rho)^{1-2\Re a-2\Re b} {}_3F_2 \left(\begin{matrix} \Re a + \Re b - \frac{1}{2}, \Re a + \Re b, a + ix \\ 2\Re a, a + \bar{b} \end{matrix}; -\frac{4\rho}{(1 - \rho)^2} \right) = \sum_{n=0}^{\infty} c_n \sum_{k=0}^n a_{n,k} p_k(x; a, c, \bar{a}, \bar{c}),$$

where

$$c_n = \frac{(2\Re a + 2\Re b - 1)_n}{(2\Re a)_n (a + \bar{b})_n} \rho^n,$$

and $a_{n,k}$ are the coefficients from (4.2). Using (4.7) and (4.11) we show that

$$\sum_{n=0}^{\infty} |c_n| \sum_{k=0}^n |a_{n,k}| |p_k(x; a, c, \bar{a}, \bar{c})| \leq K_4 \sum_{n=0}^{\infty} (1 + n)^{\sigma+\sigma_3+7/2} |\rho|^n < \infty$$

provided that $|\rho| < 1$. Using (4.5) and manipulations as in the proof of Theorem 4.3 gives the desired result (4.12). \square

5. Meixner–Pollaczek polynomials

The Meixner–Pollaczek polynomials are defined as [6, (9.7.1)]

$$P_n^{(\lambda)}(x; \phi) := \frac{(2\lambda)_n}{n!} e^{in\phi} {}_2F_1 \left(\begin{matrix} -n, \lambda + ix \\ 2\lambda \end{matrix}; 1 - e^{-2i\phi} \right). \tag{5.1}$$

We obtain a connection relation for the Meixner–Pollaczek polynomials with one free parameter.

Lemma 5.1. *Let $\lambda > 0, \phi, \psi \in (0, \pi)$. Then*

$$P_n^{(\lambda)}(x; \phi) = \frac{1}{\sin^n \psi} \sum_{k=0}^n \frac{(2\lambda + k)_{n-k}}{(n - k)!} \sin^k \phi \sin^{n-k}(\psi - \phi) P_k^{(\lambda)}(x; \psi). \tag{5.2}$$

Proof. The Meixner–Pollaczek polynomials are obtained from the continuous dual Hahn polynomials via the limit relation [6, Section 9.7, Limit Relations]

$$\lim_{t \rightarrow \infty} \frac{S_n((x - t)^2; \lambda + it, \lambda - it, t \cot \phi)}{\left(\frac{t}{\sin \phi}\right)_n n!} = P_n^{(\lambda)}(x; \phi).$$

Letting $x \mapsto x - t, a \mapsto \lambda + it, b \mapsto \lambda - it, c \mapsto \phi$, and $d \mapsto \psi$ in (3.3) gives

$$\begin{aligned} S_n((x - t)^2; \lambda + it, \lambda - it, t \cot \phi) &= \sum_{k=0}^n \binom{n}{k} (k + 2\lambda)_{n-k} (t(\cot \phi - \cot \psi))_{n-k} \\ &\quad \times S_k((x - t)^2; \lambda + it, \lambda - it, t \cot \psi). \end{aligned}$$

Dividing both sides of this equation by $\left(\frac{t}{\sin \phi}\right)_n$ and multiplying the right-hand side by $\frac{\left(\frac{t}{\sin \psi}\right)_k}{\left(\frac{t}{\sin \psi}\right)_k}$ and taking the limit as $t \rightarrow \infty$ gives the desired result. \square

We now derive an upper bound for the Meixner–Pollaczek polynomials.

Lemma 5.2. *Let $\lambda > 0$, $\phi \in (0, \pi)$ and $x \in \mathbb{R}$. Then*

$$|P_n^{(\lambda)}(x; \phi)| \leq (1 + n)^\sigma, \tag{5.3}$$

where σ is a constant independent of n .

Proof. The generating function [6, (9.7.11)]

$$(1 - e^{i\phi}\rho)^{-\lambda+ix}(1 - e^{-i\phi}\rho)^{-\lambda-ix} = \sum_{n=0}^{\infty} P_n^{(\lambda)}(x; \phi)\rho^n \tag{5.4}$$

leads to the representation

$$P_n^{(\lambda)}(x; \phi) = \sum_{k=0}^n \frac{(\lambda - ix)_k (\lambda + ix)_{n-k}}{k! (n - k)!} e^{i\phi(2k-n)}.$$

Straightforward estimation using (2.4) gives (5.3) with $\sigma = 2|\lambda + ix| + 1$. \square

Theorem 5.3. *Let $\lambda > 0$, $\psi, \phi \in (0, \pi)$, $x \in \mathbb{R}$, and $\rho \in \mathbb{C}$ such that*

$$|\rho|(\sin \phi + |\sin(\psi - \phi)|) < \sin \psi. \tag{5.5}$$

Then

$$(1 - e^{i\phi}\rho)^{-\lambda+ix}(1 - e^{-i\phi}\rho)^{-\lambda-ix} = \left(1 - \rho \frac{\sin(\psi - \phi)}{\sin \psi}\right)^{-2\lambda} \sum_{k=0}^{\infty} P_k^{(\lambda)}(x; \psi)\tilde{\rho}^k, \tag{5.6}$$

where

$$\tilde{\rho} = \frac{\rho \sin \phi}{\sin \psi - \rho \sin(\psi - \phi)}.$$

Proof. We apply (5.2) to the generating function (5.4). This yields a double sum

$$(1 - e^{i\phi}\rho)^{-\lambda+ix}(1 - e^{-i\phi}\rho)^{-\lambda-ix} = \sum_{n=0}^{\infty} \rho^n \sum_{k=0}^n a_{n,k} P_k^{(\lambda)}(x; \psi),$$

where $a_{n,k}$ are the coefficients satisfying

$$P_n^{(\lambda)}(x; \phi) = \sum_{k=0}^n a_{n,k} P_k^{(\lambda)}(x; \psi) \tag{5.7}$$

given explicitly in (5.2). In order to justify reversal of summation, we show

$$\sum_{n=0}^{\infty} |\rho|^n \sum_{k=0}^n |a_{n,k}| |P_k^{(\lambda)}(x; \psi)| < \infty.$$

Using (2.5) we determine

$$|a_{n,k}| \leq \frac{(1+n)^{2\lambda}}{\sin^n \psi} \sum_{k=0}^n \binom{n}{k} \sin^k \phi |\sin(\psi - \phi)|^{n-k} = \frac{(1+n)^{2\lambda}}{\sin^n \psi} (\sin \phi + |\sin(\psi - \phi)|)^n. \tag{5.8}$$

Combining (5.8) with (5.3), we see that

$$\sum_{n=0}^{\infty} |\rho|^n \sum_{k=0}^n |a_{n,k}| |P_k^{(\lambda)}(x; \psi)| \leq \sum_{n=0}^{\infty} (1+n)^{2\lambda+\sigma+1} \frac{|\rho|^n}{\sin^n \psi} (\sin \phi + |\sin(\psi - \phi)|)^n < \infty$$

by assumption on ρ . Reversing the summation and simplifying gives the desired result. \square

In view of (5.4), (5.6) is equivalent to the identity

$$(1 - e^{i\phi} \rho)^{-\lambda+ix} (1 - e^{-i\phi} \rho)^{-\lambda-ix} = \left(1 - \rho \frac{\sin(\psi - \phi)}{\sin \psi}\right)^{-2\lambda} (1 - e^{i\psi} \tilde{\rho})^{-\lambda+ix} (1 - e^{-i\psi} \tilde{\rho})^{-\lambda-ix}. \tag{5.9}$$

Actually, (5.9) can be verified by elementary but not completely trivial calculations. Nevertheless, (5.6) is a useful formula. Since it can be derived independently of the connection formula (5.2) it may be used to give a second proof of Lemma 5.1.

Theorem 5.4. *Let $\lambda > 0$, $\rho \in \mathbb{C}$ and $\psi, \phi \in (0, \pi)$. Then*

$$e^{\rho} {}_1F_1 \left(\begin{matrix} \lambda + ix \\ 2\lambda \end{matrix}; (e^{-2i\phi} - 1)\rho \right) = \exp \left(\frac{\rho e^{-i\phi} \sin(\psi - \phi)}{\sin \psi} \right) \sum_{k=0}^{\infty} \left(\frac{\sin^k \phi}{\sin^k \psi} \right) \frac{P_k^{(\lambda)}(x; \psi)}{(2\lambda)_k e^{ik\phi}} \rho^k. \tag{5.10}$$

Proof. A generating function for the Meixner–Pollaczek polynomials is given by [6, (9.7.12)]

$$e^{\rho} {}_1F_1 \left(\begin{matrix} \lambda + ix \\ 2\lambda \end{matrix}; (e^{-2i\phi} - 1)\rho \right) = \sum_{n=0}^{\infty} \frac{P_n^{(\lambda)}(x; \phi)}{(2\lambda)_n e^{in\phi}} \rho^n. \tag{5.11}$$

To this generating function, we apply (5.2). This yields a double sum

$$e^{\rho} {}_1F_1 \left(\begin{matrix} \lambda + ix \\ 2\lambda \end{matrix}; (e^{-2i\phi} - 1)\rho \right) = \sum_{n=0}^{\infty} c_n \sum_{k=0}^n a_{n,k} P_k^{(\lambda)}(x; \psi), \tag{5.12}$$

where

$$c_n = \frac{\rho^n}{(2\lambda)_n e^{in\phi}}$$

and $a_{n,k}$ are the coefficients satisfying (5.7).

We bound $|c_n|$ using (2.3)

$$|c_n| \leq K_1 \frac{(1+n)|\rho|^n}{n!}, \tag{5.13}$$

where $K_1 = \max\{1, (2\lambda)^{-1}\}$. We use (5.3), (5.8) and (5.13) to show

$$\sum_{n=0}^{\infty} |c_n| \sum_{k=0}^n |a_{n,k}| |P_k^{(\lambda)}(x; \psi)| \leq K_1 \sum_{n=0}^{\infty} \frac{(1+n)^{2\lambda+\sigma+1}}{n!} \left(\frac{2|\rho|}{\sin \psi} \right)^n < \infty$$

which justifies reversal of the sum in (5.12). Reversing the summation and simplifying gives the desired result. \square

Theorem 5.5. *Let $\lambda > 0, \gamma, \rho \in \mathbb{C}$ and $\psi, \phi \in (0, \pi)$ be such that (5.5) holds. Then*

$$\begin{aligned} & (1 - \rho)^{-\gamma} {}_2F_1 \left(\gamma, \lambda + ix; \frac{(1 - e^{-2i\phi})\rho}{\rho - 1} \right) \\ &= \left(1 - \frac{\rho \sin(\psi - \phi)}{e^{i\phi} \sin \psi} \right)^{-\gamma} \sum_{k=0}^{\infty} \left(\frac{\sin^k \phi}{\sin^k \psi} \right) \left(1 - \frac{\rho \sin(\psi - \phi)}{e^{i\phi} \sin \psi} \right)^{-k} \frac{(\gamma)_k P_k^{(\lambda)}(x; \psi)}{(2\lambda)_k e^{ik\phi}} \rho^k. \end{aligned} \tag{5.14}$$

Proof. A generating function for the Meixner–Pollaczek polynomials is given by [6, (9.7.13)]

$$(1 - \rho)^{-\gamma} {}_2F_1 \left(\gamma, \lambda + ix; \frac{(1 - e^{-2i\phi})\rho}{\rho - 1} \right) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{(2\lambda)_n} \frac{P_n^{(\lambda)}(x; \phi)}{e^{in\phi}} \rho^n. \tag{5.15}$$

We substitute (5.2) for $P_n^{(\lambda)}(x; \phi)$ in this generating function. This yields the double sum

$$(1 - \rho)^{-\gamma} {}_2F_1 \left(\gamma, \lambda + ix; \frac{(1 - e^{-2i\phi})\rho}{\rho - 1} \right) = \sum_{n=0}^{\infty} c_n \sum_{k=0}^n a_{n,k} P_n^{(\lambda)}(x; \psi),$$

where

$$c_n = \frac{(\gamma)_n \rho^n}{(2\lambda)_n e^{in\phi}},$$

and $a_{n,k}$ are the coefficients satisfying (5.7). In order to reverse the order of summation, we show that

$$\sum_{n=0}^{\infty} |c_n| \sum_{k=0}^n |a_{n,k}| |P_k^{(\lambda)}(x; \phi)| < \infty.$$

To bound $|c_n|$, we apply (2.3) and (2.4)

$$|c_n| \leq K_1 (1 + n)^{\sigma_1} |\rho|^n, \tag{5.16}$$

where $\sigma_1 = |\gamma| + 1$ and $K_1 = \max\{1, (2\lambda)^{-1}\}$. Combining (5.8) with (5.3) and (5.16), we see that

$$\sum_{n=0}^{\infty} |c_n| \sum_{k=0}^n |a_{n,k}| |P_k^{(\lambda)}(x; \psi)| \leq K_1 \sum_{n=0}^{\infty} (1 + n)^{\sigma_1 + \sigma + 2\lambda} \left(\frac{|\rho| (|\sin \phi| + |\sin(\psi - \phi)|)}{\sin \psi} \right)^n < \infty.$$

Reversing the summation and simplifying gives the desired result. \square

6. Definite integrals

We may apply the orthogonality relation on these continuous hypergeometric orthogonal polynomials to the generalized generating functions considered above to calculate their corresponding definite integrals. The Wilson polynomials satisfy the orthogonality relation given in [6, (9.1.2)] with weight $w : (0, \infty) \rightarrow \mathbb{R}$ defined by

$$w(x) := \left| \frac{\Gamma(a + ix)\Gamma(b + ix)\Gamma(c + ix)\Gamma(d + ix)}{\Gamma(2ix)} \right|^2. \tag{6.1}$$

Corollary 6.1. *Let $\rho \in \mathbb{C}$, $|\rho| < 1$, and a, b, c, d, h complex parameters with positive real parts, non-real parameters occurring in conjugate pairs among a, b, c, d and a, b, c, h . Then*

$$\begin{aligned} & \int_0^\infty {}_2F_1\left(\begin{matrix} a+ix, c+ix \\ a+c \end{matrix}; \rho\right) {}_2F_1\left(\begin{matrix} b-ix, d-ix \\ b+d \end{matrix}; \rho\right) \\ & \quad \times W_k(x^2; a, b, c, h) \left| \frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)\Gamma(h+ix)}{\Gamma(2ix)} \right|^2 dx \\ & = 2\pi \frac{\Gamma(a+c)\Gamma(k+a+b)\Gamma(k+a+h)\Gamma(k+b+c)\Gamma(k+c+h)\Gamma(k+b+h)}{(b+d)_k \Gamma(2k+a+b+c+h) \{(k+a+b+c+d-1)_k\}^{-1}} \\ & \quad \times {}_4F_3\left(\begin{matrix} d-h, 2k+a+b+c+d-1, k+a+b, k+b+c \\ k+a+b+c+d-1, 2k+a+b+c+h, k+b+d \end{matrix}; \rho\right) \rho^k. \end{aligned} \tag{6.2}$$

Proof. Choose some $k \in \mathbb{N}_0$. We begin by multiplying both sides of (2.7) by $W_k(x^2; a, b, c, h)w(x)$, where $w(x)$ is the weight function as defined in (6.1) with d replaced by h . Integrating over $x \in (0, \infty)$ causes the infinite sum on the right hand side to vanish except for a single term. Simplification produces the desired result.

In order to justify the interchange of sum and integral we show that the series on the right-hand side of (2.7) converges in the L^2 -sense with respect to the weight $w(x)$. This requires

$$\sum_{k=0}^\infty d_k^2 s_k^2 < \infty, \tag{6.3}$$

where $s_k > 0$ is defined by

$$s_k^2 = \int_0^\infty w(x) \{W_k(x)\}^2 dx,$$

and

$$d_k = \sum_{n=k}^\infty c_n a_{n,k},$$

where c_n and $a_{n,k}$ are defined as in the proof of Theorem 2.1. From the orthogonality relation corresponding to (6.1) we obtain

$$s_k \leq K(1+k)^\sigma k!^3,$$

where K, σ are positive constants independent of k . Since the estimate of s_k is of the same type as that of W_k , by the argument used in the proof of Theorem 2.1, we see that, for $|\rho| < 1$,

$$\sum_{n=0}^\infty |c_n| \sum_{k=0}^n |a_{n,k}| s_k < \infty.$$

This implies

$$\sum_{k=0}^\infty |d_k| s_k < \infty$$

which in turn implies (6.3) and the proof is complete. \square

Corollary 6.2. Let $\rho \in \mathbb{C}$, $|\rho| < 1$, and a, b, c, d, h complex parameters with positive real parts, non-real parameters occurring in conjugate pairs among a, b, c, d and a, b, c, h . Then

$$\begin{aligned} & \int_0^\infty (1 - \rho)^{1-a-b-c-d} {}_4F_3 \left(\begin{matrix} \frac{1}{2}(a + b + c + d - 1), \frac{1}{2}(a + b + c + d), a + ix, a - ix \\ a + b, a + c, a + d \end{matrix} ; -\frac{4\rho}{(1 - \rho)^2} \right) \\ & \quad \times W_k(x^2; a, b, c, h)w(x)dx \\ & = \frac{2\pi\Gamma(a + b)\Gamma(a + c)\Gamma(k + a + h)\Gamma(k + b + c)\Gamma(k + b + h)\Gamma(k + c + h)}{\Gamma(2k + a + b + c + h)(a + d)_k \{(k + a + b + c + d - 1)_k(a + b + c + d - 1)_k\}^{-1}} \\ & \quad \times {}_3F_2 \left(\begin{matrix} 2k + a + b + c + d - 1, d - h, k + b + c \\ 2k + a + b + c + h, a + d + k \end{matrix} ; \rho \right) \rho^k. \end{aligned} \tag{6.4}$$

Proof. The proof is the similar to the proof of Corollary 6.1, except apply to both sides of (2.13). \square

The continuous dual Hahn polynomials satisfy the orthogonality relation [6, (9.3.2)].

Corollary 6.3. Let $\rho \in \mathbb{C}$, $|\rho| < 1$, and $a, b, c, d, f > 0$, except for possibly a pair of complex conjugates with positive real parts. Then

$$\begin{aligned} & \int_0^\infty (1 - \rho)^{-d+ix} {}_2F_1 \left(\begin{matrix} a + ix, b + ix \\ a + b \end{matrix} ; \rho \right) \left| \frac{\Gamma(a + ix)\Gamma(d + ix)\Gamma(f + ix)}{\Gamma(2ix)} \right|^2 S_k(x^2; a, d, f)dx \\ & = 2\pi \frac{\Gamma(k + a + d)\Gamma(k + a + f)\Gamma(k + d + f)}{(a + b)_k k!} {}_2F_1 \left(\begin{matrix} b - f, k + a + d \\ k + a + b \end{matrix} ; \rho \right). \end{aligned} \tag{6.5}$$

Proof. Multiplying both sides of (3.7) by $S_k(x^2; a, d, f) \left| \frac{\Gamma(a+ix)\Gamma(d+ix)\Gamma(f+ix)}{\Gamma(2ix)} \right|^2$ and integrating across $x \in (0, \infty)$ causes the series to vanish except for a single term. Justification for interchanging the sum and the integral is similar to the proof of Corollary 6.1. We leave this exercise to the reader. Simplifying completes the proof. \square

Corollary 6.4. Let $\rho \in \mathbb{C}$, and $a, b, c, d > 0$ except for possibly a pair of complex conjugates with positive real parts among a, b, c and a, b, d . Then

$$\begin{aligned} & \int_0^\infty e^{\rho} {}_2F_2 \left(\begin{matrix} a + ix, a - ix \\ a + b, a + c \end{matrix} ; -\rho \right) \left| \frac{\Gamma(a + ix)\Gamma(b + ix)\Gamma(d + ix)}{\Gamma(2ix)} \right|^2 S_k(x^2; a, b, d)dx \\ & = \frac{\Gamma(a + b)\Gamma(k + a + d)\Gamma(k + b + d)}{(a + c)_k} {}_1F_1 \left(\begin{matrix} c - d \\ k + a + c \end{matrix} ; \rho \right) \rho^k. \end{aligned} \tag{6.6}$$

Proof. The proof is essentially the same as for Corollary 6.3. Multiply both sides of (3.13) by $S_k(x^2; a, b, d) \left| \frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(d+ix)}{\Gamma(2ix)} \right|^2$ and integrate over $x \in (0, \infty)$. With justification for reordering, simplification completes the proof. \square

Corollary 6.5. Let $\rho \in \mathbb{C}$ with $|\rho| < 1$, $\gamma \in \mathbb{C}$ and $a, b, c, d > 0$ except for possibly a pair of complex conjugates with positive real parts among a, b, c and a, b, d . Then

$$\int_0^\infty (1-\rho)^{-\gamma} {}_3F_2\left(\gamma, a+ix, a-ix; a+b, a+c; \frac{\rho}{\rho-1}\right) \left|\frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(d+ix)}{\Gamma(2ix)}\right|^2 S_k(x^2; a, b, d) dx$$

$$= \frac{(\gamma)_k \Gamma(a+b)\Gamma(k+a+d)\Gamma(k+b+d)}{(a+c)_k} {}_2F_1\left(\begin{matrix} -d, \gamma+k \\ k+a+c \end{matrix}; \rho\right) \rho^k. \tag{6.7}$$

Proof. Multiply both sides of (3.20) by $S_k(x^2; a, b, d) \left|\frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(d+ix)}{\Gamma(2ix)}\right|^2$, integrate over $x \in (0, \infty)$. Justification for reordering with simplification completes the proof. \square

The continuous Hahn polynomials satisfy the orthogonality relation [6, (9.4.2)].

Corollary 6.6. Let $\rho \in \mathbb{C}$, and $a, b, c \in \mathbb{C}$ such that $\Re a > 0, \Re b > 0, \Re c > 0$ and $\Im a = \Im b = \Im c$. Then

$$\int_{-\infty}^\infty {}_1F_1\left(\begin{matrix} a+ix \\ 2\Re a \end{matrix}; -i\rho\right) {}_1F_1\left(\begin{matrix} \bar{b}-ix \\ 2\Re b \end{matrix}; i\rho\right) \Gamma(a+ix)\Gamma(c+ix)\Gamma(\bar{a}-ix)\Gamma(\bar{c}-ix) p_k(x; a, c, \bar{a}, \bar{c}) dx$$

$$= 2\pi \frac{\Gamma(2\Re a)\Gamma(a+\bar{c})^2\Gamma(2\Re c)(k+2\Re a+2\Re b-1)_k(a+\bar{c})_k(2\Re c)_k(\rho/4)^k}{k!(2k+2\Re a+2\Re c-1)\Gamma(2\Re a+2\Re c-1)(2\Re b)_k(\frac{1}{2}(2\Re a+2\Re c-1))_k}$$

$$\times {}_4F_5\left(\begin{matrix} \frac{1}{2}(\Re a+\Re b+k), \frac{1}{2}(\Re a+\Re b+k+1), \Re a+\Re b+k-\frac{1}{2}, \Re b-\Re c \\ \Re a+\Re b+\frac{k-1}{2}, \Re a+\Re b+\frac{k}{2}, \Re b+\frac{k}{2}, \Re b+\frac{k+1}{2}, \Re a+\Re c+k+\frac{1}{2} \end{matrix}; \frac{-\rho^2}{4}\right). \tag{6.8}$$

Proof. Multiply both sides of (4.10) by $|\Gamma(a+ix)\Gamma(c+ix)|^2 p_k(a, c, \bar{a}, \bar{c})$. Justification for interchanging the sum and the integral is similar to the proof of Corollary 6.1. Again, we leave this to the reader. Integrating over $x \in (-\infty, \infty)$ completes the proof. \square

Corollary 6.7. Let $\rho \in \mathbb{C}, |\rho| < 1$, and $a, b, c \in \mathbb{C}$ such that $\Re a > 0, \Re b > 0, \Re c > 0$ and $\Im a = \Im b = \Im c$. Then

$$\int_{-\infty}^\infty (1-\rho)^{1-2\Re a-2\Re b} {}_3F_2\left(\begin{matrix} \Re a+\Re b-\frac{1}{2}, \Re a+\Re b, a+ix \\ 2\Re a, a+\bar{b} \end{matrix}; -\frac{4\rho}{(1-\rho)^2}\right)$$

$$\times |\Gamma(a+ix)\Gamma(c+ix)|^2 p_k(x; a, c, \bar{a}, \bar{c}) dx \tag{6.9}$$

$$= \frac{2\pi(-i\rho)^k \Gamma(2\Re a)\Gamma(2\Re c)[\Gamma(a+\bar{c})]^2 (2\Re a+2\Re b-1)_k (k+2\Re a+2\Re b-1)_k (a+\bar{c})_k (2\Re c)_k}{4^k k! (2k+2\Re a+2\Re c-1)\Gamma(2\Re a+2\Re c-1)(\Re a+\Re c-\frac{1}{2})_k (\Re a+\Re b)_k}$$

$$\times {}_2F_1\left(\begin{matrix} \Re a+\Re b+k-\frac{1}{2}, \Re b-\Re c \\ \Re a+\Re c+k+\frac{1}{2} \end{matrix}; \rho^2\right). \tag{6.10}$$

Proof. The proof is the same as Corollary 6.6, except apply to both sides of (4.12). \square

If $\lambda > 0$ and $\phi \in (0, \pi)$, then the Meixner–Pollaczek polynomials satisfy the orthogonality relation [6, (9.7.2)].

Corollary 6.8. Let $\lambda > 0, \psi, \phi \in (0, \pi)$, and $\rho \in \mathbb{C}$ such that (5.5) holds. Then

$$\int_{-\infty}^\infty (1-e^{i\phi}\rho)^{-\lambda+ix} (1-e^{-i\phi}\rho)^{-\lambda-ix} e^{(2\psi-\pi)x} |\Gamma(\lambda+ix)|^2 P_k^{(\lambda)}(x; \psi) dx$$

$$= \left(1-\rho \frac{\sin(\psi-\phi)}{\sin\psi}\right)^{-2\lambda} \frac{2\pi\Gamma(k+2\lambda)\tilde{\rho}^k}{(2\sin\psi)^{2\lambda} k!}, \tag{6.11}$$

where

$$\tilde{\rho} = \frac{\rho \sin \phi}{\sin \psi - \rho \sin(\psi - \phi)}.$$

Proof. Multiply both sides of (5.6) by $e^{(2\psi-\pi)x}|\Gamma(\lambda + ix)|^2 P_k^{(\lambda)}(x; \psi)$. Integrating over $x \in (-\infty, \infty)$ gives the desired result. The justification for interchange of the sum and the integral is similar to the proof of Corollary 6.1. This is left to the reader. \square

Corollary 6.9. Let $\lambda > 0$, $\rho \in \mathbb{C}$ and $\psi, \phi \in (0, \pi)$. Then

$$\begin{aligned} & \int_{-\infty}^{\infty} e^{\rho} {}_1F_1 \left(\begin{matrix} \lambda + ix \\ 2\lambda \end{matrix}; (e^{-2i\phi} - 1)\rho \right) e^{(2\psi-\pi)x} |\Gamma(\lambda + ix)|^2 P_k^{(\lambda)}(x; \psi) dx \\ &= \exp \left(\frac{\rho e^{-i\phi} \sin(\psi - \phi)}{\sin \psi} \right) \left(\frac{\sin \phi}{\sin \psi} \right)^k \frac{2\pi \Gamma(2\lambda) \rho^k}{e^{ik\phi} (2 \sin \psi)^{2\lambda} k!}. \end{aligned} \tag{6.12}$$

Proof. The proof is the same as Corollary 6.8, except apply to both sides of (5.10). \square

Corollary 6.10. Let $\lambda > 0$, $\gamma, \rho \in \mathbb{C}$ and $\psi, \phi \in (0, \pi)$ be such that (5.5) holds. Then

$$\begin{aligned} & \int_{-\infty}^{\infty} (1 - \rho)^{-\gamma} {}_2F_1 \left(\begin{matrix} \gamma, \lambda + ix \\ 2\lambda \end{matrix}; \frac{(1 - e^{-2i\phi})\rho}{\rho - 1} \right) e^{(2\psi-\pi)x} |\Gamma(\lambda + ix)|^2 P_k^{(\lambda)}(x; \psi) dx \\ &= \left(1 - \frac{\rho \sin(\psi - \phi)}{e^{i\phi} \sin \psi} \right)^{-\gamma-k} \left(\frac{\sin \phi}{\sin \psi} \right)^k \frac{(\gamma)_k \Gamma(2\lambda) 2\pi \rho^k}{e^{ik\phi} (2 \sin \psi)^{2\lambda} k!}. \end{aligned} \tag{6.13}$$

Proof. The proof is the same as Corollary 6.8, except apply to both sides of (5.14). \square

Acknowledgments

An anonymous referee suggested a different proof of the definite integrals in Section 6 by exploiting the symmetry of the Wilson polynomials in its parameters.

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