



# HIGHER INTEGRABILITY FOR SOLUTIONS TO PARABOLIC PROBLEMS WITH IRREGULAR OBSTACLES AND NONSTANDARD GROWTH

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**ABSTRACT.** The aim of this paper is to derive the self-improving property of integrability for the spatial gradient of solutions to degenerate parabolic obstacle problem with irregular obstacles and  $p(x, t)$ -nonstandard growth. More precisely, we prove that the spatial gradient of the solution is integrable to a larger power than the natural one determined by the structural assumptions on the involved differential operator.

## 1. INTRODUCTION

In this paper we establish higher integrability properties of solutions to degenerate parabolic obstacle problems with  $p(x, t)$ -nonstandard growth, i.e. solutions to parabolic variational inequality satisfying an obstacle constraint. In general, the idea of the self-improving property of integrability is the following: In principle the proof is based on certain reverse Hölder inequalities and an application of the Gehring's Lemma. To conclude a reverse Hölder inequality, we need a Caccioppoli estimate. Note that a Caccioppoli estimate has the structure of a reverse Poincaré inequality. Such a Caccioppoli estimate follows by considering the weak formulation of the elliptic or parabolic equation resp. system, then applying the structure condition on the vector-field and an application of Sobolev-Poincaré inequality. This yields the desired Caccioppoli estimate and therefore, the reverse Hölder inequality. In the nonstandard case, it is necessary to use additionally a localization argument, which allows to homogenize the estimates, to derive a reverse Hölder type inequality. This estimate is comparable to the one from the standard case and the key to the higher integrability.

*Historical background.* In the elliptic case with standard  $p$ -growth, it is a by now classical fact that weak solutions are locally higher integrable in the sense of Meyer's higher integrability result for the spatial gradient. This was first proved for the Jacobian of quasiconformal mapping by Gehring [28] and later on, for solutions to elliptic systems by Elcrat and Meyers [21], see also the monograph [29]. The result of higher integrability is already stated for parabolic systems with standard  $p$ -growth by Kinnunen and Lewis [35]. In the stationary nonstandard case, there are results of higher integrability by Zhikov in [43], while in the nonstandard  $p(z)$ -growth case there is the higher integrability result for the homogeneous  $p(z)$ -Laplacian, i.e.

$$\partial_t u - \operatorname{div}(|Du|^{p(z)-2} Du) = 0 \text{ in } \Omega_T$$

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by Antontsev and Zhikov in [6]. Moreover, there is the  $p(z)$ -analogue to [35] on the one hand by Bögelein and Duzaar [13] and on the other hand by Zhikov and Pastukhova in [44]. Zhikov and Pastukhova established independently and slightly earlier a higher integrability result, which is very similar to the one of Bögelein and Duzaar in [13]. Bögelein and Duzaar have shown a Meyer's type higher integrability result for the spatial gradient of weak solutions to parabolic systems of the form

$$\partial_t u - \operatorname{div} a(z, Du) = \operatorname{div}(|F|^{p(z)-2} F) \text{ in } \Omega_T. \quad (1.1)$$

Their result ensures that weak solutions of the preceding equation belong to a slightly higher Sobolev space than the natural space uniquely by the growth of the vector-field  $a(z, \cdot)$  and therefore, obey a certain self-improving property of integrability. This result we may also extend to solutions to parabolic equations of the form:

$$\partial_t u - \operatorname{div} a(z, Du) = f - \operatorname{div}(|F|^{p(z)-2} F) \text{ in } \Omega_T. \quad (1.2)$$

Finally, the higher integrability of solution to obstacle problems with  $p$ -growth, is a result by Bögelein and Scheven [15].

*Motivation of parabolic problems with variable exponents and obstacles.* Obstacle problems are interesting objects in the theory of partial differential equations and the calculus of variations. In general, the theory of obstacle problems is motivated by numerous applications, e.g. in mechanics or in control theory. We refer to [10, 34] for an overview of the classical theory and applications. Moreover, obstacle problems have been exploited in nonlinear potential theory for approximating supersolutions by solutions to obstacle problems, see [31, 33, 36]. Up to now, the theory for elliptic problems is well understood, as well the theory for elliptic obstacle problems and also the nonstandard case. Therefore, parabolic problems arouse interest more and more in mathematics during the last years. Also parabolic problems are motivated by physical aspects. In particular, evolutionary equations and systems can be used to model physical processes, e.g. heat conduction or diffusion processes. There are many open problems, e.g. the Navier-Stokes equation, the basic equation of fluid mechanics. Furthermore, some properties of solutions of the system of a modified Navier-Stokes equation, describing electro-rheological fluids are studied in [3]. Such fluids, which are recently of high technological interest, because of their ability to change the mechanical properties under the influence of exterior electro-magnetic field, see [27, 38]. For example, many electro-rheological fluids are suspensions consisting of solid particles and a carrier oil. These suspensions change their material properties dramatically if they are exposed to an electric field, see [39]. Most of the known results concern the stationary models, see for example [1, 2]. Other applications are the models for flows in porous media [5, 32].

Turning towards obstacle problems, it observes that the stationary case with standard growth is well developed, also the nonstandard case. Furthermore, in the last four till five years, a gap in the parabolic theory of obstacle problems with standard  $p$ -growth was closed, see [8, 14, 15, 16, 17, 19, 25, 41]. Moreover, in the last two till three years there were several regularity for the nonstandard growth case results developed, see [18, 22, 23, 26].

*A short overview of the theory of parabolic obstacle problems.* First existence results for parabolic problems with time-independent obstacles have been achieved in the linear case by Lions and Stampacchia [37] and for more general parabolic problems by Brezis [11]. Obstacle functions that depend in some sense continuously on time are treated in [12]. The article [4] by Alt and Luckhaus contains existence results for elliptic and parabolic problems in great generality, but the results on obstacle problems are limited to time-independent or bounded obstacle functions. However, the case of non-linear problems with general obstacle functions remained open for a long time. First results were achieved very recently by Bögelein, Duzaar and Mingione [14] and then, by Scheven [40, 41]. Here, we want to highlight that in [14] the authors established the first existence result to parabolic problems with irregular obstacles, which are not necessarily non-increasing in time. They consider general obstacles with the only additional assumption that the time derivative of the obstacle lies in  $L^{p'}$ . This is required since their method relies on a time mollification argument, combined with a maximum construction in order to recover the obstacle condition, where the pointwise maximum construction is not compatible with distributional time derivatives. Moreover, they established the Calderón-Zygmund theory for a large class of parabolic obstacle problems, i.e. they proved that the (spatial) gradient of solutions is as integrable as that of the assigned obstacles. Then, in [40, 41] Scheven introduced a new concept of solution to parabolic obstacle problems of  $p$ -Laplacian type with highly irregular obstacles, the so-called *localizable solutions*, see Definition 1.3. The main feature of localizable solutions is that they solve the obstacle problem locally, which is a priori not clear by the formulation of the problem, cf. the remarks preceding Definition 1.3. This new concept allows to consider more general settings, i.e. it is no more necessary to assume that the time derivative of the obstacle function lies in  $L^{p'}$ . It suffices to consider obstacles with distributional time derivatives. Moreover, we want to emphasize that the concept of localizable solutions allows to prove more general regularity results. Scheven also proved Calderón-Zygmund estimates for parabolic obstacle problems. The main difference between the result of Scheven and the result of Bögelein, Duzaar and Mingione is that in [14] they need an additional assumption on the boundary data, which seems to be unnatural for proving regularity in the interior. The reason for the additional assumption on the boundary data arises from the fact that the formulation of the obstacle problem is not of local nature. Bögelein, Duzaar and Mingione used a complex approximation argument to approximate the solutions by more regular ones and since the given solution was not known to be localizable, this approximation procedure had to be implemented on the whole domain. This problem could be avoided by the concept of localizable solutions. Moreover, this concept enables to prove the Hölder continuity of the spatial gradient of solutions to the parabolic obstacle problem without any additional assumption on the boundary data, see [25].

**1.1. General assumptions.** We consider a bounded domain  $\Omega \subset \mathbb{R}^n$  of dimension  $n \geq 2$  and we write  $\Omega_T := \Omega \times (0, T)$  for the space-time cylinder over  $\Omega$  of the height  $T > 0$ . In this paper,  $u_t$  respectively  $\partial_t u$  denotes the partial derivative with respect to the time variable  $t$  and  $Du$  denotes the one with respect to the space variable  $x$ .

**The setting.** First of all, we should mention that we denote by  $\partial_p \Omega_T = (\bar{\Omega} \times \{0\}) \cup (\partial\Omega \times (0, T))$  the **parabolic boundary** of  $\Omega_T$ . Furthermore, we write  $z = (x, t)$  for points in  $\mathbb{R}^{n+1}$ . We shall consider vector-fields  $a : \Omega_T \times \mathbb{R}^n \rightarrow \mathbb{R}^n$

which are assumed to be Carathéodory functions - i.e.  $a(z, w)$  is measurable in the first argument for every  $w \in \mathbb{R}^n$  and continuous in the second one for a.e.  $z \in \Omega_T$  - and satisfy the following nonstandard **growth** and **monotonicity properties**, for some growth exponent  $p : \Omega_T \rightarrow (\frac{2n}{n+2}, \infty)$  and structure constants  $0 < \nu \leq 1 \leq L$  and  $\mu \in [0, 1]$ :

$$|a(z, w)| \leq L(1 + |w|)^{p(z)-1}, \quad (1.3)$$

$$(a(z, w) - a(z, w_0)) \cdot (w - w_0) \geq \nu(\mu^2 + |w|^2 + |w_0|^2)^{\frac{p(z)-2}{2}} |w - w_0|^2 \quad (1.4)$$

for all  $z \in \Omega_T$  and  $w, w_0 \in \mathbb{R}^n$ . Furthermore, the growth exponent  $p : \Omega_T \rightarrow (\frac{2n}{n+2}, \infty)$  satisfies the following conditions: There exist constants  $\gamma_1, \gamma_2 < \infty$ , such that

$$\frac{2n}{n+2} < \gamma_1 \leq p(z) \leq \gamma_2 \text{ and } |p(z_1) - p(z_2)| \leq \omega(d_{\mathcal{P}}(z_1, z_2)) \quad (1.5)$$

holds for any choice of  $z_1, z_2 \in \Omega_T$ , where  $\omega : [0, \infty) \rightarrow [0, 1]$  denotes a modulus of continuity. More precisely, we shall assume that  $\omega(\cdot)$  is a concave, non-decreasing function with  $\lim_{\rho \downarrow 0} \omega(\rho) = 0 = \omega(0)$ . Moreover, the **parabolic distance** is given by  $d_{\mathcal{P}}(z_1, z_2) := \max\{|x_1 - x_2|, \sqrt{|t_1 - t_2|}\}$  for  $z_1 = (x_1, t_1), z_2 = (x_2, t_2) \in \mathbb{R}^{n+1}$ . In addition, for the modulus of continuity  $\omega(\cdot)$  we assume the following **weak logarithmic continuity condition** to hold:

$$\limsup_{\rho \downarrow 0} \omega(\rho) \log \left( \frac{1}{\rho} \right) < +\infty. \quad (1.6)$$

By virtue of (1.6) we may assume for a constant  $L_1 > 0$  depending on  $\omega(\cdot)$  that

$$\omega(\rho) \log \left( \frac{1}{\rho} \right) \leq L_1, \text{ for all } \rho \in (0, 1]. \quad (1.7)$$

At this stage it is worth to mention that assuming the existence of such  $\gamma_1, \gamma_2$  is not restrictive, since the results we are going to prove are of local nature. We mention that the previous lower bound on  $\gamma_1$  is a typical assumption in the regularity theory of nonlinear parabolic equations and systems.

**1.2. The function spaces.** The spaces  $L^p(\Omega)$ ,  $W^{1,p}(\Omega)$  and  $W_0^{1,p}(\Omega)$  stand for the usual Lebesgue and Sobolev spaces.

*Parabolic Lebesgue-Orlicz spaces.* We start by the definition of the nonstandard  $p(z)$ -Lebesgue space. The space  $L^{p(z)}(\Omega_T, \mathbb{R}^k)$  is defined as the set of those measurable functions  $v : \Omega_T \rightarrow \mathbb{R}^k$  for  $k \in \mathbb{N}$ , such that  $|v|^{p(\cdot)} \in L^1(\Omega_T, \mathbb{R}^k)$ , i.e.

$$L^{p(z)}(\Omega_T, \mathbb{R}^k) := \left\{ v : \Omega_T \rightarrow \mathbb{R}^k \text{ is measurable in } \Omega_T : \int_{\Omega_T} |v|^{p(\cdot)} dz < +\infty \right\}.$$

The set  $L^{p(\cdot)}(\Omega_T, \mathbb{R}^k)$  equipped with the **Luxemburg norm**

$$\|v\|_{L^{p(\cdot)}(\Omega_T)} := \inf \left\{ \lambda > 0 : \int_{\Omega_T} \left| \frac{v}{\lambda} \right|^{p(\cdot)} dz \leq 1 \right\}$$

becomes a Banach spaces. Finally, for the right-hand side of (1.2) we assume

$$F \in L^{p(\cdot)}(\Omega_T, \mathbb{R}^n) \text{ and } f \in L^{\gamma_1'}(\Omega_T). \quad (1.8)$$

*Parabolic Sobolev-Orlicz spaces.* By  $W_g^{p(\cdot)}(\Omega_T)$  we denote the Banach space

$$W_g^{p(\cdot)}(\Omega_T) := \{u \in [g + L^1(0, T; W_0^{1,1}(\Omega))] \cap L^{p(\cdot)}(\Omega_T) \mid Du \in L^{p(\cdot)}(\Omega_T, \mathbb{R}^n)\}$$

equipped by the norm

$$\|u\|_{W^{p(\cdot)}(\Omega_T)} := \|u\|_{L^{p(\cdot)}(\Omega_T)} + \|Du\|_{L^{p(\cdot)}(\Omega_T)}.$$

If  $g = 0$  we write  $W_0^{p(\cdot)}(\Omega_T)$  instead of  $W_g^{p(\cdot)}(\Omega_T)$ . Here, it is worth to mention that  $(u - g) \in W_0^{p(\cdot)}(\Omega_T)$  respectively  $u \in g + W_0^{p(\cdot)}(\Omega_T)$  to indicate that  $u$  agrees with  $g$  on the lateral boundary of the cylinder  $\Omega_T$ , i.e.  $u \in W_g^{p(\cdot)}(\Omega_T)$ . We are now ready to give the definition of a weak solution to the parabolic equation (1.2):

**Definition 1.1.** We identify a function  $u \in L^1(\Omega_T)$  as a **weak solution** of the parabolic equation (1.2), if and only if  $u \in C^0([0, T]; L^2(\Omega)) \cap W^{p(\cdot)}(\Omega_T)$  and

$$\int_{\Omega_T} [u \cdot \varphi_t - a(z, Du) \cdot D\varphi] dz = - \int_{\Omega_T} [f \cdot \varphi + |F|^{p(\cdot)-2} F \cdot D\varphi] dz \quad (1.9)$$

holds, whenever  $\varphi \in C_0^\infty(\Omega_T)$ .

Moreover, we denote by  $W^{p(\cdot)}(\Omega_T)'$  the dual of the space  $W_0^{p(\cdot)}(\Omega_T)$ . In the following, we write  $\langle \cdot, \cdot \rangle_{\Omega_T}$  for the pairing between  $W^{p(\cdot)}(\Omega_T)'$  and  $W_0^{p(\cdot)}(\Omega_T)$ , see [23, 24].

**1.2.1. Obstacle function, boundary, initial values and energy bound.** At this stage, we state the assumptions for the obstacle function, boundary data, initial values and the obstacle constraint. These assumptions we need to define the function spaces in which we will formulate the obstacle problems. Therefore, we consider on the lateral boundary  $\partial\Omega \times (0, T)$  **Dirichlet boundary data** given by

$$g \in C^0([0, T]; L^2(\Omega)) \cap W^{p(\cdot)}(\Omega_T) \text{ and } \partial_t g \in L^{\gamma_1'}(\Omega_T). \quad (1.10)$$

Moreover, we consider initial values  $u_0 \in L^2(\Omega)$ . The **obstacle constraint** will be given by a function  $\psi : \Omega_T \rightarrow \mathbb{R}$  with

$$\psi \in C^0([0, T]; L^2(\Omega)) \cap W^{p(\cdot)}(\Omega_T) \text{ and } \partial_t \psi \in L^{\gamma_1'}(\Omega_T). \quad (1.11)$$

For the boundary and initial values, we assume the **compatibility conditions**

$$g \geq \psi \text{ on } \partial\Omega \times (0, T) \text{ and } u_0 \geq \psi(\cdot, 0) \text{ a.e. on } \Omega, \quad (1.12)$$

where the first one is to be understood in the  $L^1$ - $W_0^{1,1}$ -sense, i.e.  $(\psi - g)_+ \in W_0^{p(\cdot)}(\Omega_T)$ . Now, we are in a situation to introduce the function spaces in which we will formulate the obstacle problem. These spaces are defined as follows:

$$\mathcal{K}_{\psi, g}(\Omega_T) := \left\{ u \in C^0([0, T]; L^2(\Omega)) \cap W_g^{p(\cdot)}(\Omega_T), u \geq \psi \text{ a.e. on } \Omega_T \right\},$$

and the function space

$$\mathcal{K}'_{\psi, g}(\Omega_T) := \left\{ u \in \mathcal{K}_{\psi, g}(\Omega_T) \mid \partial_t u \in W^{p(\cdot)}(\Omega_T)' \right\},$$

whose members play the role of admissible comparison functions.

**1.3. Parabolic obstacle problems with nonstandard  $p(z)$ -growth.** The main problem we are going to deal with, are the obstacle problems. More precisely, time dependent obstacles with the obstacle  $\psi : \Omega_T \rightarrow \mathbb{R}$ . It turns out that in our situation, the solution to the obstacle problem does not necessarily possess a time derivative in the distributional space  $W^{p(\cdot)}(\Omega_T)'$ , but only satisfies  $u \in \mathcal{K}_{\psi,g}(\Omega_T)$ . In this case, only the following formulation makes sense:

**Definition 1.2.** We identify a function  $u \in \mathcal{K}_{\psi,g}(\Omega_T)$  as a solution of the **weak formulation of the variational inequality** if

$$\begin{aligned} \langle \partial_t v, v - u \rangle_{\Omega_T} + \int_{\Omega_T} a(z, Du) \cdot D(v - u) dz + \|v(\cdot, 0) - u_0\|_{L^2(\Omega)}^2 \\ \geq \int_{\Omega_T} f(v - u) + |F|^{p(\cdot)-2} F \cdot D(v - u) dz \end{aligned} \quad (1.13)$$

holds for all test functions  $v \in \mathcal{K}'_{\psi,g}(\Omega_T)$ .

**1.4. The concept of localizable solutions.** The concept of localizable solutions goes back to Ch. Scheven, see [40, 41], and the idea of this concept is the following: In the general situation that we are considering, the solutions do not necessarily satisfy  $\partial_t u \in W^{p(\cdot)}(\Omega_T)'$ , so that the variational inequality can only be written in the weak formulation (1.13). However, this formulation does not seem to be the most suitable notion of solution, since it is not of local nature. More precisely, for a given parabolic cylinder  $\mathcal{O}_I = \mathcal{O} \times (t_1, t_2) \subset \Omega_T$ , it is a priori not clear that the restriction  $u|_{\mathcal{O}_I}$  of a solution  $u$  to the weak formulation of the variational inequality (1.13) again satisfies a variational inequality on  $\mathcal{O}_I$ . Even more, it is unclear if the space  $\mathcal{K}'_{\psi,u}(\mathcal{O}_I)$  of admissible comparison maps is not empty. In fact, it is not evident from the formulation (1.13) that there exists any map that agrees with  $u$  on the lateral boundary of  $\mathcal{O}_I$  and at the same time possesses a time derivative in the distributional space  $W^{p(\cdot)}(\Omega_T)'$ , which would be necessary for the construction of suitable comparison maps. These considerations motivate the following concept of a localizable solution to a parabolic obstacle problem.

**Definition 1.3.** We say that  $u \in \mathcal{K}_{\psi,g}(\Omega_T)$  is a **localizable solution** of the weak formulation (1.13) of the obstacle problem if for every parabolic cylinder  $\mathcal{O}_I := \mathcal{O} \times (t_1, t_2) \subset \Omega_T$ , where  $\mathcal{O} = \tilde{\mathcal{O}} \cap \Omega$  with a Lipschitz regular domain  $\tilde{\mathcal{O}} \subset \mathbb{R}^n$  and a time interval  $I = (t_1, t_2) \subset (0, T) \subset \mathbb{R}$ , the following two conditions hold.

- (i) The map  $u$  satisfies the extension property, i.e. there holds  $\mathcal{K}'_{\psi,u}(\mathcal{O}_I) \neq \emptyset$ .
- (ii) For all comparison maps  $v \in \mathcal{K}'_{\psi,u}(\mathcal{O}_I)$ , there holds

$$\begin{aligned} \langle \partial_t v, v - u \rangle_{\mathcal{O}_I} + \int_{\mathcal{O}_I} a(z, Du) \cdot D(v - u) dz + \|(v - u)(\cdot, t_1)\|_{L^2(\mathcal{O})}^2 \\ \geq \int_{\mathcal{O}_I} f(v - u) + |F|^{p(\cdot)-2} F \cdot D(v - u) dz, \end{aligned} \quad (1.14)$$

where  $\langle \cdot, \cdot \rangle_{\mathcal{O}_I}$  denotes the dual pairing between  $W^{p(\cdot)}(\mathcal{O}_I)'$  and  $W_0^{p(\cdot)}(\mathcal{O}_I)$ .

**1.5. Statement of the result.** The higher integrability result reads as follows.

**Theorem 1.4.** *Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain,  $\sigma > 0$  and  $p : \Omega_T \rightarrow [\gamma_1, \gamma_2]$  satisfies (1.5)-(1.7). Then, there exists a constant  $\varepsilon_0 = \varepsilon_0(n, \gamma_1, \gamma_2, \nu, L, L_1, \sigma) \in (0, \sigma]$ , such that the following is true: Whenever  $u \in \mathcal{K}_{\psi,g}(\Omega_T)$  is a localizable solution - see Definition 1.3 - to the obstacle problem (1.13) with the initial values*



$u(\cdot, 0) = u_0$ , where (1.8), (1.10) and (1.11) are in force. Moreover, the growth and monotonicity conditions (1.3)-(1.4) are valid and higher integrability assumptions to the inhomogeneities and the obstacle function

$$|F|, |D\psi| \in L_{\text{loc}}^{p(\cdot)(1+\sigma)}(\Omega_T, \mathbb{R}^n) \quad (1.15)$$

and

$$|f|^{\frac{1}{\gamma_1-1}}, |\partial_t \psi|^{\frac{1}{\gamma_1-1}} \in L_{\text{loc}}^{\gamma_1(1+\sigma)}(\Omega_T) \quad (1.16)$$

are fulfilled, then

$$Du \in L_{\text{loc}}^{p(\cdot)(1+\varepsilon_0)}(\Omega_T, \mathbb{R}^n). \quad (1.17)$$

Further, for  $M \geq 1$  there exists a radius  $r_0 = r_0(n, \gamma_1, \gamma_2, L, L_1, \omega(\cdot), M) > 0$ , such that there holds: If the energy bound  $M$  satisfies

$$\int_{\Omega_T} |Du|^{p(\cdot)} + \Psi \, dz \leq M, \quad (1.18)$$

then for any parabolic cylinder  $Q_{2r} = Q_{2r}(\mathfrak{z}_0) \subseteq \Omega_T$  with  $r \in (0, r_0]$ , there holds the following estimate:

$$\int_{Q_r} |Du|^{p(\cdot)(1+\varepsilon)} \, dz \leq c \left( \int_{Q_{2r}} |Du|^{p(\cdot)} + \Psi \, dz \right)^{1+\varepsilon d} + c \int_{Q_{2r}} \Psi^{(1+\varepsilon)} \, dz, \quad (1.19)$$

where  $c = c(n, \gamma_1, \gamma_2, \nu, L, L_1)$  and  $\varepsilon \in (0, \varepsilon_0]$ . Thereby,  $d$  is defined as follows:

$$d = d(p_0) := \begin{cases} \frac{2p_0}{p_0(n+2)-2n} & \text{if } p_0 < 2, \\ \frac{p_0}{2} & \text{if } p_0 \geq 2, \end{cases} \quad (1.20)$$

with  $p_0 = p(\mathfrak{z}_0)$ .

*Plan of the paper.* Finally, we briefly describe the strategy of the proof to our main result and the technical novelties of the paper. We start by proving a Reverse Hölder type inequality of a localizable solution  $u$  [cf. Definition 1.3] to our obstacle problem. Here, we will use a comparison argument and the Reverse Hölder type inequality stated by Bögelein and Duzaar in [13], see Section 3. Then, we will apply our Reverse Hölder type inequality to gain the higher integrability estimate (1.19) and a localization argument to prove the self-improving property of integrability for the spatial gradient of solution to degenerate parabolic obstacle problem, see Section 4. Before, we start with the proof, we mention some notations and preliminary results in the next section.

## 2. PRELIMINARIES AND NOTATIONS

### 2.1. Notations.

**Intrinsic geometry.** Furthermore, we introduce symmetric parabolic cylinders with center in  $z_0 = (x_0, t_0) \in \Omega_T$  of the form  $Q_\rho(z_0) := B_\rho(x_0) \times (t_0 - \rho^2, t_0 + \rho^2)$ , where  $(t_0 - \rho^2, t_0 + \rho^2) \subset (0, T)$  and  $B_\rho(x_0) \subset \Omega$  denotes a ball with radius  $\rho > 0$  and center  $x_0$ . To obtain the relevant (scaling invariant) local estimates we will use, in order to re-balance the non-homogeneity of parabolic problems, certain scaled cylinders, i.e. so-called **intrinsic cylinders** of the form

$$Q_\rho^{(\lambda)}(z_0) := B_\rho(x_0) \times \Lambda_\rho^{(\lambda)}(t_0), \text{ where } \Lambda_\rho^{(\lambda)}(t_0) := \left( t_0 - \lambda^{\frac{2-p_0}{p_0}} \rho^2, t_0 + \lambda^{\frac{2-p_0}{p_0}} \rho^2 \right),$$

where  $\lambda > 0$  and  $p_0 := p(z_0)$ . The reason for such scaled cylinder is based on the fact (explained by the easiest problem), that a multiple  $c \cdot u$  of a solution to



$\partial_t u - \operatorname{div}(|Du|^{p-2} Du) = 0$  is no longer a solution, except  $c \in \{0, 1\}$ ,  $p = 2$  or  $u \equiv 0$ . Such kind of intrinsic cylinders were introduced in the case  $p = \text{const.}$  in the pioneering work of DiBenedetto and Friedman [20]. The way we use the idea of intrinsic cylinders goes back to Bögelein and Duzaar in [13]. The delicate aspect in this technique relies in the fact that the cylinders will be constructed in such a way, that the scaling parameter  $\lambda > 0$  and the average of  $|Du|^{p(\cdot)}$  over  $Q_\rho^{(\lambda)}(z_0)$  are coupled in the following way:

$$\int_{Q_\rho^{(\lambda)}(z_0)} |Du|^{p(\cdot)} dz \approx \lambda.$$

## 2.2. Preliminaries.

*Localization argument.* The first important goal, is to control the variable exponent  $p(\cdot)$ . Therefore, we use the technique of localization, which is used by Bögelein and Duzaar in [13], to handle these exponents much more easier. Thus, we assume that, on a cylinder  $Q_\rho^{(\lambda)}(z_0) \subseteq \Omega_T$  with  $0 < \rho \leq 1$  and  $\lambda \geq 1$ , the **intrinsic coupling**

$$\lambda \leq \kappa \left( \int_{Q_\rho^{(\lambda)}(z_0)} |Du|^{p(\cdot)} + \Psi dz \right) \quad (2.1)$$

is fulfilled for some  $\kappa \geq 1$  with

$$\Psi := 1 + |F|^{p(\cdot)} + |f|^{\gamma'_1} + |D\psi|^{p(\cdot)} + |\partial_t \psi|^{\gamma'_1}, \quad (2.2)$$

where  $F \in L^{p(\cdot)}(\Omega_T)$ ,  $\psi \in C^0([0, T]; L^2(\Omega)) \cap W^{p(\cdot)}(\Omega_T)$  and  $f, \partial_t \psi \in L^{\gamma'_1}(\Omega_T)$ . Moreover, we know that  $|Q_\rho^{(\lambda)}(z_0)| = 2\alpha_n \rho^{n+2} \lambda^{\frac{2-p_0}{p_0}}$ , where  $\alpha_n$  denotes the measure to the unit ball of  $\mathbb{R}^n$ . Hence, it holds

$$\lambda \leq \frac{\kappa}{2\alpha_n \rho^{n+2} \lambda^{\frac{2-p_0}{p_0}}} \left( \int_{Q_\rho^{(\lambda)}(z_0)} |Du|^{p(\cdot)} + \Psi dz \right).$$

This is equivalent to

$$\lambda^{\frac{2}{p_0}} \leq \frac{\kappa}{2\alpha_n \rho^{n+2}} \left( \int_{Q_\rho^{(\lambda)}(z_0)} |Du|^{p(\cdot)} + \Psi dz \right).$$

Now, we can bound the integral by the energy bound  $M \geq 1$  from above, where  $M$  is introduced in (1.18). Consequently, we have

$$\lambda^{\frac{2}{p_0}} \leq \frac{\kappa M}{2\alpha_n \rho^{n+2}} \leq c(n) \frac{\kappa M}{\rho^{n+2}}.$$

Therefore, this yields a bound to  $\lambda$ , which only depends on the radius  $\rho$ , the energy bound  $M \geq 1$ , a constant  $\beta = \beta(n) \geq 1$ , as well as on  $\kappa$  and  $p_0$ :

$$\lambda \leq \left( \beta_n \frac{\kappa M}{\rho^{n+2}} \right)^{\frac{p_0}{2}}. \quad (2.3)$$

Next, we determine a preliminary bound for the oscillation of  $p(\cdot)$  on  $Q_\rho^{(\lambda)}(z_0)$ . Therefore, we notice that  $\lambda \geq 1$  and  $p_0 \geq \gamma_1$ . From (1.5) we can conclude that

$$p_2 - p_1 \leq \omega(d_{\mathcal{P}}(z_1, z_2)) \leq \omega(2\rho + \sqrt{2\lambda^{\frac{2-p_0}{p_0}} \rho^2}).$$

In the case  $p_0 \geq 2$ , it holds

$$p_2 - p_1 \leq \omega(4\rho), \quad (2.4)$$

since in this case  $\lambda^{\frac{2-p_0}{p_0}} \leq 1$  and  $\omega(\cdot)$  is non-decreasing. While in the case  $\frac{2n}{n+2} < p_0 < 2$ , we have

$$\begin{aligned} p_2 - p_1 &\leq \omega(4\lambda^{\frac{2-p_0}{2p_0}}\rho) \leq \omega(4(\beta_n\kappa M)^{\frac{2-\gamma_1}{4}}\rho^{1-\frac{(2-\gamma_1)(n+2)}{4}}) \\ &\leq \omega(4\sqrt{\beta_n\kappa M}\rho^{\gamma_1\frac{n+2}{4}-\frac{n}{2}}) = \omega(\Gamma\rho^{\gamma_1\frac{n+2}{4}-\frac{n}{2}}), \end{aligned} \quad (2.5)$$

where we defined  $\Gamma := 4\sqrt{\beta_n\kappa M} \geq 4$ . Note also that, the exponent of  $\rho$  is positive, since  $\gamma_1 > \frac{2n}{n+2}$ . Thus, it is essential  $\gamma_1\frac{n+2}{4} - \frac{n}{2} > 0$ . Combining (2.4) and (2.5), we finally get the following estimate:

$$p_2 - p_1 \leq \omega(\Gamma\rho^\alpha), \text{ where } \alpha := \min\left\{1, \gamma_1\frac{n+2}{4} - \frac{n}{2}\right\}. \quad (2.6)$$

By means of the weak logarithmic continuity condition (1.6), we can conclude that

$$\rho^{-(p_2-p_1)} \leq \rho^{-\omega(\Gamma\rho^\alpha)} = \exp\left[\omega(\Gamma\rho^\alpha)\log\frac{1}{\rho}\right] \leq e^{\frac{L_1}{\alpha}} \exp\left[\alpha^{-1}\omega(\Gamma\rho^\alpha)\log\Gamma\right].$$

At this stage, we choose

$$\rho_1 \leq \Gamma^{-\frac{2}{\alpha}} := \left(4\sqrt{\beta_n\kappa M}\right)^{-\frac{2}{\alpha}}. \quad (2.7)$$

Hence, we determine  $\rho_1$  as a constant, which only depends on  $n$ ,  $\gamma_1$ ,  $\kappa$  and  $M$ . Now, we suppose that  $0 < \rho \leq \rho_1$  and use (1.6) to the previous estimate. This yields

$$\rho^{-(p_2-p_1)} \leq e^{\frac{L_1}{\alpha}} \exp\left[\alpha^{-1}\omega(\Gamma\rho^\alpha)\log\Gamma\right] \leq e^{\frac{L_1}{\alpha}} \exp\left[\alpha^{-1}\omega(\Gamma\rho_1^\alpha)\log\Gamma\right] \leq e^{\frac{2L_1}{\alpha}}, \quad (2.8)$$

where we used the bound (2.7). Moreover, we can conclude from (2.3), the definition of  $\Gamma$  and (2.6) the following estimate:

$$\lambda^{\frac{p_2-p_1}{p_0}} \leq \left(\Gamma\rho^{-\frac{n+2}{2}}\right)^{p_2-p_1} \leq \Gamma^{\omega(\Gamma\rho_1^\alpha)} e^{\frac{L_1(n+2)}{\alpha}} \leq \Gamma^{\omega(\Gamma^{-1})} e^{\frac{L_1(n+2)}{\alpha}},$$

where we again used the bound (2.7). Next, we use (1.6) to estimate  $\Gamma^{\omega(\Gamma\rho_1^\alpha)}$  from above as follows

$$\Gamma^{\omega(\Gamma^{-1})} = e^{\omega(\Gamma^{-1})\log\Gamma} \leq e^{L_1},$$

where we utilized the fact, that  $\Gamma \geq 4$  and therefore, the inverse of  $\Gamma$  is smaller than 1, i.e.  $\Gamma^{-1} \in (0, 1)$ . Hence, we were able to use (1.6). All together, we get

$$\lambda^{\frac{p_2-p_1}{p_0}} \leq e^{\frac{L_1(n+2)+L_1}{\alpha}} \leq e^{\frac{4nL_1}{\alpha}}. \quad (2.9)$$

*Iteration lemma.* In order to "re-absorb" certain terms, we will use the following iteration lemma, which is a standard tool and can for instance be found in [29, p. 81]. The iteration result reads as follows.

**Lemma 2.1.** *Let  $0 < \vartheta < 1$ ,  $A, C \geq 0$  and  $\beta > 0$ . Then, there exists a constant  $c = c(\beta, \vartheta)$ , such that there holds: For any non-negative bounded function satisfying*

$$\Phi(t) \leq \vartheta\Phi(s) + A(s-t)^{-\beta} + C \text{ for all } 0 < r \leq t < s \leq \rho,$$

*we have*

$$\Phi(r) \leq c[A(\rho-r)^{-\beta} + C].$$

*An existence result.* Here, we cite from [23, 24] an existence result for parabolic equations with nonstandard growth. Therefore, we consider the local Cauchy-Dirichlet problem

$$\begin{cases} \partial_t u - \operatorname{div} a(z, Du) &= f - \operatorname{div} (|F|^{p(\cdot)-2} F) \text{ in } Q_\rho(z_0), \\ u &= g \text{ on } \partial B_\rho(x_0) \times (t_0 - \rho^2, t_0), \\ u(\cdot, t_0 - \rho^2) &= u_0 \text{ on } B_\rho(x_0) \times \{t_0 - \rho^2\}, \end{cases} \quad (2.10)$$

where initial values  $u_0$  and boundary data  $g$  are given. The precise statement reads as follows.

**Corollary 2.2** ([23], Corollary 4.5). *Let  $\Omega \subset \mathbb{R}^n$  be an open, bounded Lipschitz domain and  $Q_\rho(z_0) \subset \Omega_T$ . Assume that  $p : \Omega_T \rightarrow [\gamma_1, \gamma_2]$  satisfies (1.5)-(1.6). Then, suppose that the vector-field  $a : \Omega_T \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a Carathéodory function and satisfies (1.3)-(1.4). Moreover, assume that  $F \in L^{p(\cdot)}(\Omega_T, \mathbb{R}^n)$ ,  $f \in W^{p(\cdot)}(\Omega_T)'$ ,  $g \in W(\Omega_T)$  and  $u_0 \in L^2(\Omega)$  are in force. Then, there exist  $\theta_0 = \theta_0(n, \gamma_1) \in (0, 1)$  and a radius  $\rho_0 = \rho_0(\theta, \omega(\cdot)) \in (0, 1]$  with  $\theta \leq \theta_0$ , such that the following holds: Whenever  $0 < \rho \leq \rho_0$ , there exists a weak solution  $u \in W(Q_\rho(z_0))$  of the local parabolic boundary problem (2.10).*

*Comparison principle.* In this section we refer a comparison principle, which will be a key tool for constructing comparison maps that almost everywhere satisfy the obstacle constraint  $v \geq \psi$ . This technical tool is stated in [23].

**Lemma 2.3** ([23], Lemma 3.15). *Let  $\Omega \subset \mathbb{R}^n$  with  $n \geq 2$  and  $p : \Omega_T \rightarrow [\gamma_1, \gamma_2]$  satisfies (1.5)-(1.6). Moreover, suppose that  $\psi, v \in W(\Omega_T)$  satisfy in the weak sense*

$$\begin{cases} \partial_t \psi - \operatorname{div} a(z, D\psi) &\leq \partial_t v - \operatorname{div} a(z, Dv) \text{ in } \Omega_T, \\ \psi &\leq v \text{ on } \partial_p \Omega_T, \end{cases} \quad (2.11)$$

where (1.4) are in force. Then, there holds  $\psi \leq v$  a.e. on  $\Omega_T$ .

### 3. REVERSE HÖLDER TYPE INEQUALITY

The aim of this section is to affiliate a Reverse Hölder type inequality. This is an important step to the proof of the higher integrability, since higher integrability follows from the Reverse Hölder type inequality. The Reverse Hölder type inequality we will conclude by a comparison argument. Therefore, let  $Q_{8\rho}^{(\lambda)}(z_0) \subset \Omega_T$  with  $\lambda \geq 1$ . Here, we have to mention that by the Definition of localizable solution  $u \in \mathcal{K}_{\psi, g}(\Omega_T)$ , we know that there exists a function with  $\mathcal{K}'_{\psi, u}(Q_{8\rho}^{(\lambda)}(z_0))$  [cf. Definition 1.3], i.e. a function in  $\mathcal{K}_{\psi, u}(Q_{8\rho}^{(\lambda)}(z_0))$  with boundary datum  $u$ , which possess a time derivate in  $W^{p(\cdot)}(Q_{8\rho}^{(\lambda)}(z_0))'$ . Thus, we are allowed to use  $u$  as boundary datum. Moreover, because of the inhomogeneous behavior of the parabolic variational inequality, the Reverse Hölder type inequality can only be obtained on parabolic cylinders satisfying an intrinsic coupling of the type (3.1) and (3.2) below. The result of this section is stated in the following lemma:

**Lemma 3.1.** *Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain and  $M \geq 1$  the energy bound from (1.18). Moreover,  $p : \Omega_T \rightarrow [\gamma_1, \gamma_2]$  satisfies the conditions (1.5)-(1.7). Then, there exists a radius  $\rho_2 = \rho_2(n, \gamma_1, \gamma_2, L_1, \omega(\cdot), M) \in (0, 1]$  and a constant  $c \geq 1$ , which only depends on  $n, \gamma_1, \gamma_2, \nu, L, L_1$ , such that the following holds: Suppose that  $u \in \mathcal{K}_{\psi, g}(\Omega_T)$  is a localizable solution - see Definition 1.3 - to the*

variational inequality (1.13) with respect to the structure assumptions (1.3)-(1.4), the inhomogeneities (1.8), the initial values  $u(\cdot, 0) = u_0 \in L^2(\Omega)$ , the boundary data (1.10) and the obstacle function (1.11) which satisfy the compatibility condition (1.12) are given. Furthermore, assume that on some parabolic cylinder  $Q_{8\rho}^{(\lambda)}(z_0) \subseteq \Omega_T$  with  $0 < \rho \leq \rho_2$  and  $\lambda \geq 1$ , an intrinsic coupling is given in the sense that on  $Q_\rho^{(\lambda)}$  we have the lower bound

$$\lambda \leq \int_{Q_\rho^{(\lambda)}(z_0)} |Du|^{p(\cdot)} + \Psi dz, \quad (3.1)$$

while on  $Q_{8\rho}^{(\lambda)}$  the upper bound

$$\int_{Q_{8\rho}^{(\lambda)}(z_0)} |Du|^{p(\cdot)} + \Psi dz \leq \lambda \quad (3.2)$$

is fulfilled. Then, there holds the **Reverse Hölder type inequality**

$$\int_{Q_\rho^{(\lambda)}(z_0)} |Du|^{p(\cdot)} dz \leq c \left( \int_{Q_{2\rho}^{(\lambda)}(z_0)} |Du|^{\frac{p(\cdot)}{\vartheta}} dz \right)^\vartheta + c \left( \int_{Q_{8\rho}^{(\lambda)}(z_0)} \Psi dz \right), \quad (3.3)$$

where  $\vartheta$  is defined as follows

$$\vartheta := \min \left\{ \frac{\gamma_1(n+2)}{2n}, \sqrt{\frac{n+2}{n}}, \frac{2\gamma_2}{2\gamma_2-1} \right\} > 1 \quad (3.4)$$

and  $\Psi$  is introduced in (2.2).

*Proof.* In the following, we will use the notation  $Q_\rho^{(\lambda)}$  instead of  $Q_\rho^{(\lambda)}(z_0)$ . Now, let

$$w \in C^0(\Lambda_{8\rho}^{(\lambda)}; L^2(B_{8\rho})) \cap W^{p(\cdot)}(Q_{8\rho}^{(\lambda)}) \text{ with } \partial_t w \in W^{p(\cdot)}(Q_{8\rho}^{(\lambda)})'$$

the unique solution of the following Cauchy-Dirichlet problem

$$\begin{cases} \partial_t w - \operatorname{div}_a(z, Dw) &= \partial_t \psi - \operatorname{div}_a(z, D\psi) \text{ in } Q_{8\rho}^{(\lambda)}, \\ w &= u \text{ on } \partial_{\mathcal{P}} Q_{8\rho}^{(\lambda)}, \end{cases} \quad (3.5)$$

with  $\rho \leq \frac{\rho_0}{8}$ , where  $\rho_0 = \rho_0(\theta, \omega(\cdot)) \in (0, 1]$  with  $\theta \leq \theta_0 = \theta_0(n, \gamma_1) \in (0, 1)$  is the radius from Corollary 2.2. Therefore, we can refer that  $w$  is a solution of (3.5). Since,  $w = u \geq \psi$  on  $\partial_{\mathcal{P}} Q_{8\rho}^{(\lambda)}$ , we can utilize the comparison argument from Lemma 2.3 to conclude that  $w \geq \psi$  a.e. on  $Q_{8\rho}^{(\lambda)}$ . Since,  $u \in \mathcal{K}_{\psi, g}(\Omega_T)$  is a localizable solution of the variational inequality (1.13), we know that  $u \in \mathcal{K}_{\psi, w}(Q_{8\rho}^{(\lambda)})$  is solution of the variational inequality

$$\begin{aligned} \langle \partial_t w, w - u \rangle_{Q_{8\rho}^{(\lambda)}} + \int_{Q_{8\rho}^{(\lambda)}} a(z, Dw) \cdot D(w - u) dz + \frac{1}{2} \|w(\cdot, 0) - u_0\|_{L^2(B_{8\rho})}^2 \\ \geq \int_{\Omega_{T'}} |F|^{p(z)-2} F \cdot D(w - u) dz + \int_{Q_{8\rho}^{(\lambda)}} f(w - u) dz. \end{aligned} \quad (3.6)$$

From the equation (3.5) we have the following weak formulation

$$\langle \partial_t w, \varphi \rangle_{Q_{8\rho}^{(\lambda)}} + \int_{Q_{8\rho}^{(\lambda)}} a(z, Dw) \cdot D\varphi dz = \int_{Q_{8\rho}^{(\lambda)}} \partial_t \psi \cdot \varphi dz + \int_{Q_{8\rho}^{(\lambda)}} a(z, D\psi) \cdot D\varphi dz$$

with admissible test-function  $\varphi = w - u \in W_0^{p(\cdot)}(Q_{8\rho}^{(\lambda)})$ . This can be adapt as follows

$$\begin{aligned} \langle\langle \partial_t w, (w - u) \rangle\rangle_{Q_{8\rho}^{(\lambda)}} + \int_{Q_{8\rho}^{(\lambda)}} a(z, Dw) \cdot D(w - u) \, dz &= \int_{Q_{8\rho}^{(\lambda)}} \partial_t \psi \cdot (w - u) \, dz \\ &+ \int_{Q_{8\rho}^{(\lambda)}} a(z, D\psi) \cdot D(w - u) \, dz. \end{aligned}$$

Now, we subtract the last equation from the variational inequality (3.6) and then, we estimate the right-hand side from above by the absolute value. This yields

$$\begin{aligned} \int_{Q_{8\rho}^{(\lambda)}} (a(z, Dw) - a(z, Du)) \cdot D(w - u) \, dz &\leq \int_{Q_{8\rho}^{(\lambda)}} |\partial_t \psi| |w - u| \, dz \\ &+ \int_{Q_{8\rho}^{(\lambda)}} |f| |w - u| \, dz + \int_{Q_{8\rho}^{(\lambda)}} \left( a(z, D\psi) - |F|^{p(\cdot)-2} F \right) \cdot D(w - u) \, dz \\ &=: I + II + III \end{aligned}$$

with the obvious meaning of  $I$ ,  $II$  and  $III$ . Here, we apply the monotonicity condition (1.4) to the left-hand side. Finally, we can conclude that

$$\nu \int_{Q_{8\rho}^{(\lambda)}} (\mu^2 + |Dw|^2 + |Du|^2)^{\frac{p(\cdot)-2}{2}} |Dw - Du|^2 \, dz \leq I + II + III. \quad (3.7)$$

Next, we estimate the term  $III$  by the absolute value and utilize the growth property (1.3) from above. Therefore, we have

$$III \leq L \int_{Q_{8\rho}^{(\lambda)}} \left( (1 + |D\psi|)^{p(\cdot)-1} + |F|^{p(\cdot)-1} \right) \cdot |D(w - u)| \, dz.$$

After that, we apply the Young's inequality (in the  $\varepsilon$ -version) with exponents  $\frac{p(\cdot)}{p(\cdot)-1}$  and  $p(\cdot)$  to the last inequality. This yields

$$\begin{aligned} III &\leq L \int_{Q_{8\rho}^{(\lambda)}} \varepsilon^{-\frac{1}{p(\cdot)-1}} \frac{p(\cdot)-1}{p(\cdot)} \left[ (1 + |D\psi|)^{p(\cdot)-1} + |F|^{p(\cdot)-1} \right]^{\frac{p(\cdot)}{p(\cdot)-1}} \, dz \\ &+ L \cdot \varepsilon \int_{Q_{8\rho}^{(\lambda)}} \frac{1}{p(\cdot)} |D(w - u)|^{p(\cdot)} \, dz. \end{aligned}$$

This inequality we estimate from above and get

$$\begin{aligned} III &\leq L 2^{\frac{1}{\gamma_2-1}-1} \frac{\gamma_2-1}{\gamma_1} \int_{Q_{8\rho}^{(\lambda)}} \varepsilon^{-\frac{1}{p(\cdot)-1}} (1 + |D\psi| + |F|)^{p(\cdot)} \, dz \\ &+ \frac{L}{\gamma_1} \cdot \varepsilon \int_{Q_{8\rho}^{(\lambda)}} |Dw|^{p(\cdot)} + |Du|^{p(\cdot)} \, dz. \end{aligned}$$

The factors which are smaller or equal than 1, e.g.  $2^{\frac{1}{\gamma_2-1}-1}$ , we estimate from above by 1 and summarize the others factor by a constant, which depends on  $c(\gamma_1, \gamma_2, L) \geq 1$ . Thus, it yields

$$\begin{aligned} III &\leq c(L, \gamma_1, \gamma_2) \int_{Q_{8\rho}^{(\lambda)}} \varepsilon^{-\frac{1}{p(\cdot)-1}} (1 + |D\psi| + |F|)^{p(\cdot)} \, dz \\ &+ c(L, \gamma_1, \gamma_2) \cdot \frac{\varepsilon}{\gamma_1} \int_{Q_{8\rho}^{(\lambda)}} |Dw|^{p(\cdot)} + |Du|^{p(\cdot)} \, dz. \end{aligned} \quad (3.8)$$

Now, we use the Young's inequality to  $II$  (also in the  $\varepsilon$ -version) with exponents  $\frac{p(\cdot)}{p(\cdot)-1}$  and  $p(\cdot)$ . Therefore, we can conclude the following estimate

$$II \leq \frac{\varepsilon}{\gamma_1} c(\gamma_1) \int_{Q_{8\rho}^{(\lambda)}} |w - u|^{\gamma_1} dz + \varepsilon^{-\frac{1}{\gamma_1-1}} c(\gamma_1, \gamma_2) \int_{Q_{8\rho}^{(\lambda)}} |f|^{\gamma'_1} dz.$$

Next, we apply the standard Poincaré inequality slicewise to the first term on the right-hand side. This yields

$$\begin{aligned} II &\leq \frac{\varepsilon}{\gamma_1} c \left( \int_{Q_{8\rho}^{(\lambda)}} |D(w - u)|^{\gamma_1} dz \right) + \varepsilon^{-\frac{1}{\gamma_1-1}} c \int_{Q_{8\rho}^{(\lambda)}} |f|^{\gamma'_1} dz \\ &\leq \frac{\varepsilon}{\gamma_1} c \left( \int_{Q_{8\rho}^{(\lambda)}} |D(w - u)|^{p(\cdot)} + 1 dz \right) + \varepsilon^{-\frac{1}{\gamma_1-1}} c \int_{Q_{8\rho}^{(\lambda)}} |f|^{\gamma'_1} dz \end{aligned} \quad (3.9)$$

with  $c = c(n, \gamma_1, \gamma_2)$ . Note that we used the fact that  $8\rho \leq \rho_0 = \rho_0(n, \gamma_1, \omega(\cdot)) \in (0, 1]$ . Moreover, we use the Young's inequality as above to  $I$ , such that

$$\begin{aligned} I &\leq \varepsilon^{-\frac{1}{\gamma_1-1}} \int_{Q_{8\rho}^{(\lambda)}} |\partial_t \psi|^{\gamma'_1} dz + \frac{\varepsilon}{\gamma_1} \int_{Q_{8\rho}^{(\lambda)}} |w - u|^{\gamma_1} dz \\ &\leq \varepsilon^{-\frac{1}{\gamma_1-1}} \int_{Q_{8\rho}^{(\lambda)}} |\partial_t \psi|^{\gamma'_1} dz + \frac{\varepsilon}{\gamma_1} \left( \int_{Q_{8\rho}^{(\lambda)}} |D(w - u)|^{p(\cdot)} + 1 dz \right) \end{aligned} \quad (3.10)$$

with  $c = c(n, \gamma_1, \gamma_2)$ , where we again used the standard Poincaré inequality slice-wise. Plugging (3.8), (3.9) and (3.10) into (3.7) and estimating the left-hand side from below, this yields

$$\nu \int_{Q_{8\rho}^{(\lambda)}} |Dw - Du|^{p(\cdot)} dz \leq \varepsilon \cdot c \int_{Q_{8\rho}^{(\lambda)}} |Dw|^{p(\cdot)} + |Du|^{p(\cdot)} dz + c_\varepsilon \int_{Q_{8\rho}^{(\lambda)}} \Psi dz \quad (3.11)$$

for any  $Q_{8\rho}^{(\lambda)} \subset \Omega_T$  with  $8\rho \leq \rho_0$  and constants  $c_\varepsilon = c_\varepsilon(\varepsilon, n, \gamma_1, \gamma_2, L)$  and  $c = c(n, \gamma_1, \gamma_2, L)$  with  $\varepsilon \in (0, 1)$ , where we used  $\Psi$  which is introduced in (2.2). Hereby, we can along estimate the inequality (3.11) from below. Here, we can deduce the following **energy estimate**

$$\int_{Q_{8\rho}^{(\lambda)}} |Dw|^{p(\cdot)} dz \leq c \left( \int_{Q_{8\rho}^{(\lambda)}} |Du|^{p(\cdot)} + \Psi dz \right) \quad (3.12)$$

for any  $Q_{8\rho}^{(\lambda)} \subset \Omega_T$  with  $8\rho \leq \rho_0$  with a constant  $c(n, \gamma_1, \gamma_2, \nu, L) \geq 1$ , where we had chosen  $\varepsilon$  depending on  $(n, \gamma_1, \gamma_2, \nu, L)$  in (3.11). Next, we combine (3.12) with (3.11) and get, where we have to choose conveniently  $\varepsilon$  (e.g.  $\varepsilon = \delta/c$ ), the following **comparison estimate**

$$\int_{Q_{8\rho}^{(\lambda)}} |Dw - Du|^{p(\cdot)} dz \leq \delta \int_{Q_{8\rho}^{(\lambda)}} |Du|^{p(\cdot)} dz + c_\delta \left( \int_{Q_{8\rho}^{(\lambda)}} \Psi dz \right) \quad (3.13)$$

for any  $Q_{8\rho}^{(\lambda)} \subset \Omega_T$  with  $8\rho \leq \rho_0$  with  $\delta \in (0, 1]$  and a constant  $c_\delta = (c_\delta, n, \gamma_1, \gamma_2, \nu, L)$ . Our next aim is to show that the cylinder  $Q_{\rho}^{(\lambda)}$  is also, with respect to the comparison function  $w$ , an intrinsic cylinder, i.e. there holds also the lower bound (3.1)

and the upper bound (3.2) for  $Dw$  instead of  $Du$ . We start with the lower bound for  $\int_{Q_\rho^{(\lambda)}} |Dw|^{p(\cdot)} dz$ . Therefore, we consider the following relation

$$|Du|^{p(\cdot)} = |Du - Dw + Dw|^{p(\cdot)} \leq 2^{p(\cdot)-1} |Du - Dw|^{p(\cdot)} + 2^{p(\cdot)-1} |Dw|^{p(\cdot)}.$$

This is equivalent to  $|Dw|^{p(\cdot)} \geq 2^{1-p(\cdot)} |Du|^{p(\cdot)} - |Du - Dw|^{p(\cdot)}$ . Thus, we can conclude

$$\int_{Q_\rho^{(\lambda)}} |Dw|^{p(\cdot)} dz \geq 2^{1-\gamma_1} \int_{Q_\rho^{(\lambda)}} |Du|^{p(\cdot)} dz - \int_{Q_\rho^{(\lambda)}} |Du - Dw|^{p(\cdot)} dz.$$

The first term on the right-hand side can be estimated from below by the lower bound (3.1). Then, we gain

$$\int_{Q_\rho^{(\lambda)}} |Dw|^{p(\cdot)} dz \geq 2^{1-\gamma_1} \lambda - \int_{Q_\rho^{(\lambda)}} |Du - Dw|^{p(\cdot)} dz - 2^{1-\gamma_1} \left( \int_{Q_\rho^{(\lambda)}} \Psi dz \right). \quad (3.14)$$

Now, we want to apply the comparison estimate (3.13) to the third term on the right-hand side. This is only possible, if we exchange from the smaller cylinder  $Q_\rho^{(\lambda)}$  into the bigger cylinder  $Q_{8\rho}^{(\lambda)}$ . Since, we consider the mean value, we conserve an additional factor, i.e.  $\frac{|Q_{8\rho}^{(\lambda)}|}{|Q_\rho^{(\lambda)}|} = \frac{\alpha_n (8\rho)^n}{\alpha_n \rho^n} \cdot \frac{\lambda^{\frac{2-p_0}{p_0}} (8\rho)^2}{\lambda^{\frac{2-p_0}{p_0}} \rho^2} = 8^{n+2}$ , where  $\alpha_n$  denotes the measure of the unity ball. Hence, it yields

$$\begin{aligned} \int_{Q_\rho^{(\lambda)}} |Dw|^{p(\cdot)} dz &\geq 2^{1-\gamma_1} \lambda - 2^{1-\gamma_1} \int_{Q_\rho^{(\lambda)}} \Psi dz - \delta 8^{n+2} \int_{Q_{8\rho}^{(\lambda)}} |Du|^{p(\cdot)} dz \\ &\quad - c_\delta \left( \int_{Q_{8\rho}^{(\lambda)}} \Psi dz \right) \end{aligned}$$

with a constant  $c_\delta = c(\delta, n, \gamma_1, \gamma_2, \nu, L)$ . At this stage, we determinate  $\delta = 2^{-\gamma_1-3(n+2)}$ . Then, in the second term on the right-hand side of (3.14) we also exchange from the smaller cylinder  $Q_\rho^{(\lambda)}$  into the bigger cylinder  $Q_{8\rho}^{(\lambda)}$  and use the upper bound (3.2). Therefore, we can conclude that

$$\begin{aligned} \int_{Q_\rho^{(\lambda)}} |Dw|^{p(\cdot)} dz + c \left( \int_{Q_{8\rho}^{(\lambda)}} \Psi dz \right) &\geq 2^{1-\gamma_1} \lambda - 2^{-\gamma_1} \int_{Q_{8\rho}^{(\lambda)}} |Du|^{p(\cdot)} dz \\ &\quad - 2^{1-\gamma_1} 8^{n+2} \left( \int_{Q_{8\rho}^{(\lambda)}} \Psi dz \right) \geq 8^{n+2} 2^{-\gamma_1} \lambda, \end{aligned}$$

where  $c = c(\delta, n, \gamma_1, \gamma_2, \nu, L)$ . This yields the following

$$\int_{Q_\rho^{(\lambda)}} |Dw|^{p(\cdot)} dz + \int_{Q_{8\rho}^{(\lambda)}} \Psi dz \geq c^{-1} \lambda,$$

for any  $Q_{8\rho}^{(\lambda)} \subset \Omega_T$  with  $8\rho \leq \rho_0$  with a constant  $c = c(n, \gamma_1, \gamma_2, \nu, L)$ . Furthermore, we can conclude from the energy estimate (3.12) and the bound (3.2), the upper bound for the spatial gradient of the comparison function  $w$ . Even, it is valid

$$\int_{Q_{8\rho}^{(\lambda)}} |Dw|^{p(\cdot)} + \Psi dz \leq c \left( \int_{Q_{8\rho}^{(\lambda)}} |Du|^{p(\cdot)} + \Psi dz \right) \leq c\lambda$$



for any  $Q_{8\rho}^{(\lambda)} \subset \Omega_T$  with  $8\rho \leq \rho_0$ , where  $c = c(n, \gamma_1, \gamma_2, \nu, L) \geq 1$ . Now, we compose  $\kappa := c(n, \gamma_1, \gamma_2, \nu, L)$ . Hence, we have shown that the comparison function  $w$  satisfies the condition of Lemma 3.2. The next tool we need, is stated by Bögelein and Duzaar in [13, Lemma 6.1]. This Lemma we will use to prove the Reverse Hölder type inequality by a comparison argument which allows to carry over the estimate for solutions to certain parabolic equations from [35] to the solutions of the variational inequality.

**Lemma 3.2.** *Let  $M, \kappa \geq 1$ , where  $M$  is introduced in (1.18) and the exponent function  $p : \Omega_T \rightarrow [\gamma_1, \gamma_2]$  satisfies the conditions (1.5)-(1.7). Then, there exists a radius  $\rho_1 = \rho_1(n, \gamma_1, \gamma_2, L_1, M) \in (0, 1)$  and a constant  $c = c(n, \gamma_1, \gamma_2, \nu, L, L_1, \kappa)$ , such that the following holds: Suppose that (1.8) is valid. Further, assume that the boundary data and the obstacle function satisfy (1.10) and (1.11). Moreover, let  $w \in C^0([0, T]; L^2(\Omega)) \cap W^{p(\cdot)}(\Omega_T)$  a weak solution of the parabolic equation*

$$\partial_t w - \operatorname{div} a(z, Dw) = \partial_t \psi - \operatorname{div} a(z, D\psi) \text{ on } \Omega_T, \quad (3.15)$$

where the vector-field  $a$  satisfies the growth condition (1.3) and the monotonicity condition (1.4). Moreover,  $w$  satisfies the energy bound (1.18) for the given number  $M$ . Then, on any parabolic cylinder  $Q_{8\rho}^{(\lambda)}(z_0) \subseteq \Omega_T$  with  $0 < \rho \leq \rho_1$  and  $\lambda \geq 1$ , on which an intrinsic coupling is given, in the sense that on  $Q_\rho^{(\lambda)}(z_0)$  the lower bound

$$\kappa^{-1} \lambda \leq \int_{Q_\rho^{(\lambda)}(z_0)} |Dw|^{p(\cdot)} dz + \int_{Q_{8\rho}^{(\lambda)}(z_0)} \Psi dz \quad (3.16)$$

holds, while on  $Q_{8\rho}^{(\lambda)}(z_0)$  the upper bound

$$\int_{Q_{8\rho}^{(\lambda)}(z_0)} |Dw|^{p(\cdot)} + \Psi dz \leq \kappa \lambda \quad (3.17)$$

is fulfilled, there holds the following Reverse Hölder type inequality

$$\int_{Q_\rho^{(\lambda)}(z_0)} |Dw|^{p(\cdot)} dz \leq c \left( \int_{Q_{2\rho}^{(\lambda)}(z_0)} |Dw|^{\frac{p(\cdot)}{\vartheta}} dz \right)^\vartheta + c \left( \int_{Q_{8\rho}^{(\lambda)}(z_0)} \Psi dz \right), \quad (3.18)$$

where  $\vartheta$  is defined as in (3.4) and  $\Psi$  is defined as in (2.2).

**Remark 3.3.** At this stage, we will explain how we have to modify the proof of Lemma 6.1 in [13], to prove Lemma 3.2. First, notice that  $a(z, D\psi)$  from (3.15) play the role of  $F$  in the equation (1.1) from [13]. Moreover, the additional term  $\partial_t \psi$  in (3.15) - which does not occur in the preceding equation - can be treat in the standard way. More precisely, in the proof of the Caccoppoli inequality, see [13, Lemma 4.1], we get an additional term because we have to include the term  $\partial_t \psi$  as follows

$$\int_{Q_\rho^{(\lambda)}(z_0)} \partial_t \psi \cdot \varphi dz,$$

where  $\varphi = \theta(u - A)$  with  $A \in \mathbb{R}^n$  and  $\theta$  is a admissible cut-off function. This expression can be estimate Young inequality from above, this yields

$$\int_{Q_\rho^{(\lambda)}(z_0)} \partial_t \psi \cdot \varphi dz \leq c(\gamma_1, \gamma_2) \int_{Q_\rho^{(\lambda)}(z_0)} |\partial_t \psi|^{\gamma'_1} + |\varphi|^{\gamma_1} dz.$$

Therefore, we have with respect to the requirements of Lemma 4.1 in [13] the following **Caccoppoli inequality** to the problem (3.15)

$$\begin{aligned} & \sup_{t \in \Lambda_r^{(\lambda)}(z_0)} \int_{B_r(x_0)} \lambda^{\frac{p_0-2}{p_0}} \left| \frac{u(\cdot, t) - A}{r} \right|^2 dx + \int_{Q_r^{(\lambda)}(x_0)} |Du|^{p(\cdot)} dz \\ & \leq c \int_{Q_\rho^{(\lambda)}(x_0)} \left[ \lambda^{\frac{p_0-2}{p_0}} \left| \frac{u - A}{\rho - r} \right|^2 + \left| \frac{u - A}{\rho - r} \right|^{p(\cdot)} + |D\psi|^{p(\cdot)} + |\partial_t \psi|^{\gamma'_1 + 1} \right] dz. \end{aligned}$$

The necessary modifications in the proof of the Poincaré-type inequality [13, Lemma 5.1] are similar the same. Therefore, we get with respect to the requirements of Lemma 5.1 in [13] the following **Poincaré-type inequality** to the problem (3.15)

$$\begin{aligned} & \int_{Q_\rho^{(\lambda)}(x_0)} \left| \frac{u - (u)^{(\lambda)}_{z_0, \rho}}{\rho} \right|^\vartheta dz \leq \int_{Q_\rho^{(\lambda)}(x_0)} |Du|^\vartheta dz \\ & + \lambda^{\frac{2-p_0}{p_0}} \int_{Q_\rho^{(\lambda)}(x_0)} \left( |Du|^{p(\cdot)} + |D\psi|^{p(\cdot)} + |\partial_t \psi|^{\gamma'_1 + 1} \right)^{p(\cdot)} dz. \end{aligned}$$

□

Now, we will convert the inequality (3.18) into a Reverse Hölder type inequality to the solution  $u$ . For this aim, we have to choose a radius  $\rho_2$  depending on  $(n, \gamma_1, \gamma_2, L_1, \omega(\cdot))$ , such that

$$\rho_2 := \min \left\{ \frac{\rho_0}{8}, \rho_1 \right\},$$

where  $\rho_0$  is the radius of Corollary 2.2 and  $\rho_1$  the radius of Lemma 3.2. This yields

$$\begin{aligned} & \int_{Q_\rho^{(\lambda)}} |Du|^{p(\cdot)} dz \leq 2^{\gamma_2-1} \int_{Q_\rho^{(\lambda)}} |Dw|^{p(\cdot)} dz + 2^{\gamma_2-1} \int_{Q_\rho^{(\lambda)}} |Dw - Du|^{p(\cdot)} dz \\ & \leq c \left( \int_{Q_{2\rho}^{(\lambda)}} |Du|^{\frac{p(\cdot)}{\vartheta}} dz \right)^\vartheta + c \left( \int_{Q_{2\rho}^{(\lambda)}} |Dw - Du|^{\frac{p(\cdot)}{\vartheta}} dz \right)^\vartheta \\ & + c \left( \int_{Q_{8\rho}^{(\lambda)}} \Psi dz \right) + 2^{\gamma_2-1} \int_{Q_\rho^{(\lambda)}} |Dw - Du|^{p(\cdot)} dz, \end{aligned}$$

where  $c = c(n, \gamma_1, \gamma_2, \nu, L, L_1) \geq 1$ . Next, we use the standard Hölder's inequality with exponents  $\vartheta$  and  $\vartheta/(\vartheta - 1)$  to the right-hand side of the last inequality. All in all, we get the following estimate

$$\begin{aligned} & \int_{Q_\rho^{(\lambda)}} |Du|^{p(\cdot)} dz \leq c \left( \int_{Q_{2\rho}^{(\lambda)}} |Du|^{\frac{p(\cdot)}{\vartheta}} dz \right)^\vartheta + c \left( \int_{Q_{8\rho}^{(\lambda)}} \Psi dz \right) \\ & + c \int_{Q_{8\rho}^{(\lambda)}} |Dw - Du|^{p(\cdot)} dz, \end{aligned} \tag{3.19}$$

where  $c = c(n, \gamma_1, \gamma_2, \nu, L, L_1) \geq 1$ . At this stage, we estimate the last term on the right-hand side from above by a version of the comparison estimate (3.13). Notice here, that the lower bound (3.1) and the upper bound (3.2) imply the following

$$\int_{Q_{8\rho}^{(\lambda)}} |Du|^{p(\cdot)} dz \leq \lambda \leq \int_{Q_\rho^{(\lambda)}} |Du|^{p(\cdot)} dz + \int_{Q_{8\rho}^{(\lambda)}} \Psi dz.$$

Combining this with the comparison estimate (3.13), then we get for  $0 < \delta \leq 1$  the following estimate

$$\int_{Q_{8\rho}^{(\lambda)}} |Dw - Du|^{p(\cdot)} dz \leq \delta \int_{Q_{\rho}^{(\lambda)}} |Du|^{p(\cdot)} dz + c_{\delta} \left( \int_{Q_{8\rho}^{(\lambda)}} \Psi dz \right) \quad (3.20)$$

with a constant  $c_{\delta} = c(\delta, n, \gamma_1, \gamma_2, \nu, L, L_1) \geq 1$ . Plugging now (3.20) into (3.19) and choosing  $\delta$  small enough, e.g.  $\delta = 1/2$  or similarly, we can reabsorb the term  $\delta \int_{Q_{\rho}^{(\lambda)}} |Du|^{p(\cdot)} dz$  on the left-hand side. Therefore, it follows the desired Reverse Hölder's type inequality to the solution  $u$

$$\int_{Q_{\rho}^{(\lambda)}} |Du|^{p(\cdot)} dz \leq c \left( \int_{Q_{2\rho}^{(\lambda)}} |Du|^{\frac{p(\cdot)}{\vartheta}} dz \right)^{\vartheta} + c \left( \int_{Q_{8\rho}^{(\lambda)}} \Psi dz \right)$$

with a constant  $c = c(n, \gamma_1, \gamma_2, \nu, L, L_1) \geq 1$  and radius  $\rho \leq \rho_2$ . This proves the conclusion of the Lemma.  $\square$

#### 4. PROOF OF THEOREM 1.4

At this stage, we will now show the higher integrability of  $|Du|$ . The Reverse Hölder type inequality plays an important role to show that

$$|Du| \in L_{\text{loc}}^{p(\cdot)(1+\epsilon_0)}(\Omega_T, \mathbb{R}^n)$$

under the conditions, which will be mentioned in Theorem 1.4. In Theorem 1.4 we will establish the quantitative gradient estimate, which implies the higher integrability of a localizable solution, see Definition 1.3. A further tool we will need, is a covering argument. As we have seen in the previous chapter a Reverse Hölder type inequality is only available on intrinsic parabolic cylinders, see Lemma 3.1. Therefore, the main objective in the proof of the higher integrability theorem is to find parabolic cylinders covering the upper-level set of the spatial gradient in the sense of a Vitali-type covering, such that on each cylinder the intrinsic coupling in the form of (3.1) and (3.2) holds. Here, we will apply a version of the Vitali's covering Theorem for non-uniformly intrinsic parabolic cylinders, which is stated by Bögelein and Duzaar in [13, Lemma 7.1]. The result is the following:

**Lemma 4.1.** *Assume that  $M \geq 1$ ,  $\lambda \geq 1$  and  $p : \Omega_T \rightarrow (\gamma_1, \gamma_2)$  satisfies the conditions (1.5)-(1.7). Then, there exists a constant  $\chi = \chi(n, L_1, \gamma_1) \geq 5$ , such that the following is true:*

*Let  $\mathcal{F} = \{Q_i\}_{i \in \mathcal{J}}$  be a family of axially parallel parabolic cylinders of the form*

$$Q_i = Q_{\rho_i}^{(\lambda)}(z_i) := B_{\rho_i}(x_i) \times \left( t_i - \lambda^{\frac{2-p(z_i)}{p(z_i)}} \rho_i^2, t_i + \lambda^{\frac{2-p(z_i)}{p(z_i)}} \rho_i^2 \right)$$

*with uniformly bounded radii, uniformly in the sense that*

$$\rho_i \leq \min \left\{ \rho_3, \left[ (\beta_n M)^{\frac{p(z_i)}{2}} \lambda^{-1} \right]^{\frac{2}{p(z_i)(n+2)}} \right\} \quad \forall i \in \mathcal{J}, \quad (4.1)$$

*where*

$$\rho_3 := \left[ 6\sqrt{\beta_n M} \right]^{-\frac{2}{\alpha}} \leq 1.$$

Then, there exists a countable sub-collection  $\mathcal{G} \subseteq \mathcal{F}$  of disjoint parabolic cylinders, such that

$$\bigcup_{Q \in \mathcal{F}} Q \subset \bigcup_{Q \in \mathcal{G}} \chi Q,$$

where  $\chi Q$  denotes the  $\chi$ -time enlarged cylinder  $Q$ , i.e. if  $Q = Q_\rho^{(\lambda)}(z)$  then  $\chi Q = Q_{\chi\rho}^{(\lambda)}(z)$ .

Now, we have the essential components, on which the proof of the higher integrability is based. Therefore, the next part of this section is treated with the proof of the higher integrability. As we mentioned above, we have to find cylinders on which the intrinsic coupling in the form of (3.2) and (3.1) holds. Hence, we will use a **stopping time argument**, which exploits in a certain way the continuous dependence of the integral on the domain of integration. The system of intrinsic cylinders covering the upper-level set of  $|Du|$  however does not yield a Vitali-type covering in the usual sense. The peculiarity thereby is that the family of intrinsic cylinders is not uniform - even if the scaling factor  $\lambda$  is fixed - in the sense that the scaling of a cylinder around a point  $z_0$  is  $\lambda^{\frac{2-p_0}{p_0}}$  and hence depends on  $p_0 = p(z_0)$ . Our main result of this chapter reads as follows.

*Proof of Theorem 1.4.* The proof of Theorem 1.4 is divided in several steps. First, we define

$$\lambda_0^{\frac{1}{d(p_m)} + \frac{2}{p_M} - \frac{2}{p_m}} := \oint_{Q_{2r}} |Du|^{p(\cdot)} + \Psi dz \geq 1 \quad (4.2)$$

with  $p_m := \inf_{Q_{2r}} p(\cdot)$  and  $p_M := \sup_{Q_{2r}} p(\cdot)$ . Furthermore, we use the following

$$d(p_m) := \begin{cases} \frac{2p_m}{p_m(n+2)-p_m} & \text{if } p_m < 2 \\ \frac{p_m}{2} & \text{if } p_m \geq 2. \end{cases}$$

Moreover, we consider the concentric parabolic cylinders

$$Q_r \subseteq Q_{r_1} \subset Q_{r_2} \subseteq Q_{2r}$$

for fixed radii  $r \leq r_1 < r_2 \leq 2r$ . Then, the parabolic cylinders of the type

$$Q_s^{(\lambda)}(z_0) \text{ with } 0 < s \leq \min \left\{ \lambda^{\frac{p_m-2}{2p_m}}, 1 \right\} \left( \frac{r_2 - r_1}{2} \right) = R_0$$

are contained in  $Q_{r_2}$ , where  $\lambda \geq \lambda_0$  and  $z_0 \in Q_{r_1}$ .

**Step 1: Choice of the intrinsic cylinders.** Now, with the help of a **stopping time argument**, we want to construct an appropriate cylinder with center in  $z_0$ , which satisfies the lower bound (3.1) and the upper bound (3.2) from Lemma 3.1. We start by showing the availability of the lower bound (3.1). By the Lebesgue's differentiation theorem, we can conclude that for a.e. points  $z_0 \in Q_{r_1}$ , which satisfy the condition  $|Du(z_0)|^{p_0} > \lambda$ , we have the following estimate

$$\lim_{s \downarrow 0} \left( \oint_{Q_s^{(\lambda)}(z_0)} |Du|^{p(\cdot)} + \Psi dz \right) \geq |Du(z_0)|^{p_0} > \lambda. \quad (4.3)$$

This can be shown by the following fact: We know that in the Lebesgue points  $z_0$  of  $Du$  we have (see for example [30], see also [13])

$$\lim_{s \downarrow 0} \int_{Q_s(z_0)} |Du|^{p(\cdot)} dz = |Du(z_0)|^{p_0}.$$

At this stage, it is worth to mention that the averages are taken with respect to the usual cylinders while in (4.3) we need to have the averages on cylinders  $Q_s^{(\lambda)}(z_0)$ . Since,

$$Q_s^{(\lambda)}(z_0) \subseteq Q_{\mu s}(z_0) \text{ for } \mu := \max \left\{ \lambda^{\frac{2-p_0}{2p_0}}, 1 \right\},$$

we get

$$\begin{aligned} & \lim_{s \downarrow 0} \int_{Q_s^{(\lambda)}(z_0)} \left| |Du|^{p(\cdot)} - |Du(z_0)|^{p_0} \right| dz \\ & \leq \frac{\mu^{n+2}}{\lambda^{\frac{2-p_0}{2p_0}}} \lim_{s \downarrow 0} \int_{Q_{\mu s}(z_0)} \left| |Du|^{p(\cdot)} - |Du(z_0)|^{p_0} \right| dz = 0. \end{aligned}$$

Hence (4.3) is valid. Next, we want to deduce the upper bound (3.2). Therefore, we consider  $\lambda$ , which satisfies

$$\lambda > B\lambda_0 \quad (4.4)$$

with

$$B^{\frac{1}{d(p_m)} + \frac{2}{p_M} - \frac{2}{p_m}} := \left( \frac{8\chi r}{r_2 - r_1} \right)^{n+2}, \quad (4.5)$$

where  $\chi = \chi(n, L_1, \gamma_1) \geq 5$  denote the corresponding constant from Lemma 4.1. In addition, we observe radii  $s$ , which are conform to

$$\frac{1}{2\chi} R_0 = \frac{\min \left\{ \lambda^{\frac{p_m-2}{2p_m}}, 1 \right\} (r_2 - r_1)}{4\chi} \leq s \leq \frac{\min \left\{ \lambda^{\frac{p_m-2}{2p_m}}, 1 \right\} (r_2 - r_1)}{2} = R_0. \quad (4.6)$$

Notice that the maximal radius  $R_0$  is chosen, such that for all points  $z_0 \in Q_{r_1}$  and radii  $s \leq R_0$  the inclusion  $Q_s^{(\lambda)}(z_0) \subset Q_{r_2}$  is fulfilled. From this fact and the definition of  $\lambda_0$  we can conclude that

$$\begin{aligned} \int_{Q_s^{(\lambda)}(z_0)} |Du|^{p(\cdot)} + \Psi dz & \leq \frac{|Q_{2r}|}{|Q_s^{(\lambda)}(z_0)|} \left( \int_{Q_{2r}} |Du|^{p(\cdot)} + \Psi dz \right) \\ & \leq \frac{|Q_{2r}|}{|Q_s^{(\lambda)}(z_0)|} \left( \int_{Q_{2r}} |Du|^{p(\cdot)} + \Psi dz \right) \\ & \leq \left( \frac{2r}{s} \right)^{n+2} \lambda^{\frac{p_0-2}{p_0}} \lambda_0^{\frac{1}{d(p_m)} + \frac{2}{p_M} - \frac{2}{p_m}} \end{aligned} \quad (4.7)$$

holds, where we used (4.2) for the last estimate. Now, we are in the situation to estimate the right-hand side from above. For this aim, we have to treat the two cases  $2 \leq p_m \leq \gamma_2$  and  $\gamma_1 \leq p_m < 2$  separated. In the case  $p_m \geq 2$ , we have  $d(p_m) = \frac{p_m}{2}$  and  $\min \left\{ \lambda^{\frac{p_m-2}{2p_m}}, 1 \right\} = 1$ . Hence, this yields

$$\int_{Q_s^{(\lambda)}(z_0)} |Du|^{p(\cdot)} + \Psi dz \stackrel{(4.7)}{\leq} \left( \frac{2r}{s} \right)^{n+2} \lambda^{\frac{p_0-2}{p_0}} \lambda_0^{\frac{2}{p_M}}$$

$$\begin{aligned}
 &\stackrel{(4.6)}{\leq} \left( \frac{8\chi r}{r_2 - r_1} \right)^{n+2} \lambda^{\frac{p_0-2}{p_0}} \lambda_0^{\frac{2}{p_M}} \\
 &\stackrel{(4.4)}{<} \left( \frac{8\chi r}{r_2 - r_1} \right)^{n+2} \lambda^{\frac{p_0-2}{p_0}} \left( \frac{\lambda}{B} \right)^{\frac{2}{p_M}} \\
 &\stackrel{(4.5)}{=} \lambda.
 \end{aligned}$$

Therefore, we have

$$\oint_{Q_s^{(\lambda)}(z_0)} |Du|^{p(\cdot)} + \Psi \, dz < \lambda. \quad (4.8)$$

In the case  $\gamma_1 \leq p_m < 2$ , there is  $d(p_m) = \frac{2p_m}{p_m(n+2)-n}$  respectively  $\frac{1}{d(p_m)} = \frac{n+2}{2} - \frac{n}{p_m}$  and  $\min \left\{ \lambda^{\frac{p_m-2}{2p_m}}, 1 \right\} = \lambda^{\frac{p_m-2}{2p_m}}$ , we can conclude, in the same way as in (4.8), that

$$\begin{aligned}
 \oint_{Q_s^{(\lambda)}(z_0)} |Du|^{p(\cdot)} + \Psi \, dz &\stackrel{(4.6)}{\leq} \left( \frac{8\chi r}{\lambda^{\frac{p_m-2}{2p_m}}(r_2 - r_1)} \right)^{n+2} \lambda^{\frac{p_0-2}{p_0}} \lambda_0^{\frac{1}{d(p_m)} + \frac{2}{p_M} - \frac{2}{p_m}} \\
 &< \left( \frac{8\chi r}{r_2 - r_1} \right)^{n+2} \frac{\lambda}{B^{\frac{1}{d(p_m)} + \frac{2}{p_M} - \frac{2}{p_m}}} \\
 &\stackrel{(4.5)}{=} \lambda.
 \end{aligned}$$

This addict the same estimate as in (4.8), even

$$\oint_{Q_s^{(\lambda)}(z_0)} |Du|^{p(\cdot)} + \Psi \, dz < \lambda. \quad (4.9)$$

The availability can be shown easily. For that purpose, we consider the following calculation

$$\begin{aligned}
 \left( \frac{1}{\lambda^{\frac{p_m-2}{2p_m}}} \right)^{n+2} \lambda^{\frac{p_0-2}{p_0}} \lambda_0^{\frac{1}{d(p_m)} + \frac{2}{p_M} - \frac{2}{p_m}} &= \lambda^{\frac{n+2}{p_m} - \frac{n+2}{2}} \lambda^{\frac{p_0-2}{p_0}} \lambda_0^{\frac{1}{d(p_m)} + \frac{2}{p_M} - \frac{2}{p_m}} \\
 &= \lambda^{\frac{2}{p_m} - \frac{1}{d(p_m)}} \lambda^{\frac{p_0-2}{p_0}} \lambda_0^{\frac{1}{d(p_m)} + \frac{2}{p_M} - \frac{2}{p_m}} \\
 &\leq \lambda^{1 + \frac{2}{p_m} - \frac{2}{p_M} - \frac{1}{d(p_m)}} \lambda_0^{\frac{1}{d(p_m)} + \frac{2}{p_M} - \frac{2}{p_m}} \\
 &\stackrel{(4.4)}{\leq} \frac{\lambda}{B^{\frac{1}{d(p_m)} + \frac{2}{p_M} - \frac{2}{p_m}}}.
 \end{aligned}$$

Now, we have on the one hand a cylinder  $Q_s^{(\lambda)}(z_0)$ , on which the integral is smaller than  $\lambda$ , see (4.8) respectively (4.9), and on the other hand we have shown that the integral over this cylinder is bigger than  $\lambda$ , see (4.3). More precisely, by the absolute continuity of the integral, we can conclude from (4.3) and (4.8) in the case  $p_m \geq 2$  respectively from (4.3) and (4.9) in the case  $p_m < 2$  that, there exists a maximal radius  $0 < \rho_{z_0} < \frac{1}{2\chi} R_0 \Leftrightarrow 0 < 2\chi\rho_{z_0} < R_0 < r_2$ , such that

$$\oint_{Q_{\rho_{z_0}}^{(\lambda)}(z_0)} |Du|^{p(\cdot)} + \Psi \, dz = \lambda, \quad (4.10)$$

while for any  $s \in (\rho_{z_0}, R_0)$  we have the following estimate

$$\oint_{Q_s^{(\lambda)}(z_0)} |Du|^{p(\cdot)} + \Psi \, dz < \lambda. \quad (4.11)$$

At this stage, notice that  $Q_{2\chi\rho_{z_0}}^{(\lambda)}(z_0) \subseteq Q_{r_2}$ . Thus, we can deduce from (4.10) and (4.11) with  $s = 8\rho_{z_0}$ , that the lower bound (3.1) and the upper bound (3.2) from Lemma 3.1 are fulfilled. Furthermore, note that  $8 \leq 2\chi$  and therefore

$$8\rho_{z_0} \in (\rho_{z_0}, R_0) = \left( \rho_{z_0}, \min \left\{ \lambda^{\frac{p_m-2}{p_m}}, 1 \right\} \frac{(r_2 - r_1)}{2} \right)$$

are in force. The next desired goal, is to have a situation, where we are able to use Lemma 3.1. Hence, we still require a upper bound to the radius  $r$ , such that

$$r \leq r_0 = r_0(n, \gamma_1, \gamma_2, L_1, \omega(\cdot), M).$$

Thereby,  $M$  is the energy bound from (1.18) and  $r_0$  denotes the radius bound from Lemma 3.1, i.e.  $r_0 \equiv \rho_2$ . Now, we can use Lemma 3.1. This yields the following Reverse Hölder type inequality

$$\int_{Q_{\rho_{z_0}}^{(\lambda)}(z_0)} |Du|^{p(\cdot)} dz \leq c \left( \int_{Q_{2\rho_{z_0}}^{(\lambda)}(z_0)} |Du|^{\frac{p(\cdot)}{\vartheta}} dz \right)^{\vartheta} + c \left( \int_{Q_{8\rho_{z_0}}^{(\lambda)}(z_0)} \Psi dz \right), \quad (4.12)$$

where  $\vartheta$  satisfies the condition (3.4) and  $c = c(n, \gamma_1, \gamma_2, \nu, L, L_1) \geq 1$ .

**Step 2: Estimates on the level sets.** In the following, we will use the notation

$$E(\rho, \lambda) := \left\{ z \in Q_\rho : |Du|^{p(z)} > \lambda \right\}$$

and

$$G(\rho, \lambda) := \{ z \in Q_\rho : \Psi > \lambda \}$$

for the upper level sets of  $|Du|^{p(\cdot)}$  and  $\Psi$  on cylinders  $Q_\rho$  with  $\rho \in [r, 2r]$ . For  $\eta \in (0, 1)$ , which we have to fix later, we consider the level sets  $E(\rho, \eta\lambda)$  and  $G(\rho, \eta\lambda)$ . In the case  $\eta\lambda > B\lambda_0$ , there exists for a.e.  $z_0 \in E(r_1, \eta\lambda)$  a parabolic cylinder  $Q_{\rho_{z_0}}^{(\lambda)}(z_0)$ , on which (4.10), (4.11) and (4.12) are in force. Moreover, we have  $Q_{2\chi\rho_{z_0}}^{(\lambda)}(z_0) \subseteq Q_{r_2}$ . In addition, we define  $p_0 := p(z_0)$ ,  $p_1 := \inf_{Q_{2\chi\rho_{z_0}}^{(\lambda)}(z_0)} p(\cdot)$  and  $p_2 := \sup_{Q_{2\chi\rho_{z_0}}^{(\lambda)}(z_0)} p(\cdot)$ . Our next aim is to infer a suitable estimate for the  $L^{p(\cdot)}$ -norm of  $Du$  on the cylinder  $Q_{2\chi\rho_{z_0}}^{(\lambda)}(z_0)$ . We start with the Reverse Hölder type inequality (4.12), where  $\rho_{z_0} \leq \rho_2$  ( $r \leq r_0 \leq [6\sqrt{\beta_n M}]^{-\frac{2}{\alpha}}$ )

$$\begin{aligned} \int_{Q_{\rho_{z_0}}^{(\lambda)}(z_0)} |Du|^{p(\cdot)} + \Psi dz &\leq c \left( \int_{Q_{2\rho_{z_0}}^{(\lambda)}(z_0)} |Du|^{\frac{p(\cdot)}{\vartheta}} dz \right)^{\vartheta} + c \left( \int_{Q_{8\rho_{z_0}}^{(\lambda)}(z_0)} \Psi dz \right) \\ &= c \left( \frac{1}{|Q_{2\rho_{z_0}}^{(\lambda)}(z_0)|} \int_{Q_{2\rho_{z_0}}^{(\lambda)}(z_0) \cap E(r_2, \eta\lambda)} |Du|^{\frac{p(\cdot)}{\vartheta}} dz \right. \\ &\quad \left. + \frac{1}{|Q_{2\rho_{z_0}}^{(\lambda)}(z_0)|} \int_{Q_{2\rho_{z_0}}^{(\lambda)}(z_0) \setminus E(r_2, \eta\lambda)} |Du|^{\frac{p(\cdot)}{\vartheta}} dz \right)^{\vartheta} \\ &\quad + \frac{c}{|Q_{8\rho_{z_0}}^{(\lambda)}(z_0)|} \left( \int_{Q_{8\rho_{z_0}}^{(\lambda)}(z_0) \cap G(r_2, \eta\lambda)} \Psi dz \right) \end{aligned}$$



$$+ \frac{c}{|Q_{8\rho_{z_0}}^{(\lambda)}(z_0)|} \left( \int_{Q_{8\rho_{z_0}}^{(\lambda)}(z_0) \setminus G(r_2, \eta\lambda)} \Psi dz \right).$$

Further, we use the facts

$$Q_{2\rho_{z_0}}^{(\lambda)}(z_0) \setminus E(r_2, \eta\lambda) = \left\{ z \in Q_{2\rho_{z_0}}^{(\lambda)}(z_0) : |Du|^{p(z)} \leq \eta\lambda \right\}$$

and

$$Q_{8\rho_{z_0}}^{(\lambda)}(z_0) \setminus G(r_2, \eta\lambda) = \left\{ z \in Q_{8\rho_{z_0}}^{(\lambda)}(z_0) : \Psi \leq \eta\lambda \right\},$$

since  $Q_{8\rho_{z_0}}^{(\lambda)}(z_0) \subset Q_{2\chi\rho_{z_0}}^{(\lambda)}(z_0) \subseteq Q_{r_2}$ . This yields

$$\begin{aligned} \int_{Q_{\rho_{z_0}}^{(\lambda)}(z_0)} |Du|^{p(\cdot)} + \Psi dz &\leq c \left( \frac{1}{|Q_{2\rho_{z_0}}^{(\lambda)}(z_0)|} \int_{Q_{2\rho_{z_0}}^{(\lambda)}(z_0) \cap E(r_2, \eta\lambda)} |Du|^{\frac{p(\cdot)}{\vartheta}} dz \right)^{\vartheta} \\ &\quad + c\eta\lambda + \frac{c}{|Q_{8\rho_{z_0}}^{(\lambda)}(z_0)|} \left( \int_{Q_{8\rho_{z_0}}^{(\lambda)}(z_0) \cap G(r_2, \eta\lambda)} \Psi dz \right), \end{aligned}$$

where  $c = c(n, \gamma_1, \gamma_2, \nu, L, L_1) \geq 1$ . Therefore, we can conclude that

$$\begin{aligned} &\int_{Q_{\rho_{z_0}}^{(\lambda)}(z_0)} |Du|^{p(\cdot)} + \Psi dz \leq c\eta\lambda \\ &\quad + \frac{c}{|Q_{2\rho_{z_0}}^{(\lambda)}(z_0)|} \int_{Q_{2\rho_{z_0}}^{(\lambda)}(z_0) \cap E(r_2, \eta\lambda)} |Du|^{\frac{p(\cdot)}{\vartheta}} dz \left( \int_{Q_{2\rho_{z_0}}^{(\lambda)}(z_0)} |Du|^{\frac{p(\cdot)}{\vartheta}} dz \right)^{\vartheta-1} \\ &\quad + \frac{c}{|Q_{8\rho_{z_0}}^{(\lambda)}(z_0)|} \left( \int_{Q_{8\rho_{z_0}}^{(\lambda)}(z_0) \cap G(r_2, \eta\lambda)} \Psi dz \right) \\ &= c\eta \left( \int_{Q_{\rho_{z_0}}^{(\lambda)}(z_0)} |Du|^{p(\cdot)} + \Psi dz \right) + \frac{c}{|Q_{8\rho_{z_0}}^{(\lambda)}(z_0)|} \left( \int_{Q_{8\rho_{z_0}}^{(\lambda)}(z_0) \cap G(r_2, \eta\lambda)} \Psi dz \right) \\ &\quad + \frac{c}{|Q_{2\rho_{z_0}}^{(\lambda)}(z_0)|} \int_{Q_{2\rho_{z_0}}^{(\lambda)}(z_0) \cap E(r_2, \eta\lambda)} |Du|^{\frac{p(\cdot)}{\vartheta}} dz \left( \int_{Q_{2\rho_{z_0}}^{(\lambda)}(z_0)} |Du|^{\frac{p(\cdot)}{\vartheta}} dz \right)^{\vartheta-1}, \end{aligned}$$

where  $c = c(n, \gamma_1, \gamma_2, \nu, L, L_1)$ . Next, we have to choose  $\eta = \eta(n, \gamma_1, \gamma_2, \nu, L, L_1) > 0$  small enough, such that we can reabsorb the first integral on the right-hand side. In addition, (4.11) yields that

$$\int_{Q_{2\rho_{z_0}}^{(\lambda)}(z_0)} |Du|^{p(\cdot)} dz < \lambda.$$

Now, we multiply the resulting estimate by  $|Q_{2\rho_{z_0}}^{(\lambda)}(z_0)|$ . Therefore, we have

$$\begin{aligned} \int_{Q_{\rho_{z_0}}^{(\lambda)}(z_0)} |Du|^{p(\cdot)} + \Psi dz &\leq c \int_{Q_{2\rho_{z_0}}^{(\lambda)}(z_0) \cap E(r_2, \eta\lambda)} \lambda^{\frac{\vartheta-1}{\vartheta}} |Du|^{\frac{p(\cdot)}{\vartheta}} dz \\ &\quad + c4^{-(n+2)} \left( \int_{Q_{8\rho_{z_0}}^{(\lambda)}(z_0) \cap G(r_2, \eta\lambda)} \Psi dz \right), \end{aligned} \tag{4.13}$$

with a constant  $c = c(n, \gamma_1, \gamma_2, \nu, L, L_1)$ . Next, we use (4.10) and (4.11) with  $s = 8\chi\rho_{z_0}$ . Thus, we can bound the left-hand side of the previous estimate from below by  $\int_{Q_{8\chi\rho_{z_0}}^{(\lambda)}(z_0)} |Du|^{p(\cdot)} dz$ . This yields

$$\begin{aligned} \int_{Q_{8\chi\rho_{z_0}}^{(\lambda)}(z_0)} |Du|^{p(\cdot)} dz &\leq c \int_{Q_{2\rho_{z_0}}^{(\lambda)}(z_0) \cap E(r_2, \eta\lambda)} \lambda^{\frac{\vartheta-1}{\vartheta}} |Du|^{\frac{p(\cdot)}{\vartheta}} dz \\ &\quad + c \left( \int_{Q_{8\rho_{z_0}}^{(\lambda)}(z_0) \cap G(r_2, \eta\lambda)} \Psi dz \right), \end{aligned} \quad (4.14)$$

where  $c = c(\theta, n, \gamma_1, \gamma_2, \nu, L, L_1)$ . At this stage, we establish a further bound for the radii  $r \leq r_0$

$$r \leq r_0 \leq (6\sqrt{\beta_n M})^{-\frac{2}{\alpha}},$$

where  $\alpha$  is defined as in (2.6). Moreover, notice that from (2.1) with  $\kappa = 1$  we have the equation (4.10) and thus, we can conclude from (2.3) that

$$\lambda \leq \left( \frac{\beta_n M}{\rho_{z_0}^{n+2}} \right)^{\frac{p_0}{2}}, \quad (4.15)$$

where  $\beta_n = \beta_n(n) \geq 1$ .

**Step 3: The final gradient estimate.** Up to now we have indicate that for arbitrary  $\lambda > B\lambda_0$  the level set  $E(r_1, \lambda)$  can be covered by a family  $\mathcal{F} = \{Q_{2\chi\rho_{z_0}}^{(\lambda)}(z_0)\}$  of parabolic cylinders with center in  $z_0 \in E(r_1, \lambda)$ . Thereby, the radii  $\rho_{z_0}$  are bounded by  $\left[ (\beta_n M)^{\frac{p_0}{2}} \lambda^{-1} \right]^{\frac{2}{p_0(n+2)}}$ . This follows from (4.14) and (4.15). Moreover, on each cylinders of the covering the inequality (4.13) holds. By the meaning of the Covering Theorem of Vitali, i.e. the version for non-uniformly parabolic cylinders, provided in Lemma 4.1, we can conclude that, there exists a countable subfamily  $\{Q_{2\rho_{z_i}}^{(\lambda)}\}_{i=1}^\infty \subseteq \mathcal{F}$  of pairwise disjoint parabolic cylinders, such that the  $\chi$ -times enlarged cylinders  $Q_{2\chi\rho_{z_i}}^{(\lambda)}$  cover the set  $E(r_1, \lambda)$ . That means up to a set of measure zero, there holds

$$E(r_1, \lambda) \subseteq \bigcup_{i=1}^\infty Q_{2\chi\rho_{z_i}}^{(\lambda)} \subseteq Q_{r_2}.$$

Notice that, by our construction the relation  $Q_{2\chi\rho_{z_i}}^{(\lambda)} \subseteq Q_{r_2}$  holds. Since, the cylinders of the subfamily  $\{Q_{2\rho_{z_i}}^{(\lambda)}\}_{i=1}^\infty \subseteq \mathcal{F}$  are pairwise disjoint, we can conclude from (4.13) that

$$\begin{aligned} \int_{E(r_1, \lambda)} |Du|^{p(\cdot)} dz &\leq \sum_{i=1}^\infty \int_{Q_{2\rho_{z_i}}^{(\lambda)}} |Du|^{p(\cdot)} dz \\ &\leq c \sum_{i=1}^\infty \left[ \int_{Q_{2\rho_{z_i}}^{(\lambda)} \cap E(r_2, \eta\lambda)} \lambda^{\frac{\vartheta-1}{\vartheta}} |Du|^{\frac{p(\cdot)}{\vartheta}} dz + \int_{Q_{8\rho_{z_i}}^{(\lambda)} \cap G(r_2, \eta\lambda)} \Psi dz \right] \\ &\leq c \int_{E(r_2, \eta\lambda)} \lambda^{\frac{\vartheta-1}{\vartheta}} |Du|^{\frac{p(\cdot)}{\vartheta}} dz + c \left( \int_{G(r_2, \eta\lambda)} \Psi dz \right) \end{aligned} \quad (4.16)$$

with a constant  $c = c(n, \gamma_1, \gamma_2, \nu, L, L_1)$ . Furthermore, on the set  $E(r_1, \eta\lambda) \setminus E(r_1, \lambda)$  the relation  $|Du|^{p(\cdot)} \leq \lambda$  is fulfilled and therefore, we have

$$\begin{aligned} \int_{E(r_1, \eta\lambda) \setminus E(r_1, \lambda)} |Du|^{p(\cdot)} dz &\leq \int_{E(r_1, \eta\lambda) \setminus E(r_1, \lambda)} \lambda^{\frac{\vartheta-1}{\vartheta}} |Du|^{\frac{p(\cdot)}{\vartheta}} dz \\ &\leq \int_{E(r_1, \eta\lambda)} \lambda^{\frac{\vartheta-1}{\vartheta}} |Du|^{\frac{p(\cdot)}{\vartheta}} dz. \end{aligned}$$

Now, we combine the last estimate with (4.16), then we get the following Reverse Hölder type inequality for super-level sets with increasing domains:

$$\int_{E(r_1, \eta\lambda)} |Du|^{p(\cdot)} dz \leq c \int_{E(r_2, \eta\lambda)} \lambda^{\frac{\vartheta-1}{\vartheta}} |Du|^{\frac{p(\cdot)}{\vartheta}} dz + c \left( \int_{G(r_2, \eta\lambda)} \Psi dz \right),$$

where  $c = c(n, \gamma_1, \gamma_2, \nu, L, L_1)$ . Here, we replace  $\eta\lambda$  by  $\lambda$  and recall that  $\eta < 1$  depends only on  $n, \gamma_1, \gamma_2, \nu, L$  and  $L_1$ . For this reason, we gain for arbitrary  $\lambda \leq B\lambda_0/\eta =: \lambda_1$  the following estimate

$$\begin{aligned} \int_{E(r_1, \lambda)} |Du|^{p(\cdot)} dz &\leq c\eta^{-\frac{\vartheta-1}{\vartheta}} \int_{E(r_2, \lambda)} \lambda^{\frac{\vartheta-1}{\vartheta}} |Du|^{\frac{p(\cdot)}{\vartheta}} dz + c \left( \int_{G(r_2, \lambda)} \Psi dz \right) \\ &\leq c \int_{E(r_2, \lambda)} \lambda^{\frac{\vartheta-1}{\vartheta}} |Du|^{\frac{p(\cdot)}{\vartheta}} dz + c \left( \int_{G(r_2, \lambda)} \Psi dz \right) \end{aligned} \quad (4.17)$$

with a constant  $c = c(n, \gamma_1, \gamma_2, \nu, L, L_1)$ . At this stage, we multiply the preceding inequality by  $\lambda^{\varepsilon-1}$ , where  $\varepsilon \in (0, 1]$ . Then, we integrate the resulting inequality with respect to  $\lambda$  over  $(\lambda_1, \infty)$ . This, formally yields the desired higher integrability of  $|Du|$ , where  $\varepsilon$  has to be chosen small enough in order to reabsorb a certain terms on the left-hand side. However, there is a difficulty in moving terms to the left-hand side, since they may be infinite. This technical problem can be treated, for example, by truncating of  $|Du|$ , see for example [33]. The precise argument is as follows: For  $k > \lambda_1$  we define the truncation operator

$$T_k : [0, \infty) \rightarrow [0, k], \quad T_k(s) := \min\{s, k\}$$

and

$$E_k(r_i, \lambda) := \left\{ z \in Q_{r_i} : T_k(|Du|^{p(z)}) > \lambda \right\}, \quad i = 1, 2.$$

In order that, we can conclude from (4.17) the following

$$\begin{aligned} \int_{E_k(r_1, \lambda)} T_k(|Du|^{p(z)})^{\frac{\vartheta-1}{\vartheta}} |Du|^{\frac{p(\cdot)}{\vartheta}} dz &\leq c \int_{E_k(r_2, \lambda)} \lambda^{\frac{\vartheta-1}{\vartheta}} |Du|^{\frac{p(\cdot)}{\vartheta}} dz \\ &\quad + c \left( \int_{G(r_2, \lambda)} \Psi dz \right). \end{aligned} \quad (4.18)$$

Notice that (4.18) trivially holds when  $k \leq \lambda$ , since in this cases  $E_k(r_1, \lambda) = \emptyset$ . Whereas, in the case  $k > \lambda$  we can conclude (4.18) from the inequality (4.17), since

$$E_k(r_2, \lambda) = E(r_2, \lambda) \text{ and } T_k(|Du|^{p(z)}) \leq |Du|^{p(z)}.$$

Next, we multiply (4.18) by  $\lambda^{\varepsilon-1}$ , where  $\varepsilon \in (0, 1]$ . The parameter  $\varepsilon$  have to be fixed later. Then, we integrate the resulting inequality with respect to  $\lambda$  over  $(\lambda_1, \infty)$ .

Consequently, it follows

$$\begin{aligned} & \int_{\lambda_1}^{\infty} \lambda^{\varepsilon-1} \int_{E_k(r_1, \lambda)} T_k(|Du|^{p(z)})^{\frac{(\vartheta-1)}{\vartheta}} |Du|^{\frac{p(\cdot)}{\vartheta}} dz d\lambda \\ & \leq c \int_{\lambda_1}^{\infty} \int_{E_k(r_2, \lambda)} \lambda^{\frac{\vartheta-1}{\vartheta} + \varepsilon - 1} |Du|^{\frac{p(\cdot)}{\vartheta}} dz d\lambda + c \int_{\lambda_1}^{\infty} \lambda^{\varepsilon-1} \int_{G(r_2, \lambda)} \Psi dz d\lambda. \end{aligned} \quad (4.19)$$

Now, we use the Theorem of Fubini to the left-hand side. This yields

$$\begin{aligned} & \int_{\lambda_1}^{\infty} \lambda^{\varepsilon-1} \int_{E_k(r_1, \lambda)} T_k(|Du|^{p(z)})^{\frac{(\vartheta-1)}{\vartheta}} |Du|^{\frac{p(\cdot)}{\vartheta}} dz d\lambda \\ & = \int_{E_k(r_1, \lambda_1)} T_k(|Du|^{p(z)})^{\frac{(\vartheta-1)}{\vartheta}} |Du|^{\frac{p(\cdot)}{\vartheta}} \int_{\lambda_1}^{\infty} \lambda^{\varepsilon-1} d\lambda dz \\ & = \frac{1}{\varepsilon} \int_{E_k(r_1, \lambda_1)} \left[ T_k(|Du|^{p(z)})^{\frac{(\vartheta-1)}{\vartheta} + \varepsilon} |Du|^{\frac{p(\cdot)}{\vartheta}} - \lambda_1^{\varepsilon} T_k(|Du|^{p(z)})^{\frac{(\vartheta-1)}{\vartheta}} |Du|^{\frac{p(\cdot)}{\vartheta}} \right] dz. \end{aligned}$$

The first integral on the right-hand side is treated similarly. Therefore, we get

$$\begin{aligned} & \int_{\lambda_1}^{\infty} \int_{E_k(r_2, \lambda)} \lambda^{\frac{\vartheta-1}{\vartheta} + \varepsilon - 1} |Du|^{\frac{p(\cdot)}{\vartheta}} dz d\lambda \\ & = \int_{E_k(r_2, \lambda_1)} |Du|^{\frac{p(\cdot)}{\vartheta}} \int_{\lambda_1}^{\infty} T_k(|Du|^{p(z)})^{\frac{(\vartheta-1)}{\vartheta}} \lambda^{\frac{\vartheta-1}{\vartheta} + \varepsilon - 1} dz d\lambda \\ & \leq \int_{E_k(r_2, \lambda_1)} \frac{1}{\frac{\vartheta-1}{\vartheta} + \varepsilon} T_k(|Du|^{p(z)})^{\frac{(\vartheta-1)}{\vartheta} + \varepsilon} |Du|^{\frac{p(\cdot)}{\vartheta}} dz d\lambda \\ & \leq \frac{\vartheta}{\vartheta-1} \int_{E_k(r_2, \lambda_1)} T_k(|Du|^{p(z)})^{\frac{(\vartheta-1)}{\vartheta} + \varepsilon} |Du|^{\frac{p(\cdot)}{\vartheta}} dz d\lambda. \end{aligned}$$

Finally, for the second integral on the right-hand side we obtain

$$\begin{aligned} & \int_{\lambda_1}^{\infty} \lambda^{\varepsilon-1} \int_{G(r_2, \lambda)} \Psi dz d\lambda = \int_{G(r_2, \lambda_1)} \Psi \int_{\lambda_1}^{\infty} \lambda^{\varepsilon-1} d\lambda dz \\ & \leq \frac{\vartheta}{\vartheta-1} \int_{G(r_2, \lambda_1)} \Psi^{(1+\varepsilon)} dz. \end{aligned}$$

The next step is to plug the preceding estimates in (4.19) and multiply the resulting inequality by  $\varepsilon$ . Then, this yields

$$\begin{aligned} & \int_{E_k(r_1, \lambda_1)} T_k(|Du|^{p(z)})^{\frac{(\vartheta-1)}{\vartheta} + \varepsilon} |Du|^{\frac{p(\cdot)}{\vartheta}} dz \\ & \leq \lambda_1^{\varepsilon} \int_{E_k(r_1, \lambda_1)} T_k(|Du|^{p(z)})^{\frac{(\vartheta-1)}{\vartheta} + \varepsilon} |Du|^{\frac{p(\cdot)}{\vartheta}} dz + c \frac{\vartheta}{\vartheta-1} \left( \int_{G(r_2, \lambda_1)} \Psi^{(1+\varepsilon)} dz \right) \\ & + \frac{\varepsilon c \vartheta}{\vartheta-1} \int_{E_k(r_2, \lambda_1)} T_k(|Du|^{p(z)})^{\frac{(\vartheta-1)}{\vartheta} + \varepsilon} |Du|^{\frac{p(\cdot)}{\vartheta}} dz \end{aligned}$$

with a constant  $c = c(n, \gamma_1, \gamma_2, \nu, L, L_1)$ . Additionally, we utilize

$$\begin{aligned} & \int_{Q_{r_1} \setminus E_k(r_1, \lambda_1)} T_k(|Du|^{p(z)})^{\frac{(\vartheta-1)}{\vartheta} + \varepsilon} |Du|^{\frac{p(\cdot)}{\vartheta}} dz \\ & \leq \lambda_1^{\varepsilon} \int_{Q_{r_1} \setminus E_k(r_1, \lambda_1)} T_k(|Du|^{p(z)})^{\frac{(\vartheta-1)}{\vartheta}} |Du|^{\frac{p(\cdot)}{\vartheta}} dz \end{aligned}$$

and gain

$$\begin{aligned} \int_{Q_{r_1}} T_k(|Du|^{p(z)})^{\frac{(\vartheta-1)}{\vartheta}+\varepsilon} |Du|^{\frac{p(\cdot)}{\vartheta}} dz &\leq \frac{\varepsilon c \vartheta}{\vartheta-1} \int_{Q_{r_2}} T_k(|Du|^{p(z)})^{\frac{(\vartheta-1)}{\vartheta}+\varepsilon} |Du|^{\frac{p(\cdot)}{\vartheta}} dz \\ &\quad + \lambda_1^\varepsilon \int_{Q_{2r}} T_k(|Du|^{p(z)})^{\frac{(\vartheta-1)}{\vartheta}} |Du|^{\frac{p(\cdot)}{\vartheta}} dz \\ &\quad + c \left( \int_{Q_{2r}} \Psi^{(1+\varepsilon)} dz \right). \end{aligned}$$

At this stage, we identify the choice of  $\varepsilon$ . Therefore, we choose

$$0 < \varepsilon \leq \varepsilon_0 = \varepsilon_0(n, \gamma_1, \gamma_2, \nu, L, L_1, \sigma) := \min \left\{ \sigma, \frac{\vartheta-1}{2c\vartheta} \right\},$$

thereby, we have the definition of  $\lambda_1$  in mind, i.e.

$$\lambda_1^\varepsilon = \left( \frac{B\lambda_0}{\eta} \right)^\varepsilon \leq \frac{B\lambda_0^\varepsilon}{\eta},$$

since  $B/\eta \geq 1$  and  $\varepsilon \leq 1$ . From (4.5) we obtain

$$\begin{aligned} \int_{Q_{r_1}} T_k(|Du|^{p(z)})^{\frac{(\vartheta-1)}{\vartheta}+\varepsilon} |Du|^{\frac{p(\cdot)}{\vartheta}} dz &\leq \frac{1}{2} \int_{Q_{r_2}} T_k(|Du|^{p(z)})^{\frac{(\vartheta-1)}{\vartheta}+\varepsilon} |Du|^{\frac{p(\cdot)}{\vartheta}} dz \\ &\quad + \frac{c_*(2r)^\beta \lambda_0^\varepsilon}{(r_2-r_1)^\beta} \int_{Q_{2r}} T_k(|Du|^{p(z)})^{\frac{(\vartheta-1)}{\vartheta}} |Du|^{\frac{p(\cdot)}{\vartheta}} dz \\ &\quad + c \left( \int_{Q_{2r}} \Psi^{(1+\varepsilon)} dz \right), \end{aligned}$$

where

$$c_* := \frac{(4\chi)^\beta}{\eta} \text{ and } \beta := \frac{n+2}{\frac{1}{d(p_m)} + \frac{2}{p_M} - \frac{2}{p_m}}.$$

Since  $r \leq r_1 < r_2 \leq 2r$  are arbitrary, we are in the position to use the following iteration lemma - which is stated in [29, p. 81] and can be read in Lemma 2.1 - with the following choices  $\beta, \vartheta = \frac{1}{2}$ ,

$$\Phi(s) := \int_{Q_s} T_k(|Du|^{p(z)})^{\frac{(\vartheta-1)}{\vartheta}+\varepsilon} |Du|^{\frac{p(\cdot)}{\vartheta}} dz$$

and

$$A := c_*(2r)^\beta \lambda_0^\varepsilon \int_{Q_{2r}} T_k(|Du|^{p(z)})^{\frac{(\vartheta-1)}{\vartheta}} |Du|^{\frac{p(\cdot)}{\vartheta}} dz, \quad C := c \left( \int_{Q_{2r}} \Psi^{(1+\varepsilon)} dz \right).$$

This allows to move the first integral from the right-hand side to the left (by the iteration argument from the proof of Lemma 2.1). Consequently, we get

$$\begin{aligned} \int_{Q_r} T_k(|Du|^{p(z)})^{\frac{(\vartheta-1)}{\vartheta}+\varepsilon} |Du|^{\frac{p(\cdot)}{\vartheta}} dz \\ \leq c(\beta) \left[ 2^\beta c_* \lambda_0^\varepsilon \int_{Q_{2r}} T_k(|Du|^{p(z)})^{\frac{(\vartheta-1)}{\vartheta}} |Du|^{\frac{p(\cdot)}{\vartheta}} dz + c \left( \int_{Q_{2r}} \Psi^{(1+\varepsilon)} dz \right) \right]. \end{aligned}$$

Passing the limit  $k \rightarrow \infty$ , where we use the Lemma of Fatou to the left-hand side and the Dominated Convergence Theorem of Lebesgue to the right-hand side. Then, we obtain

$$\int_{Q_r} |Du|^{p(\cdot)(1+\varepsilon)} dz \leq c \left[ \lambda_0^\varepsilon \int_{Q_{2r}} |Du|^{p(\cdot)} dz + \int_{Q_{2r}} \Psi dz \right]$$

with a constant  $c = c(n, \gamma_1, \gamma_2, \nu, L, L_1)$ . At this stage, we remember that the dependence of  $\beta$  can be eliminated, since  $p_m \in [\gamma_1, \gamma_2]$  and  $p \rightarrow d(p)$  is a continuous map. In this cases, we get  $c(\beta) = c(n, \gamma_1, \gamma_2)$ . Finally, we apply the averages and use the definition of  $\lambda_0$ , i.e. (4.2). This yields

$$\begin{aligned} \int_{Q_r} |Du|^{p(\cdot)(1+\varepsilon)} dz &\leq c \left( \int_{Q_{2r}} |Du|^{p(\cdot)} + \Psi dz \right)^{1+\varepsilon \left[ \frac{1}{d(p_m)} + \frac{2}{p_M} - \frac{2}{p_m} \right]^{-1}} \\ &\quad + c \left( \int_{Q_{2r}} \Psi^{(1+\varepsilon)} dz \right). \end{aligned} \quad (4.20)$$

After all, we want to replace the term  $\left[ \frac{1}{d(p_m)} + \frac{2}{p_M} - \frac{2}{p_m} \right]^{-1}$  by  $d(p_0)$ , where  $p_0 = p(\mathfrak{z}_0)$  denotes the value of  $p(\cdot)$  at the center  $\mathfrak{z}_0$  of the cylinder  $Q_{2r} := Q_{2r}(\mathfrak{z}_0)$ . Next, we consider the difference of this two terms. This difference can be estimated from above as follows

$$\begin{aligned} 0 &\leq \left[ \frac{1}{d(p_m)} + \frac{2}{p_M} - \frac{2}{p_m} \right]^{-1} - d(p_0) \leq \left[ \frac{1}{d(p_m)} - \frac{2\omega(4r)}{\gamma_1^2} \right]^{-1} - d(p_0) \\ &= \frac{d(p_m) - d(p_0) + 2\gamma_1^{-2}d(p_m)d(p_0)\omega(4r)}{1 - 2\gamma_1^{-2}d(p_m)\omega(4r)} \leq \frac{d(p_m) - d(p_0) + 2\gamma_1^{-2}\mathcal{D}^2\omega(4r)}{1 - 2\gamma_1^{-2}\mathcal{D}\omega(4r)}, \end{aligned}$$

where we defined  $\mathcal{D} := \max\{d(\gamma_1), d(\gamma_2)\}$ . Now, we rewrite and use the restriction of  $r_0$  in the following form

$$\omega(4r_0) \leq \min \left\{ \frac{\gamma_1^2}{4\mathcal{D}}; \frac{2}{n+2} \right\}. \quad (4.21)$$

In order that, we get the bound

$$\left[ \frac{1}{d(p_m)} + \frac{2}{p_M} - \frac{2}{p_m} \right]^{-1} - d(p_0) \leq 2 \left[ d(p_m) - d(p_0) + 2 \frac{\mathcal{D}^2}{\gamma_1^2} \omega(4r) \right]. \quad (4.22)$$

It only remains to bound the difference  $d(p_m) - d(p_0)$ . In the cases  $2 \leq p_m \leq p_0$ , it yields  $d(p_m) - d(p_0) = \frac{p_m}{2} - \frac{p_0}{2} \leq 0$ . In the case  $p_m < p_0 \leq 2$ , we have

$$\begin{aligned} d(p_m) - d(p_0) &= \frac{2p_m}{p_m(n+2) - 2n} - \frac{2p_0}{p_0(n+2) - 2n} \\ &= \frac{4n(p_0 - p_m)}{(p_m(n+2) - 2n)(p_0(n+2) - 2n)} \leq \frac{4n\omega(4r)}{[\gamma_1(n+2) - 2n]^2}. \end{aligned}$$

In the third cases  $p_m < 2 \leq p_0$  we use the fact, that  $p_m \geq p_0 - \omega(4r) \geq 2 - \omega(4r)$  and conclude from (4.20), that

$$\begin{aligned} d(p_m) - d(p_0) &= \frac{2p_m}{p_m(n+2) - 2n} - \frac{p_0}{2} \leq \frac{2p_m}{(2 - \omega(4r))(n+2) - 2n} - \frac{p_m}{2} \\ &= \frac{2p_m}{(4 - \omega(4r))(n+2)} - \frac{p_m}{2} \leq \frac{\gamma_2(n+2)\omega(4r)}{4}. \end{aligned}$$

For this reason, we obtain in all case, that  $d(p_m) - d(p_0) \leq c(n, \gamma_1, \gamma_2)\omega(4r)$ . Plugging this into (4.22), this yields

$$0 \leq \left[ \frac{1}{d(p_m)} + \frac{2}{p_M} - \frac{2}{p_m} \right]^{-1} - d(p_0) \leq c(n, \gamma_1, \gamma_2)\omega(4r).$$

The preceding inequality implies with respect to  $\varepsilon \in (0, 1]$  and the energy bound  $M \geq 1$  from (1.18) the following estimate

$$\begin{aligned} \left( \int_{Q_{2r}} |Du|^{p(\cdot)} + \Psi dz + 1 \right)^\varepsilon \left[ \left[ \frac{1}{d(p_m)} + \frac{2}{p_M} - \frac{2}{p_m} \right]^{-1} - d(p_0) \right] \\ \leq \left( \int_{Q_{2r}} |Du|^{p(\cdot)} + \Psi dz \right)^{c(n, \gamma_1, \gamma_2)\omega(4r)} \\ \leq \left( c(n)(2r)^{-(n+2)\omega(4r)} M^{\omega(4r)} \right)^{c(n, \gamma_1, \gamma_2)} \\ \leq c(n, \gamma_1, \gamma_2) \left( (4r)^{-\omega(4r)} M^{\omega(4r)} \right)^{c(n, \gamma_1, \gamma_2)}. \end{aligned}$$

In order to proceed further, we use the logarithmic continuity condition (1.7) twice to infer for the two terms in the last bracket, that

$$(4r)^{-\omega(4r)} \leq c(L_1) \text{ and } M^{\omega(4r)} \leq c(L_1).$$

The second assumption is fulfilled, since

$$M^{\omega(4r)} = \exp[\omega(4r) \log M] \leq \exp \left[ \omega \left( \frac{1}{M} \right) \log M \right] \leq e^{L_1},$$

provided  $r \leq r_0 \leq \frac{1}{4M}$ . The restriction of  $r_0$  holds even because of the condition

$$r_0 \leq (6\sqrt{\beta_n M})^{-\frac{2}{\alpha}},$$

where  $\alpha$  is defined as in (2.6). All together, we get

$$\left( \int_{Q_{2r}} |Du|^{p(\cdot)} + \Psi dz \right)^\varepsilon \left[ \left[ \frac{1}{d(p_m)} + \frac{2}{p_M} - \frac{2}{p_m} \right]^{-1} - d(p_0) \right] \leq c(n, L_1, \gamma_1, \gamma_2). \quad (4.23)$$

From (4.23) together with (4.20), it yields the desired inequality (1.19), since

$$1 + \varepsilon \left[ \frac{1}{d(p_m)} + \frac{2}{p_M} - \frac{2}{p_m} \right]^{-1} = 1 + \varepsilon d(p_0) + \left( \varepsilon \left[ \frac{1}{d(p_m)} + \frac{2}{p_M} - \frac{2}{p_m} \right]^{-1} - d(p_0) \right)$$

and therefore

$$\begin{aligned} \left( \int_{Q_{2r}} |Du|^{p(\cdot)} + \Psi dz \right)^{1+\varepsilon \left[ \frac{1}{d(p_m)} + \frac{2}{p_M} - \frac{2}{p_m} \right]^{-1}} \\ = \left( \int_{Q_{2r}} |Du|^{p(\cdot)} + \Psi dz \right)^{1+\varepsilon d(p_0) + \varepsilon \left[ \left[ \frac{1}{d(p_m)} + \frac{2}{p_M} - \frac{2}{p_m} \right]^{-1} - d(p_0) \right]} \\ \leq c(n, L_1, \gamma_1, \gamma_2) \left( \int_{Q_{2r}} |Du|^{p(\cdot)} + \Psi dz \right)^{1+\varepsilon d(p_0)}. \end{aligned}$$

This completes the proof of the Theorem 1.4.  $\square$



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