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Asymptotic boundary estimates to infinity Laplace equations with Γ-varying nonlinearity ☆

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ABSTRACT

In a previous paper of the authors (Wang et al. (2014) [40]), the asymptotic estimates of boundary blow-up solutions were established to the infinity Laplace equation $\Delta_\infty u = b(x)f(u)$ in $\Omega \subset \mathbb{R}^N$, with the nonlinearity $0 \le f \in C[0,\infty)$ regularly varying at ∞, and the weighted function $b \in C(\bar{\Omega})$ positive in Ω and vanishing on the boundary. The present paper gives a further investigation on the asymptotic behavior of boundary blow-up solutions to the same equation by assuming f to be Γ-varying. Note that a Γ-varying function grows faster than any regularly varying function. We first quantitatively determine the boundary blow-up estimates with the first expansion, relying on the decay rate of b near the boundary and the growth rate of f at infinity, and further characterize these results via examples possessing various decay rates for b and growth rates for f. In particular, we pay attention to the second-order estimates of boundary blow-up solutions. It was observed in our previous work that the second expansion of solutions to the infinity Laplace equation is independent of the geometry of the domain, quite different from the classical Laplacian. The second expansion obtained in this paper furthermore shows a substantial difference on the asymptotic behavior of boundary blow-up solutions between the infinity Laplacian and the classical Laplacian.

1. Introduction

In this paper we study the boundary asymptotic behavior of solutions to the infinity Laplace equation

$$\begin{cases} \Delta_\infty u = b(x)f(u), & x \in \Omega, \\ u = \infty, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded C^1 domain in $\mathbb{R}^N$ with $N \ge 2$, the weighted function b and the nonlinearity f satisfy

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(H-b) $b \in C(\bar{\Omega})$, $b > 0$ in Ω;
(H-f) $f \in W^{1,\infty}_{\mathrm{loc}}[0,\infty)$, $f(0) = 0$, $f(s)/s$ is increasing in $(0,\infty)$.

The infinity Laplacian, defined as

$$\Delta_\infty u := \sum_{i,j=1}^{N} u_{x_i} u_{x_j} u_{x_i x_j} = \langle D^2 u(x) Du(x), Du(x) \rangle,$$

was proposed by Aronsson [4], and the infinity Laplace equation $\Delta_\infty u = 0$ just is the Euler–Lagrange equation for smooth absolute minimizers. Notice that the infinity Laplace equation is nonlinear and highly degenerate, and does not have smooth solutions in general. The equivalence of absolute minimizers and viscosity solutions (in the sense of [16]) of the Dirichlet problem to the infinity harmonic equation, as well as the uniqueness of solutions were proved by Jensen [25]. Refer to e.g. [8,9,15,19,26,30–33,37,39] and the survey [5] for important results of the infinity Laplace equations.

By a solution to the problem (1.1), we mean a nonnegative function $u \in C(\Omega)$ that satisfies the equation in the viscosity sense (see Section 2 for the definition) and the boundary condition with $u(x) \to \infty$ as the distance function $d(x) := \mathrm{dist}(x, \partial\Omega) \to 0$. Such a solution is usually called a boundary blow-up solution or a large solution.

The boundary blow-up problems have been studied extensively in the context of the classical Laplace operator and other elliptic operators. Recently, the boundary blow-up problems have been extended to the elliptic problems involving the infinity Laplacian. Juutinen and Rossi [27] investigated the existence and uniqueness of solutions to the infinity Laplace problem

$$\begin{cases} \Delta_\infty^N u = u^q, & x \in \Omega, \\ u = \infty, & x \in \partial\Omega \end{cases} \tag{1.2}$$

with the normalized ∞-Laplacian

$$\Delta_\infty^N u := \frac{1}{|Du(x)|^2} \langle D^2 u(x) Du(x), Du(x) \rangle,$$

and proved that (1.2) admits a solution if and only if $q > 1$. They also obtained the boundary asymptotic estimates, and thus the uniqueness of solutions. The existence or nonexistence of boundary blow-up solutions to the problem

$$\begin{cases} \Delta_\infty u = h(x,u), & x \in \Omega, \\ u = \infty, & x \in \partial\Omega \end{cases} \tag{1.3}$$

with $h : \Omega \times [0,\infty) \to [0,\infty)$ continuous and nondecreasing in u for each $x \in \Omega$, was considered in [35]. In particular, it was shown that (1.3) with $h(x,u) = b(x)f(u)$ (i.e. the problem (1.1)) admits a nonnegative solution if and only if the Keller–Osserman condition

$$\int_1^\infty \frac{ds}{\sqrt[4]{F(s)}} < \infty, \quad F(s) = \int_0^s f(\tau)d\tau \tag{1.4}$$

holds, where a boundary asymptotic estimate was obtained as well with $b > 0$ on $\bar{\Omega}$ and some additional assumptions on Ω and b. The problem (1.3) for the case without the monotonicity restriction of h has been also studied, see [36].

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In a recent work of ours [40], we established the asymptotic behavior of solutions to the problem (1.1) with the weighted function b maybe vanishing on the boundary and the nonlinearity f regularly varying at infinity:

Definition 1.1. A positive measurable function f defined on $[a, \infty)$, for some $a > 0$, is called regularly varying at infinity with index p, written as $f \in RV_p$, if for some $p \in \mathbb{R}$,

$$\lim_{s\to\infty} \frac{f(\xi s)}{f(s)} = \xi^p, \quad \forall \xi > 0.$$

In particular, f is slowly varying at infinity if $p = 0$. Similarly, f is rapidly varying at infinity if

$$\lim_{s\to\infty} \frac{f(\xi s)}{f(s)} = \infty, \quad \forall \xi > 1.$$

Additionally, a function $h(t)$ is said to be regularly varying at zero with index p if $s \longmapsto h(1/s) \in RV_{-p}$, and rapidly varying at zero if $s \longmapsto h(1/s)$ is rapidly varying at infinity.

Denote by $\mathcal{K}_\lambda$ the class of positive nondecreasing functions $k(t) \in C^1(0, t_0)$ $(t_0 > 0)$ satisfying

$$\lim_{t\to 0} \frac{d}{dt}\left(\frac{K(t)}{k(t)}\right) = \lambda \in [0, \infty), \quad K(t) = \int_0^t k(\tau)d\tau.$$

It is easy to see that $\lambda \in [0, 1]$ due to the nondecreasing of k, and $\lim_{t\to 0} K(t)/k(t) = 0$ for any $k \in \mathcal{K}_\lambda$. When $f \in RV_p$ with index $p > 3$ and $\lim_{d(x)\to 0} b(x)/k^4(d(x)) = B_0$ for some $B_0 > 0$ and $k \in \mathcal{K}_\lambda$, the exact boundary blow-up estimates to (1.1) were obtained with the first expansion

$$\lim_{d(x)\to 0} \frac{u(x)}{\xi_0 \kappa(K(d(x)))} = 1,$$

where $\kappa : (0, \varsigma(0^+)) \to (0, \infty)$ is the inverse of

$$\varsigma(s) = \int_s^\infty \frac{d\tau}{\sqrt[4]{4F(\tau)}}, \quad s > 0,$$

and

$$\xi_0 = \left[\frac{4 + \lambda(p-3)}{B_0(p+1)}\right]^{\frac{1}{p-3}}.$$

Furthermore, for the case of $f(s) = s^p(1 + \tilde{c}g(s))$ with g *normalized regularly varying at infinity* (defined after Proposition 2.2), we gave the second expansion of solutions near the boundary as

$$u(x) = \xi_1 (K(d(x)))^{-\frac{4}{p-3}} \big(1 + c_1 d(x) + o(d(x))\big),$$

where $\xi_1 > 0$ and c_1 are two constants. It is interesting that the second term in the asymptotic expansion of boundary blow-up solutions to the infinity Laplace equation is independent of the geometry of the domain, quite different from the boundary blow-up problems involving the classical Laplacian.

The main purpose of this paper is to give a further investigation on the asymptotic behavior of boundary blow-up solutions, including the first and second expansions, to the problem (1.1) with f growing faster than any *regularly varying function*, such as

$$f(s) = \mathrm{e}^{(\ln s)^p} \ (p > 1), \quad f(s) = \mathrm{e}^{s^p} \ (p > 0), \quad \text{or} \quad f(s) = \mathrm{e}^{\mathrm{e}^{s^p}} \ (p > 0) \quad \text{for } s > 0 \text{ large}.$$

All these examples are covered by the so called Γ-*varying functions*:

Definition 1.2. A nondecreasing function f defined on (a, ∞) is Γ-varying at ∞, written as $f \in \Gamma$, if $\lim_{s\to\infty} f(s) = \infty$ and there exists a function $\chi : (a, \infty) \to (0, \infty)$ such that

$$\lim_{s\to\infty} \frac{f(s + \mu\chi(s))}{f(s)} = \mathrm{e}^{\mu} \quad \text{for any } \mu \in \mathbb{R}. \tag{1.5}$$

Here the auxiliary function χ is unique up to asymptotic equivalence. By Theorem 1.28 in [22] we know that under (H-f), f is Γ-varying at ∞ if and only if

$$f(s) \sim \hat{f}(s) = \mathrm{e}^{\int_B^s \frac{d\tau}{G(\tau)}} \quad \text{as } s \to \infty,$$

for some $B > 0$ and positive $G \in C^1[B, \infty)$ with $\lim_{s\to\infty} G'(s) = 0$. Obviously, if f in (H-f) is Γ-varying at ∞, then f is *rapidly varying* and grows faster than any *regularly varying function*. In addition, it will be shown that the Keller–Osserman condition (1.4) must be satisfied by any (H-f) function $f \in \Gamma$ (see Lemma 2.2 for details), and this ensures the existence of nonnegative solutions to the problem (1.1) [35].

As early as 1916, Bieberbach [10] has studied the following boundary blow-up problem with the usual Laplacian:

$$\begin{cases} \Delta u = b(x)f(u), & x \in \Omega, \\ u = \infty, & x \in \partial\Omega, \end{cases} \tag{1.6}$$

where $b(x) = 1$, $f(u) = \mathrm{e}^u$ and $\Omega \subset \mathbb{R}^2$ is a smooth bounded domain. It was proved that the problem (1.6) has a unique positive solution $u \in C^2(\Omega)$ with $u(x) - \ln(1/d^2(x))$ bounded as $d(x) \to 0$. Later, Rademacher [38] extended these results to smooth bounded domains in $\mathbb{R}^3$. For the general N-dimensional case, the problem (1.6) was considered by Lazer and McKenna [28] with $f(u) = \mathrm{e}^u$, $b(x)$ continuous and strictly positive on $\bar{\Omega}$.

In [29], it was shown for the boundary blow-up solution of $\Delta u = \mathrm{e}^u$ in $\Omega \subset \mathbb{R}^N$ that $u(x) - \ln(2/d^2(x)) \to 0$ as $d(x) \to 0$. Furthermore, Bandle [6] improved this estimate by proving the expansion

$$u(x) = \ln(2/d^2(x)) + (N-1)H(\bar{x})d(x) + o(d(x)),$$

where $H(\bar{x})$ denotes the mean curvature of $\partial\Omega$ at the point $\bar{x}$ nearest to x. Subsequently, Anedda et al. [1] established the second-order estimates for boundary blow-up solutions of $\Delta u = \mathrm{e}^{u|u|^{p-1}}$ in $\Omega \subset \mathbb{R}^N$ with $p > 0$. It was proved that

$$u(x) = \omega(d(x)) + p^{-1}(N-1)H(x)d(x)[\omega(d(x))]^{1-p} + O(1)d(x)[\omega(d(x))]^{1-2p} \tag{1.7}$$

near the boundary, where $\omega(t)$ is defined by

$$\int_{\omega(t)}^{\infty} \frac{ds}{\sqrt{2F(s)}} = t \quad \text{with } F(s) = \int_{-\infty}^{s} \mathrm{e}^{\tau|\tau|^{p-1}} d\tau,$$

and $H(x)$ is the mean curvature of the surface $\{x \in \Omega : d(x) = \text{constant}\}$. One can see from (1.7) that

$$p^{-1}(N-1)H_m \leq \liminf_{d(x)\to 0} \frac{u(x) - \omega(d(x))}{d(x)[\omega(d(x))]^{1-p}} \leq \limsup_{d(x)\to 0} \frac{u(x) - \omega(d(x))}{d(x)[\omega(d(x))]^{1-p}} \leq p^{-1}(N-1)H_M \tag{1.8}$$

with H_m and H_M standing for the minimum and the maximum of the mean curvature of the boundary $\partial\Omega$ respectively, which implies $u(x) - \omega(d(x))$ and $d(x)[\omega(d(x))]^{1-p}$ maybe share the same decay order, and the quotient of them is related to the geometry of the domain. See also [2,3,7,17,21,41] for the effect of the domain geometry in the asymptotic behavior of boundary blow-up solutions.

In [14], Cîrstea studied the asymptotic behavior of boundary blow-up solutions to the problem

$$\begin{cases} \Delta u + au = b(x)f(u), & x \in \Omega, \\ u = \infty, & x \in \partial\Omega, \end{cases} \tag{1.9}$$

where f satisfies (H-f) and the weighted function $b \in C^\mu(\bar{\Omega})$ is allowed to vanish on the boundary. When the nonlinearity $f(s)$ grows faster at ∞ than any power function s^p $(p > 1)$ and the weighted function $b(x)$ is controlled on the boundary in some manner, the asymptotic estimates of boundary blow-up solutions to (1.9) were determined. The boundary blow-up problems with singular weighted functions [12,13,20] and other elliptic operators instead of the Laplace operator [18,24,34] have been studied as well.

Now, we state our main results. First, the exact asymptotic estimates with the first expansion can be formulated as follows:

Theorem 1. *Suppose f in* (H-f) *is Γ-varying at ∞, and b satisfies* (H-b) *with*

$$\lim_{d(x)\to 0} \frac{b(x)}{k^4(d(x))} = B_0 \tag{1.10}$$

for some $B_0 > 0$ and $k \in \mathcal{K}_\lambda$. Let $u(x)$ be a solution of (1.1).

(i) *If $0 < \lambda \leq 1$, then*

$$\lim_{d(x)\to 0} \frac{u(x)}{\theta(\Psi(d(x)))} = 1, \tag{1.11}$$

with θ defined by

$$\int_{\theta(s)}^{\infty} \frac{d\tau}{\sqrt[3]{\hat{f}(\tau)}} = \frac{1}{\sqrt[3]{s}} \tag{1.12}$$

and $\Psi(t) = 1/(tk(t)K^3(t))$.

(ii) *Assume $\lambda = 0$, and $\lim_{t\to 0} \frac{tW'(t)}{W(t)} = 1$ with $W(t) := \frac{K(t)}{k(t)}$ for small $t > 0$. If*

$$\lim_{s\to\infty} G'(s) \ln \hat{f}(s) = \beta \in \mathbb{R},$$

then the same estimate as (1.11) *holds.*

(iii) *Assume* $\lambda = 0$, *and* $\lim_{t\to 0} W'(t)\ln K(t) = \alpha \in \mathbb{R}$. *If*

$$\text{(a)}\ \lim_{s\to\infty}\frac{sG'(s)}{G(s)} = 1, \quad \textit{or} \quad \text{(b)}\ \lim_{s\to\infty} G'(s)\ln \hat{f}(s) = \beta \in \mathbb{R} \textit{ with } \beta \neq \alpha,$$

then

$$\lim_{d(x)\to 0}\frac{u(x)}{\vartheta(K^{-4}(d(x)))} = 1, \tag{1.13}$$

with $\vartheta(s)$ *given by*

$$\int_{\vartheta(s)}^{\infty}\frac{d\tau}{\sqrt[3]{\hat{f}(\tau)g(\tau)}} = \frac{1}{\sqrt[3]{s}} \tag{1.14}$$

and

$$g(s) := \begin{cases} \frac{s}{G(s)} & \textit{for case (a)}, \\ \ln \hat{f}(s) & \textit{for case (b)}. \end{cases} \tag{1.15}$$

Remark 1.1. By Lemmas 3.1 and 3.3 in [14], if $\lim_{t\to 0} W'(t)\ln K(t) = \alpha \in \mathbb{R}$, then $\alpha \leq -1$. Moreover, $\lim_{t\to 0} W'(t)\ln K(t) = \alpha < -1$ if and only if $\lim_{t\to 0}\frac{tW'(t)}{W(t)} = \frac{\alpha}{1+\alpha} \in (1,\infty)$, and $\lim_{t\to 0} W'(t)\ln K(t) = -1$ implies $\lim_{t\to 0}\frac{tW'(t)}{W(t)} = \infty$. It follows that if $\lim_{s\to\infty} G'(s)\ln \hat{f}(s) = \beta \in \mathbb{R}$, then $\beta \geq -1$. Likewise, we also have that $\lim_{s\to\infty} G'(s)\ln\hat{f}(s) = \beta > -1$ is equivalent to $\lim_{s\to\infty}\frac{sG'(s)}{G(s)} = \frac{\beta}{1+\beta} \in (-\infty, 1)$, and $\lim_{s\to\infty} G'(s)\ln\hat{f}(s) = -1$ leads to $\lim_{s\to\infty}\frac{sG'(s)}{G(s)} = -\infty$. All these restrictions on the functions k and $\hat{f}$ indicate the rates of the weighted function b decaying to zero near the boundary and the nonlinearity f growing at ∞. So, the three items of Theorem 1 quantitatively determine the boundary asymptotic behavior of solutions in different cases for the decay of b and the growth of f. In Section 5, we will characterize these results via an explicit description with examples possessing various decay rates for b and growth rates for f.

Next, a special attention is paid to the second-order estimates of boundary blow-up solutions.

Theorem 2. *Let* $b(x) \equiv 1$ *in* Ω. *Assume* $f \in C^1[0,\infty)$ *is increasing in* $(0,\infty)$ *with* $f(0) = 0$ *and*

$$(\Gamma\text{-}1)\ f(s) = \mathrm{e}^{(\ln s)^p}\ (p > 1), \quad \textit{or} \quad (\Gamma\text{-}2)\ \ f(s) = \mathrm{e}^{s^p}\ (0 < p < 1)$$

in $[s_*,\infty)$ *for some* $s_* > 0$. *Then for any solution of* (1.1),

$$u(x) = \begin{cases} \psi(d(x)) + O(1)d(x)\psi(d(x))[\ln\psi(d(x))]^{1-2p} & \textit{in case } (\Gamma\text{-}1), \\ \psi(d(x)) + O(1)d(x)[\psi(d(x))]^{1-2p} & \textit{in case } (\Gamma\text{-}2) \end{cases} \tag{1.16}$$

as $d(x) \to 0$, *where* $\psi(t) : (0, \nu(0+)) \to (0,\infty)$ *is the inverse of the decreasing function*

$$\nu(t) = \int_t^{\infty}\frac{ds}{\sqrt[4]{4F(s)}}, \quad t > 0 \textit{ with } F(s) = \int_0^s f(\tau)d\tau,$$

and $O(1)$ *denotes a bounded quantity.*

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Remark 1.2. Theorem 2 indicates a substantial difference on the asymptotic behavior of boundary blow-up solutions between the infinity Laplacian and the classical Laplacian. The second expansion of solutions for the classical $\Delta u = \mathrm{e}^{u|u|^{p-1}}$ was known (see e.g. [1,6]), corresponding to case (Γ-2) here. We conclude from $(1.16)_2$ that

$$\lim_{d(x)\to 0} \frac{u(x) - \psi(d(x))}{d(x)[\psi(d(x))]^{1-p}} = 0,$$

i.e., $u(x) - \psi(d(x))$ must be an infinitely small quantity of higher order than $d(x)[\psi(d(x))]^{1-p}$ (not only independent of the domain geometry), substantially different from (1.8) for the classical $\Delta u = \mathrm{e}^{u|u|^{p-1}}$.

Remark 1.3. Note that in Theorem 2, $f(s) = \mathrm{e}^{\int_0^s \frac{d\tau}{G(\tau)}}$ on $[B, \infty)$ for some $B > 0$ with

$$G(s) = \frac{f(s)}{f'(s)} = \begin{cases} s(\ln s)^{1-p}/p & \text{for } (\Gamma\text{-}1), \\ s^{1-p}/p & \text{for } (\Gamma\text{-}2), \end{cases} \quad s \in [B, \infty). \tag{1.17}$$

One can readily check that in both cases $\lim_{s\to\infty} G'(s) = \lim_{s\to\infty} sG''(s) = 0$, and it is obvious that $\lim_{s\to\infty} G(s) = \infty$. In addition,

$$\lim_{s\to\infty} \frac{s}{G(s)\ln f(s)} = \begin{cases} 0 & \text{for } (\Gamma\text{-}1), \\ p & \text{for } (\Gamma\text{-}2). \end{cases}$$

These simple facts will be used in the proof of Proposition 6.2 and Theorem 2.

The rest of the present paper is organized as follows. In Section 2, as preliminaries, we recall the definition of viscosity solutions with the comparison principle and the results of the Karamata regular variation theory, and check the required Keller–Osserman condition. Then we prove Theorem 1 (i)–(ii) and (iii) in the next two sections respectively. In Section 5, we give remarks and examples to illustrate the results of Theorem 1. Finally, Section 6 is devoted to the proof of Theorem 2 on the second expansion of solutions.

2. Preliminaries

In this section, we give some definitions and auxiliary results that will be used throughout the paper. We first state the concept of viscosity solutions for problem (1.1).

Definition 2.1. A function $u \in C(\Omega)$ is a viscosity subsolution of the PDE $\Delta_\infty u = b(x)f(u)$ in Ω if for every $\varphi \in C^2(\Omega)$, with the property that $u - \varphi$ has a local maximum at some $x_0 \in \Omega$, then

$$\Delta_\infty \varphi(x_0) \geq b(x_0)f(u(x_0)).$$

We say a function $u \in C(\Omega)$ is a viscosity supersolution of the PDE $\Delta_\infty u = b(x)f(u)$ in Ω if for every $\varphi \in C^2(\Omega)$, with the property that $u - \varphi$ has a local minimum at some $x_0 \in \Omega$, then

$$\Delta_\infty \varphi(x_0) \leq b(x_0)f(u(x_0)).$$

A function $u \in C(\Omega)$ is a viscosity solution of the PDE $\Delta_\infty u = b(x)f(u)$ in Ω if it is both a subsolution and a supersolution. Finally, by a solution of (1.1), we mean a function u that is a solution of the PDE $\Delta_\infty u = b(x)f(u)$ such that $u = \infty$ on $\partial\Omega$.

In the proof of the boundary asymptotic estimates, we need the following comparison principle (see [35]):

Lemma 2.1 *(Comparison principle). Let b satisfy* (H-b)*, f satisfy* (H-f)*. Suppose $u, v \in C(\bar{\Omega})$ such that*

$$\Delta_\infty u \geq b(x)f(u) \quad \text{in } \Omega \quad \text{and} \quad \Delta_\infty v \leq b(x)f(v) \quad \text{in } \Omega$$

in the viscosity sense. If $u \leq v$ on $\partial\Omega$ and $0 \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω.

As a preparation, let us now recall some basic results from the Karamata regular variation theory (see [11]).

Proposition 2.1 *(Uniform convergence theorem). If $f \in RV_p$, then (in the case $p > 0$, assuming f bounded on each interval $(0, \kappa_1]$),*

$$\lim_{s\to\infty} \frac{f(\xi s)}{f(s)} = \xi^p \quad \text{uniformly in } \xi$$

on each

$$\begin{cases} [\kappa_1, \kappa_2] \ (0 < \kappa_1 \leq \kappa_2 < \infty) & \text{if } p = 0, \\ (0, \kappa_1] \ (0 < \kappa_1 < \infty) & \text{if } p > 0, \\ [\kappa_2, \infty) \ (0 < \kappa_2 < \infty) & \text{if } p < 0. \end{cases}$$

Proposition 2.2 *(Representation theorem). A function L is slowly varying at infinity if and only if it may be written in the form*

$$L(s) = c(s) \exp\left\{ \int_{\hat{a}}^{s} \frac{y(\tau)}{\tau} d\tau \right\}, \quad s \geq \hat{a},$$

for some $\hat{a} \geq a$, where the functions $c(\cdot)$ and $y(\cdot)$ are measurable and continuous respectively, and $c(s) \to c_0 \in (0, \infty)$, $y(s) \to 0$ as $s \to \infty$.

Replacing $c(s)$ by c_0 in Proposition 2.2, we call

$$\ell(s) = c_0 \exp\left\{ \int_{\hat{a}}^{s} \frac{y(\tau)}{\tau} d\tau \right\}, \quad s \geq \hat{a}$$

normalized slowly varying at infinity, and furthermore define

$$f(s) = s^p \ell(s), \quad s \geq \hat{a}$$

normalized regularly varying at infinity with index p, written as $f \in NRV_p$.

It is easy to see that for a positive function f on $[a, \infty)$, $f \in NRV_p$ if and only if $sf'(s)/f(s)$ is continuous and tends to $p \in \mathbb{R}$ as $s \to \infty$. In addition, for any $f \in RV_p$, it is clear that $L(s) := f(s)/s^p$ is slowly varying at ∞, and hence is asymptotically equivalent to a normalized slowly varying function $\ell(s)$. Denote $\tilde{f}(s) := s^p \ell(s)$. Then $\tilde{f}(s) \in NRV_p$ and $\tilde{f}(s)/f(s) \to 1$ as $s \to \infty$.

Proposition 2.3.

(i) *If $f \in RV_p$, then as $s \to \infty$,*

$$f(s) \to \begin{cases} \infty & \text{if } p > 0, \\ 0 & \text{if } p < 0. \end{cases}$$

(ii) *If* $f \in NRV_p$*, then* $f^\gamma \in NRV_{\gamma p}$ *for every* $\gamma \in \mathbb{R}$*.*
(iii) *If* $f_i \in NRV_{p_i}$ $(i = 1, 2)$*, then* $f_1(s) \cdot f_2(s) \in NRV_{p_1+p_2}$*.*
(iv) *If* $f_i \in NRV_{p_i}$ $(i = 1, 2)$*, and* $f_2(s) \to \infty$ *as* $s \to \infty$*, then* $f_1(f_2(s)) \in NRV_{p_1 p_2}$*.*
(v) *If* $f \in NRV_p$ *is strictly increasing with* $p > 0$*, then* $f^{-1} \in NRV_{1/p}$*, where* f^{-1} *is the inverse of* f*.*

Proposition 2.4 *(Asymptotic behavior). Suppose* L *is slowly varying at infinity, and let* $\hat{a} \geq a$ *be the constant such that* $L(s)$ *is locally bounded in* $[\hat{a}, \infty)$*. Then as* $s \to \infty$*,*

(i) $\int_{\hat{a}}^s \tau^p L(\tau) d\tau \sim (p + 1)^{-1} s^{1+p} L(s)$ *if* $p > -1$*;*
(ii) $\int_s^\infty \tau^p L(\tau) d\tau \sim (-p - 1)^{-1} s^{1+p} L(s)$ *if* $p < -1$*.*

At last, we give the following lemma to end this section.

Lemma 2.2. *If* $f \in \Gamma$ *satisfies* (H-f)*, then the Keller–Osserman condition* (1.4) *holds.*

Proof. Fix $\sigma > 3$. Since $\lim_{s\to\infty} G'(s) = 0$,

$$\frac{s\hat{f}'(s)}{\hat{f}(s)} = \frac{s e^{\int_B^s \frac{d\tau}{G(\tau)}} \frac{1}{G(s)}}{e^{\int_B^s \frac{d\tau}{G(\tau)}}} = \frac{s}{G(s)} \to \infty \quad \text{as } s \to \infty,$$

and hence there exists a positive constant c_0 such that $s\hat{f}'(s)/\hat{f}(s) > \sigma$ for $s \geq c_0$. Integrate to get that $\hat{f}(s) \geq \hat{f}(c_0)(s/c_0)^\sigma$, $s \geq c_0$. Thus

$$\lim_{s\to\infty} \frac{f(s)}{s^\sigma} = \lim_{s\to\infty} \frac{\hat{f}'(s)}{\sigma s^{\sigma-1}} = \lim_{s\to\infty} \left(\frac{s\hat{f}'(s)}{\hat{f}(s)} \cdot \frac{\hat{f}(s)}{\sigma s^\sigma} \right) = \infty,$$

which implies the Keller–Osserman condition (1.4) is true. □

3. The proof of Theorem 1 (i)–(ii)

This section is devoted to the proof of Theorem 1 (i)–(ii). We begin with a crucial lemma.

Lemma 3.1. *Assume* $0 < \lambda \leq 1$*, or* $\lambda = 0$ *and* $\lim_{t\to 0} \frac{tW'(t)}{W(t)} = 1$*. Then there exists a decreasing* C^2*-function* A *on* $(0, \epsilon)$ *with* $\epsilon > 0$ *such that as* $t \to 0$*,* $\mathrm{A}(t) \sim \frac{1}{tk(t)K^3(t)}$ *and*

(i) $\ln \mathrm{A}(t) \sim -4 \ln K(t)$*;*
(ii) $[\ln \mathrm{A}(t)]' \sim -\frac{4}{W(t)}$*;*
(iii) $[\ln \mathrm{A}(t)]'' \sim \frac{4}{tW(t)}$*.*

Proof. Note that for $0 < \lambda \leq 1$, it is automatically satisfied that

$$\lim_{t\to 0} \frac{tW'(t)}{W(t)} = 1.$$

Set $\omega(s) = 1/W(1/s)$ for large s. Then $\omega \in NRV_1$, and hence ω' is slowly varying at infinity. Accordingly, there exists a C^2-function $\tilde{\omega}$ such that $\tilde{\omega}' \in NRV_0$, and $\tilde{\omega}'(s) \sim \omega'(s)$ as $s \to \infty$. Moreover, $\tilde{\omega}(s) \sim \omega(s)$ as $s \to \infty$ and $\tilde{\omega} \in NRV_1$. Thus,

$$-(\ln \tilde{\omega}(s))'' \sim \frac{\tilde{\omega}'(s)}{s\tilde{\omega}(s)} \sim \frac{1}{s^2} \quad \text{as } s \to \infty.$$

Define

$$\mathrm{A}(t) = \frac{1}{tK^4(t)\tilde{\omega}(\frac{1}{t})}.$$

One can readily verify that $\mathrm{A}(t) \sim \frac{1}{tk(t)K^3(t)}$ with (i)–(iii) valid as $t \to 0$. $\square$

The next lemma can be deduced via similar arguments to the proof of Lemma 4.1 (a more complicated one), and refer to those in Section 4.

Lemma 3.2. *Let $\theta(s)$ be the function in (1.12). Then*

(i) $\theta(s) \to \infty$ *and* $27s^4[\theta'(s)]^3 = \hat{f}(\theta(s)) \sim f(\theta(s))$ *as* $s \to \infty$*;*
(ii) $\hat{f}(\theta(s)) \in NRV_1$*;*
(iii) θ *is slowly varying at* ∞ *and* $\theta' \in NRV_{-1}$*.*

Moreover, if

$$\lim_{s\to\infty} G'(s)\ln \hat{f}(s) = \beta \in \mathbb{R},$$

then

(iv) $\lim_{s\to\infty} \ln \hat{f}(\theta(s))\Big(1 + \frac{s\theta''(s)}{\theta'(s)}\Big) = \beta.$

In what follows, we will establish the exact boundary blow-up estimates in Theorem 1 (i)–(ii). The primary technique is the comparison principle. Denote

$$\Omega_\delta := \{x \in \Omega : d(x) < \delta\}.$$

Since $\partial\Omega \in C^1$, it follows from [23] that $d(x) \in C^1(\Omega_{\delta_0})$ for some $\delta_0 > 0$. Moreover, $|Dd(x)| = 1$ in Ω_{δ_0}, and consequently $\Delta_\infty d = 0$ in Ω_{δ_0} in the viscosity sense.

The key proposition to prove Theorem 1 (i)–(ii) is as follows.

Proposition 3.1. *Under the assumptions of Theorem 1 (i)–(ii), for any small $\varepsilon > 0$, there exists $\delta \in (0, \delta_0/2)$ such that for each $\rho \in (0, \delta)$,*

$$\bar{u}_\varepsilon = \theta\big((\xi_1 + \varepsilon)\mathrm{A}(d(x) - \rho)\big), \quad x \in \Omega_{2\delta} \setminus \bar{\Omega}_\rho =: \Omega_\rho^-$$

and

$$\underline{u}_\varepsilon = \theta\big((\xi_1 - \varepsilon)\mathrm{A}(d(x) + \rho)\big), \quad x \in \Omega_{2\delta-\rho} =: \Omega_\rho^+$$

are a supersolution and a subsolution of Eq. $(1.1)_1$ in Ω_ρ^- and in Ω_ρ^+, respectively, where $\xi_1 = 64/(27B_0)$ and A is the function in Lemma 3.1.

Proof. Fix a small $\varepsilon > 0$. Let $\delta \in (0, \delta_0/2)$, $\rho \in (0, \delta)$. For simplicity, denote

$$d^-(x) := d(x) - \rho, \quad d^+(x) := d(x) + \rho.$$

Next, we only prove that

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$$\bar{u}_\varepsilon = \theta\big((\xi_1+\varepsilon)\mathrm{A}(d^-(x))\big)$$

is a supersolution of Eq. (1.1)$_1$ in Ω_ρ^-. The proof for $\underline{u}_\varepsilon$ is similar.

Define

$$\eta(t) = \theta\big((\xi_1+\varepsilon)\mathrm{A}(t)\big), \quad t \in (0, 2\delta-\rho).$$

Note that θ and A are increasing and decreasing respectively by Lemmas 3.1 and 3.2. Therefore, η is decreasing in $(0, 2\delta-\rho)$ for δ small enough. Let ζ be the inverse of η. It is easy to check that

$$\begin{aligned}
\zeta'(t) &= \frac{1}{\eta'(\zeta(t))} = \big[(\xi_1+\varepsilon)\theta'\big((\xi_1+\varepsilon)\mathrm{A}(\zeta(t))\big)\mathrm{A}'(\zeta(t))\big]^{-1},\\
\zeta''(t) &= -(\xi_1+\varepsilon)^{-2}\big[\theta'\big((\xi_1+\varepsilon)\mathrm{A}(\zeta(t))\big)\mathrm{A}'(\zeta(t))\big]^{-3}\\
&\quad \times \big[(\xi_1+\varepsilon)\theta''\big((\xi_1+\varepsilon)\mathrm{A}(\zeta(t))\big)\big(\mathrm{A}'(\zeta(t))\big)^2 + \theta'\big((\xi_1+\varepsilon)\mathrm{A}(\zeta(t))\big)\mathrm{A}''(\zeta(t))\big].
\end{aligned} \tag{3.1}$$

Let $(x_0, \varphi) \in \Omega_\rho^- \times C^2(\Omega_\rho^-)$ be a pair such that $\bar{u}_\varepsilon \geq \varphi$ in a neighborhood $\mathcal{N}$ of x_0 and $\bar{u}_\varepsilon(x_0) = \varphi(x_0)$. Then $\phi = \zeta(\varphi) \in C^2(\Omega_\rho^-)$, and

$$d^-(x) \leq \phi(x) \quad \text{in } \mathcal{N}, \quad d^-(x_0) = \phi(x_0).$$

Since $\Delta_\infty d = 0$ in Ω_ρ^-, we have $\Delta_\infty \phi(x_0) \geq 0$. A simple computation shows that

$$\Delta_\infty \phi = \zeta''(\varphi)(\zeta'(\varphi))^2|D\varphi|^4 + (\zeta'(\varphi))^3\Delta_\infty\varphi,$$

which together with $\Delta_\infty\phi(x_0) \geq 0$ and $\zeta' < 0$ yields that

$$\Delta_\infty\varphi(x_0) \leq -\zeta''(\varphi(x_0))(\zeta'(\varphi(x_0)))^{-1}|D\varphi(x_0)|^4.$$

Also, since $|Dd(x)| = 1$ for $x \in \Omega_\rho^-$ and $d^- - \phi$ attains a local maximum at x_0, it follows that

$$|Dd^-(x_0)| = |\zeta'(\varphi(x_0))D\varphi(x_0)| = 1.$$

Therefore

$$\Delta_\infty\varphi(x_0) \leq -\zeta''(\varphi(x_0))(\zeta'(\varphi(x_0)))^{-5}.$$

By (3.1), we further have

$$\begin{aligned}
\Delta_\infty\varphi(x_0) &\leq (\xi_1+\varepsilon)^3\big[\theta'\big((\xi_1+\varepsilon)\mathrm{A}(d^-(x_0))\big)\mathrm{A}'(d^-(x_0))\big]^2\\
&\quad \times \big[(\xi_1+\varepsilon)\theta''\big((\xi_1+\varepsilon)\mathrm{A}(d^-(x_0))\big)\big(\mathrm{A}'(d^-(x_0))\big)^2 + \theta'\big((\xi_1+\varepsilon)\mathrm{A}(d^-(x_0))\big)\mathrm{A}''(d^-(x_0))\big]\\
&= \left(\frac{\mathrm{A}'(d^-(x_0))}{\mathrm{A}(d^-(x_0))}\right)^4 \Big\{\big((\xi_1+\varepsilon)\mathrm{A}(d^-(x_0))\big)^4\theta''\big((\xi_1+\varepsilon)\mathrm{A}(d^-(x_0))\big)\big[\theta'\big((\xi_1+\varepsilon)\mathrm{A}(d^-(x_0))\big)\big]^2\\
&\quad + \big((\xi_1+\varepsilon)\mathrm{A}(d^-(x_0))\big)^3\big[\theta'\big((\xi_1+\varepsilon)\mathrm{A}(d^-(x_0))\big)\big]^3\frac{\mathrm{A}''(d^-(x_0))\mathrm{A}(d^-(x_0))}{[\mathrm{A}'(d^-(x_0))]^2}\Big\}\\
&= \big([\ln\mathrm{A}(t_0)]'\big)^4\Big\{s_0^4\theta''(s_0)[\theta'(s_0)]^2 + s_0^3[\theta'(s_0)]^3\frac{\mathrm{A}''(t_0)\mathrm{A}(t_0)}{[\mathrm{A}'(t_0)]^2}\Big\}
\end{aligned}$$

with $t_0 := d^-(x_0)$ and $s_0 := (\xi_1+\varepsilon)\mathrm{A}(t_0)$. Note that x_0 will approach to the boundary of Ω when δ is taken small enough. Thus, $t_0 \to 0$ and $s_0 \to \infty$ as $\delta \to 0$. We now proceed to get

$$\begin{aligned}
&\frac{1}{k^4(t_0)f(\theta(s_0))}\left(\Delta_\infty\varphi(x_0) - b(x_0)f(\bar{u}_\varepsilon(x_0))\right)\\
&\le \frac{\left([\ln \mathrm{A}(t_0)]'\right)^4}{k^4(t_0)f(\theta(s_0))}\left\{s_0^4\theta''(s_0)[\theta'(s_0)]^2 + s_0^3[\theta'(s_0)]^3\frac{\mathrm{A}''(t_0)\mathrm{A}(t_0)}{[\mathrm{A}'(t_0)]^2}\right\} - \frac{b(x_0)}{k^4(d(x_0))}\\
&=: I(x_0) - \frac{b(x_0)}{k^4(d(x_0))}.
\end{aligned}$$

According to Lemma 3.2 (i), we have if $0 < \lambda \le 1$, then

$$\begin{aligned}
I(x_0) &= \frac{\left([\ln \mathrm{A}(t_0)]'\right)^4}{k^4(t_0)f(\theta(s_0))}\left\{\frac{(\hat{f}(\theta(s_0)))'}{81} - \frac{4\hat{f}(\theta(s_0))}{81 s_0} + \frac{\hat{f}(\theta(s_0))}{27 s_0}\frac{\mathrm{A}''(t_0)\mathrm{A}(t_0)}{[\mathrm{A}'(t_0)]^2}\right\}\\
&= \frac{\hat{f}(\theta(s_0))}{(\xi_1+\varepsilon)f(\theta(s_0))}\frac{\left([\ln \mathrm{A}(t_0)]'\right)^4}{\mathrm{A}(t_0)k^4(t_0)}\left\{\frac{s_0(\hat{f}(\theta(s_0)))'}{81\hat{f}(\theta(s_0))} - \frac{4}{81} + \frac{1}{27}\frac{\mathrm{A}''(t_0)\mathrm{A}(t_0)}{[\mathrm{A}'(t_0)]^2}\right\},
\end{aligned}$$

and for $\lambda = 0$,

$$\begin{aligned}
I(x_0) &= \frac{\left([\ln \mathrm{A}(t_0)]'\right)^4}{k^4(t_0)f(\theta(s_0))}\left\{s_0^4\theta''(s_0)[\theta'(s_0)]^2 + s_0^3[\theta'(s_0)]^3\right\}\\
&\quad + \frac{\left([\ln \mathrm{A}(t_0)]'\right)^4}{k^4(t_0)f(\theta(s_0))}\left\{s_0^3[\theta'(s_0)]^3\frac{\mathrm{A}''(t_0)\mathrm{A}(t_0)}{[\mathrm{A}'(t_0)]^2} - s_0^3[\theta'(s_0)]^3\right\}\\
&= \frac{\left([\ln \mathrm{A}(t_0)]'\right)^4 s_0^3[\theta'(s_0)]^3}{k^4(t_0)f(\theta(s_0))}\left(1 + \frac{s_0\theta''(s_0)}{\theta'(s_0)}\right) + \frac{[\ln \mathrm{A}(t_0)]''\left([\ln \mathrm{A}(t_0)]'\right)^2 s_0^3[\theta'(s_0)]^3}{k^4(t_0)f(\theta(s_0))}\\
&= \frac{\hat{f}(\theta(s_0))\ln \mathrm{A}(t_0)}{27(\xi_1+\varepsilon)f(\theta(s_0))\ln\hat{f}(\theta(s_0))}\cdot\frac{\left([\ln \mathrm{A}(t_0)]'\right)^4}{k^4(t_0)\mathrm{A}(t_0)\ln \mathrm{A}(t_0)}\cdot \ln\hat{f}(\theta(s_0))\left(1 + \frac{s_0\theta''(s_0)}{\theta'(s_0)}\right)\\
&\quad + \frac{\hat{f}(\theta(s_0))}{27(\xi_1+\varepsilon)f(\theta(s_0))}\cdot\frac{[\ln \mathrm{A}(t_0)]''\left([\ln \mathrm{A}(t_0)]'\right)^2}{k^4(t_0)\mathrm{A}(t_0)}.
\end{aligned}$$

It follows from Lemmas 3.1 and 3.2 that

$$I(x_0) - \frac{b(x_0)}{k^4(d(x_0))} \to \frac{64}{27(\xi_1+\varepsilon)} - B_0, \quad \delta \to 0,$$

in both cases of $0 < \lambda \le 1$ and $\lambda = 0$, where we point out separately that

$$\lim_{\delta\to 0}\frac{\ln \mathrm{A}(t_0)}{\ln\hat{f}(\theta(s_0))} = \lim_{s_0\to\infty}\frac{\ln\left(\frac{1}{\xi_1+\varepsilon}s_0\right)}{\ln\hat{f}(\theta(s_0))} = \lim_{s_0\to\infty}\frac{\hat{f}(\theta(s_0))}{s_0[\hat{f}(\theta(s_0))]'} = 1.$$

By the choice of ξ_1, we have $I(x_0) - b(x_0)/k^4(d(x_0)) < 0$ provided $\delta \in (0, \delta_0/2)$ small enough. Thus

$$\Delta_\infty\varphi(x_0) \le b(x_0)f(\bar{u}_\varepsilon(x_0)).$$

The proof is complete. $\square$

We now give the proof of Theorem 1 (i)–(ii).

Proof of Theorem 1 (i)–(ii). Given a small $\varepsilon > 0$, by Proposition 3.1, there exists $\delta \in (0, \delta_0/2)$ such that for each $\rho \in (0, \delta)$,

$$\bar{u}_\varepsilon(x) = \theta\big((\xi_1 + \varepsilon)\mathrm{A}(d(x) - \rho)\big) \quad \text{and} \quad \underline{u}_\varepsilon(x) = \theta\big((\xi_1 - \varepsilon)\mathrm{A}(d(x) + \rho)\big)$$

are a supersolution and a subsolution of Eq. $(1.1)_1$ in Ω_ρ^- and in Ω_ρ^+, respectively. Take $M = M(\delta)$ large enough such that

$$u \le \bar{u}_\varepsilon + M \quad \text{on } \{x \in \Omega : d(x) = 2\delta\}$$

and

$$\underline{u}_\varepsilon \le u + M \quad \text{on } \{x \in \Omega : d(x) = 2\delta - \rho\}.$$

Consequently, $u \le \bar{u}_\varepsilon + M$ on $\partial\Omega_\rho^-$ and $\underline{u}_\varepsilon \le u + M$ on $\partial\Omega_\rho^+$. Due to the monotonicity of f, we can also conclude that $\bar{u}_\varepsilon + M$ and $u + M$ are two supersolutions of Eq. $(1.1)_1$ in Ω_ρ^- and in Ω. By comparison (Lemma 2.1), we get

$$u \le \bar{u}_\varepsilon + M \quad \text{in } \Omega_\rho^-, \quad \text{and} \quad \underline{u}_\varepsilon \le u + M \quad \text{in } \Omega_\rho^+.$$

Hence, for $x \in \Omega_\rho^- \cap \Omega_\rho^+$, we have

$$\frac{\theta\big((\xi_1 - \varepsilon)\mathrm{A}(d(x) + \rho)\big) - M}{\theta(\Psi(d(x)))} \le \frac{u(x)}{\theta(\Psi(d(x)))} \le \frac{\theta\big((\xi_1 + \varepsilon)\mathrm{A}(d(x) - \rho)\big) + M}{\theta(\Psi(d(x)))}$$

with $\Psi(t) = 1/(tk(t)K^3(t))$. Let $\rho \to 0$ to obtain

$$\frac{\theta\big((\xi_1 - \varepsilon)\mathrm{A}(d(x))\big)}{\theta(\Psi(d(x)))} - \frac{M}{\theta(\Psi(d(x)))} \le \frac{u(x)}{\theta(\Psi(d(x)))} \le \frac{\theta\big((\xi_1 + \varepsilon)\mathrm{A}(d(x))\big)}{\theta(\Psi(d(x)))} + \frac{M}{\theta(\Psi(d(x)))},$$

for any $x \in \Omega_{2\delta}$. Recalling that $\theta(s) \to \infty$ as $s \to \infty$ and θ is slowly varying at ∞, we further get by Lemma 3.1 and Proposition 2.1 (the uniform convergence theorem) that

$$1 \le \liminf_{d(x)\to 0} \frac{u(x)}{\theta(\Psi(d(x)))} \le \limsup_{d(x)\to 0} \frac{u(x)}{\theta(\Psi(d(x)))} \le 1.$$

The proof is complete. □

4. The proof of Theorem 1 (iii)

In this section, we give the proof of Theorem 1 (iii). The following lemma plays an important role in the proof.

Lemma 4.1. *Suppose*

$$\text{(a)} \lim_{s\to\infty} \frac{sG'(s)}{G(s)} = 1, \quad \textit{or} \quad \text{(b)} \lim_{s\to\infty} G'(s) \ln \hat{f}(s) = \beta \in \mathbb{R}. \tag{4.1}$$

Let $g(s)$ and $\vartheta(s)$ be the functions in (1.15) and in (1.14) respectively. Then

(i) *$\vartheta(s) \to \infty$ and $27s^4[\vartheta'(s)]^3 = \hat{f}(\vartheta(s))g(\vartheta(s)) \sim f(\vartheta(s))g(\vartheta(s))$ as $s \to \infty$;*
(ii) *$\hat{f}(\vartheta(s)) \in NRV_1$;*

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(iii) ϑ *is slowly varying at* ∞ *and* $\vartheta' \in NRV_{-1}$;

(iv) $\lim_{s\to\infty} s[g(\vartheta(s))]' = \begin{cases} 0 & \textit{for case } \text{(a)}, \\ 1 & \textit{for case } \text{(b)}; \end{cases}$

(v) $\lim_{s\to\infty} g(\vartheta(s))\Big(1+\frac{s\vartheta''(s)}{\vartheta'(s)}\Big) = \begin{cases} 1 & \textit{for case } \text{(a)}, \\ \beta & \textit{for case } \text{(b)}. \end{cases}$

Proof. Assertion (i) can be derived directly from the definition of $\vartheta(s)$. By (1.14), we have

$$\left(\int_{\hat{f}(\vartheta(s))}^{\infty} [\tau \cdot g(\hat{f}^{-1}(\tau))]^{-\frac{1}{3}}[\hat{f}^{-1}(\tau)]' d\tau\right)^{-3} = s \quad \text{for } s > 0 \text{ large},$$

where $\hat{f}^{-1}$ is the inverse of $\hat{f}$. Define

$$\mathcal{H}(r) = \left(\int_r^{\infty} [\tau \cdot g(\hat{f}^{-1}(\tau))]^{-\frac{1}{3}}[\hat{f}^{-1}(\tau)]' d\tau\right)^{-3}.$$

Then $\mathcal{H}\big(\hat{f}(\vartheta(s))\big) = s \in NRV_1$. A simple computation yields

$$\lim_{s\to\infty} \frac{\hat{f}(s)\hat{f}''(s)}{[\hat{f}'(s)]^2} = 1,$$

whence

$$\lim_{s\to\infty} \frac{s[\hat{f}^{-1}(s)]''}{[\hat{f}^{-1}(s)]'} = \lim_{s\to\infty} \frac{-s\hat{f}''(\hat{f}^{-1}(s))}{[\hat{f}'(\hat{f}^{-1}(s))]^2} = -1,$$

which shows $[\hat{f}^{-1}(s)]' \in NRV_{-1}$, and thereby $\hat{f}^{-1}(s) \in NRV_0$. We claim that $g(\hat{f}^{-1}(s)) \in NRV_0$. Indeed, for case (a), it is clear that $g \in NRV_0$ and the claim follows from Proposition 2.3 (iv); for case (b), $g(\hat{f}^{-1}(s)) = \ln s$ is certainly normalized slowly varying at infinity. By Proposition 2.3 (ii) and (iii), we have $[g(\hat{f}^{-1}(s))]^{-\frac{1}{3}}[\hat{f}^{-1}(s)]' \in NRV_{-1}$. Denote

$$L(s) := s[g(\hat{f}^{-1}(s))]^{-\frac{1}{3}}[\hat{f}^{-1}(s)]'.$$

Then $L(s)$ is slowly varying at ∞. In view of Proposition 2.4 (ii), we get

$$\lim_{r\to\infty} \frac{r\mathcal{H}'(r)}{\mathcal{H}(r)} = \lim_{r\to\infty} \frac{3r^{-\frac{1}{3}}L(r)}{\int_r^{\infty} \tau^{-\frac{4}{3}}L(\tau)d\tau} = 1,$$

that is, $\mathcal{H}(r) \in NRV_1$. Recalling that $\mathcal{H}\big(\hat{f}(\vartheta(s))\big) \in NRV_1$, we deduce from Proposition 2.3 (iv) and (v) that $\hat{f}(\vartheta(s)) \in NRV_1$.

Since $\hat{f}^{-1}(s) \in NRV_0$ and $\hat{f}(\vartheta(s)) \in NRV_1$, it follows by Proposition 2.3 (iv) that $\vartheta \in NRV_0$. Noticing that

$$\hat{f}'(\vartheta(s)) = \frac{\hat{f}(\vartheta(s))}{G(\vartheta(s))}, \tag{4.2}$$

we obtain

$$\lim_{s\to\infty}\frac{s\vartheta'(s)}{G(\vartheta(s))}=\lim_{s\to\infty}\frac{s[\hat{f}(\vartheta(s))]'}{\hat{f}(\vartheta(s))}=1 \tag{4.3}$$

due to $\hat{f}(\vartheta(s))\in NRV_1$. Observe that

$$\lim_{s\to\infty}g'(s)G(s)=\begin{cases}0 & \text{for case } (a),\\ 1 & \text{for case (b).}\end{cases} \tag{4.4}$$

Combining this with (4.3), we arrive at assertion (iv). As a result, $g(\vartheta(s))\in NRV_0$. Assertion (i) implies that

$$\vartheta'(s)=\frac{1}{3}s^{-\frac{4}{3}}[\hat{f}(\vartheta(s))g(\vartheta(s))]^{\frac{1}{3}}, \tag{4.5}$$

whence, again by Proposition 2.3 (ii) and (iii), we get $\vartheta'\in NRV_{-1}$.

From (4.2) and (4.5), we have

$$\begin{aligned}g(\vartheta(s))\left(1+\frac{s\vartheta''(s)}{\vartheta'(s)}\right)&=g(\vartheta(s))\left(\frac{s[\hat{f}(\vartheta(s))g(\vartheta(s))]'}{3\hat{f}(\vartheta(s))g(\vartheta(s))}-\frac{1}{3}\right)\\&=\frac{s[\hat{f}(\vartheta(s))]'}{\hat{f}(\vartheta(s))}\left(\frac{g(\vartheta(s))}{3}-\frac{\hat{f}(\vartheta(s))g(\vartheta(s))}{3s[\hat{f}(\vartheta(s))]'}\right)+\frac{1}{3}s[g(\vartheta(s))]'\\&=\frac{s[\hat{f}(\vartheta(s))]'}{\hat{f}(\vartheta(s))}\left(\frac{g(\vartheta(s))}{3}-\frac{s^{\frac{1}{3}}G(\vartheta(s))[g(\vartheta(s))]^{\frac{2}{3}}}{[\hat{f}(\vartheta(s))]^{\frac{1}{3}}}\right)+\frac{1}{3}s[g(\vartheta(s))]'.\end{aligned}$$

Since $\hat{f}(\vartheta(s))\in NRV_1$,

$$\begin{aligned}&\lim_{s\to\infty}g(\vartheta(s))\left(1+\frac{s\vartheta''(s)}{\vartheta'(s)}\right)\\&=\lim_{s\to\infty}\left(\frac{g(\vartheta(s))}{3}-\frac{s^{\frac{1}{3}}G(\vartheta(s))[g(\vartheta(s))]^{\frac{2}{3}}}{[\hat{f}(\vartheta(s))]^{\frac{1}{3}}}\right)+\frac{1}{3}\lim_{s\to\infty}s[g(\vartheta(s))]'.\end{aligned}$$

Using l'Hôpital's rule, we obtain

$$\begin{aligned}&\lim_{s\to\infty}\left(\frac{g(\vartheta(s))}{3}-\frac{s^{\frac{1}{3}}G(\vartheta(s))[g(\vartheta(s))]^{\frac{2}{3}}}{[\hat{f}(\vartheta(s))]^{\frac{1}{3}}}\right)\\&=\lim_{s\to\infty}\frac{G(\vartheta(s))[\hat{f}(\vartheta(s))]^{-\frac{1}{3}}[g(\vartheta(s))]^{\frac{2}{3}}-\frac{1}{3}s^{-\frac{1}{3}}g(\vartheta(s))}{-s^{-\frac{1}{3}}}\\&=\lim_{s\to\infty}\left(G'(\vartheta(s))g(\vartheta(s))+\frac{2}{3}g'(\vartheta(s))G(\vartheta(s))-s[g(\vartheta(s))]'\right).\end{aligned}$$

Thus, assertion (v) is established due to (4.1), (4.4) and assertion (iv). □

We proceed to give the following proposition.

Proposition 4.1. *Under the assumptions of Theorem 1 (iii), for any small $\varepsilon>0$, there exists $\delta\in(0,\delta_0/2)$ such that for each $\rho\in(0,\delta)$,*

$$\bar{u}_\varepsilon=\vartheta\big((\xi_2+\varepsilon)\Phi(d(x)-\rho)\big),\quad x\in\Omega_{2\delta}\setminus\bar{\Omega}_\rho=:\Omega_\rho^-$$

and

$$\underline{u}_\varepsilon = \vartheta\big((\xi_2 - \varepsilon)\Phi(d(x) + \rho)\big), \quad x \in \Omega_{2\delta-\rho} =: \Omega_\rho^+$$

are a supersolution and a subsolution to Eq. (1.1)$_1$ *in* Ω_ρ^- *and in* Ω_ρ^+*, respectively, where the function* $\Phi(t) = K^{-4}(t)$ *and the constant* ξ_2 *is determined by*

$$\xi_2 = \begin{cases} \frac{256}{27B_0} & \text{for case (a)}, \\ \frac{256(\beta-\alpha)}{27B_0} & \text{for case (b)}. \end{cases}$$

Proof. Fix a small $\varepsilon > 0$. Let $\delta \in (0, \delta_0/2)$ and $\rho \in (0, \delta)$. As before, a detailed discussion will be given only for $\bar{u}_\varepsilon$.

Let $(x_0, \varphi) \in \Omega_\rho^- \times C^2(\Omega_\rho^-)$ satisfy $\bar{u}_\varepsilon \geq \varphi$ in a neighborhood $\mathcal{N}$ of x_0 and $\bar{u}_\varepsilon(x_0) = \varphi(x_0)$. Similar to the proof of Proposition 3.1, we obtain

$$\begin{aligned} \Delta_\infty\varphi(x_0) &\leq (\xi_2+\varepsilon)^3\big[\vartheta'\big((\xi_2+\varepsilon)\Phi(d^-(x_0))\big)\Phi'(d^-(x_0))\big]^2 \\ &\quad \times \big[(\xi_2+\varepsilon)\vartheta''\big((\xi_2+\varepsilon)\Phi(d^-(x_0))\big)\big(\Phi'(d^-(x_0))\big)^2 + \vartheta'\big((\xi_2+\varepsilon)\Phi(d^-(x_0))\big)\Phi''(d^-(x_0))\big] \\ &= \big([\ln\Phi(t_0)]'\big)^4\left\{s_0^4\vartheta''(s_0)[\vartheta'(s_0)]^2 + s_0^3[\vartheta'(s_0)]^3\frac{\Phi''(t_0)\Phi(t_0)}{[\Phi'(t_0)]^2}\right\} \end{aligned}$$

with $t_0 := d^-(x_0) = d(x_0) - \rho$ and $s_0 := (\xi_2+\varepsilon)\Phi(t_0)$. Furthermore, we have

$$\begin{aligned} &\frac{1}{k^4(t_0)f(\vartheta(s_0))}\big(\Delta_\infty\varphi(x_0) - b(x_0)f(\bar{u}_\varepsilon(x_0))\big) \\ &\leq \frac{\big([\ln\Phi(t_0)]'\big)^4}{k^4(t_0)f(\vartheta(s_0))}\left\{s_0^4\vartheta''(s_0)[\vartheta'(s_0)]^2 + s_0^3[\vartheta'(s_0)]^3\frac{\Phi''(t_0)\Phi(t_0)}{[\Phi'(t_0)]^2}\right\} - \frac{b(x_0)}{k^4(d(x_0))} \\ &=: J(x_0) - \frac{b(x_0)}{k^4(d(x_0))} \end{aligned}$$

with

$$\begin{aligned} J(x_0) &= \frac{\big([\ln\Phi(t_0)]'\big)^4}{k^4(t_0)f(\vartheta(s_0))}\left\{s_0^4\vartheta''(s_0)[\vartheta'(s_0)]^2 + s_0^3[\vartheta'(s_0)]^3\right\} \\ &\quad + \frac{\big([\ln\Phi(t_0)]'\big)^4}{k^4(t_0)f(\vartheta(s_0))}\left\{s_0^3[\vartheta'(s_0)]^3\frac{\Phi''(t_0)\Phi(t_0)}{[\Phi'(t_0)]^2} - s_0^3[\vartheta'(s_0)]^3\right\} \\ &= \frac{\big([\ln\Phi(t_0)]'\big)^4 s_0^3[\vartheta'(s_0)]^3}{k^4(t_0)f(\vartheta(s_0))}\left(1 + \frac{s_0\vartheta''(s_0)}{\vartheta'(s_0)}\right) + \frac{[\ln\Phi(t_0)]''\big([\ln\Phi(t_0)]'\big)^2 s_0^3[\vartheta'(s_0)]^3}{k^4(t_0)f(\vartheta(s_0))} \\ &= \frac{\hat{f}(\vartheta(s_0))}{27(\xi_2+\varepsilon)f(\vartheta(s_0))}\cdot\frac{\big([\ln\Phi(t_0)]'\big)^4}{k^4(t_0)\Phi(t_0)}\cdot g(\vartheta(s_0))\left(1 + \frac{s_0\vartheta''(s_0)}{\vartheta'(s_0)}\right) \\ &\quad + \frac{\hat{f}(\vartheta(s_0))}{27(\xi_2+\varepsilon)f(\vartheta(s_0))}\cdot\frac{g(\vartheta(s_0))[\ln\Phi(t_0)]''\big([\ln\Phi(t_0)]'\big)^2}{k^4(t_0)\Phi(t_0)} \\ &= \frac{256\hat{f}(\vartheta(s_0))}{27(\xi_2+\varepsilon)f(\vartheta(s_0))}\cdot g(\vartheta(s_0))\left(1 + \frac{s_0\vartheta''(s_0)}{\vartheta'(s_0)}\right) \\ &\quad - \frac{256\hat{f}(\vartheta(s_0))}{27(\xi_2+\varepsilon)f(\vartheta(s_0))}\cdot\frac{g(\vartheta(s_0))W'(t_0)\ln K(t_0)}{\ln\Phi(t_0)}, \end{aligned}$$

where Lemma 4.1 (i) is used. Therefore, it follows from the assumptions and Lemma 4.1 that

$$J(x_0) - \frac{b(x_0)}{k^4(d(x_0))} \to \begin{cases} \frac{256}{27(\xi_2+\varepsilon)} - B_0 & \text{for case (a)}, \\ \frac{256(\beta-\alpha)}{27(\xi_2+\varepsilon)} - B_0 & \text{for case (b)}, \end{cases} \quad \text{as } \delta \to 0,$$

where we specifically mention that

$$\lim_{\delta\to 0} \frac{g(\vartheta(s_0))}{\ln \Phi(t_0)} = \lim_{s_0\to\infty} \frac{g(\vartheta(s_0))}{\ln\left(\frac{1}{\xi_2+\varepsilon} s_0\right)} = \lim_{s_0\to\infty} s_0[g(\vartheta(s_0))]' = \begin{cases} 0 & \text{for case (a)}, \\ 1 & \text{for case (b)}. \end{cases}$$

In view of the choice of ξ_2, we obtain $J(x_0) - \frac{b(x_0)}{k^4(d(x_0))} < 0$, and thereby $\Delta_\infty \varphi(x_0) \le b(x_0) f(\bar{u}_\varepsilon(x_0))$ provided δ small enough. The proof is complete. □

Proof of Theorem 1 (iii). By using the procedures performed in the proof of Theorem 1 (i)–(ii), we can arrive at the desired result. □

5. Remarks and examples of Theorem 1

In this section, we give an explicit description for the results of Theorem 1. First, we illustrate the manner of the boundary blow-up solutions going to infinity. That is the following proposition.

Proposition 5.1.

(i) *Under the conditions of Theorem 1 (i)–(ii),*

$$\lim_{t\to 0} \frac{t[\theta(\Psi(t))]'}{\theta(\Psi(t))} = 0,$$

where $\theta(s)$ is the function given by (1.12) and $\Psi(t) = 1/(tk(t)K^3(t))$.

(ii) *Let $\vartheta(s)$ be the function defined in Theorem 1 (iii) and $\Phi(t) = K^{-4}(t)$. Then*

$$\lim_{t\to 0} \frac{t[\vartheta(\Phi(t))]'}{\vartheta(\Phi(t))} = \begin{cases} -\infty & \textit{for case } (a), \\ \frac{1+\beta}{1+\alpha} & \textit{for case } (b), \end{cases}$$

where $1/(1+\alpha)$ denotes $-\infty$ if $\alpha = -1$.

Proof. For $0 < \lambda \le 1$, since $\theta(s) \in NRV_0$ and $\lim_{t\to 0} tW'(t)/W(t) = 1$, we have

$$\lim_{t\to 0} \frac{t[\theta(\Psi(t))]'}{\theta(\Psi(t))} = \lim_{t\to 0} \left[\frac{\Psi(t)\theta'(\Psi(t))}{\theta(\Psi(t))} \cdot \frac{t\Psi'(t)}{\Psi(t)}\right] = 0.$$

When $\lambda = 0$ and $\lim_{t\to 0} tW'(t)/W(t) = 1$, note that

$$\frac{t[\theta(\Psi(t))]'}{\theta(\Psi(t))} = \frac{\Psi(t)\theta'(\Psi(t))}{G(\theta(\Psi(t)))} \cdot \frac{G(\theta(\Psi(t))) \ln \Psi(t)}{\theta(\Psi(t))} \cdot \frac{t[\ln \Psi(t)]'}{\ln \Psi(t)}.$$

Since

$$\hat{f}'(\theta(s)) = \frac{\hat{f}(\theta(s))}{G(\theta(s))},$$

and $\hat{f}(\theta(s)) \in NRV_1$ by Lemma 3.2 (ii), we have

$$\lim_{s\to\infty} \frac{s\theta'(s)}{G(\theta(s))} = \lim_{s\to\infty} \frac{s[\hat{f}(\theta(s))]'}{\hat{f}(\theta(s))} = 1$$

and

$$\lim_{s\to\infty} \frac{G(\theta(s))\ln s}{\theta(s)} = \lim_{s\to\infty} \frac{G(\theta(s))\ln \hat{f}(\theta(s))}{\theta(s)} = \lim_{s\to\infty} \frac{G(s)\ln \hat{f}(s)}{s} = 1+\beta.$$

Moreover, a direct calculation yields

$$[\ln \Psi(t)]' \sim -\frac{4}{W(t)} \quad \text{and} \quad \ln \Psi(t) \sim -4\ln K(t) \quad \text{as } t \to 0,$$

which together with l'Hôpital's rule results in

$$\lim_{t\to 0} \frac{t[\ln \Psi(t)]'}{\ln \Psi(t)} = \lim_{t\to 0} \frac{t/W(t)}{\ln K(t)} = 0.$$

Consequently, assertion (i) follows.

We proceed to prove assertion (ii). For now, we have

$$\frac{t[\vartheta(\Phi(t))]'}{\vartheta(\Phi(t))} = \frac{\Phi(t)\vartheta'(\Phi(t))}{G(\vartheta(\Phi(t)))} \cdot \frac{G(\vartheta(\Phi(t)))g(\vartheta(\Phi(t)))}{\vartheta(\Phi(t))} \cdot \frac{\ln \Phi(t)}{g(\vartheta(\Phi(t)))} \cdot \frac{t[\ln \Phi(t)]'}{\ln \Phi(t)}.$$

Similar to the proof of assertion (i), we get

$$\lim_{s\to\infty} \frac{s\vartheta'(s)}{G(\vartheta(s))} = \lim_{s\to\infty} \frac{s[\hat{f}(\vartheta(s))]'}{\hat{f}(\vartheta(s))} = 1.$$

Observe that

$$\lim_{s\to\infty} \frac{g(s)G(s)}{s} = \begin{cases} 1 & \text{for case (a)}, \\ 1+\beta & \text{for case (b)}, \end{cases}$$

and

$$\lim_{s\to\infty} \frac{\ln s}{g(\vartheta(s))} = \begin{cases} \infty & \text{for case (a)}, \\ 1 & \text{for case (b)} \end{cases}$$

due to Lemma 4.1 (iv). By l'Hôpital's rule and Remark 1.1, it follows that

$$\lim_{t\to 0} \frac{t[\ln \Phi(t)]'}{\ln \Phi(t)} = 1 - \lim_{t\to 0} \frac{tW'(t)}{W(t)} = \frac{1}{1+\alpha},$$

where $1/(1+\alpha)$ denotes $-\infty$ if $\alpha = -1$. Thus we arrive at the desired result. □

From the above proposition, we know that $\theta(\Psi(t))$ in Theorem 1 (i)–(ii) is slowly varying at zero, whereas $\vartheta(K^{-4}(t))$ in Theorem 1 (iii) is regularly varying at zero (with index $(1+\beta)/(1+\alpha)$) for case (b) with $\alpha < -1$, but is rapidly varying at zero for case (b) with $\alpha = -1$ and case (a). This reveals in what behavior the boundary blow-up solution u goes to ∞ near the boundary since as $d(x) \to 0$, $u(x) \sim \theta(\Psi(d(x)))$ for Theorem 1 (i)–(ii) and $u(x) \sim \vartheta(K^{-4}(d(x)))$ for Theorem 1 (iii).

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Next, we list a series of functions required by the items (i)–(iii) of Theorem 1 respectively as examples to give a picture sketching these results.

We start from the following facts. Firstly, it is true for Theorem 1 (i)–(ii) that

$$\ln \hat{f}(\theta(\Psi(t))) \sim \ln \Psi(t) \sim -4 \ln K(t) \sim \begin{cases} -\frac{4}{\lambda} \ln t & \text{if } 0 < \lambda \le 1, \\ -4 \ln k(t) & \text{if } \lambda = 0 \end{cases} \tag{5.1}$$

as $t \to 0$ by Lemma 3.2 (ii). Secondly, it holds for Theorem 1 (iii) that

$$\ln \hat{f}(\vartheta(K^{-4}(t))) \sim -4 \ln K(t) \sim -4 \ln k(t) \quad \text{as } t \to 0 \tag{5.2}$$

by Lemma 4.1 (ii).

Let the (H-f) function $f \in \Gamma$ satisfy one of the following

(1) $f(s) \sim \hat{f}(s) = \mathrm{e}^{(\ln s)^{p_1}}$, $p_1 > 1$ (so $\lim_{s\to\infty} \frac{sG'(s)}{G(s)} = 1$);
(2) $f(s) \sim \hat{f}(s) = \mathrm{e}^{s^{p_2}}$, $p_2 > 0$ (thus $\lim_{s\to\infty} G'(s) \ln \hat{f}(s) = \frac{1-p_2}{p_2} > -1$);
(3) $f(s) \sim \hat{f}(s) = \mathrm{e}^{\mathrm{e}^{s^{p_3}}}$, $p_3 > 0$ (here $\lim_{s\to\infty} G'(s) \ln \hat{f}(s) = -1$).

Based upon (5.1) and (5.2), we obtain immediately the following explicit representation for the boundary asymptotic behavior of solutions with various decays of b (described via k as in (1.10) of Theorem 1):

(I-1) Theorem 1 (i) If $k(t) = \mathrm{e}^{-(-\ln t)^{q_1}}$ with $0 < q_1 < 1$, then $\lim_{t\to 0} \left(\frac{K(t)}{k(t)}\right)' = 1$. We conclude

$$\begin{cases} \ln u(x) \sim \big[-4 \ln d(x)\big]^{\frac{1}{p_1}} & \text{for case (1)}, \\ u(x) \sim \big[-4 \ln d(x)\big]^{\frac{1}{p_2}} & \text{for case (2)}, \\ u(x) \sim \big[\ln(-4 \ln d(x))\big]^{\frac{1}{p_3}} & \text{for case (3)} \end{cases}$$

as $d(x) \to 0$.

(I-2) Theorem 1 (i) If $k(t) = t^{q_2}$ with $q_2 > 0$, then $\lim_{t\to 0} \left(\frac{K(t)}{k(t)}\right)' = \frac{1}{q_2+1} \in (0,1)$, and hence

$$\begin{cases} \ln u(x) \sim \big[-4(q_2+1) \ln d(x)\big]^{\frac{1}{p_1}} & \text{for case (1)}, \\ u(x) \sim \big[-4(q_2+1) \ln d(x)\big]^{\frac{1}{p_2}} & \text{for case (2)}, \\ u(x) \sim \big[\ln(-4(q_2+1) \ln d(x))\big]^{\frac{1}{p_3}} & \text{for case (3)} \end{cases}$$

as $d(x) \to 0$.

(II) Theorem 1 (ii) If $k(t) = \mathrm{e}^{-(-\ln t)^{q_3}}$ with $q_3 > 1$, then $\lim_{t\to 0} \left(\frac{K(t)}{k(t)}\right)' = 0$ and

$$\lim_{t\to 0} \frac{tW'(t)}{W(t)} = 1.$$

Therefore,

$$\begin{cases} \text{not considered here} & \text{for case (1)}, \\ u(x) \sim \big[4(-\ln d(x))^{q_3}\big]^{\frac{1}{p_2}} & \text{for case (2)}, \\ u(x) \sim \big[\ln(4(-\ln d(x))^{q_3})\big]^{\frac{1}{p_3}} & \text{for case (3)} \end{cases}$$

as $d(x) \to 0$.

(III-1) Theorem 1 (iii) If $k(t) = \mathrm{e}^{-t^{-q_4}}$ with $q_4 > 0$, then $\lim_{t\to 0}\left(\frac{K(t)}{k(t)}\right)' = 0$ and

$$\lim_{t\to 0} W'(t)\ln K(t) = -\frac{1+q_4}{q_4} < -1.$$

We have

$$\begin{cases} \ln u(x) \sim \left[4(d(x))^{-q_4}\right]^{\frac{1}{p_1}} & \text{for case (1)}, \\ u(x) \sim \left[4(d(x))^{-q_4}\right]^{\frac{1}{p_2}} & \text{for case (2)}, \\ u(x) \sim \left[\ln(4(d(x))^{-q_4})\right]^{\frac{1}{p_3}} & \text{for case (3)} \end{cases}$$

as $d(x) \to 0$.

(III-2) Theorem 1 (iii) If $k(t) = \mathrm{e}^{-\mathrm{e}^{t^{-q_5}}}$ with $q_5 > 0$, then $\lim_{t\to 0}\left(\frac{K(t)}{k(t)}\right)' = 0$ and

$$\lim_{t\to 0} W'(t)\ln K(t) = -1.$$

So,

$$\begin{cases} \ln u(x) \sim \left[4\mathrm{e}^{(d(x))^{-q_5}}\right]^{\frac{1}{p_1}} & \text{for case (1)}, \\ u(x) \sim \left[4\mathrm{e}^{(d(x))^{-q_5}}\right]^{\frac{1}{p_2}} & \text{for case (2)}, \\ \text{not established} & \text{for case (3)} \end{cases}$$

as $d(x) \to 0$.

The above examples clearly show the contribution of the decay of the weighted function b (near the boundary) and the growth of the nonlinearity f (at infinity) to the boundary blow-up rate of the solutions: more rapidly b decays to zero, or more slowly f grows at ∞, more rapidly the solutions go to infinity near the boundary.

6. The proof of Theorem 2

The last section is devoted to the proof of Theorem 2 concerning the second expansion of solutions near the boundary. We give a crucial lemma at first.

Lemma 6.1. *Suppose the C^1-function f on $[0,\infty)$ satisfies $f(s) > 0$ for $s > 0$, and*

$$\lim_{s\to\infty} \frac{F(s)f'(s)}{f^2(s)} = 1 \quad \text{with } F(s) = \int_0^s f(\tau)d\tau. \tag{6.1}$$

Let $\psi(t) : (0, \nu(0+)) \to (0,\infty)$ be the inverse of

$$\nu(t) = \int_t^\infty \frac{d\tau}{\sqrt[4]{4F(\tau)}}, \quad t > 0.$$

Then

(i) $\lim_{t\to 0}\psi(t)=\infty$, *and* $\psi'(t)=-\sqrt[4]{4F(\psi(t))}$ *with* $\psi''(t)=f(\psi(t))/\sqrt{4F(\psi(t))}$;

(ii) $\lim_{t\to 0}\frac{t\psi'(t)}{\psi(t)}=0$;

(iii) $\lim_{t\to 0}t[\ln f(\psi(t))]'=-4$ *and* $\lim_{t\to 0}\frac{t\psi''(t)}{\psi'(t)}=-1$.

Proof. Note that $F(s)=\mathrm{e}^{\int_0^s\frac{d\tau}{G_1(\tau)}}$ on $[B,\infty)$ for some $B>0$ with $G_1(s)=F(s)/f(s)$, $s>B$. By (6.1), $G_1'(s)\to 0$ as $s\to\infty$. Similar to the proof of Lemma 2.2, we get that for fixed $\sigma>4$, there exists $c_0>0$ such that $F(s)\geq F(c_0)(s/c_0)^\sigma$, $s\geq c_0$, and thereby

$$\int_t^\infty\frac{d\tau}{\sqrt[4]{4F(\tau)}}<\infty,\quad t>0.$$

Also, it follows from L'Hôpital's rule and (6.1) that

$$\lim_{s\to\infty}\frac{F(s)}{sf(s)}=0.$$

A simple computation yields assertion (i). Noticing that

$$t=\int_{\psi(t)}^\infty\frac{d\tau}{\sqrt[4]{4F(\tau)}},\quad \forall t\in(0,\nu(0+))\tag{6.2}$$

and using L'Hôpital's rule, we have with assertion (i) that

$$\lim_{t\to 0}\frac{t\psi'(t)}{\psi(t)}=\lim_{s\to\infty}\frac{-\sqrt[4]{4F(s)}\int_s^\infty\frac{d\tau}{\sqrt[4]{4F(\tau)}}}{s}=\lim_{s\to\infty}\frac{4}{4-\frac{sf(s)}{F(s)}}=0.$$

By (6.2),

$$\left(\int_{F(\psi(1/s))}^\infty(4\tau)^{-\frac{1}{4}}[F^{-1}(\tau)]'d\tau\right)^{-1}=s\quad\text{for large } s>0,$$

where F^{-1} is the inverse of F. Following the procedures used in the proof of Lemma 4.1 (ii), we arrive at $F(\psi(1/s))\in NRV_4$, whence

$$\lim_{t\to 0}t[\ln F(\psi(t))]'=-\lim_{s\to\infty}\frac{s[F(\psi(1/s))]'}{F(\psi(1/s))}=-4,$$

and

$$\lim_{t\to 0}\frac{t\psi''(t)}{\psi'(t)}=\lim_{t\to 0}\frac{tf(\psi(t))/\sqrt{4F(\psi(t))}}{-\sqrt[4]{4F(\psi(t))}}=\lim_{t\to 0}\frac{t[\ln F(\psi(t))]'}{4}=-1.$$

Since $[\ln f(s)]'\sim[\ln F(s)]'$ as $s\to\infty$ due to (6.1),

$$\lim_{t\to 0}t[\ln f(\psi(t))]'=\lim_{t\to 0}t[\ln F(\psi(t))]'=-4.$$

Now, assertions (ii) and (iii) have been established. $\square$

Remark 6.1. $\lim_{t\to 0} t[\ln f(\psi(t))]' = -4$ can be rewritten as

$$\lim_{t\to 0} \frac{G(\psi(t))}{t\psi'(t)} = -4 \tag{6.3}$$

with $G(s) = f(s)/f'(s)$ for $s > 0$ large. In addition, we point out that Lemma 6.1 is valid for the functions f in Theorem 2, since here (6.1) is satisfied.

In addition, we need two propositions:

Proposition 6.1. *Let $b(x) \equiv 1$. Suppose $f \in C[0,\infty)$ is increasing in $(0,\infty)$ with $f(0) = 0$, and satisfies the Keller–Osserman condition (1.4). Then for any solution u of (1.1), there are constants $\delta \in (0, \delta_0/2)$ and $M > 0$ such that*

$$\psi(d(x)) - M \le u(x) \le \psi(d(x)) + M, \quad x \in \Omega_{2\delta}$$

with $\psi(t)$ defined in Lemma 6.1.

Proof. The claimed result follows with the same proof of Theorem 1 if we can find some $\delta \in (0, \delta_0/2)$ such that for each $\rho \in (0, \delta)$,

$$w(x) = \psi(d(x) - \rho), \quad x \in \Omega_{2\delta} \setminus \bar{\Omega}_\rho =: \Omega_\rho^-$$

and

$$z(x) = \psi(d(x) + \rho), \quad x \in \Omega_{2\delta-\rho} =: \Omega_\rho^+$$

are a supersolution and a subsolution of Eq. $(1.1)_1$ in Ω_ρ^- and in Ω_ρ^+, respectively.

Next, we take z for instance to show that it indeed can be a subsolution of $(1.1)_1$. Let $\delta \in (0, \delta_0/2)$, $\rho \in (0, \delta)$. It is obvious that ψ is decreasing in $(\rho, 2\delta)$ provided δ small enough. Denote by ζ the inverse of ψ. Then

$$\zeta'(t) = [\psi'(\zeta(t))]^{-1}, \quad \zeta''(t) = -[\psi'(\zeta(t))]^{-3}\psi''(\zeta(t)). \tag{6.4}$$

Let $(x_0, \varphi) \in \Omega_\rho^+ \times C^2(\Omega_\rho^+)$ be a pair such that $z \le \varphi$ in a neighborhood $\mathcal{N}$ of x_0 and $z(x_0) = \varphi(x_0)$ (i.e. $\zeta(\varphi(x_0)) = d(x_0) + \rho =: d^+(x_0)$). Similar to the proof of Proposition 3.1, we get by (6.4) that

$$\begin{aligned}
\Delta_\infty \varphi(x_0) &\ge -\zeta''(\varphi(x_0))(\zeta'(\varphi(x_0)))^{-5} \\
&= [\psi'(d^+(x_0))]^2 \psi''(d^+(x_0)) \\
&= f(\psi(d^+(x_0))) \\
&= f(z(x_0)).
\end{aligned}$$

The proof is complete. □

Remark 6.2. Clearly, $u(x) \sim \psi(d(x))$ as $d(x) \to 0$ in Proposition 6.1. Generally, we can prove that if b in (H-b) satisfies (1.10) with $0 < \lambda \le 1$, $f \in C^1[0,\infty)$ is increasing in $(0,\infty)$ with $f(0) = 0$, and (6.1) holds, then for any solution u of (1.1), $u(x) \sim \psi(K(d(x)))$ as $d(x) \to 0$.

Proposition 6.2. *Under the assumptions of Theorem 2, there exists $\delta \in (0, \delta_0)$ such that for each $\rho \in (0, \delta)$ and each $\Lambda > 0$ with $\Lambda\rho[\ln f(\psi(\rho))]^{-1} \le \delta$, the functions*

$$\bar{u} = \psi(d(x)) + \Lambda d(x)[\ln f(\psi(d(x)))]^{-1} G(\psi(d(x)))$$

and

$$\underline{u} = \psi(d(x)) - \Lambda d(x)[\ln f(\psi(d(x)))]^{-1} G(\psi(d(x)))$$

are a supersolution and a subsolution of Eq. (1.1)$_1$ in Ω_ρ, respectively. Here $G(s)$ is the function given by (1.17).

Proof. Let $\rho \in (0, \delta)$ and $\Lambda > 0$ satisfy $\Lambda\rho[\ln f(\psi(\rho))]^{-1} \le \delta$ with $\delta \in (0, \delta_0)$ to be determined. We only give the proof for subsolutions.

Define

$$\eta(t) = \psi(t) - \Lambda t[\ln f(\psi(t))]^{-1} G(\psi(t)), \quad t \in (0, \rho).$$

Then

$$\begin{aligned}
\eta'(t) &= \psi'(t) - \Lambda[\ln f(\psi(t))]^{-1} G(\psi(t)) \\
&\quad + \Lambda t[\ln f(\psi(t))]^{-2} \cdot \psi'(t) - \Lambda t[\ln f(\psi(t))]^{-1} G'(\psi(t)) \cdot \psi'(t) \\
&= \psi'(t)\Big\{1 - \Lambda t[\ln f(\psi(t))]^{-1}\Big(\frac{G(\psi(t))}{t\psi'(t)} - [\ln f(\psi(t))]^{-1} + G'(\psi(t))\Big)\Big\} \\
&\le \psi'(t)\Big\{1 - \delta\Big|\frac{G(\psi(t))}{t\psi'(t)} - [\ln f(\psi(t))]^{-1} + G'(\psi(t))\Big|\Big\} \\
&< 0
\end{aligned}$$

with δ small enough, since $\lim_{t\to 0}\psi(t) = \infty$, $\lim_{s\to\infty} G'(s) = 0$ and $\lim_{t\to 0} G(\psi(t))/(t\psi'(t)) = -4$ due to (6.3). Hence η is decreasing in $(0, \rho)$. Denote by ζ the inverse of η. A direct but tedious computation shows that

$$\begin{aligned}
\zeta'(t) &= \frac{1}{\eta'(\zeta(t))} \\
&= \big\{\psi'(\zeta(t)) - \Lambda[\ln f(\psi(\zeta(t)))]^{-1} G(\psi(\zeta(t))) + \Lambda\zeta(t)[\ln f(\psi(\zeta(t)))]^{-2} \cdot \psi'(\zeta(t)) \\
&\quad - \Lambda\zeta(t)[\ln f(\psi(\zeta(t)))]^{-1} G'(\psi(\zeta(t))) \cdot \psi'(\zeta(t))\big\}^{-1}, \\
\zeta''(t) &= -\big\{\psi'(\zeta(t)) - \Lambda[\ln f(\psi(\zeta(t)))]^{-1} G(\psi(\zeta(t))) + \Lambda\zeta(t)[\ln f(\psi(\zeta(t)))]^{-2} \cdot \psi'(\zeta(t)) \\
&\quad - \Lambda\zeta(t)[\ln f(\psi(\zeta(t)))]^{-1} G'(\psi(\zeta(t))) \cdot \psi'(\zeta(t))\big\}^{-3} \\
&\quad \times \big\{\psi''(\zeta(t)) + 2\Lambda[\ln f(\psi(\zeta(t)))]^{-2} \cdot \psi'(\zeta(t)) \\
&\quad - 2\Lambda[\ln f(\psi(\zeta(t)))]^{-1} G'(\psi(\zeta(t))) \cdot \psi'(\zeta(t)) \\
&\quad - 2\Lambda\zeta(t)\frac{[\ln f(\psi(\zeta(t)))]^{-3}}{G(\psi(\zeta(t)))} \cdot [\psi'(\zeta(t))]^2 + \Lambda\zeta(t)[\ln f(\psi(\zeta(t)))]^{-2} \cdot \psi''(\zeta(t)) \\
&\quad + \Lambda\zeta(t)[\ln f(\psi(\zeta(t)))]^{-2}\frac{G'(\psi(\zeta(t)))}{G(\psi(\zeta(t)))} \cdot [\psi'(\zeta(t))]^2 \\
&\quad - \Lambda\zeta(t)[\ln f(\psi(\zeta(t)))]^{-1} G''(\psi(\zeta(t))) \cdot [\psi'(\zeta(t))]^2 \\
&\quad - \Lambda\zeta(t)[\ln f(\psi(\zeta(t)))]^{-1} G'(\psi(\zeta(t))) \cdot \psi''(\zeta(t))\big\}.
\end{aligned} \tag{6.5}$$

Let $(x_0, \varphi) \in \Omega_\rho \times C^2(\Omega_\rho)$ be a pair such that $\underline{u} \le \varphi$ in a neighborhood $\mathcal{N}$ of x_0 and $\underline{u}(x_0) = \varphi(x_0)$ (i.e. $d(x_0) = \zeta(\varphi(x_0))$). Similar to the proof of Proposition 3.1, we get

$$\Delta_\infty \varphi(x_0) \ge -\zeta''(\varphi(x_0))(\zeta'(\varphi(x_0)))^{-5},$$

which together with (6.5) yields that

$$\begin{aligned}
\Delta_\infty \varphi(x_0) \ge{}& \psi''(t_0)[\psi'(t_0)]^2 \Big\{1 - \Lambda t_0[\ln f(\psi(t_0))]^{-1}\Big(\frac{G(\psi(t_0))}{t_0\psi'(t_0)} - [\ln f(s_0)]^{-1} + G'(s_0)\Big)\Big\}^2 \\
&\times \Big\{1 + \Lambda t_0[\ln f(\psi(t_0))]^{-1}\Big(\frac{2[\ln f(s_0)]^{-1}\psi'(t_0)}{t_0\psi''(t_0)} - \frac{2G'(s_0)\psi'(t_0)}{t_0\psi''(t_0)} \\
&- \frac{2s_0[\ln f(s_0)]^{-1}}{G(s_0)\ln f(s_0)}\frac{\psi'(t_0)}{t_0\psi''(t_0)}\frac{t_0\psi'(t_0)}{\psi(t_0)} + [\ln f(s_0)]^{-1} \\
&+ \frac{s_0 G'(s_0)}{G(s_0)\ln f(s_0)}\frac{\psi'(t_0)}{t_0\psi''(t_0)}\frac{t_0\psi'(t_0)}{\psi(t_0)} - s_0 G''(s_0)\frac{\psi'(t_0)}{t_0\psi''(t_0)}\frac{t_0\psi'(t_0)}{\psi(t_0)} - G'(s_0)\Big)\Big\},
\end{aligned}$$

where $t_0 := d(x_0)$ and $s_0 := \psi(d(x_0)) = \psi(t_0)$. Note that since we require $\rho, \Lambda > 0$ such that ρ and $\Lambda\rho[\ln f(\psi(\rho))]^{-1}$ are not more than δ, t_0 and $\Lambda t_0[\ln f(\psi(t_0))]^{-1}$ can be arbitrarily small whenever δ is taken small enough. In the following, we denote with $o(1)$ an infinitely small quantity with $t_0 \to 0$ that is independent of Λ. Notice that s_0 is a function of t_0 satisfying $s_0 \to \infty$ as $t_0 \to 0$. It follows from Lemma 6.1 and Remark 1.3 that

$$\begin{aligned}
\Delta_\infty \varphi(x_0) &\ge f(\psi(t_0))\big\{1 + \Lambda t_0[\ln f(\psi(t_0))]^{-1}(4 + o(1))\big\}^2\big\{1 + \Lambda t_0[\ln f(\psi(t_0))]^{-1}\cdot o(1)\big\} \\
&= f(\psi(t_0))\big\{1 + \Lambda t_0[\ln f(\psi(t_0))]^{-1} \\
&\quad \times (8 + \Lambda t_0[\ln f(\psi(t_0))]^{-1}(16 + o(1)) + (\Lambda t_0[\ln f(\psi(t_0))]^{-1})^2\cdot o(1) + o(1))\big\} \\
&\ge f(\psi(t_0))\big\{1 + \Lambda t_0[\ln f(\psi(t_0))]^{-1}(8 - 16\delta + o(1))\big\}
\end{aligned} \tag{6.6}$$

provided δ small enough.

Let us continue to estimate $f(\underline{u}(x_0))$. Noticing that $\lim_{s\to\infty} G(s)/s = \lim_{s\to\infty} G'(s) = 0$, we can take δ small so that

$$-\Lambda t_0[\ln f(\psi(t_0))]^{-1}\frac{G(s_0)}{s_0} > -\frac{1}{2},$$

and thereby invoking Taylor's expansion and further taking δ small if necessary lead to

$$\begin{aligned}
\mathrm{e}^{[\ln \underline{u}(x_0)]^p} &= \mathrm{e}^{[\ln\psi(t_0)]^p\left[1+\frac{\ln\left(1-\Lambda t_0[\ln f(\psi(t_0))]^{-1}\frac{G(s_0)}{s_0}\right)}{\ln s_0}\right]^p} \\
&\le \mathrm{e}^{[\ln\psi(t_0)]^p\left[1-p\Lambda t_0[\ln f(\psi(t_0))]^{-1}\frac{G(s_0)}{s_0\ln s_0}+\frac{C_1}{\ln s_0}\left(-\Lambda t_0[\ln f(\psi(t_0))]^{-1}\frac{G(s_0)}{s_0}\right)^2\right]} \\
&= \mathrm{e}^{[\ln\psi(t_0)]^p+\Lambda t_0[\ln f(\psi(t_0))]^{-1}(-1+\Lambda t_0[\ln f(\psi(t_0))]^{-1}\cdot o(1))} \\
&\le \mathrm{e}^{[\ln\psi(t_0)]^p+\Lambda t_0[\ln f(\psi(t_0))]^{-1}(-1+o(1))} \\
&\le \mathrm{e}^{[\ln\psi(t_0)]^p}\{1 + \Lambda t_0[\ln f(\psi(t_0))]^{-1}(-1 + \Lambda t_0[\ln f(\psi(t_0))]^{-1}\cdot C_2 + o(1))\}
\end{aligned}$$

for case (Γ-1), and

$$\begin{aligned}
\mathrm{e}^{[\underline{u}(x_0)]^p} &= \mathrm{e}^{[\psi(t_0)]^p\left[1-\Lambda t_0[\ln f(\psi(t_0))]^{-1}\frac{G(s_0)}{s_0}\right]^p} \\
&\le \mathrm{e}^{[\psi(t_0)]^p\left[1-p\Lambda t_0[\ln f(\psi(t_0))]^{-1}\frac{G(s_0)}{s_0}+C_3\left(-\Lambda t_0[\ln f(\psi(t_0))]^{-1}\frac{G(s_0)}{s_0}\right)^2\right]} \\
&= \mathrm{e}^{[\psi(t_0)]^p+\Lambda t_0[\ln f(\psi(t_0))]^{-1}(-1+\Lambda t_0[\ln f(\psi(t_0))]^{-1}\cdot o(1))} \\
&\le \mathrm{e}^{[\psi(t_0)]^p+\Lambda t_0[\ln f(\psi(t_0))]^{-1}(-1+o(1))} \\
&\le \mathrm{e}^{[\psi(t_0)]^p}\left\{1+\Lambda t_0[\ln f(\psi(t_0))]^{-1}(-1+\Lambda t_0[\ln f(\psi(t_0))]^{-1}\cdot C_4+o(1))\right\}
\end{aligned}$$

for case (Γ-2), where C_i $(i = 1, 2, 3, 4)$ are positive constants independent of Λ. Therefore, in both cases, $f(\underline{u}(x_0))$ admits the same estimate of the form

$$f(\underline{u}(x_0)) \le f(\psi(t_0))\left\{1+\Lambda t_0[\ln f(\psi(t_0))]^{-1}(-1+D_1\delta+o(1))\right\} \tag{6.7}$$

with δ small enough. Hence we obtain from (6.6) and (6.7) that

$$\Delta_\infty\varphi(x_0) - f(\underline{u}(x_0)) \ge f(\psi(t_0))\left\{\Lambda t_0[\ln f(\psi(t_0))]^{-1}(9-D_2\delta+o(1))\right\} > 0$$

provided δ small enough. The proof is complete. □

At last, we give the proof of Theorem 2.

Proof of Theorem 2. Let u be a solution of (1.1). By Proposition 6.1, there are constants $\delta^* \in (0, \delta_0)$ and $M > 0$ such that

$$\psi(d(x)) - M \le u(x) \le \psi(d(x)) + M, \quad x \in \Omega_{\delta^*}.$$

In view of Proposition 6.2, there exists $\delta \in (0, \delta^*)$ such that for each $\rho \in (0, \delta)$ and each $\Lambda > 0$ with $\Lambda\rho[\ln f(\psi(\rho))]^{-1} \le \delta$, the functions

$$\bar{u} = \psi(d(x)) + \Lambda d(x)[\ln f(\psi(d(x)))]^{-1}G(\psi(d(x)))$$

and

$$\underline{u} = \psi(d(x)) - \Lambda d(x)[\ln f(\psi(d(x)))]^{-1}G(\psi(d(x)))$$

are a supersolution and a subsolution of Eq. $(1.1)_1$ in Ω_ρ, respectively. Here $G(s)$ is the function given by (1.17). In what follows we will fix ρ and Λ to suit our purpose.

It is easy to see that $f(s)/s^3$ is ultimately increasing, i.e., there exists a constant $s_0 > s_*$ such that $f(s)/s^3$ is increasing in (s_0, ∞), which shows

$$\frac{1}{\sigma^3}f(s) \le f\left(\frac{s}{\sigma}\right) \quad \text{and} \quad \sigma^3 f(s) \ge f(\sigma s)$$

for any $s > 2s_0$ and any $\sigma \in (1/2, 1)$. Noticing that $u(x) \to \infty$ as $d(x) \to 0$, we can take $\rho_1 \in (0, \delta)$ small to guarantee that $u(x) > 2s_0$ for $d(x) < \rho_1$. Hence for any $\sigma \in (1/2, 1)$,

$$\frac{1}{\sigma^3}f(u(x)) \le f\left(\frac{1}{\sigma}u(x)\right) \quad \text{and} \quad \sigma^3 f(u(x)) \ge f(\sigma u(x))$$

in $\{x \in \Omega : d(x) < \rho_1\}$, whence

$$\Delta_\infty\left(\frac{1}{\sigma}u\right) \le f\left(\frac{1}{\sigma}u\right), \quad \Delta_\infty(\sigma u) \ge f(\sigma u) \quad \text{in } \{x \in \Omega : d(x) < \rho_1\}$$

in the viscosity sense. Since $G(s) \to \infty$ as $s \to \infty$ indicated in Remark 1.3, we can let ρ_1 be small such that $\delta G(\psi(d(x))) > M$ on $\{x \in \Omega : d(x) \le \rho_1\}$. Let $\Lambda_1 \rho_1 [\ln f(\psi(\rho_1))]^{-1} = \delta$. Then $\underline{u}$ and $\bar{u}$ (with ρ_1 and Λ_1 taken above) are a subsolution and a supersolution of Eq. $(1.1)_1$ in Ω_{ρ_1}, with

$$\underline{u}(x) < u(x) < \bar{u}(x) \quad \text{on } \{x \in \Omega : d(x) = \rho_1\}.$$

Certainly,

$$\underline{u}(x) < \frac{1}{\sigma} u(x) \quad \text{and} \quad \bar{u}(x) > \sigma u(x) \quad \text{on } \{x \in \Omega : d(x) = \rho_1\}.$$

From Proposition 6.1 and the expressions of $\underline{u}$ and $\bar{u}$, we have

$$\lim_{d(x) \to 0} \frac{\underline{u}(x)}{u(x)} = \lim_{d(x) \to 0} \frac{\bar{u}(x)}{u(x)} = 1.$$

Hence

$$\underline{u}(x) \le \frac{1}{\sigma} u(x) \quad \text{and} \quad \bar{u}(x) \ge \sigma u(x) \quad \text{near } \partial\Omega.$$

By comparison, we get

$$\underline{u}(x) \le \frac{1}{\sigma} u(x) \quad \text{and} \quad \bar{u}(x) \ge \sigma u(x) \quad \text{in } \Omega_{\rho_1}.$$

Since $\sigma \in (1/2, 1)$ is arbitrary, we have

$$\underline{u}(x) \le u(x) \le \bar{u}(x) \quad \text{in } \Omega_{\rho_1},$$

that is,

$$\left| \frac{u(x) - \psi(d(x))}{d(x)[\ln f(\psi(d(x)))]^{-1} G(\psi(d(x)))} \right| \le \Lambda_1 \quad \text{in } \Omega_{\rho_1}.$$

Inserting the expressions of f and G, we get the desired result. □

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