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Existence, uniqueness and multiplicity of positive solutions for Schrödinger–Poisson system with singularity ☆

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ABSTRACT

In this paper, we consider the following Schrödinger–Poisson system with singularity

$$\begin{cases} -\Delta u + \eta\phi u = \mu u^{-r}, & \text{in } \Omega, \\ -\Delta\phi = u^2, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = \phi = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^3$ is a smooth bounded domain with boundary $\partial\Omega$, $\eta = \pm 1$, $r \in (0,1)$ is a constant, $\mu > 0$ is a parameter. We obtain the existence and uniqueness of positive solution for $\eta = 1$ and any $\mu > 0$ by using the variational method. The existence and multiplicity of solutions for the system are also considered for $\eta = -1$ and $\mu > 0$ small enough by using the method of Nehari manifold.

1. Introduction

In this paper, we consider the following singular Schrödinger–Poisson system

$$\begin{cases} -\Delta u + \eta\phi u = \mu u^{-r}, & \text{in } \Omega, \\ -\Delta\phi = u^2, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = \phi = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^3$ is a smooth bounded domain with boundary $\partial\Omega$, $\eta = \pm 1$, $r \in (0,1)$ is a constant, $\mu > 0$ is a parameter.

This problem is derived from the recent research on the following Schrödinger–Poisson system

$$\begin{cases} -\Delta u + u + q\phi f(u) = g(x,u), & \text{in } \mathbb{R}^3, \\ -\Delta \phi = 2F(u), & \text{in } \mathbb{R}^3. \end{cases} \tag{1.2}$$

Recently, the existence, nonexistence, multiplicity results, ground state and sign-changing solutions of system (1.2) have been studied widely by using the modern variational method and critical point theory under various assumptions of nonlocal term f and nonlinear term g, see [9,11,12,22,16,1,6,5,30,19,18,28,31,20], etc.

There are also many references which investigated Schrödinger–Poisson system in bounded domain, see [7,3,4]. In [3], the following system involving the critical growing nonlocal term was considered

$$\begin{cases} -\Delta u = \lambda u + q|u|^3 u\phi, & \text{in } B_R, \\ -\Delta \phi = q|u|^5, & \text{in } B_R, \\ u = \phi = 0, & \text{on } \partial B_R, \end{cases}$$

where B_R is a ball in $\mathbb{R}^3$ centered at the origin and with radius R. The existence and nonexistence results were obtained by discussing the scope of the parameter λ. By using the methods of a cut-off function and the variational arguments, in [4], the authors studied the following Schrödinger–Poisson system in bounded domain

$$\begin{cases} -\Delta u + \varepsilon q\phi f(u) = \eta|u|^{p-1}u, & \text{in } \Omega, \\ -\Delta \phi = 2qF(u), & \text{in } \Omega, \\ u = \phi = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$, $p \in (1,5)$, $q > 0$, ε, $\eta = \pm 1$, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and $F(t) = \int_0^t f(s)ds$. They obtained the existence and multiplicity results assuming on f a subcritical growth condition and they also considered the existence and nonexistence results under the critical case.

The following singular semilinear elliptic problem

$$\begin{cases} -\Delta u = \lambda u^p + \mu u^{-r}, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $N \geqslant 3$, $p > 0$, $r \in (0,1)$, has been extensively studied, see [17,25,24,29,15,8,26,2,27], etc. When $\lambda \equiv 0$, the existence of solutions has been studied in [10,13,17]. When $p \in (1, 2^* - 1)$, the existence and multiplicity of solutions for (1.3) have been studied in [25,29,15,2,27] for all $\mu > 0$ and $\lambda > 0$ small enough. The existence of multiple solutions of (1.3) for $p = 2^* - 1$ and $\lambda > 0$ small enough has been considered in [2]. In [24], the existence result for $p \in (0,1)$ was considered.

Recently, in [21], the following singular Kirchhoff type problem which possesses the nonlocal term $(b\int_\Omega |\nabla u|)\Delta u$ has been considered

$$\begin{cases} -(a + b\int_\Omega |\nabla u|^2)\Delta u = \lambda u^3 + \mu u^{-r}, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded smooth domain with boundary $\partial\Omega$, $a, b, \lambda, \mu > 0$ are four parameters and $r \in (0, 1)$. By using the variational method and the Nehari manifold, the existence and multiplicity of solutions for problem (1.4) have been considered.

To our knowledge, the singular Schrödinger–Poisson system has not been studied up to now. Motivated by above references, especially by [25,21], in this paper, we consider the singular Schrödinger–Poisson system (1.1).

Before we state the main results about Schrödinger–Poisson system (1.1), we give several definitions.

Denote by H the Sobolev space $H_0^1(\Omega)$ with the inner product and norm

$$(u, v) = \int_\Omega \nabla u \cdot \nabla v, \ \ \|u\| = (u, u)^{1/2},$$

and denote $|\cdot|_s$ the usual norm of $L^s(\Omega)$ for $s \in [1, \infty)$. By Sobolev embedding theorem, H can be compactly embedded into $L^s(\Omega)$ for $s \in [1, 6)$, and the embedding $H \hookrightarrow L^6(\Omega)$ is continuous. Let $S > 0$ be the embedding constant, that is

$$|u|_6^2 \leqslant S^{-1}\|u\|^2, \quad u \in H. \tag{1.5}$$

Since $r \in (0, 1)$, $\left(\int_\Omega |u|^{1-r}\right)^{\frac{1}{1-r}}$ is not a norm, but by Hölder inequality and (1.5) the following holds

$$\int_\Omega |u|^{1-r} \leqslant |u|_4^{1-r}|\Omega|^{(3+r)/4} \leqslant |u|_6^{1-r}|\Omega|^{(5+r)/6} \leqslant S^{-(1-r)/2}\|u\|^{1-r}|\Omega|^{(5+r)/6}, \tag{1.6}$$

where $|\Omega|$ denotes the Lebesgue measure of the domain Ω.

By using the Lax–Milgram theorem, for each $u \in H$, the second equation of system (1.1) with Dirichlet boundary condition has a unique solution $\phi_u \in H$, substituting ϕ_u to the first equation of (1.1), the system can be transformed into one variable problem

$$\begin{cases} -\Delta u + \eta\phi_u u = \mu u^{-r}, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

For problem (1.7), we define the functional

$$J_\mu(u) = \frac{1}{2}\int_\Omega \|u\|^2 + \frac{\eta}{4}\int_\Omega \phi_u u^2 - \frac{\mu}{1-r}\int_\Omega |u|^{1-r}, \quad u \in H. \tag{1.8}$$

Since $r \in (0, 1)$, for any $u, v \in H$, by (1.6),

$$\left|\int_\Omega |u|^{1-r} - |v|^{1-r}\right| \leqslant \int_\Omega |u - v|^{1-r} \leqslant |u - v|_4^{1-r}|\Omega|^{(3+r)/4} \leqslant S^{-(1-r)/2}\|u - v\|^{1-r}|\Omega|^{(5+r)/6}. \tag{1.9}$$

Then, by Lemma 2.1 in the following section, J_μ is well defined and continuous on H.

In general, a function $u \in H$ is called a solution of (1.7), that is (u, ϕ_u) is a solution of (1.1) and $u > 0$ satisfying

$$\int_\Omega \nabla u \cdot \nabla\phi + \eta\int_\Omega \phi_u u\phi - \mu\int_\Omega u^{-r}\phi = 0, \phi \in H. \tag{1.10}$$

We denote $\Psi(u) = \int_\Omega \phi_u u^2$, it follows from Lemma 2.1 in the following section that $\Psi : H \to \mathbb{R}$ is C^1 and $\Psi(tu) = t^4\Psi(u)$. Although Ψ is 4 homogeneous function, it is not equivalent to the nonlinear function $f(u) = u^4$. Therefore our problem (1.1) is totally different from the semilinear problem (1.3) with $p = 4$. Our problem is also different from Kirchhoff problem (1.4). Although the two problems possess nonlocal term, the nonlocal terms $\phi_u u$ in (1.7) and $b(\int_\Omega |\nabla u|)\Delta u$ in (1.4) are essentially different. In fact, the existence and multiplicity of solutions for (1.4) are related to the following eigenvalue problem

$$\begin{cases} -b(\int_\Omega |\nabla u|^2)\Delta u = \nu u^3, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

However, Schrödinger–Poisson system doesn't have similar eigenvalue problem. To our knowledge, the singular Schrödinger–Poisson system has not been considered, and the study on the existence, uniqueness and multiplicity of solutions for singular Schrödinger–Poisson system (1.1) is meaningful in mathematics.

We consider system (1.1) in two cases: $\eta = 1$ and $\eta = -1$. When $\eta = 1$, we can prove that system (1.1) possesses a unique solution which is the global minimum of the functional J_μ on H for any $\mu > 0$ by using the variational method. For $\eta = -1$, two solutions of system (1.1) can be obtained by using the Nehari manifold for $\mu \in (0, \mu^*)$, where

$$\mu^* = \frac{2}{3+r}\left(\frac{1+r}{3+r}\right)^{(1+r)/2} S^{2+r}|\Omega|^{-(4+2r)/3},$$

$S > 0$ in (1.5). Specifically, we decompose Nehari manifold N_μ into three parts: N_μ^+, N_μ^- and N_μ^0, meanwhile, we can prove that $N_\mu^\pm \neq \emptyset$, $N_\mu^0 = \{0\}$ and N_μ^- is closed for $\mu \in (0, \mu^*)$. Consequently, we consider the minimums of the functional J_μ on $N_\mu^+ \cup N_\mu^0$ and N_μ^- respectively by using Ekeland's variational principle.

Our main results can be described as follows.

Theorem 1.1. *Assume $r \in (0, 1)$, then system (1.1) has a unique solution for $\eta = 1$ and $\mu > 0$.*

Theorem 1.2. *Assume $r \in (0, 1)$, when $\eta = -1$, there exists*

$$\mu^* = \frac{2}{3+r}\left(\frac{1+r}{3+r}\right)^{(1+r)/2} S^{2+r}|\Omega|^{-(4+2r)/3}$$

such that system (1.1) has at least two solutions for each $\mu \in (0, \mu^*)$.

This paper is organized as follows. Some preliminaries and the proof of Theorem 1.1 are given in Section 2. In Section 3, we give the proof of Theorem 1.2.

In this paper, C, C_i denote various positive constants, which may vary from line to line.

2. Preliminaries and proof of Theorem 1.1

For given $u \in H$, the second equation of system (1.1) is a Poisson equation for ϕ which is uniquely solved. Then system (1.1) can be reduced to the first equation with ϕ represented by the solution of the Poisson equation. This is a basic strategy of solving Schrödinger–Poisson system. To be more precise about the solution of the Poisson equation, we recall the following lemma from [7,11,22], etc.

Lemma 2.1. *For each $u \in H$, there exists a unique $\phi_u \in H$ solution of*

$$\begin{cases} -\Delta\phi = u^2, & \textit{in } \Omega, \\ \phi = 0, & \textit{on } \partial\Omega. \end{cases}$$

Moreover,

(i) $\|\phi_u\|^2 = \int_\Omega \phi_u u^2$;
(ii) $\phi_u \geqslant 0$. *Moreover,* $\phi_u > 0$ *when* $u \neq 0$;
(iii) *for each* $t \neq 0$, $\phi_{tu} = t^2\phi_u$;
(iv)

$$\int_\Omega \phi_u u^2 = \int_\Omega |\nabla\phi_u|^2 \leqslant S^{-1}|u|^4_{12/5} \leqslant S^{-1}|u|_4^4|\Omega|^{2/3} \leqslant S^{-3}\|u\|^4|\Omega|, \quad u \in H,$$

where $S > 0$ *is the embedding constant in* (1.5);
(v) *assume that* $u_n \rightharpoonup u$ *in* H, *then* $\phi_{u_n} \to \phi_u$ *in* H *and* $\int_\Omega \phi_{u_n} u_n v \to \int_\Omega \phi_u uv$ *for any* $v \in H$;
(vi) *we denote* $\Psi(u) = \int_\Omega \phi_u u^2$, *then* $\Psi : H \to H$ *is* C^1 *and for any* $v \in H$,

$$(\Psi'(u), v) = 4\int_\Omega \phi_u uv.$$

(vii) $\phi_u \in W^{2,3}_{loc}(\Omega) \cap C^0(\bar{\Omega})$;
(viii) *for* $u, v \in H$, $\int_\Omega (\phi_u u - \phi_v v)(u - v) \geqslant \frac{1}{2}\|\phi_u - \phi_v\|^2$.

Proof. The proofs of (i)–(vi) of Lemma 2.1 are standard, see [7,11,22], etc. (vii) gives the regularity of ϕ_u, which can be obtained by Theorem 9.30 in [14]. We only give the proof of (viii). By the definitions of ϕ_u, ϕ_v, we obtain that

$$\begin{cases} -\Delta(\phi_u - \phi_v) = u^2 - v^2, & \text{in } \Omega, \\ \phi_u - \phi_v = 0, & \text{on } \partial\Omega, \end{cases}$$

then, we have

$$\|\phi_u - \phi_v\|^2 = \int_\Omega |\nabla(\phi_u - \phi_v)|^2 = \int_\Omega (\phi_u - \phi_v)(u^2 - v^2). \tag{2.1}$$

Thus

$$\begin{aligned} \int_\Omega (\phi_u u - \phi_v v)(u - v) &= \int_\Omega [\phi_u u^2 + \phi_v v^2 - (\phi_u + \phi_v)uv] \\ &\geqslant \int_\Omega \phi_u u^2 + \phi_v v^2 - (\phi_u + \phi_v)\frac{u^2 + v^2}{2} \\ &= \frac{1}{2}\int_\Omega (\phi_u - \phi_v)(u^2 - v^2). \end{aligned}$$

It follows from (2.1) that

$$\int_\Omega (\phi_u u - \phi_v v)(u - v) \geqslant \frac{1}{2}\|\phi_u - \phi_v\|^2.$$

The proof is completed. □

By Lemma 2.1, system (1.1) can be reduced to a semilinear nonlocal elliptic equation (1.7), the corresponding functional J_μ is defined in (1.8). In the following of this section, we show that for any $\mu > 0$, the functional J_μ attains the global minimizer in H, which is the unique solution of system (1.1) for $\eta = 1$.

Lemma 2.2. *For any $\mu > 0$ and $\eta = 1$, the functional J_μ defined in (1.8) attains the global minimizer in H, that is, there exists $u_* \in H$ such that $J_\mu(u_*) = m_\mu = \inf_H J_\mu < 0$.*

Proof. For $u \in H$, by Lemma 2.1 (ii) and (1.6),

$$\begin{aligned} J_\mu(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{4}\int_\Omega \phi_u u^2 - \frac{\mu}{1-r}\int_\Omega |u|^{1-r} \\ &\geqslant \frac{1}{2}\|u\|^2 - \frac{\mu}{1-r} S^{-\frac{1-r}{2}} |\Omega|^{\frac{5+r}{6}} \|u\|^{1-r}, \end{aligned} \tag{2.2}$$

since $r \in (0,1)$, J_μ is coercive and bounded from below on H for any $\mu > 0$. Thus $m_\mu = \inf_H J_\mu$ is well defined. For $t > 0$ and given $u \in H \setminus \{0\}$,

$$J_\mu(tu) = \frac{t^2}{2}\|u\|^2 + \frac{t^4}{4}\int_\Omega \phi_u u^2 - \frac{\mu}{1-r} t^{1-r} \int_\Omega |u|^{1-r},$$

we can see that for $t > 0$ small enough, $J_\mu(tu) < 0$. It follows that $m_\mu = \inf_H J_\mu < 0$.

According to the definition of m_μ, there exists a minimizing sequence $\{u_n\} \subset H$ such that $\lim_{n\to\infty} J_\mu(u_n) = m_\mu < 0$. Since $J_\mu(u_n) = J_\mu(|u_n|)$, we may assume that $u_n \geqslant 0$. It follows from (2.2) that $\{u_n\}$ is bounded in H. Going if necessary to a subsequence, we can assume that

$$\begin{aligned} &u_n \rightharpoonup u_*, \quad \text{in} \quad H, \\ &u_n \to u_*, \quad \text{in} \quad L^p(\Omega), p \in [1,6), \\ &u_n(x) \to u_*(x), \quad \text{a.e. in} \quad \Omega. \end{aligned}$$

Then by the weakly lower semi-continuity of the norm, Lemma 2.1 (v) and (1.9), we have

$$\begin{aligned} J_\mu(u_*) &= \frac{1}{2}\|u_*\|^2 + \frac{1}{4}\int_\Omega \phi_{u_*} u_*^2 - \frac{\mu}{1-r}\int_\Omega |u_*|^{1-r} \\ &\leqslant \liminf_{n\to\infty} J_\mu(u_n) = m_\mu. \end{aligned}$$

On the other hand, $J_\mu(u_*) \geqslant m_\mu$, therefore $J_\mu(u_*) = m_\mu < 0$. The proof is completed. □

Proof of Theorem 1.1. We divide three steps to prove Theorem 1.1.

Firstly, we show $u_* > 0$ in Ω. From Lemma 2.2, $u_* \geqslant 0$ and $u_* \neq 0$. Fix $\phi \in H$, $\phi > 0$ and $t \geqslant 0$, by Lemma 2.1 (vi), we have

$$\begin{aligned} 0 &\leqslant \liminf_{t\to 0} \frac{J_\mu(u_* + t\phi) - J_\mu(u_*)}{t} \\ &= \int_\Omega [\nabla u_* \nabla \phi + \phi_{u_*} u_* \phi] - \frac{\mu}{1-r} \limsup_{t\to 0} \int_\Omega \frac{(u_* + t\phi)^{1-r} - u_*^{1-r}}{t}, \end{aligned}$$

that is

$$\frac{\mu}{1-r} \limsup_{t \to 0} \int_{\Omega} \frac{(u_* + t\phi)^{1-r} - u_*^{1-r}}{t} \leqslant \int_{\Omega} [\nabla u_* \nabla \phi + \phi_{u_*} u_* \phi]. \tag{2.3}$$

Notice that

$$\int_{\Omega} \frac{(u_* + t\phi)^{1-r} - u_*^{1-r}}{t} = (1-r) \int_{\Omega} (u_* + t\phi\zeta)^{-r} \phi,$$

where $\zeta(x) \in (0,1)$ and $(u_*(x) + t\phi(x)\zeta(x))^{-r}\phi(x) \to u_*(x)^{-r}\phi(x)$, a.e. $x \in \Omega$, $t \to 0$. Since $(u_*(x) + t\phi(x)\zeta(x))^{-r}\phi(x) \geqslant 0$, by using Fatou's Lemma, from (2.3), we have

$$\mu \int_{\Omega} u_*^{-r} \phi \leqslant \int_{\Omega} [\nabla u_* \nabla \phi + \phi_{u_*} u_* \phi].$$

By the idea of approximation, the above expression also holds for $\phi \in H$, $\phi \geqslant 0$, that is

$$\int_{\Omega} [\nabla u_* \nabla \phi + \phi_{u_*} u_* \phi] - \mu \int_{\Omega} u_*^{-r} \phi \geqslant 0, \phi \in H, \phi \geqslant 0. \tag{2.4}$$

Therefore,

$$-\Delta u_* + \phi_{u_*} u_* \geqslant 0 \quad \text{in the weak sense.}$$

Since $u_* \geqslant 0$ and $u_* \neq 0$, by Lemma 2.1 (ii) and (vii), $\phi_{u_*} > 0$ and $\phi_{u_*} \in C^0(\bar{\Omega})$, it follows from the maximum principle (Theorem 3.5 in [14]) that $u_* > 0$ in Ω.

Secondly, we show that u_* is a solution of system (1.1), that is, we prove u_* satisfies (1.10) for $\eta = 1$.

For given $\delta > 0$, define $h : [-\delta, \delta] \to \mathbb{R}$ by $h(t) = J_\mu(u_* + tu_*)$, then h attains its minimum at $t = 0$ by Lemma 2.2. It implies that

$$h'(0) = \|u_*\|^2 + \int_{\Omega} \phi_{u_*} u_*^2 - \mu \int_{\Omega} |u_*|^{1-r} = 0. \tag{2.5}$$

We take $\phi \in H \setminus \{0\}$, $\varepsilon > 0$ and define $\Psi = (u_* + \varepsilon\phi)^+$. Let

$$\Omega_1 = \{x \in \Omega : u_*(x) + \varepsilon\phi(x) > 0\}, \quad \Omega_2 = \{x \in \Omega : u_*(x) + \varepsilon\phi(x) \leqslant 0\}.$$

Then $\Psi|_{\Omega_1} = u_* + \varepsilon\phi$, $\Psi|_{\Omega_2} = 0$. Inserting Ψ into (2.4) and using (2.5), we can obtain that

$$\begin{aligned}
0 &\leqslant \int_{\Omega} [\nabla u_* \cdot \nabla \Psi + \phi_{u_*} u_* \Psi - \mu u_*^{-r} \Psi] \\
&= \int_{\Omega_1} [\nabla u_* \cdot \nabla (u_* + \varepsilon\phi) + \phi_{u_*} u_* (u_* + \varepsilon\phi) - \mu u_*^{-r} (u_* + \varepsilon\phi)] \\
&= \int_{\Omega \setminus \Omega_2} [\nabla u_* \cdot \nabla (u_* + \varepsilon\phi) + \phi_{u_*} u_* (u_* + \varepsilon\phi) - \mu u_*^{-r} (u_* + \varepsilon\phi)] \\
&= \varepsilon \int_{\Omega} [\nabla u_* \cdot \nabla \phi + \phi_{u_*} u_* \phi - \mu u_*^{-r} \phi] - \int_{\Omega_2} [\nabla u_* \cdot \nabla (u_* + \varepsilon\phi) + \phi_{u_*} u_* (u_* + \varepsilon\phi) - \mu u_*^{-r} (u_* + \varepsilon\phi)] \\
&\leqslant \varepsilon \int_{\Omega} [\nabla u_* \cdot \nabla \phi + \phi_{u_*} u_* \phi - \mu u_*^{-r} \phi - \varepsilon \int_{\Omega_2} \nabla u_* \cdot \nabla \phi + \phi_{u_*} u_* \phi].
\end{aligned} \tag{2.6}$$

Since $u_* > 0$ and the measure of the domain $\Omega_2 = \{x \in \Omega : u_*(x) + \varepsilon\phi(x) \leqslant 0\}$ tends to zero as $\varepsilon \to 0$, it follows that

$$\int_{\Omega_2} [\nabla u_* \cdot \nabla \phi + \phi_{u_*} u_* \phi] \to 0.$$

Then dividing by $\varepsilon > 0$ and letting $\varepsilon \to 0$ in (2.6), we see that

$$\int_{\Omega} [\nabla u_* \cdot \nabla \phi + \phi_{u_*} u_* \phi - \mu u_*^{-r} \phi] \geqslant 0, \ \phi \in H.$$

This inequality also holds for $-\phi$, so we get

$$\int_{\Omega} [\nabla u_* \cdot \nabla \phi + \phi_{u_*} u_* \phi - \mu u_*^{-r} \phi] = 0, \quad \phi \in H.$$

Then $u_* \in H$ is a solution of system (1.1) for $\mu > 0$ and $\eta = 1$.

Finally, we show that u_* is the unique solution of system (1.1) for $\eta = 1$. Assume that $v_* \in H$ is also a solution of system (1.1), it follows from (1.10) that

$$\int_{\Omega} [\nabla u_* \nabla (u_* - v_*) + \phi_{u_*} u_* (u_* - v_*)] - \mu \int_{\Omega} u_*^{-r} (u_* - v_*) = 0 \tag{2.7}$$

and

$$\int_{\Omega} [\nabla v_* \nabla (u_* - v_*) + \phi_{v_*} v_* (u_* - v_*)] - \mu \int_{\Omega} v_*^{-r} (u_* - v_*) = 0. \tag{2.8}$$

Subtracting (2.7) and (2.8), we obtain that

$$\|u_* - v_*\|^2 + \int_{\Omega} (\phi_{u_*} u_* - \phi_{v_*} v_*)(u_* - v_*) = \mu \int_{\Omega} (u_*^{-r} - v_*^{-r})(u_* - v_*). \tag{2.9}$$

Since $r \in (0, 1)$, $u_*, v_* > 0$ in Ω, the following inequality holds

$$\int_{\Omega} (u_*^{-r} - v_*^{-r})(u_* - v_*) \leqslant 0.$$

Consequently, it follows from Lemma 2.1 (viii) and (2.9) that

$$\|u_* - v_*\|^2 \leqslant 0,$$

which implies that

$$\|u_* - v_*\|^2 = 0,$$

that is $u_* = v_*$. Therefore, $u_* \in H$ is the unique solution of system (1.1). This completes the proof of Theorem 1.1. □

3. Proof of Theorem 1.2

In this section, we consider the existence and multiplicity of solutions to system (1.1) with $\eta = -1$. We shall prove that system (1.1) possesses two solutions by using the method of Nehari manifold for $\mu > 0$ small enough.

We firstly define the Nehari manifold by

$$N_\mu = \left\{ u \in H : \|u\|^2 - \int_\Omega \phi_u u^2 - \mu \int_\Omega |u|^{1-r} = 0 \right\}.$$

Lemma 2.1 (vi) and (1.9) imply that N_μ is a closed set in H. It is obvious that solutions of system (1.1) lie in N_μ by (1.10). In order to state our results, we decompose N_μ with N_μ^+, N_μ^- and N_μ^0 defined as follows

$$N_\mu^+ = \left\{ u \in N_\mu : 2\|u\|^2 - 4\int_\Omega \phi_u u^2 - \mu(1-r)\int_\Omega |u|^{1-r} > 0 \right\},$$

$$N_\mu^- = \left\{ u \in N_\mu : 2\|u\|^2 - 4\int_\Omega \phi_u u^2 - \mu(1-r)\int_\Omega |u|^{1-r} < 0 \right\},$$

$$N_\mu^0 = \left\{ u \in N_\mu : 2\|u\|^2 - 4\int_\Omega \phi_u u^2 - \mu(1-r)\int_\Omega |u|^{1-r} = 0 \right\}.$$

It is easy to see that for $u \in N_\mu$,

$$\begin{aligned} 2\|u\|^2 - 4\int_\Omega \phi_u u^2 - \mu(1-r)\int_\Omega |u|^{1-r} &= -2\int_\Omega \phi_u u^2 + \mu(1+r)\int_\Omega |u|^{1-r} \\ &= -2\|u\|^2 + \mu(3+r)\int_\Omega |u|^{1-r} \\ &= (1+r)\|u\|^2 - (3+r)\int_\Omega \phi_u u^2. \end{aligned} \tag{3.1}$$

We firstly show that $N_\mu^\pm$ are nonempty and $N_\mu^0 = \{0\}$ for $\mu > 0$ small enough.

Lemma 3.1. *There exists $\mu^* > 0$ such that for $\mu \in (0, \mu^*)$, $N_\mu^\pm \neq \emptyset$ and $N_\mu^0 = \{0\}$, where*

$$\mu^* = \frac{2}{3+r}\left(\frac{1+r}{3+r}\right)^{(1+r)/2} S^{2+r}|\Omega|^{-(4+2r)/3}.$$

Proof. (1) For any given $u \in H \setminus \{0\}$, $t \geqslant 0$, by calculating, we can get that

$$\begin{aligned} t\frac{\mathrm{d}}{\mathrm{d}t}[J_\mu(tu)] &= t^2\|u\|^2 - t^4\int_\Omega \phi_u u^2 - \mu t^{1-r}\int_\Omega |u|^{1-r} \\ &= t^{1-r}\left(t^{1+r}\|u\|^2 - t^{3+r}\int_\Omega \phi_u u^2 - \mu\int_\Omega |u|^{1-r} \right) \\ &= t^{1-r}\left(\Phi_u(t) - \mu\int_\Omega |u|^{1-r} \right), \end{aligned} \tag{3.2}$$

where

$$\Phi_u(t) = t^{1+r}\|u\|^2 - t^{3+r}\int_\Omega \phi_u u^2, \quad t \geqslant 0.$$

By Lemma 2.1 (ii), $\int_\Omega \phi_u u^2 > 0$, we can see that $\Phi_u(0) = 0$, $\lim_{t\to\infty} \Phi_u(t) = -\infty$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_u(t) = t^r\left((1+r)\|u\|^2 - (3+r)t^2\int_\Omega \phi_u u^2\right).$$

Then Φ_u achieves its maximum at

$$t_u = \left(\frac{(1+r)\|u\|^2}{(3+r)\int_\Omega \phi_u u^2}\right)^{1/2}$$

and

$$\max_{t\in[0,\infty)} \Phi_u(t) = \Phi_u(t_u) = \frac{2}{3+r}\|u\|^2\left(\frac{(1+r)\|u\|^2}{(3+r)\int_\Omega \phi_u u^2}\right)^{(1+r)/2}.$$

Then by Lemma 2.1 (iv) and (1.6),

$$\begin{aligned}
&\Phi_u(t_u) - \mu\int_\Omega |u|^{1-r}\\
&= \frac{2}{3+r}\|u\|^2\left(\frac{(1+r)\|u\|^2}{(3+r)\int_\Omega \phi_u u^2}\right)^{(1+r)/2} - \mu\int_\Omega |u|^{1-r}\\
&\geqslant \left[\frac{2}{3+r}\left(\frac{1+r}{3+r}\right)^{(1+r)/2} S^{3(1+r)/2}|\Omega|^{-(1+r)/2} - \mu S^{-(1-r)/2}|\Omega|^{(5+r)/6}\right]\|u\|^{1-r}\\
&=: E(\mu)\|u\|^{1-r},
\end{aligned} \tag{3.3}$$

where

$$E(\mu) = \frac{2}{3+r}\left(\frac{1+r}{3+r}\right)^{(1+r)/2} S^{3(1+r)/2}|\Omega|^{-(1+r)/2} - \mu S^{-(1-r)/2}|\Omega|^{(5+r)/6}.$$

By direct calculation, $E(\mu) = 0$ is equivalent to $\mu = \mu^*$, where

$$\mu^* = \frac{2}{3+r}\left(\frac{1+r}{3+r}\right)^{(1+r)/2} S^{2+r}|\Omega|^{-(4+2r)/3}.$$

Then by (3.3), for $\mu \in (0, \mu^*)$,

$$\Phi_u(t_u) - \mu\int_\Omega |u|^{1-r} \geqslant E(\mu)\|u\|^{1-r} > 0.$$

Thus there exactly exist two points $0 < t_u^+ < t_u < t_u^-$ such that

$$\Phi_u(t_u^+) = \Phi_u(t_u^-) = \mu \int_\Omega |u|^{1-r}$$

and

$$\Phi_u'(t_u^+) > 0 > \Phi_u'(t_u^-),$$

which imply that $t_u^+ u \in N_\mu^+$, $t_u^- u \in N_\mu^-$ by (3.2). Hence, both N_μ^+ and N_μ^- are nonempty for $\mu \in (0, \mu^*)$.

(2) By contradiction, assume that there exists $u_0 \in N_\mu^0 \subset N_\mu$ and $u_0 \neq 0$ for $\mu \in (0, \mu^*)$. Similar to (3.3), by (3.1), we have

$$\begin{aligned} 0 &< E(\mu)\|u_0\|^{1-r} \\ &\leqslant \frac{2}{3+r}\|u_0\|^2 \left(\frac{(1+r)\|u_0\|^2}{(3+r)\int_\Omega \phi_{u_0} u_0^2} \right)^{(1+r)/2} - \mu \int_\Omega |u_0|^{1-r} = 0. \end{aligned}$$

This is impossible. Hence, $N_\mu^0 = \{0\}$ for $\mu \in (0, \mu^*)$. $\square$

The next lemma shows that μ^* is also related to a gap structure in N_μ.

Lemma 3.2. *Suppose that $\mu \in (0, \mu^*)$, where μ^* is defined in Lemma 3.1. Then we have*

$$\|u\| < A_\mu < A_{\mu^*} < \|v\|, \quad |u|_4 < B_\mu < B_{\mu^*} < |v|_4, \quad u \in N_\mu^+, \ v \in N_\mu^-,$$

where

$$A_\mu = \left(\mu \frac{3+r}{2} S^{-(1-r)/2} |\Omega|^{(5+r)/6} \right)^{1/(1+r)}, \quad B_\mu = \left(\mu \frac{3+r}{2} S^{-1} |\Omega|^{(11+3r)/12} \right)^{1/(1+r)}, \quad \mu \in (0, \mu^*].$$

Proof. For $\mu \in (0, \mu^*)$, $u \in N_\mu^+$, by Hölder inequality, (1.5), (1.6) and (3.1), we have

$$|u|_4^2 S |\Omega|^{-1/6} \leqslant \|u\|^2 < \mu \frac{3+r}{2} \int_\Omega |u|^{1-r} \leqslant \mu \frac{3+r}{2} |u|_4^{1-r} |\Omega|^{(3+r)/4} \leqslant \mu \frac{3+r}{2} S^{-(1-r)/2} \|u\|^{1-r} |\Omega|^{(5+r)/6}.$$

It is easy to see that $\|u\| < A_\mu$ and $|u|_4 < B_\mu$.

For $v \in N_\mu^-$, by Hölder inequality, (1.5), (3.1) and Lemma 2.1 (iv), we have

$$(1+r)|v|_4^2 S |\Omega|^{-1/6} \leqslant (1+r)\|v\|^2 < (3+r) \int_\Omega \phi_v v^2 \leqslant (3+r) S^{-1} |v|_4^4 |\Omega|^{2/3} \leqslant (3+r) S^{-3} \|v\|^4 |\Omega|.$$

It follows from the definition of μ^* in Lemma 3.1 and the expressions of A_μ, B_μ that $\|v\| > A_{\mu^*}$, $|v|_4 > B_{\mu^*}$ and $A_\mu < A_{\mu^*}$, $B_\mu < B_{\mu^*}$ for $\mu \in (0, \mu^*)$. The proof is completed. $\square$

From the gap structure in N_μ and Lemma 3.1, we can easily obtain that N_μ^- is closed for $\mu \in (0, \mu^*)$. In fact, we have the following lemma.

Lemma 3.3. *N_μ^- is a closed set in H for $\mu \in (0, \mu^*)$.*

Proof. By Lemma 3.1, $N_\mu^- \neq \emptyset$ for $\mu \in (0, \mu^*)$. Let $\{v_n\}$ be a sequence in N_μ^- with $v_n \to v$ in H, it follows that $v \in N_\mu$. By Lemma 3.2,

$$\|v\| \geqslant A_{\mu^*} > A_\mu > \|u\|, \quad \forall u \in N_\mu^+,$$

that is, $v \notin N_\mu^+$ and $v \neq 0$. Hence by Lemma 3.1, $v \in N_\mu^-$. The proof is completed. □

Since $N_\mu^+ \cup N_\mu^0$ and N_μ^- are two closed subsets in H for $\mu \in (0, \mu^*)$, we shall use Ekeland's variational principle to discuss the minimums of the functional J_μ on $N_\mu^+ \cup N_\mu^0$ and N_μ^- respectively. For this purpose, we need the following two lemmas.

Lemma 3.4. *Assume $\mu \in (0, \mu^*)$, J_μ is coercive, bounded from below on N_μ and $\inf_{N_\mu^+ \cup N_\mu^0} J_\mu < 0$.*

Proof. For $u \in N_\mu$ and $\mu \in (0, \mu^*)$, we have,

$$\begin{aligned} J_\mu(u) &= \frac{1}{2}\|u\|^2 - \frac{1}{4}\int_\Omega \phi_u u^2 - \frac{\mu}{1-r}\int_\Omega |u|^{1-r} \\ &= \frac{1}{4}\|u\|^2 - \mu\frac{3+r}{4(1-r)}\int_\Omega |u|^{1-r} \\ &\geqslant \frac{1}{4}\|u\|^2 - \mu^*\frac{3+r}{4(1-r)}S^{-(1-r)/2}\|u\|^{1-r}|\Omega|^{(5+r)/6}. \end{aligned} \tag{3.4}$$

It follows from $r \in (0,1)$ that J_μ is coercive and bounded from below on N_μ. Since $N_\mu^+ \cup N_\mu^0$ and N_μ^- are two closed subsets of H from Lemma 3.3, $\inf_{N_\mu^+ \cup N_\mu^0} J_\mu$ and $\inf_{N_\mu^-} J_\mu$ are well defined. For any given $u_0 \in N_\mu^+$, by (3.1), we have

$$\begin{aligned} J_\mu(u_0) &= \frac{1}{4}\|u_0\|^2 - \mu\frac{3+r}{4(1-r)}\int_\Omega |u_0|^{1-r} \\ &< \frac{1}{4}\|u_0\|^2 - \frac{1}{2(1-r)}\|u_0\|^2 \\ &= -\frac{1+r}{4(1-r)}\|u_0\|^2, \end{aligned}$$

which means that

$$\inf_{N_\mu^+} J_\mu \leqslant -\frac{1+r}{4(1-r)}\|u_0\|^2 < 0.$$

Then, by Lemma 3.1, $\inf_{N_\mu^+ \cup N_\mu^0} J_\mu = \inf_{N_\mu^+} J_\mu < 0$ for given $\mu \in (0, \mu^*)$. The proof is completed. □

Lemma 3.5. *Assume that $\mu \in (0, \mu^*)$, given $u \in N_\mu^+$ (resp. N_μ^-), there exist $\varepsilon > 0$ and a continuous function $f = f(w) > 0$, $w \in H$, $\|w\| < \varepsilon$ satisfying that*

$$f(0) = 1, \quad f(w)(u+w) \in N_\mu^+ \ (\textit{resp. } N_\mu^-), \ w \in H, \ \|w\| < \varepsilon.$$

Proof. We only consider the case $u \in N_\mu^+$. Define $F : H \times \mathbb{R} \to \mathbb{R}$ by

$$F(w,t) = t^2\|u+w\|^2 - t^4\int_\Omega \phi_{u+w}(u+w)^2 - \mu t^{1-r}\int_\Omega |u+w|^{1-r}.$$

It is obvious that F is continuous on some neighborhood of $(0,1) \in H \times \mathbb{R}$ and F_t is continuous at $(0,1)$. It follows from $u \in N_\mu^+ \subset N_\mu$ that

$$F(0,1) = \|u\|^2 - \int_\Omega \phi_u u^2 - \mu \int_\Omega |u|^{1-r} = 0$$

and

$$F_t(0,1) = 2\|u\|^2 - 4\int_\Omega \phi_u u^2 - \mu(1-r)\int_\Omega |u|^{1-r} > 0.$$

Then by the implicit function theorem, there exists $\bar{\varepsilon} > 0$ such that for $w \in H$, $\|w\| < \bar{\varepsilon}$, the equation $F(w,t) = 0$ has a unique continuous function $t = f(w) > 0$ satisfying that $f(0) = 1$ and $F(w, f(w)) = 0$. That is

$$f(w)(u+w) \in N_\mu.$$

Since $F_t(0,1) > 0$, by the continuity of F_t at $(0,1)$, we can take $\varepsilon \in (0, \bar{\varepsilon})$ such that $F_t(w, f(w)) > 0$ for $\|w\| < \varepsilon$, that is

$$f(w)(u+w) \in N_\mu^+, \quad w \in H, \|w\| < \varepsilon.$$

The proof is completed. □

Proof of Theorem 1.2. From Lemma 3.4, $\inf_{N_\mu^+ \cup N_\mu^0} J_\mu$ and $\inf_{N_\mu^-} J_\mu$ are well defined, we divide two steps to consider the minimums of functional J_μ on $N_\mu^+ \cup N_\mu^0$ and N_μ^- respectively by using Ekeland's variational principle [23].

Step 1. We prove that there exists $u_0 > 0$ in Ω such that $J_\mu(u_0) = \inf_{N_\mu^+ \cup N_\mu^0} J_\mu$ and u_0 is a solution of system (1.1) for $\mu \in (0, \mu^*)$.

By Ekeland's variational principle, there exists a minimizing sequence $\{u_n\} \subset N_\mu^+ \cup N_\mu^0$ satisfying

(i) $J_\mu(u_n) \leqslant \inf_{N_\mu^+ \cup N_\mu^0} J_\mu + 1/n^2$,
(ii) $J_\mu(u) \geqslant J_\mu(u_n) - \frac{1}{n}\|u - u_n\|$, $u \in N_\mu^+ \cup N_\mu^0$.

From $J_\mu(|u|) = J_\mu(u)$ and Lemma 3.4, we may assume that $u_n \in N_\mu^+$ and $u_n \geqslant 0$ for any $n \in \mathbb{N}$. Since $r \in (0,1)$, it follows from (3.4) that $\{u_n\}$ is bounded in H, we assume that $\|u_n\| \leqslant C_0$. Going if necessary to a subsequence, we can assume that

$$\begin{aligned} &u_n \rightharpoonup u_0, \quad \text{in } H, \\ &u_n \to u_0, \quad \text{in } L^p(\Omega), p \in [1,6), \\ &u_n(x) \to u(x), \quad \text{a.e. in } \Omega. \end{aligned}$$

Since $\{u_n\} \subset N_\mu^+ \subset N_\mu$, by the weakly lower semi-continuity of the norm, Lemma 2.1 (v), (1.9) and (3.1), we have

$$\begin{aligned} &\|u_0\|^2 - \int_\Omega \phi_{u_0} u_0^2 - \mu \int_\Omega |u_0|^{1-r} \leqslant 0, \\ &-2\int_\Omega \phi_{u_0} u_0^2 + \mu(1+r)\int_\Omega |u_0|^{1-r} \geqslant 0, \end{aligned} \tag{3.5}$$

and

$$J_\mu(u_0) \leqslant \lim_{n\to\infty} J_\mu(u_n) = \inf_{N_\mu^+ \cup N_\mu^0} J_\mu < 0. \tag{3.6}$$

By (3.6), $u_0 \geqslant 0$ and $u_0 \neq 0$.

We firstly claim that for $\mu \in (0, \mu^*)$,

$$-2\int\limits_\Omega \phi_{u_0} u_0^2 + \mu(1+r)\int\limits_\Omega |u_0|^{1-r} > 0. \tag{3.7}$$

We argue indirectly and assume that

$$-2\int\limits_\Omega \phi_{u_0} u_0^2 + \mu(1+r)\int\limits_\Omega |u_0|^{1-r} = 0. \tag{3.8}$$

Then (3.5) and (3.8) imply that $\|u_0\|^2 \leqslant \mu\frac{3+r}{2}\int_\Omega |u_0|^{1-r}$. For $\mu \in (0, \mu^*)$ and $u_0 \neq 0$, by similar arguments as (3.3), we obtain that

$$\begin{aligned} 0 &< E(\mu)\|u_0\|^{1-r} \\ &\leqslant \frac{2}{3+r}\|u_0\|^2 \left[\frac{(1+r)\|u_0\|^2}{(3+r)\int_\Omega \phi_{u_0} u_0^2}\right]^{(1+r)/2} - \mu\int\limits_\Omega |u_0|^{1-r} \leqslant 0, \end{aligned}$$

which is clearly impossible. Hence (3.7) holds. Since

$$\lim_{n\to\infty}\left(-2\int\limits_\Omega \phi_{u_n} u_n^2 + \mu(1+r)\int\limits_\Omega |u_n|^{1-r}\right) = -2\int\limits_\Omega \phi_{u_0} u_0^2 + \mu(1+r)\int\limits_\Omega |u_0|^{1-r} > 0,$$

then there exist $C_1 > 0$ (independent of n) and $N_1 > 0$ such that for any $n \geqslant N_1$,

$$-2\int\limits_\Omega \phi_{u_n} u_n^2 + \mu(1+r)\int\limits_\Omega |u_n|^{1-r} \geqslant C_1.$$

By (3.1), we also get

$$-2\|u_n\|^2 + \mu(3+r)\int\limits_\Omega |u_n|^{1-r} \geqslant C_1. \tag{3.9}$$

We take $N = \max\{N_1, \frac{2(1-r)}{1+r}C_0C_1^{-1}\}$ and fix $\phi \in H$, $\phi > 0$. For given $n \geqslant N$, we apply Lemma 3.5 with $u = u_n$ and $w = t\phi$, $t > 0$ small enough, we can find $g_n(t) = f_n(t\phi)$ such that $g_n(0) = 1$, $g_n(t)(u_n + t\phi) \in N_\mu^+$. It follows from $u_n, g_n(t)(u_n + t\phi) \in N_\mu^+ \subset N_\mu$ that

$$\|u_n\|^2 - \int\limits_\Omega \phi_{u_n} u_n^2 - \mu\int\limits_\Omega |u_n|^{1-r} = 0$$

and

$$g_n^2(t)\|u_n+t\phi\|^2 - g_n^4(t)\int_\Omega \phi_{u_n+t\phi}(u_n+t\phi)^2 - \mu g_n^{1-r}(t)\int_\Omega |u_n+t\phi|^{1-r} = 0.$$

By above two equalities, we have

$$\begin{aligned}
0 &= \left(g_n^2(t)-1\right)\|u_n+t\phi\|^2 + \left(\|u_n+t\phi\|^2 - \|u_n\|^2\right) - \left(g_n^4(t)-1\right)\int_\Omega \phi_{u_n+t\phi}(u_n+t\phi)^2 \\
&\quad - \int_\Omega [\phi_{u_n+t\phi}(u_n+t\phi)^2 - \phi_{u_n}u_n^2] - \mu\left(g_n^{1-r}(t)-1\right)\int_\Omega |u_n+t\phi|^{1-r} \\
&\quad - \mu\int_\Omega [|u_n+t\phi|^{1-r} - |u_n|^{1-r}] \\
&\leqslant \left(g_n^2(t)-1\right)\|u_n+t\phi\|^2 + \left(\|u_n+t\phi\|^2 - \|u_n\|^2\right) - \left(g_n^4(t)-1\right)\int_\Omega \phi_{u_n+t\phi}(u_n+t\phi)^2 \\
&\quad - \int_\Omega [\phi_{u_n+t\phi}(u_n+t\phi)^2 - \phi_{u_n}u_n^2] - \mu\left(g_n^{1-r}(t)-1\right)\int_\Omega |u_n+t\phi|^{1-r} \\
&= (g_n(t)-1)\left[(g_n(t)+1)\|u_n+t\phi\|^2 - \left(g_n^2(t)+1\right)(g_n(t)+1)\int_\Omega \phi_{u_n+t\phi}(u_n+t\phi)^2 \right. \\
&\quad \left. - \frac{\mu\left(g_n^{1-r}(t)-1\right)}{g_n(t)-1}\int_\Omega |u_n+t\phi|^{1-r}\right] + \left(\|u_n+t\phi\|^2 - \|u_n\|^2\right) - \int_\Omega [\phi_{u_n+t\phi}(u_n+t\phi)^2 - \phi_{u_n}u_n^2].
\end{aligned} \tag{3.10}$$

Denote $D^+g_n(0)$ the right upper Dini derivative of g_n at zero. By the definition of $D^+g_n(0) = \limsup_{t\to 0^+}\frac{g_n(t)-1}{t}$, there exists a sequence $\{t_k\}$ with $t_k>0$ and $t_k\to 0$ as $k\to\infty$ such that

$$D^+g_n(0) = \lim_{k\to\infty}\frac{g_n(t_k)-1}{t_k}. \tag{3.11}$$

Then replacing t in (3.10) with t_k, dividing $t_k>0$ and letting $k\to\infty$, by Lemma 2.1 (vi) and (3.1), we deduce that

$$D^+g_n(0)\left[-2\|u_n\|^2 + \mu(3+r)\int_\Omega |u_n|^{1-r}\right] + 2\int_\Omega \nabla u_n\cdot\nabla\phi - 4\int_\Omega \phi_{u_n}u_n\phi \geqslant 0. \tag{3.12}$$

By $\|u_n\|\leqslant C_0$ and Lemma 2.1 (iv), there exist $C_2, C_3>0$ independent of n such that

$$\left|2\int_\Omega \nabla u_n\cdot\nabla\phi - 4\int_\Omega \phi_{u_n}u_n\phi\right| \leqslant C_2 \tag{3.13}$$

and

$$\left|\frac{1}{1+r}[(1+r)\int_\Omega \nabla u_n\cdot\nabla\phi - (3+r)\int_\Omega \phi_{u_n}u_n\phi]\right| \leqslant C_3. \tag{3.14}$$

It follows from (3.12), (3.9) and (3.13) that

$$D^+g_n(0) \geqslant -C_4, \quad \forall n \geqslant N, \tag{3.15}$$

where $C_4 = C_1^{-1}C_2$ is independent of n.

In the following, we prove

$$D^+g_n(0) \leqslant 2(\|\phi\| + C_3)C_1^{-1}, \quad \forall n \geqslant N. \tag{3.16}$$

Arguing by contradiction, if there exists $n_0 \geqslant N$ such that $D^+g_{n_0}(0) > 2(\|\phi\| + C_3)C_1^{-1}$. By (3.11), for k (which is dependent of n_0) large enough, $g_{n_0}(t_k) > 1$. By (ii) of Ekeland's variational principle with $u = g_{n_0}(t_k)(u_{n_0} + t_k\phi)$, we clearly have that

$$\begin{aligned}
&\frac{(g_{n_0}(t_k) - 1)\|u_{n_0}\| + t_k g_{n_0}(t_k)\|\phi\|}{n_0} \\
&\geqslant \frac{\|g_{n_0}(t_k)(u_n + t_k\phi) - u_{n_0}\|}{n_0} \\
&\geqslant J_\mu(u_{n_0}) - J_\mu(g_{n_0}(t_k)(u_{n_0} + t_k\phi)) \\
&= -\frac{1+r}{2(1-r)}\|u_{n_0}\|^2 + \frac{3+r}{4(1-r)}\int_\Omega \phi_{u_{n_0}} u_{n_0}^2 + \frac{1+r}{2(1-r)} g_{n_0}^2(t_k)\|u_{n_0} + t_k\phi\|^2 \\
&\quad - \frac{3+r}{4(1-r)} g_{n_0}^4(t_k) \int_\Omega \phi_{u_{n_0}+t_k\phi}(u_{n_0} + t_k\phi)^2 \\
&= \frac{1+r}{2(1-r)}\left[(g_{n_0}^2(t_k) - 1)\|u_{n_0} + t_k\phi\|^2 + (\|u_{n_0} + t_k\phi\|^2 - \|u_{n_0}\|^2)\right] \\
&\quad - \frac{3+r}{4(1-r)}\left[(g_{n_0}^4(t_k) - 1)\int_\Omega \phi_{u_{n_0}+t_k\phi}(u_{n_0} + t_k\phi)^2 + \int_\Omega [\phi_{u_{n_0}+t_k\phi}(u_{n_0} + t_k\phi)^2 - \phi_{u_{n_0}} u_{n_0}^2]\right],
\end{aligned}$$

then

$$\begin{aligned}
&\frac{t_k g_{n_0}(t_k)\|\phi\|}{n} \\
&\geqslant \frac{g_{n_0}(t_k) - 1}{1-r}\left[\frac{1+r}{2}(g_{n_0}(t_k) + 1)\|u_{n_0} + t_k\phi\|^2 - \frac{(3+r)(g_{n_0}^2(t_k) + 1)(g_{n_0}(t_k) + 1)}{4}\int_\Omega \phi_{u_{n_0}+t_k\phi}(u_{n_0} + t_k\phi)^2\right. \\
&\quad \left. - \frac{(1-r)\|u_{n_0}\|}{n_0}\right] + \frac{1}{1-r}\left[\frac{1+r}{2}\left(\|u_{n_0} + t_k\phi\|^2 - \|u_{n_0}\|^2\right) - \frac{3+r}{4}\int_\Omega [\phi_{u_{n_0}+t_k\phi}(u_{n_0} + t_k\phi)^2 - \phi_{u_{n_0}} u_{n_0}^2]\right]
\end{aligned}$$

Dividing by $t_k > 0$ and letting $k \to \infty$, it follows from $u_{n_0} \in N_\mu^+ \subset N_\mu$ and (3.1) that

$$\begin{aligned}
\|\phi\| \geqslant \frac{\|\phi\|}{n_0} \geqslant \frac{D^+g_{n_0}(0)}{1-r}&\left[-2\|u_{n_0}\|^2 + \mu(3+r)\int_\Omega |u_{n_0}|^{1-r} - \frac{(1-r)\|u_{n_0}\|}{n_0}\right] \\
&+ \frac{1}{1-r}\left[(1+r)\int_\Omega \nabla u_{n_0}\cdot\nabla\phi - (3+r)\int_\Omega \phi_{u_{n_0}} u_{n_0}\phi\right].
\end{aligned}$$

By the choice of $n_0 \geqslant N$, (3.9) and (3.14), we deduce that

$$\|\phi\| \geqslant \frac{C_1}{2} D^+ g_{n_0}(0) - C_3 > \|\phi\|,$$

this is impossible. Hence (3.16) holds. Thus the two inequalities (3.15) and (3.16) imply that

$$|D^+ g_n(0)| \leqslant C, \ \forall\, n \geqslant N, \tag{3.17}$$

where $C > 0$ is independent of n.

In the following, we still denote by $\{u_n\}$ the subsequence of $\{u_n\}_{n\geqslant N}$ and show that $u_0 \in N_\mu^+$ and $u_n \to u_0$ in H.

From (ii) of Ekeland variational principle, we can deduce that

$$\begin{aligned}
&\frac{|g_n(t)-1|\|u_n\|^2 + t g_n(t)\|\phi\|}{n} \\
&\geqslant \frac{\|g_n(t)(u_n+t\phi) - u_n\|}{n} \\
&\geqslant J_\mu(u_n) - J_\mu(g_n(t)(u_n+t\phi)) \\
&= -\frac{g_n^2(t)}{2}\left(\|u_n+t\phi\|^2 - \|u_n\|^2\right) - \frac{g_n^2(t)-1}{2}\|u_n\|^2 \\
&\quad + \frac{g_n^4(t)}{4}\int_\Omega \left[\phi_{u_n+t\phi}(u_n+t\phi)^2 - \phi_{u_n}u_n^2\right] + \frac{g_n^4(t)-1}{4}\int_\Omega \phi_{u_n}u_n^2 \\
&\quad + \frac{\mu}{1-r} g_n^{1-r}(t)\int_\Omega \left[|u_n+t\phi|^{1-r} - |u_n|^{1-r}\right] + \frac{\mu}{1-r}\left(g_n^{1-r}(t)-1\right)\int_\Omega |u_n|^{1-r},
\end{aligned}$$

it follows that

$$\begin{aligned}
&\frac{\mu}{1-r} g_n^{1-r}(t)\int_\Omega \left[|u_n+t\phi|^{1-r} - |u_n|^{1-r}\right] \\
&\leqslant \frac{|g_n(t)-1|\|u_n\|^2 + t g_n(t)\|\phi\|}{n} + \frac{g_n^2(t)}{2}\left(\|u_n+t\phi\|^2 - \|u_n\|^2\right) \\
&\quad - \frac{g_n^4(t)}{4}\int_\Omega \left[\phi_{u_n+t\phi}(u_n+t\phi)^2 - \phi_{u_n}u_n^2\right] + \frac{g_n^2(t)-1}{2}\|u_n\|^2 \\
&\quad - \frac{g_n^4(t)-1}{4}\int_\Omega \phi_{u_n}u_n^2 - \frac{\mu}{1-r}\left(g_n^{1-r}(t)-1\right)\int_\Omega |u_n|^{1-r}.
\end{aligned}$$

The above inequality also holds for $t = t_k$ (t_k is dependent of n), then dividing by $t_k > 0$ and passing to the limit as $k \to \infty$, then combing $u_n \in N_\mu^+ \subset N_\mu$ and (3.17), we obtain that

$$\begin{aligned}
&\lim_{k\to\infty} \frac{\mu}{1-r}\int_\Omega \frac{(u_n+t_k\phi)^{1-r} - u_n^{1-r}}{t_k} \\
&\leqslant \frac{|D^+ g_n(0)|\|u_n\|^2 + \|\phi\|}{n} + \int_\Omega \nabla u_n\cdot\nabla\phi - \int_\Omega \phi_{u_n}u_n\phi + D^+ g_n(0)\left(\|u_n\|^2 - \int_\Omega \phi_{u_n}u_n^2 - \mu\int_\Omega |u_n|^{1-r}\right) \\
&\leqslant \frac{C + \|\phi\|}{n} + \int_\Omega \nabla u_n\cdot\nabla\phi - \int_\Omega \phi_{u_n}u_n\phi.
\end{aligned}$$

Then

$$\liminf_{t\to 0} \frac{\mu}{1-r} \int_\Omega \frac{(u_n+t\phi)^{1-r}-u_n^{1-r}}{t} \leqslant \lim_{k\to\infty} \frac{\mu}{1-r} \int_\Omega \frac{(u_n+t_k\phi)^{1-r}-u_n^{1-r}}{t_k}$$
$$\leqslant \frac{C+\|\phi\|}{n} + \int_\Omega \nabla u_n \cdot \nabla \phi - \int_\Omega \phi_{u_n} u_n \phi. \tag{3.18}$$

Notice that

$$\int_\Omega \frac{(u_n+t\phi)^{1-r}-u_n^{1-r}}{t} = (1-r)\int_\Omega (u_n+t\phi\xi_n)^{-r}\phi,$$

where $\xi_n(x) \in (0,1)$, $(u_n(x)+t\phi(x)\xi_n(x))^{-r}\phi(x) \to u_n^{-r}(x)\phi(x)$, a.e. $x \in \Omega$ as $t \to 0$. Since $(u_n(x)+t\phi(x)\xi_n(x))^{-r}\phi(x) \geqslant 0$, then by using Fatou's Lemma in (3.18), we have

$$\mu \int_\Omega u_n^{-r}\phi \leqslant \frac{C+\|\phi\|}{n} + \int_\Omega \nabla u_n \cdot \nabla\phi - \int_\Omega \phi_{u_n} u_n \phi.$$

Using Fatou's Lemma again, we get that

$$\mu \int_\Omega u_0^{-r}\phi \leqslant \liminf_{n\to\infty} \mu \int_\Omega u_n^{-r}\phi \leqslant \int_\Omega \nabla u_0 \cdot \nabla \phi - \int_\Omega \phi_{u_0} u_0 \phi, \;\; \phi \in H, \;\; \phi > 0.$$

By the idea of the approximation, the above expression also holds for $\phi \in H$, $\phi \geqslant 0$, that is

$$\int_\Omega \nabla u_0 \cdot \nabla\phi - \int_\Omega \phi_{u_0} u_0 \phi - \mu \int_\Omega u_0^{-r}\phi \geqslant 0, \;\; \phi \in H, \;\; \phi \geqslant 0. \tag{3.19}$$

Therefore, (3.19) implies that

$$\int_\Omega \nabla u_0 \cdot \nabla \phi \geqslant 0, \;\; \phi \in H, \;\; \phi \geqslant 0,$$

that is

$$-\Delta u_0 \geqslant 0, \quad \text{in the weak sense.}$$

Since $u_0 \geqslant 0$ and $u_0 \not\equiv 0$, by the maximum principle (Theorem 3.5 in [14]), $u_0 > 0$ in Ω. We take $\phi = u_0$ in (3.19) and get

$$\|u_0\|^2 - \int_\Omega \phi_{u_0} u_0^2 - \mu \int_\Omega |u_0|^{1-r} \geqslant 0. \tag{3.20}$$

On the other hand, by the weakly lower semi-continuity of the norm, (1.9) and Lemma 2.1 (v), we have

$$\|u_0\|^2 \leqslant \liminf_{n\to\infty} \|u_n\|^2 = \liminf_{n\to\infty} \left(\mu \int_\Omega |u_n|^{1-r} + \int_\Omega \phi_{u_n} u_n^2 \right)$$
$$= \mu \int_\Omega |u_0|^{1-r} + \int_\Omega \phi_{u_0} u_0^2. \tag{3.21}$$

Thus, (3.20) and (3.21) imply that

$$\|u_0\|^2 \leqslant \liminf_{n\to\infty} \|u_n\|^2 = \mu \int_\Omega |u_0|^{1-r} + \int_\Omega \phi_{u_0} u_0^2 \leqslant \|u_0\|^2, \tag{3.22}$$

this combining with $u_n \rightharpoonup u$ implies that $u_n \to u_0$ in H. According to (3.22) and (3.7), $u_0 \in N_\mu^+$.

Finally, we show that u_0 is a solution of system (1.1).

We take $\phi \in H \setminus \{0\}$, $\varepsilon > 0$ and define $\Psi = (u_0 + \varepsilon\phi)^+$. Let

$$\Omega_1 = \{x \in \Omega : u_0(x) + \varepsilon\phi(x) > 0\}, \quad \Omega_2 = \{x \in \Omega : u_0(x) + \varepsilon\phi(x) \leqslant 0\}.$$

Then $\Psi|_{\Omega_1} = u_0 + \varepsilon\phi$, $\Psi|_{\Omega_2} = 0$. Similar to the arguments in the proof of Theorem 1.1, inserting Ψ into (3.19) and using (3.22), we get

$$\int_\Omega \nabla u_0 \cdot \nabla\phi - \phi_{u_0} u_0 \phi - \mu u_0^{-r}\phi = 0, \quad \phi \in H.$$

Therefore, $u_0 \in N_\mu^+$ is a solution of (1.1) with $J_\mu(u_0) = \inf_{N_\mu^+} J < 0$ for any $\mu \in (0, \mu^*)$.

Step 2. We prove that there exists a solution of system (1.1) in N_μ^-.

From Lemma 3.3, N_μ^- is closed for any $\mu \in (0, \mu^*)$. Applying Ekeland's variational principle to the minimization problem $\inf_{N_\mu^-} J_\mu$, there exists a sequence $\{v_n\} \subset N_\mu^-$ satisfying

(i) $J_\mu(v_n) \leqslant \inf_{N_\mu^-} J_\mu + 1/n^2$,
(ii) $J_\mu(v) \geqslant J_\mu(v_n) - \frac{1}{n}\|v - v_n\|$, $v \in N_\mu^-$.

Since $J_\mu(|v|) = J_\mu(v)$, we may assume that $v_n \geqslant 0$. Obviously, $\{v_n\} \subset N_\mu^- \subset N_\mu$ is bounded, going if necessary to a subsequence, still denoted by $\{v_n\}$, then there exists $v_0 \geqslant 0$ such that

$$\begin{aligned} & v_n \rightharpoonup v_0, \quad \text{in} \quad H, \\ & v_n \to v_0, \quad \text{in} \quad L^p(\Omega),\ p \in [1, 6), \\ & v_n(x) \to v_0(x), \quad \text{a.e. in} \quad \Omega. \end{aligned}$$

By Lemma 3.2, $|v_0|_4 > B_{\mu^*} > 0$, then $v_0 \geqslant 0$ and $v_0 \neq 0$.

We can repeat the arguments used in (3.7)–(3.9) to derive

$$-2\int_\Omega \phi_{v_0} v_0^2 + \mu(1+r)\int_\Omega |v_0|^{1-r} < 0 \tag{3.23}$$

and

$$-2\int_\Omega \phi_{v_n} v_n^2 + \mu(1+r)\int_\Omega |v_n|^{1-r} = -2\|v_n\|^2 + \mu(3+r)\int_\Omega |v_n|^{1-r} \leqslant -C$$

for n large enough and $C > 0$.

At this point, we may proceed exactly as in the arguments of Step 1 and conclude that $v_0 > 0$, $v_n \to v_0$ in H and v_0 is a solution of system (1.1). Hence by Lemma 3.2 and (3.23), $v_0 \in N_\mu^-$ and $v_0 \neq u_0$ is a solution of system (1.1) with $J_\mu(v_0) = \inf_{N_\mu^-} J_\mu$. $\square$

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