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Additive maps onto matrix spaces compressing the spectrum

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ABSTRACT

We prove that given a unital C^*-algebra $\mathcal{A}$ and an additive and surjective map $T : \mathcal{A} \to \mathcal{M}_n$ such that the spectrum of $T(x)$ is a subset of the spectrum of x for each $x \in \mathcal{A}$, then T is either an algebra morphism, or an algebra anti-morphism. We arrive at the same conclusion for an arbitrary unital, complex Banach algebra $\mathcal{A}$, by imposing an extra surjectivity condition on the map T.

1. Introduction and statement of results

Let $\mathcal{A}$ be a (complex) unital Banach algebra, and denote its unit by $\mathbf{1}$. By $\sigma(a)$ we shall denote the spectrum of the element $a \in \mathcal{A}$ and $\rho(a)$ will be its spectral radius. A well-known result in the theory of Banach algebras, the Gleason–Kahane–Żelazko theorem, states that if $f : \mathcal{A} \to \mathbf{C}$ is $\mathbf{C}$-linear (that is, additive and homogeneous with respect to complex scalars) and $f(a) \in \sigma(a)$ for every $a \in \mathcal{A}$, then f is multiplicative. (See e.g. [5] and [6].) Kowalski and Slodkowski generalized their result in [7], by proving that if $f : \mathcal{A} \to \mathbf{C}$ with $f(0) = 0$ satisfies

$$f(x) - f(y) \in \sigma(x - y) \qquad (x, y \in \mathcal{A}), \tag{1}$$

then f is automatically $\mathbf{C}$-linear, and therefore also multiplicative. (That f is $\mathbf{R}$-linear and the fact that $f(ia) = if(a)$ for all $a \in \mathcal{A}$ come automatically from the inclusions (1), which combine spectrum-preserving properties and additivity properties on the functional f.) In particular, if $f : \mathcal{A} \to \mathbf{C}$ is additive and $f(x) \in \sigma(x)$ for every $x \in \mathcal{A}$, then f is a character of $\mathcal{A}$.

The natural extension of the Gleason–Kahane–Żelazko theorem for the case when the range $\mathbf{C}$ of f is replaced by $\mathcal{M}_n$, the algebra of $n \times n$ matrices over $\mathbf{C}$, was obtained by Aupetit in [1].

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Theorem 1. *(See [1, Theorem 1].) If $T : \mathcal{A} \to \mathcal{M}_n$ is a surjective* **C***-linear map such that*

$$\sigma(T(x)) \subseteq \sigma(x) \qquad (x \in \mathcal{A}), \tag{2}$$

then either

$$T(xy) = T(x)T(y) \qquad (x, y \in \mathcal{A}) \quad or \quad T(xy) = T(y)T(x) \qquad (x, y \in \mathcal{A}). \tag{3}$$

In fact, [1, Theorem 1] states that if $T : \mathcal{A} \to \mathcal{M}_n$ is linear, unital and onto, sending invertible elements from $\mathcal{A}$ into invertible elements of $\mathcal{M}_n$, then T is of the form (3). If (2) holds, then $x \in \mathcal{A}$ invertible implies $0 \notin \sigma(x)$, thus by (2) we have $0 \notin \sigma(T(x))$, which means that the matrix $T(x)$ is invertible. By Lemma 6 we also have that T sends the unit element of $\mathcal{A}$ into the unit element of $\mathcal{M}_n$. (See also [4, Theorem 2.2].) Thus, under the hypothesis of Theorem 1 we have that T is unital and invertibility-preserving.

Under the hypothesis of Theorem 1, the map T is either an algebra morphism, or an algebra anti-morphism. In this paper, we study the same type of problem as the one considered by Theorem 1, assuming only additivity instead of linearity over the complex field **C**. Our first result states that if $\mathcal{A}$ is supposed to be a C^*-algebra, then we arrive at the same conclusion by assuming only additivity on T.

Theorem 2. *Let $\mathcal{A}$ be a unital C^*-algebra and suppose $T : \mathcal{A} \to \mathcal{M}_n$ is a surjective additive map such that (2) holds. Then T is of the form (3).*

As a corollary, we obtain the following generalization of [1, Theorem 2] for the case of additive maps defined on C^*-algebras which compress the spectrum.

Theorem 3. *Let $\mathcal{A}$ be a unital C^*-algebra, and let $\mathcal{B}$ be a complex, unital Banach algebra having a separating family of irreducible finite-dimensional representations. Suppose $T : \mathcal{A} \to \mathcal{B}$ is additive and onto such that (2) holds. Then T is a Jordan morphism, that is*

$$T(x^2) = T(x)^2 \qquad (x \in \mathcal{A}).$$

For the general case of an arbitrary Banach algebra $\mathcal{A}$, we shall impose an extra surjectivity assumption on the map T in order to obtain the same type of result.

Theorem 4. *Let $\mathcal{A}$ be a unital Banach algebra and suppose $T : \mathcal{A} \to \mathcal{M}_n$ is a surjective additive map such that (2) holds. Suppose also that there exist $x_1, \ldots, x_{n^2} \in \mathcal{A}$ such that*

$$\{T(x_1) + T(ix_1)/i, \ldots, T(x_{n^2}) + T(ix_{n^2})/i\} \subseteq \mathcal{M}_n \tag{4}$$

are linearly independent over **C**. *Then T is of the form (3).*

We do not know whether the assumption that the matrices in (4) span $\mathcal{M}_n$ over the complex field may be removed from the statement of Theorem 4. We believe that this hypothesis can be eliminated, being a consequence of the fact that T is surjective and that (2) holds, but we have not been able to prove it. An important part of the proof of Theorem 4 can be carried out without the surjectivity hypothesis given by (4) being assumed, using only the surjectivity of the map T. See also the final remark in Section 3.

2. Preliminaries

Throughout this section, $\mathcal{A}$ will denote an arbitrary unital Banach algebra. The first result shows that, as in the **C**-linear case [2, Theorem 5.5.2], under the hypothesis of Theorem 2 we have that the continuity of the map T is automatic.

Theorem 5. *Let T be an additive map from $\mathcal{A}$ onto $\mathcal{M}_n$ such that*

$$\rho\left(T\left(a\right)\right) \leq \rho\left(a\right) \qquad (a \in \mathcal{A}). \tag{5}$$

Then T is continuous, and therefore also $\mathbf{R}$-linear.

Proof. Since T is supposed to be additive, it is sufficient to prove the continuity at $0 \in \mathcal{A}$. Suppose that $a_k \to 0$ in $\mathcal{A}$ and let us prove first that $\left(T\left(a_k\right)\right)_k \subseteq \mathcal{M}_n$ is bounded. Using the surjectivity of T, it is sufficient to prove that given any $x \in \mathcal{A}$ then $(\mathrm{tr}(T\left(a_k\right) T\left(x\right)))_k \subseteq \mathbf{C}$ is bounded, where $\mathrm{tr}\left(\cdot\right)$ denotes the usual trace on $\mathcal{M}_n$. By (5), for each k we have that

$$\begin{aligned}\rho((T(a_k+x))^2) = (\rho(T(a_k+x)))^2 &\leq (\rho(a_k+x))^2 \leq \left\|a_k+x\right\|^2 \\ &\leq \left(\left\|a_k\right\|+\left\|x\right\|\right)^2,\end{aligned}$$

which implies

$$|\mathrm{tr}(T\left(a_k\right)^2) + 2\mathrm{tr}\left(T\left(a_k\right) T\left(x\right)\right) + \mathrm{tr}(T\left(x\right)^2)| \leq n(\left\|a_k\right\|^2 + 2\left\|a_k\right\|\left\|x\right\| + \left\|x\right\|^2).$$

Since $a_k \to 0$ and $\rho(T(a_k)) \leq \rho(a_k) \leq \|a_k\|$ for each k, this gives $\rho(T(a_k)) \to 0$ and therefore $\mathrm{tr}(T\left(a_k\right)^2) \to 0$. Thus

$$2\limsup_{k\to\infty}\left|\mathrm{tr}\left(T\left(a_k\right) T\left(x\right)\right)\right| \leq n\left\|x\right\|^2 + |\mathrm{tr}(T\left(x\right)^2)|,$$

and therefore $\left(\mathrm{tr}\left(T\left(a_k\right) T\left(x\right)\right)\right)_k$ is bounded, as desired.

Since $\mathcal{M}_n$ is finite dimensional, without loss of generality we may suppose that $T\left(a_k\right) \to w \in \mathcal{M}_n$, and let us prove that $w = 0$. We shall use the fact that the spectral radius on a general Banach algebra is upper semicontinuous [2, Theorem 3.4.2] and the fact that on $\mathcal{M}_n$ the spectral radius is continuous [2, Corollary 3.4.5]. Given any $a \in \mathcal{A}$ and $m \in \mathbf{N}$, by (5) we have $\rho(T(ma_k + a)) \leq \rho(ma_k + a)$. Using that T is additive, this gives $\rho(mT(a_k) + T(a)) \leq \rho(ma_k + a)$. Therefore

$$\limsup_{k\to\infty}\rho(mT(a_k)+T(a)) \leq \limsup_{k\to\infty}\rho(ma_k+a).$$

Since the spectral radius is continuous on $\mathcal{M}_n$, that $T\left(a_k\right) \to w$ in $\mathcal{M}_n$ gives

$$\begin{aligned}\limsup_{k\to\infty}\rho(mT(a_k)+T(a)) &= \lim_{k\to\infty}\rho(mT(a_k)+T(a)) \\ &= \rho\left(mw+T\left(a\right)\right).\end{aligned}$$

Since the spectral radius is upper semicontinuous on $\mathcal{A}$, that $a_k \to 0$ in $\mathcal{A}$ gives

$$\limsup_{k\to\infty}\rho(ma_k+a) \leq \rho\left(a\right).$$

Hence given any $a \in \mathcal{A}$ we have that $\rho(mw + T\left(a\right)) \leq \rho\left(a\right)$ for all $m \in \mathbf{N}$. Since T is supposed to be surjective, we deduce that given any $b \in \mathcal{M}_n$ we can find $M_b \geq 0$ such that

$$\rho(mw+b) \leq M_b \qquad (m \in \mathbf{N}). \tag{6}$$

Taking $b = 0$ in (6) we get $\rho(w) = 0$. If $w \in \mathcal{M}_n$ were not zero, we may write it as

$$w = y^{-1} \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & * & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & * \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix} y$$

for some invertible $y \in \mathcal{M}_n$. For

$$b = y^{-1} \begin{bmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} y \in \mathcal{M}_n$$

we have that $\lambda^2 - m$ divides the characteristic polynomial of $mw + b$. Hence $\rho(mw + b) \geq \sqrt{m}$ for all $m \in \mathbf{N}$, contradicting (6). □

The following lemma and Theorem 5 show that an additive surjective map $T : \mathcal{A} \to \mathcal{M}_n$ satisfying (2) is automatically unital.

Lemma 6. *Let $T : \mathcal{A} \to \mathcal{M}_n$ be additive and onto such that (2) holds. Then $T(\lambda \mathbf{1}) = \lambda I_n$ for every $\lambda \in \mathbf{C}$, where I_n is the unit matrix of $\mathcal{M}_n$.*

Proof. By Theorem 5 we have that T is continuous, and therefore also $\mathbf{R}$-linear. Since T is onto, by the open mapping theorem for surjective $\mathbf{R}$-linear maps we find $N > 0$ such that $y \in \mathcal{M}_n$ implies the existence of $x \in \mathcal{A}$ such that $T(x) = y$ and $\|x\| \leq N \|y\|$. Let $\lambda \in \mathbf{C}$ and denote $u = T(\lambda \mathbf{1}) \in \mathcal{M}_n$. Then given any $y \in \mathcal{M}_n$, we have

$$\begin{aligned} \sigma(\lambda I_n - (u + y)) &= \lambda - \sigma(u + y) = \lambda - \sigma(T(\lambda \mathbf{1} + x)) \\ &\subseteq \lambda - \sigma(\lambda \mathbf{1} + x) = \lambda - (\lambda + \sigma(x)) \\ &= -\sigma(x), \end{aligned}$$

where $x \in \mathcal{A}$ was such that $T(x) = y$ and $\|x\| \leq N \|y\|$. Thus

$$\rho(\lambda I_n - (u + y)) \leq \rho(x) \leq \|x\| \leq N \|y\|,$$

that is

$$\rho((\lambda I_n - u) - y) \leq N \|y\| \qquad (y \in \mathcal{M}_n).$$

The Zemánek characterization of the radical [2, Theorem 5.3.1] implies that $\lambda I_n - u$ belongs to the radical of $\mathcal{M}_n$. That is, $u = \lambda I_n$, since $\mathcal{M}_n$ is semisimple. □

Suppose now that $T : \mathcal{A} \to \mathcal{M}_n$ is a surjective additive map such that (2) holds. By Theorem 5, we have that T is $\mathbf{R}$-linear. Following an idea from [7], given any $r \in \mathbf{R}$ we have

$$\sigma(e^{ir} T(e^{-ir} x)) \subseteq e^{ir} \sigma((e^{-ir} x)) = \sigma(x)$$

for every $x \in \mathcal{A}$. From the $\mathbf{R}$-linearity of T we also have

$$\begin{aligned} e^{ir}T(e^{-ir}x) &= (\cos r + i\sin r)(\cos r \cdot T(x) - \sin r \cdot T(ix)) \\ &= T(x)(\cos^2 r + i\sin r \cdot \cos r) - T(ix)(\cos r \cdot \sin r + i\sin^2 r) \\ &= (T(x) + T(ix)/i)/2 + e^{2ir}(T(x) - T(ix)/i)/2. \end{aligned}$$

Thus

$$\sigma(R(x) + \xi S(x)) \subseteq \sigma(x) \qquad (x \in \mathcal{A};\ \xi \in \mathbf{C},\ |\xi| = 1), \tag{7}$$

where we have denoted

$$R(x) = \frac{T(x) + T(ix)/i}{2} \qquad (x \in \mathcal{A})$$

and

$$S(x) = \frac{T(x) - T(ix)/i}{2} \qquad (x \in \mathcal{A}).$$

Since T is $\mathbf{R}$-linear, one can easily check that R and S are both $\mathbf{R}$-linear transformations from $\mathcal{A}$ into $\mathcal{M}_n$. More than that, $R(ix) = iR(x)$ for every $x \in \mathcal{A}$, and therefore R is $\mathbf{C}$-linear. Also, $S(ix) = -iS(x)$ for every $x \in \mathcal{A}$, and therefore S is conjugate-linear. Thus

$$T(x) = R(x) + S(x) \qquad (x \in \mathcal{A}),$$

where R is $\mathbf{C}$-linear and S is $\overline{\mathbf{C}}$-linear. Observe also that by Lemma 6 we have $R(\mathbf{1}) = I_n$ and $S(\mathbf{1}) = 0 \in \mathcal{M}_n$.

The inclusions (7) imply the following spectral inequalities for the maps R and S.

Theorem 7. *Suppose $T : \mathcal{A} \to \mathcal{M}_n$ is a surjective additive map such that (2) holds. Then $R(e^a) \in \mathcal{M}_n$ is invertible for each $a \in \mathcal{A}$, and*

$$\rho(S(xe^a)(R(e^a))^{-1}) \le \rho(x) \qquad (a, x \in \mathcal{A}). \tag{8}$$

Proof. Consider $x \in \mathcal{A}$ with $\rho(x) < 1$ and an arbitrary $a \in \mathcal{A}$. Let $\xi \in \mathbf{C}$ with $|\xi| = 1$. Then for each $r \in \mathbf{R}$ we have that $(\overline{\xi} - x)e^{ra} \in \mathcal{A}$ is invertible, and (2) gives $\sigma(T((\overline{\xi} - x)e^{ra})) \subseteq \mathbf{C}\backslash\{0\}$. Then

$$\sigma(\overline{\xi}R(e^{ra}) - T(xe^{ra}) + \xi S(e^{ra})) \subseteq \mathbf{C}\backslash\{0\},$$

and therefore

$$\sigma(R(e^{ra}) - \xi T(xe^{ra}) + \xi^2 S(e^{ra})) \subseteq \mathbf{C}\backslash\{0\} \quad (a \in \mathcal{A},\ \rho(x) < 1,\ |\xi| = 1,\ r \in \mathbf{R}). \tag{9}$$

This leads us to consider the family of analytic multivalued functions $(K_r)_{r\in[0,1]}$ given by

$$K_r(\lambda) = \sigma(R(e^{ra}) - \lambda T(xe^{ra}) + \lambda^2 S(e^{ra})) \qquad (\lambda \in \mathbf{C}).$$

Since by Theorem 5 we have that T, R and S are continuous and since the spectrum function is continuous on matrices, for each $\lambda \in \mathbf{C}$ we have that the function $r \mapsto K_r(\lambda)$ is continuous with respect to r. We apply now the multivalued form of Rouché's Theorem given by [8, Theorem 2.2] to see that

$$(K_0(\mathbf{D})\backslash K_1(\mathbf{D}))\bigcup(K_1(\mathbf{D})\backslash K_0(\mathbf{D})) \subseteq \bigcup\{K_r(\xi) : r \in [0,1],\ |\xi| = 1\}.$$

(By $\mathbf{D}$ we have denoted the open unit disk in $\mathbf{C}$.) Now (9) implies that $0 \notin K_r(\xi)$ for $r \in [0,1]$ and $|\xi| = 1$, and therefore $(K_1(\mathbf{D})\backslash K_0(\mathbf{D})) \subseteq \mathbf{C}\backslash\{0\}$. That $R(\mathbf{1}) = I_n$ and $S(\mathbf{1}) = 0$ imply $K_0(\lambda) = \sigma(I_n - \lambda T(x))$. But $\sigma(T(x)) \subseteq \sigma(x) \subseteq \mathbf{D}$, and therefore $K_0(\lambda) \subseteq \mathbf{C}\backslash\{0\}$ for all $\lambda \in \mathbf{D}$. That $K_1(\mathbf{D})\backslash K_0(\mathbf{D})$ does not contain $0 \in \mathbf{C}$ implies then $K_1(\mathbf{D}) \subseteq \mathbf{C}\backslash\{0\}$, and therefore

$$\sigma(R(e^a) - \lambda T(xe^a) + \lambda^2 S(e^a)) \subseteq \mathbf{C}\backslash\{0\} \qquad (a \in \mathcal{A},\ \rho(x) < 1,\ |\lambda| < 1). \tag{10}$$

Taking $\lambda = 0$ in (10), we see that $R(e^a)$ is an invertible matrix. Denoting $s = T(xe^a)(R(e^a))^{-1} \in \mathcal{M}_n$ and $p = S(e^a)(R(e^a))^{-1} \in \mathcal{M}_n$, we infer from (10) that $\det(\mu^2 I_n - \mu s + p) \neq 0$ for $|\mu| > 1$. Let us observe now that $\mu \mapsto \det(\mu^2 I_n - \mu s + p)$ is just the characteristic polynomial of

$$\begin{bmatrix} 0 & I_n \\ -p & s \end{bmatrix} \in \mathcal{M}_{2n},$$

and therefore

$$\rho\left(\begin{bmatrix} 0 & I_n \\ -S(e^a)R(e^a)^{-1} & T(xe^a)R(e^a)^{-1} \end{bmatrix}\right) \leq 1 \qquad (a \in \mathcal{A},\ \rho(x) < 1).$$

For $x \mapsto \eta x$ with $|\eta| = 1$ we get

$$\rho\left(\begin{bmatrix} 0 & I_n \\ -S(e^a)R(e^a)^{-1} & 0 \end{bmatrix} + \eta\begin{bmatrix} 0 & 0 \\ 0 & R(xe^a)R(e^a)^{-1} \end{bmatrix} + \overline{\eta}\begin{bmatrix} 0 & 0 \\ 0 & S(xe^a)R(e^a)^{-1} \end{bmatrix}\right) \leq 1,$$

and therefore

$$\rho\left(\eta\begin{bmatrix} 0 & I_n \\ -S(e^a)R(e^a)^{-1} & 0 \end{bmatrix} + \eta^2\begin{bmatrix} 0 & 0 \\ 0 & R(xe^a)R(e^a)^{-1} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & S(xe^a)R(e^a)^{-1} \end{bmatrix}\right) \leq 1$$

for all $|\eta| = 1$. Using Vesentini's theorem [2, Theorem 3.4.7] and the maximum principle for subharmonic functions we infer that

$$\rho\left(\begin{bmatrix} 0 & 0 \\ 0 & S(xe^a)R(e^a)^{-1} \end{bmatrix}\right) \leq 1.$$

Therefore $\rho(S(xe^a)R(e^a)^{-1}) \leq 1$ for all $a \in \mathcal{A}$ and for all $x \in \mathcal{A}$ with $\rho(x) < 1$. Using the fact that S is conjugate-homogeneous, we obtain (8). □

Let us remark that, under the hypothesis of Theorem 7, we have $\det S(x) = 0$ for every $x \in \mathcal{A}$. Indeed, taking $x = \mathbf{1}$ in (8) we see that $\rho(S(e^a)(R(e^a))^{-1}) \leq 1$ for every $a \in \mathcal{A}$. Therefore $\rho(S(e^{\lambda a})(R(e^{\lambda a}))^{-1}) \leq 1$ for every $a \in \mathcal{A}$ and $\lambda \in \mathbf{C}$. This implies that the analytic function $\lambda \mapsto \overline{\det S(e^{\lambda a})}\det(R(e^{\lambda a}))^{-1}$ is bounded on $\mathbf{C}$. Classical Liouville's theorem implies that it is therefore constant. Since $S(\mathbf{1}) = 0$, then $\det S(e^{\lambda a}) = 0$ for $\lambda = 0$, and therefore $\overline{\det S(e^{\lambda a})}\det(R(e^{\lambda a}))^{-1} = 0$ for every $\lambda \in \mathbf{C}$. Thus $\det S(e^{\lambda a}) = 0$ on $\mathbf{C}$, and in particular $\det S(e^a) = 0$. Now if $x \in \mathcal{A}$ is arbitrary, the holomorphic functional calculus shows that $\xi\mathbf{1} + x \in \mathcal{A}$ is an exponential for $|\xi| > \rho(x)$. Then $\det(S(x)) = \det(S(\xi\mathbf{1} + x)) = 0$, where ξ was chosen such that $|\xi| > \rho(x)$.

Let us observe that until now in this section, the only surjectivity assumption that was used in the proofs is the one we have on the map T. By Theorem 5 we have that R is continuous, and by Lemma 6 we have that R is unital. By Theorem 7, the map R sends exponentials from $\mathcal{A}$ into invertible matrices. Then the proof of [3, Theorem 3.5] shows that given any complex polynomial p, we have

$$\operatorname{tr}(R(p(x)\,y)) = \operatorname{tr}(p(R(x))\,R(y)) \qquad (x, y \in \mathcal{A}).$$

In particular, $\operatorname{tr}(R(xy)) = \operatorname{tr}(R(x)\,R(y))$ for every x and y, and

$$\operatorname{tr}(((R(x)^2 - R(x^2))R(y)) = 0 \qquad (x, y \in \mathcal{A}). \tag{11}$$

If we further suppose (4) to be true, then R is also surjective and (11) implies that $R(x)^2 = R(x^2)$ for every $x \in \mathcal{A}$. (See also [1].) Thus R is a Jordan morphism and therefore, since $\mathcal{M}_n$ is prime, of the form (3). We shall use this property in the proofs of both Theorem 2 and Theorem 4.

3. The case of C^*-algebras

Throughout this section, $\mathcal{A}$ will denote a unital C^*-algebra.

Proof of Theorem 2. By Theorem 7, we have that (8) holds. Let $a \in \mathcal{A}$ be a self-adjoint element. Then for every $r \in \mathbf{R}$ we have that $e^{ira} \in \mathcal{A}$ is a unitary element. In particular, $||e^{ira}|| = \rho(e^{ira}) = 1$. For an arbitrary $y \in \mathcal{A}$ and $\lambda = \alpha + i\beta \in \mathbf{C}$, where $\alpha, \beta \in \mathbf{R}$, by taking $x = ye^{-2i\beta a} \in \mathcal{A}$ in (8) we see that

$$\begin{aligned}\rho(S(ye^{\overline{\lambda}a})(R(e^{\lambda a}))^{-1}) &= \rho(S(ye^{-2i\beta a}e^{(\alpha+i\beta)a})(R(e^{(\alpha+i\beta)a}))^{-1}) \le \rho(ye^{-2i\beta a}) \\ &\le ||ye^{-2i\beta a}|| \le \|y\|\,||e^{-2i\beta a}|| \\ &= \|y\|\,.\end{aligned}$$

The continuity of S and R, together with the facts that S is conjugate-linear and R is $\mathbf{C}$-linear imply that $\lambda \mapsto S(ye^{\overline{\lambda}a})(R(e^{\lambda a}))^{-1}$ is analytic from $\mathbf{C}$ into $\mathcal{M}_n$. Then Liouville's Spectral Theorem [2, Theorem 3.4.14] implies that $\lambda \mapsto \sigma(S(ye^{\overline{\lambda}a})(R(e^{\lambda a}))^{-1})$ is constant on $\mathbf{C}$. In particular, for every λ we have

$$\sigma(S(ye^{\overline{\lambda}a})(R(e^{\lambda a}))^{-1}) = \sigma(S(y\mathbf{1})(R(\mathbf{1}))^{-1}) = \sigma(S(y)),$$

the last equality being true since by Lemma 6 we have that $R(\mathbf{1})$ is the $n \times n$ identity matrix. Thus

$$\sigma((S(y) + \lambda S(ya) + \lambda^2 S\left(ya^2\right)/2 + \cdots)(I_n - \lambda R(a) + \cdots)) = \sigma(S(y)) \quad (\lambda \in \mathbf{C}). \tag{12}$$

Taking $y = \mathbf{1}$ in (12), since by Lemma 6 we have that $S(\mathbf{1})$ is the $n \times n$ zero matrix we obtain that

$$\rho((\lambda S(a) + \lambda^2 S\left(a^2\right)/2 + \cdots)(I_n - \lambda R(a) + \cdots)) = 0 \qquad (\lambda \in \mathbf{C}).$$

Dividing the last equality by $\lambda \neq 0$ and letting $\lambda \to 0$ we see that $\rho(S(a)) = 0$. This holds for any arbitrary self-adjoint element $a \in \mathcal{A}$; if $x \in \mathcal{A}$ is now arbitrary, with $x = a+ib$ where $a, b \in \mathcal{A}$ are self-adjoint elements, then $\rho(S(a+rb)) = 0$ for every $r \in \mathbf{R}$, the element $a+rb \in \mathcal{A}$ being self-adjoint. Thus $\rho(S(a) + rS(b)) = 0$ for every $r \in \mathbf{R}$, and for the analytic function $\lambda \mapsto S(a) + \lambda S(b)$ this implies that $\rho(S(a) + \lambda S(b)) = 0$ for every $\lambda \in \mathbf{C}$. Taking $\lambda = -i$ we infer that $\rho(S(x)) = 0$, equality which holds for every $x \in \mathcal{A}$. Now if $y, z \in \mathcal{A}$ are arbitrary elements, we have $\rho(S(y) + \lambda S(z)) = \rho(S(y+\overline{\lambda}z)) = 0$ for every $\lambda \in \mathbf{C}$. In particular $\operatorname{tr}((S(y) + \lambda S(z))^2) = 0$ for every $\lambda \in \mathbf{C}$, and therefore

$$\operatorname{tr}(S(y)S(z)) = 0 \qquad (y, z \in \mathcal{A})\,. \tag{13}$$

Equation (12) implies that given any $y \in \mathcal{A}$ and any self-adjoint element $a \in \mathcal{A}$ we have

$$\mathrm{tr}((S(y) + \lambda S(ya) + \lambda^2 S(ya^2)/2 + \cdots)(I_n - \lambda R(a) + \cdots)) = \mathrm{tr}(S(y)) \quad (\lambda \in \mathbf{C}).$$

Computing the coefficient of λ, we see that $\mathrm{tr}(S(ya)) = \mathrm{tr}(S(y)R(a))$. That $\rho(S(ya)) = 0$ gives $\mathrm{tr}(S(ya)) = 0$, and therefore $\mathrm{tr}(S(y)R(a)) = 0$. By (13) we also have $\mathrm{tr}(S(y)S(a)) = 0$. Now if $x = a + ib$ is arbitrary, where $a, b \in \mathcal{A}$ are self-adjoint elements, then

$$\begin{aligned} \mathrm{tr}(S(y)T(x)) &= \mathrm{tr}(S(y)R(x)) + \mathrm{tr}(S(y)S(x)) \\ &= \mathrm{tr}(S(y)R(a)) + i\mathrm{tr}(S(y)R(b)) + \mathrm{tr}(S(y)S(a)) - i\mathrm{tr}(S(y)S(b)) \\ &= 0. \end{aligned}$$

Thus $\mathrm{tr}(S(y)T(x)) = 0$ for every $x, y \in \mathcal{A}$. The surjectivity of T implies that S is identically zero. Thus $T = R$. In particular R is surjective, and then (11) implies that R is a Jordan morphism and therefore of the form (3). Thus, the same is true for $T = R$ too.

As a corollary, we obtain the characterization of additive, surjective, spectrum-compressing maps into Banach algebras having a separating family of irreducible finite-dimensional representations.

Proof of Theorem 3. Let π be a finite-dimensional irreducible representation of $\mathcal{B}$. Using the Jacobson density theorem, we have that $\pi : \mathcal{B} \to \mathcal{M}_n$ is surjective, for some $n \geq 1$. (See [1].) Define $T_\pi : \mathcal{A} \to \mathcal{M}_n$ by putting $T_\pi = \pi \circ T$. Then T_π is additive and onto, and

$$\sigma(T_\pi(x)) = \sigma(\pi(T(x))) \subseteq \sigma(T(x)) \subseteq \sigma(x) \qquad (x \in \mathcal{A}).$$

We use then Theorem 2 to see that T_π is a Jordan morphism. Thus

$$\pi(T(x^2) - T(x)^2) = 0 \qquad (x \in \mathcal{A}),$$

and using now the fact that $\mathcal{B}$ has a *separating family* of irreducible finite-dimensional representations we conclude that $T(x^2) = T(x)^2$ for all $x \in \mathcal{A}$.

4. Proof of Theorem 4

We have seen that for $R(x) = (T(x) + T(ix)/i)/2$ and $S(x) = (T(x) - T(ix)/i)/2$, the map $R : \mathcal{A} \to \mathcal{M}_n$ is $\mathbf{C}$-linear, while $S : \mathcal{A} \to \mathcal{M}_n$ is conjugate-linear. Also, the hypothesis (4) implies that R is also onto, and therefore the final remark in Section 3 implies that R is either an algebra morphism, or an algebra anti-morphism. Let us suppose, for example, that R is a morphism.

Consider an arbitrary $y \in \mathcal{A}$ with $\rho(y) < 1$ and an arbitrary $\xi \in \mathbf{C}$ with $|\xi| = 1$. The holomorphic functional calculus shows that $\xi\mathbf{1} - y \in \mathcal{A}$ is an exponential, and then (8) implies that $\rho(S(x(\xi\mathbf{1} - y))(R(\xi\mathbf{1} - y))^{-1}) \leq \rho(x)$ for all x. That is, $\rho((\overline{\xi}S(x) - S(xy))(I_n - \overline{\xi}R(y))^{-1}) \leq \rho(x)$. Since R is an algebra morphism, then $\rho(R(y)) \leq \rho(y) < 1$, and we then have $(I_n - \overline{\xi}R(y))^{-1} = I_n + \overline{\xi}R(y) + \overline{\xi}^2 R(y)^2 + \cdots$. The subharmonic function

$$\mu \mapsto \rho((\mu S(x) - S(xy))(I_n + \mu R(y) + \mu^2 R(y)^2 + \cdots))$$

is well-defined on a neighborhood of the closed unit disk and is bounded by $\rho(x)$ for $|\mu| = 1$. Using the maximum principle we see that, for $\rho(y) < 1$ and $x \in \mathcal{A}$ we have

$$\rho((\mu S(x) - S(xy))(I_n + \mu R(y) + \mu^2 R(y)^2 + \cdots)) \leq \rho(x) \qquad (|\mu| \leq 1). \tag{14}$$

For $\mu = 0$ in (14) we get $\rho(S(xy)) \le \rho(x)$ for $\rho(y) < 1$, and using once more the conjugate-homogeneity of S we infer that

$$\rho(S(xy)) \le \rho(x)\,\rho(y) \qquad (x, y \in \mathcal{A}). \tag{15}$$

Taking the trace of the analytic function in the left hand side of the inequality from (14) and computing the coefficients of μ and μ^2, the Cauchy inequalities imply the existence of $c_1 > 0$ and $c_2 > 0$ such that

$$|\mathrm{tr}(S(x)) - \mathrm{tr}(S(xy)R(y))| \le c_1\rho(x) \qquad (x \in \mathcal{A},\ \rho(y) < 1) \tag{16}$$

and

$$\left|\mathrm{tr}(S(x)R(y)) - \mathrm{tr}(S(xy)R(y)^2)\right| \le c_2\rho(x) \qquad (x \in \mathcal{A},\ \rho(y) < 1). \tag{17}$$

Taking $y = \mathbf{1}$ in (15) we get $|\mathrm{tr}(S(x))| \le n\rho(x)$, and then from (16) we infer the existence of $c_3 > 0$ such that $|\mathrm{tr}(S(xy)R(y))| \le c_3\rho(x)$ for all $x, y \in \mathcal{A}$, with $\rho(y) < 1$. Since S is conjugate-homogeneous and R is homogeneous, this gives

$$|\mathrm{tr}(S(xy)R(y))| \le c_3\rho(x)\,\rho(y)^2 \qquad (x, y \in \mathcal{A}). \tag{18}$$

Using the homogeneity of S and R in (17), we have

$$|\mathrm{tr}(S(x)R(y))\rho(y)^2 - \mathrm{tr}(S(xy)R(y)^2)| \le c_2\rho(x)\,\rho(y)^3 \qquad (x, y \in \mathcal{A}). \tag{19}$$

(For arbitrary $x, y \in \mathcal{A}$ and $\varepsilon > 0$, applying (17) to x and $y/(\rho(y) + \varepsilon)$ we see that

$$|\mathrm{tr}(S(x)R(y))(\rho(y) + \varepsilon)^2 - \mathrm{tr}(S(xy)R(y)^2)| \le c_2\rho(x)\,(\rho(y) + \varepsilon)^3,$$

and then we let $\varepsilon \to 0$.) If $R(y)^2 = 0$, then (19) gives $|\mathrm{tr}(S(x)R(y))| \le c_2\rho(x)\,\rho(y)$ for all $x \in \mathcal{A}$. If $R(y)^2 = R(y)$, then using (18) in (19) we obtain that for all $x \in \mathcal{A}$ we have

$$|\mathrm{tr}(S(x)R(y))\rho(y)^2\,| \le c_3\rho(x)\,\rho(y)^2 + c_2\rho(x)\,\rho(y)^3\,.$$

Thus, there exist $c_4, c_5 \ge 0$ such that for $y \in \mathcal{A}$ if we have either $R(y)^2 = 0$ or $R(y)^2 = R(y)$, then

$$|\mathrm{tr}(S(x)R(y))| \le \rho(x)\,(c_4 + c_5\rho(y)) \qquad (x \in \mathcal{A}). \tag{20}$$

Let now $x, u \in \mathcal{A}$ be arbitrary and $y \in \mathcal{A}$ such that either $R(y)^2 = 0$, or $R(y)^2 = R(y)$. Since R is a morphism, for each invertible element $w \in \mathcal{A}$ we have that $R(y)^2 = 0$ implies $\left(R(w^{-1}yw)\right)^2 = R(w^{-1}y^2w) = R(w^{-1})R(y)^2R(w) = 0$, and analogously $R(y)^2 = R(y)$ implies $\left(R(w^{-1}yw)\right)^2 = R(w^{-1}yw)$. By (20), the entire function

$$\lambda \mapsto \mathrm{tr}(S(x)R(e^{-\lambda u}ye^{\lambda u}))$$

is then bounded on $\mathbf{C}$, and therefore, by classical Liouville's theorem, is constant. The coefficient of λ for its Taylor series is therefore zero, and using one more the fact that R is a morphism we infer that $\mathrm{tr}(S(x)R(u)R(y)) - \mathrm{tr}(S(x)R(y)R(u)) = 0$. Thus

$$\mathrm{tr}((S(x)R(u) - R(u)S(x))\,R(y)) = 0,$$

for all $y \in \mathcal{A}$ such that either $R(y)^2 = 0$, or $R(y)^2 = R(y)$. Since R is surjective, by taking y such that $R(y)$ has 1 on the (j,k) entry and zeroes everywhere else, we obtain that $S(x)R(u) - R(u)S(x) = 0$ for all $x, u \in \mathcal{A}$. We use once more the surjectivity of R to infer that $S(\mathcal{A}) \subseteq \mathbf{C}I_n$. Since $\det S(x)$ is always zero (see the remark following the proof of Theorem 7), we obtain that S itself is identically zero. Therefore $T = R$, and the theorem is proved.

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