

Uniform Fatou's Lemma

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Abstract

Fatou's lemma is a classic fact in real analysis stating that the limit inferior of integrals of functions is greater than or equal to the integral of the inferior limit. This paper introduces a stronger inequality that holds uniformly for integrals on measurable subsets of a measurable space. The necessary and sufficient condition, under which this inequality holds for a sequence of finite measures converging in total variation, is provided. This statement is called the uniform Fatou lemma, and it holds under the minor assumption that all the integrals in the inequality are well-defined. The uniform Fatou lemma improves the classic Fatou lemma in the following directions: the uniform Fatou lemma states a more precise inequality, it provides the necessary and sufficient condition, and it deals with variable measures. Various corollaries of the uniform Fatou lemma are formulated. The examples in this paper demonstrate that: (a) the uniform Fatou lemma may indeed provide a more accurate inequality than the classic Fatou lemma; (b) the uniform Fatou lemma does not hold if convergence of measures in total variation is relaxed to setwise convergence.

Keywords: finite measure, convergence, Fatou's lemma, Radon-Nikodym derivative

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1 Introduction

Fatou's lemma is an important fact in real analysis that has significant applications in various fields. It provides the inequality that relates the limit inferior of integrals of functions and the integral of the inferior limit. This paper introduces the uniform Fatou lemma for a sequence of finite measures converging in total variation, describes the necessary and sufficient condition for the validity of this statement, and provides corollaries and counterexamples. A statement, that is more particular than one of the formulated corollaries, was originally introduced by the authors [11, Theorem 5.2] for the analysis of control problems with incomplete information.

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The formulation of the uniform Fatou lemma, which is Theorem 2.1, is based on the following observation. Instead of the integral of the lower limit of the functions defined in Fatou's lemma, the integral can be equivalently written for an arbitrary measurable function bounded above by this lower limit; see (2.6). In particular, this function can be equal to the lower limit defined in Fatou's lemma. The next step is to consider the lower limit of the infimums, taken over all measurable subsets, of the differences of integrals of the functions in the sequence and this function; see (2.8). The inequality, that this lower limit is nonnegative, is stronger than the inequality in Fatou's lemma in the sense that the former implies the latter. A similar inequality can be written for variable measures; see (2.1). As mentioned above, for integrable functions this stronger inequality holds for a sequence of variable measures converging in total variation, if the functions are nonnegative or satisfy the condition guaranteeing the validity of Fatou's lemma. Theorem 2.1 provides the necessary and sufficient condition for the validity of this inequality. This necessary and sufficient condition is weaker than the classic pointwise semi-convergence taken together with the existence of an integrable minorant. In addition, the uniform Fatou lemma provides tighter lower bounds for the lower limit of integrals than the classic Fatou lemma, and Example 4.1 demonstrates that the bounds provided by the former can be strictly tighter.

The results of this paper are relevant to Fatou's lemma for varying measures. In particular, Fatou's lemma holds for setwise and weakly converging measures. For setwise converging measures it is presented in Royden [18, p. 231] for nonnegative functions and in Feinberg et al. [10] for functions that can take negative values. For weak convergence, it was introduced by Serfozo [21] for locally compact spaces. As was observed by Schäl [19], the local compactness assumption is not needed. Feinberg et al. [10] established Fatou's lemma for weakly converging measures and possibly negative functions. In the case of weak convergence, the lower limit of functions should be defined in a stronger sense than in the setwise convergence case and in the classic case of a single measure. Fatou's lemma for weakly converging measures is broadly used in stochastic control [7, 9, 19, 29], game theory [15], and in other applications [6, 23].

A particular case of the necessary statement in the uniform Fatou lemma was introduced and used in Feinberg et al. [11, Theorem 5.2] for the analysis of Partially Observable Markov Decision Processes (POMDPs). As is well-known, such processes can be reduced to Completely Observable Markov Decision Processes (COMDPs) whose states are posterior probabilities of states in the original problem. This reduction was introduced by Aoki [1], Åström [3], Dynkin [8], and Shiryaev [22]. For problems with Borel state and action spaces, this reduction was independently justified by Rhenius [17] and Yushkevich [25]. It reduces finding an optimal policy for a POMDP to finding an optimal policy for the corresponding COMDP, but it says nothing about the existence of optimal policies for the COMDP. Optimal policies may not exist, if transition probabilities are not continuous, and [11, Theorem 5.2] is useful for verifying the weak continuity of transition probabilities for COMDPs and for proving the existence of optimal policies for POMDPs with Borel state and action spaces. Based on this application, we think that statistics of random processes (Liptser and Shiryaev [16]) and limit theorems for stochastic processes (Jacod and Shiryaev [14]) are natural areas for potential applications of the uniform Fatou lemma. Another potentially natural area for applications is qualitative analysis of solutions for nonlinear PDEs and related topics; see Gorban et al. [13] and Zgurovsky et al. [26]–[28].

We remark that there is a significant literature on generalizations of Fatou's lemma to more general

objects than numerical functions, including multi-dimensional and set-valued functions. The relevant references include [2, 4, 5, 20, 24]. This paper deals only with numerical functions.

The main results of this paper are described in the following section and the proof of the main theorem is given in Section 3. Section 4 describes examples showing that the inequality in the uniform Fatou lemma may indeed be more precise than the inequality in Fatou's lemma and that the uniform Fatou lemma does not hold for setwise converging measures.

2 Main Results

For a measurable space (\mathbb{S}, Σ) , let $\mathcal{M}(\mathbb{S})$ denote the family of finite measures on (\mathbb{S}, Σ) . Let \mathbb{R} be the real line and $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$. The σ -field Σ is fixed, and we often write \mathbb{S} instead of (\mathbb{S}, Σ) . A function $f : \mathbb{S} \rightarrow \overline{\mathbb{R}}$ is called measurable if $\{s \in \mathbb{S} : f(s) < \alpha\} \in \Sigma$ for each $\alpha \in \mathbb{R}$. We recall that a sequence of measures $\{\mu^{(n)}\}_{n=1,2,\dots}$ from $\mathcal{M}(\mathbb{S})$ converges setwise to a measure μ on \mathbb{S} if for each bounded measurable function f on \mathbb{S}

$$\int_{\mathbb{S}} f(s) \mu^{(n)}(ds) \rightarrow \int_{\mathbb{S}} f(s) \mu(ds) \quad \text{as } n \rightarrow \infty.$$

A sequence of finite measures $\{\mu^{(n)}\}_{n=1,2,\dots}$ on \mathbb{S} converges in total variation to a measure μ on \mathbb{S} if $\lim_{n \rightarrow \infty} \text{dist}(\mu^{(n)}, \mu) = 0$, where $\text{dist}(\nu, \mu)$ denotes the total variation distance between a finite measure ν on \mathbb{S} and a measure μ on \mathbb{S} ,

$$\text{dist}(\nu, \mu) := \sup \{\mu(A) - \nu(A) : A \in \Sigma\} + \sup \{\nu(A) - \mu(A) : A \in \Sigma\}.$$

In view of Hahn's decomposition,

$$\text{dist}(\nu, \mu) := \sup \left\{ \left| \int_{\mathbb{S}} f(s) \mu(ds) - \int_{\mathbb{S}} f(s) \nu(ds) \right| : f : \mathbb{S} \rightarrow [-1, 1] \text{ is measurable} \right\}.$$

Setwise convergence is weaker than convergence in total variation. Therefore, if a sequence of finite measures $\{\mu^{(n)}\}_{n=1,2,\dots}$ on \mathbb{S} converges in total variation to a measure μ on \mathbb{S} , then $\mu^{(n)}(S) \rightarrow \mu(S)$ for all $S \in \Sigma$ as $n \rightarrow \infty$. In particular, convergence in total variation implies that $\lim_{n \rightarrow \infty} \mu_n(\mathbb{S}) = \mu(\mathbb{S}) < \infty$.

For $\mu \in \mathcal{M}(\mathbb{S})$ consider the vector space $L^1(\mathbb{S}; \mu)$ of all measurable functions $f : \mathbb{S} \rightarrow \overline{\mathbb{R}}$, whose absolute values have finite integrals, that is, $\int_{\mathbb{S}} |f(s)| \mu(ds) < +\infty$. The symbol \mathbf{I} denotes an indicator function. The following theorem is the main result of this paper.

Theorem 2.1. (Uniform Fatou's Lemma for Variable Measures and Unbounded Below Functions) *Let (\mathbb{S}, Σ) be a measurable space, $\{\mu^{(n)}\}_{n=1,2,\dots} \subset \mathcal{M}(\mathbb{S})$ converge in total variation to a measure μ on \mathbb{S} , $f \in L^1(\mathbb{S}; \mu)$, and $f^{(n)} \in L^1(\mathbb{S}; \mu^{(n)})$ for each $n = 1, 2, \dots$. Then the inequality*

$$\liminf_{n \rightarrow \infty} \inf_{S \in \Sigma} \left(\int_S f^{(n)}(s) \mu^{(n)}(ds) - \int_S f(s) \mu(ds) \right) \geq 0 \quad (2.1)$$

holds if and only if the following two statements hold:

(i) *for each $\varepsilon > 0$*

$$\mu(\{s \in \mathbb{S} : f^{(n)}(s) \leq f(s) - \varepsilon\}) \rightarrow 0 \text{ as } n \rightarrow \infty, \quad (2.2)$$

and, therefore, there exists a subsequence $\{f^{(n_k)}\}_{k=1,2,\dots} \subseteq \{f^{(n)}\}_{n=1,2,\dots}$ such that

$$\liminf_{k \rightarrow \infty} f^{(n_k)}(s) \geq f(s) \quad \text{for } \mu\text{-a.e. } s \in \mathbb{S}; \quad (2.3)$$

(ii) the inequality

$$\liminf_{K \rightarrow +\infty} \inf_{n=1,2,\dots} \int_{\mathbb{S}} f^{(n)}(s) \mathbf{I}\{s \in \mathbb{S} : f^{(n)}(s) \leq -K\} \mu^{(n)}(ds) \geq 0 \quad (2.4)$$

holds.

Remark 2.2. Let (\mathbb{S}, Σ) be a measurable space, $\{f^{(n)}, f\}_{n=1,2,\dots}$ be a sequence of measurable functions, and μ be a measure on \mathbb{S} . We note that if (2.2) holds for each $\varepsilon > 0$, then (2.3) holds; see Lemma 3.1. Since the functions $\{f^{(n)}\}_{n=1,2,\dots}$ can take negative values, condition (ii) cannot be omitted from the theorem.

We recall that the classic Fatou lemma can be formulated in the following form.

Fatou's lemma. Let (\mathbb{S}, Σ) be a measurable space, μ be a measure on (\mathbb{S}, Σ) and $\{f, f^{(n)}\}_{n=1,2,\dots}$ be a sequence of measurable nonnegative functions. Then the inequality

$$\liminf_{n \rightarrow \infty} f^{(n)}(s) \geq f(s) \text{ for } \mu\text{-a.e. } s \in \mathbb{S} \quad (2.5)$$

implies

$$\liminf_{n \rightarrow \infty} \int_{\mathbb{S}} f^{(n)}(s) \mu(ds) \geq \int_{\mathbb{S}} f(s) \mu(ds). \quad (2.6)$$

Note that there are generalizations of Fatou's lemma to functions that can take negative values. For example, the conclusions of Fatou's lemma hold if all the functions have a common integrable minorant.

Even in the case of a single measure, that is $\mu^{(n)} = \mu$, $n = 1, 2, \dots$, Theorem 2.1 is a more general statement than Fatou's lemma, if this measure is finite and $\{f, f^{(n)}\}_{n=1,2,\dots} \subset L^1(\mathbb{S}; \mu)$; see Corollaries 2.4, 2.7 and Remark 2.8 for details. As explained in Remark 2.6 and demonstrated in Example 4.1, Theorem 2.1 can be used to improve the lower bounds for the lower limit of integrals provided by inequality (2.6) in Fatou's lemma.

If $\{f, f^{(n)}\}_{n=1,2,\dots} \subset L^1(\mathbb{S}; \mu)$, then for $\mu^{(n)} = \mu$, $n = 1, 2, \dots$, inequality (2.1) is the uniform version of inequality (2.6) of Fatou's lemma. There are generalized versions of Fatou's lemma for weakly and setwise converging sequences of measures; see Royden [18, p. 231], Serfozo [21], Feinberg et al. [10], and the references in [10]. In particular, according to Royden [18, p. 231], for measures $\mu^{(n)}$ converging to μ setwise, Fatou's lemma has the same formulation as the classic Fatou lemma with inequality (2.6) modified to

$$\liminf_{n \rightarrow \infty} \int_{\mathbb{S}} f^{(n)}(s) \mu^{(n)}(ds) \geq \int_{\mathbb{S}} f(s) \mu(ds). \quad (2.7)$$

We remark that in the case of weak convergence, a stronger definition of \liminf is required, and \mathbb{S} is assumed to be a metric space with Σ being its Borel σ -field.

Theorem 2.1 provides the necessary and sufficient condition for the validity of inequality (2.1), when variable finite measures $\{\mu^{(n)}\}_{n=1,2,\dots}$ converge in total variation to μ and the functions are integrable

with respect to the corresponding measures. We note that inequality (2.4) always holds for nonnegative functions $\{f^{(n)}\}_{n=1,2,\dots}$ and inequality (2.5) implies statement (i) from Theorem 2.1, but not vice versa (see Example 4.1). Thus the sufficient statement of Theorem 2.1 claims a stronger inequality than those in Fatou's lemma under weaker assumptions. Example 4.1 demonstrates that Theorem 2.1 indeed can imply a stronger inequality than the inequality in Fatou's lemma. Examples 4.2 – 4.4 demonstrate that the assumption that the convergence of $\{\mu^{(n)}\}_{n=1,2,\dots} \subset \mathcal{M}(\mathbb{S})$ to a measure μ on \mathbb{S} takes place in total variation is essential and cannot be relaxed to setwise convergence.

Theorem 2.1 implies the following two corollaries.

Corollary 2.3. (Uniform Fatou's Lemma for Variable Measures and Nonnegative Functions) *Let (\mathbb{S}, Σ) be a measurable space, the sequence $\{\mu^{(n)}\}_{n=1,2,\dots} \subset \mathcal{M}(\mathbb{S})$ converge in total variation to a measure μ on \mathbb{S} , $f^{(n)} \in L^1(\mathbb{S}; \mu^{(n)})$, $n = 1, 2, \dots$, be nonnegative functions and $f \in L^1(\mathbb{S}; \mu)$. Then inequality (2.1) holds if and only if statement (i) from Theorem 2.1 takes place.*

Proof. Corollary 2.3 follows directly from Theorem 2.1 because inequality (2.4) holds for the sequence of nonnegative functions $\{f^{(n)}\}_{n=1,2,\dots}$. \square

Corollary 2.4. (Uniform Fatou's Lemma for Unbounded Below Functions) *Let (\mathbb{S}, Σ) be a measurable space, $\mu \in \mathcal{M}(\mathbb{S})$, and $\{f, f^{(n)}\}_{n=1,2,\dots} \subset L^1(\mathbb{S}; \mu)$. Then the inequality*

$$\liminf_{n \rightarrow \infty} \inf_{S \in \Sigma} \left(\int_S f^{(n)}(s) \mu(ds) - \int_S f(s) \mu(ds) \right) \geq 0 \quad (2.8)$$

holds if and only if statement (i) from Theorem 2.1 takes place and

$$\liminf_{K \rightarrow +\infty} \inf_{n=1,2,\dots} \int_{\mathbb{S}} f^{(n)}(s) \mathbf{I}\{s \in \mathbb{S} : f^{(n)}(s) \leq -K\} \mu(ds) \geq 0. \quad (2.9)$$

Proof. Corollary 2.4 follows directly from Theorem 2.1. \square

Remark 2.5. For each $a \in \mathbb{R}$ we denote $a^+ := \max\{a, 0\}$ and $a^- := a^+ - a$. Note that $a = a^+ - a^-$ and $|a| = a^+ + a^-$. For a measure μ on \mathbb{S} and functions $f, g \in L^1(\mathbb{S}; \mu)$,

$$\inf_{S \in \Sigma} \left(\int_S g(s) \mu(ds) - \int_S f(s) \mu(ds) \right) = - \int_{\mathbb{S}} (g(s) - f(s))^- \mu(ds) \leq 0.$$

Therefore, inequality (2.8) is equivalent to

$$\lim_{n \rightarrow \infty} \int_{\mathbb{S}} (f^{(n)}(s) - f(s))^- \mu(ds) = 0.$$

Remark 2.6. Let the conditions of Corollary 2.8 hold. Then inequality (2.8) can be rewritten as

$$\liminf_{n \rightarrow \infty} \int_{\mathbb{S}} f^{(n)}(s) \mu(ds) \geq \int_{\mathbb{S}} f(s) \mu(ds), \quad (2.10)$$

and for $f(s) := \liminf_{n \rightarrow \infty} f^{(n)}(s)$, $s \in \mathbb{S}$, formula (2.10) provides the lower bound for its left-hand side. However, as demonstrated in Example 4.1, it may be possible to choose a function $f \in L^1(\mathbb{S}; \mu)$ satisfying statement (i) from Theorem 2.1 and such that

$$\int_{\mathbb{S}} f(s) \mu(ds) > \int_{\mathbb{S}} \liminf_{n \rightarrow \infty} f^{(n)}(s) \mu(ds).$$

Therefore, the uniform Fatou lemma can be used to obtain better lower bounds for lower limits of integrals than those provided by Fatou's lemma.

Each of the Corollaries 2.3 and 2.4 implies the following statement.

Corollary 2.7. (Uniform Fatou's Lemma for Nonnegative Functions) *Let (\mathbb{S}, Σ) be a measurable space, $\mu \in \mathcal{M}(\mathbb{S})$, $f \in L^1(\mathbb{S}; \mu)$, and $\{f^{(n)}\}_{n=1,2,\dots} \subset L^1(\mathbb{S}; \mu)$ be a sequence of nonnegative functions. Then inequality (2.8) holds if and only if statement (i) from Theorem 2.1 takes place.*

Proof. Corollary 2.7 follows directly from Corollary 2.3. \square

Remark 2.8. Under the assumptions of Corollary 2.7, inequality (2.8) is equivalent to $(f^{(n)} - f)^- \xrightarrow{\mu} 0$ as $n \rightarrow \infty$. This follows from Remark 2.5, the dominated convergence theorem, Chebyshev's inequality, and because each function $(f^{(n)} - f)^-$, $n = 1, 2, \dots$, is bounded above by $f^+ \in L^1(\mathbb{S}, \mu)$. Statement (i) from Theorem 2.1 holds if and only if $(f^{(n)} - f)^- \xrightarrow{\mu} 0$, $n \rightarrow \infty$. Therefore, Corollary 2.7 also follows from classic results. Furthermore, the assumption, that the measure μ is finite, can be omitted from Corollary 2.7.

Corollary 2.9. (Uniform Dominated Convergence Theorem for Variable Measures) *Let (\mathbb{S}, Σ) be a measurable space, the sequence $\{\mu^{(n)}\}_{n=1,2,\dots} \subset \mathcal{M}(\mathbb{S})$ converge in total variation to a measure μ on \mathbb{S} , $f \in L^1(\mathbb{S}; \mu)$, and $f^{(n)} \in L^1(\mathbb{S}; \mu^{(n)})$ for each $n = 1, 2, \dots$. Then the equality*

$$\lim_{n \rightarrow \infty} \sup_{S \in \Sigma} \left| \int_S f^{(n)}(s) \mu^{(n)}(ds) - \int_S f(s) \mu(ds) \right| = 0 \quad (2.11)$$

holds if and only if the following two statements hold:

- (i) *the sequence $\{f^{(n)}\}_{n=1,2,\dots}$ converges in measure μ to f , and, therefore, there is a subsequence $\{f^{(n_k)}\}_{k=1,2,\dots} \subseteq \{f^{(n)}\}_{n=1,2,\dots}$ that converges μ -a.e. to f ;*
- (ii) *the following equality holds:*

$$\lim_{K \rightarrow +\infty} \sup_{n=1,2,\dots} \int_{\mathbb{S}} |f^{(n)}(s)| \mathbf{I}\{|f^{(n)}(s)| \geq K\} \mu^{(n)}(ds) = 0. \quad (2.12)$$

Proof. Theorem 2.1, being applied to the functions $\{f, f^{(n)}\}_{n=1,2,\dots}$ and $\{-f, -f^{(n)}\}_{n=1,2,\dots}$, yields Corollary 2.9. \square

We remark that, for uniformly bounded functions $\{f^{(n)}\}_{n=1,2,\dots}$, condition (ii) from Corollary 2.9 always holds and therefore is not needed. The necessary part of Corollary 2.9 for probability measures $\{\mu^{(n)}, \mu\}_{n=1,2,\dots}$ and uniformly bounded measurable functions $\{f^{(n)}, f\}_{n=1,2,\dots}$, defined on a standard Borel space \mathbb{S} , was introduced in Feinberg et al. [11, Theorem 5.2]. This necessary condition was used in Feinberg et al. [11, 12] for the analysis of control problems with incomplete observations. Theorem 5.2 from [11] can be interpreted as a converse to a version of Lebesgue's dominated convergence theorem for a sequence of measures converging in total variation. This was the starting point for formulating and investigating the uniform Fatou lemma.

Corollary 2.10. (Uniform Dominated Convergence Theorem) *Let (\mathbb{S}, Σ) be a measurable space, $\mu \in \mathcal{M}(\mathbb{S})$, and $\{f, f^{(n)}\}_{n=1,2,\dots} \subset L^1(\mathbb{S}; \mu)$. Then the equality*

$$\lim_{n \rightarrow \infty} \sup_{S \in \Sigma} \left| \int_S f^{(n)}(s) \mu(ds) - \int_S f(s) \mu(ds) \right| = 0 \quad (2.13)$$

holds if and only if statement (i) from Corollary 2.9 holds and the sequence $\{f^{(n)}\}_{n=1,2,\dots}$ is uniformly integrable, that is,

$$\lim_{K \rightarrow +\infty} \sup_{n=1,2,\dots} \int_{\mathbb{S}} |f^{(n)}(s)| \mathbf{I}\{s \in \mathbb{S} : |f^{(n)}(s)| \geq K\} \mu(ds) = 0. \quad (2.14)$$

Proof. Corollary 2.4, applied to the functions $\{f, f^{(n)}\}_{n=1,2,\dots}$ and $\{-f, -f^{(n)}\}_{n=1,2,\dots}$, yields Corollary 2.10. \square

Remark 2.11. Under the assumptions of Corollary 2.10

$$\begin{aligned} & \sup_{S \in \Sigma} \left| \int_S f^{(n)}(s) \mu(ds) - \int_S f(s) \mu(ds) \right| \\ &= \max \left\{ \int_{\mathbb{S}} (f^{(n)}(s) - f(s))^- \mu(ds), \int_{\mathbb{S}} (f^{(n)}(s) - f(s))^+ \mu(ds) \right\} \geq 0. \end{aligned}$$

Therefore, equality (2.13) is equivalent to

$$\lim_{n \rightarrow \infty} \int_{\mathbb{S}} |f^{(n)}(s) - f(s)| \mu(ds) = 0,$$

and Corollary 2.10 coincides with the classic criterion of strong convergence in $L^1(\mathbb{S}; \mu)$.

The following two corollaries describe the relation between convergence properties of a sequence of finite signed measures $\{\tilde{\mu}^{(n)}\}_{n=1,2,\dots}$ and the sequence of their Radon-Nikodym derivatives $\{\frac{d\tilde{\mu}^{(n)}}{d\mu^{(n)}}\}_{n=1,2,\dots}$ with respect to finite measures $\{\mu^{(n)}\}_{n=1,2,\dots}$ converging in total variation.

Corollary 2.12. *Let (\mathbb{S}, Σ) be a measurable space, $\{\mu^{(n)}\}_{n=1,2,\dots} \subset \mathcal{M}(\mathbb{S})$, μ be a measure on \mathbb{S} , and $\{\tilde{\mu}, \tilde{\mu}^{(n)}\}_{n=1,2,\dots}$ be a sequence of finite signed measures on \mathbb{S} . Assume that $\tilde{\mu} \ll \mu$ and $\tilde{\mu}^{(n)} \ll \mu^{(n)}$ for each $n = 1, 2, \dots$. If the sequence $\{\mu^{(n)}\}_{n=1,2,\dots}$ converges in total variation to μ , then the inequality*

$$\liminf_{n \rightarrow \infty} \inf_{S \in \Sigma} \left(\tilde{\mu}^{(n)}(S) - \tilde{\mu}(S) \right) \geq 0$$

holds if and only if the following two statements hold:

(i) *for each $\varepsilon > 0$*

$$\mu(\{s \in \mathbb{S} : \frac{d\tilde{\mu}^{(n)}}{d\mu^{(n)}}(s) \leq \frac{d\tilde{\mu}}{d\mu}(s) - \varepsilon\}) \rightarrow 0 \text{ as } n \rightarrow \infty,$$

and, therefore, there exists a subsequence $\{\frac{d\tilde{\mu}^{(n_k)}}{d\mu^{(n_k)}}\}_{k=1,2,\dots} \subseteq \{\frac{d\tilde{\mu}^{(n)}}{d\mu^{(n)}}\}_{n=1,2,\dots}$ such that

$$\liminf_{k \rightarrow \infty} \frac{d\tilde{\mu}^{(n_k)}}{d\mu^{(n_k)}}(s) \geq \frac{d\tilde{\mu}}{d\mu}(s) \quad \text{for } \mu\text{-a.e. } s \in \mathbb{S};$$

(ii) the inequality

$$\liminf_{K \rightarrow +\infty} \inf_{n=1,2,\dots} \tilde{\mu}^{(n)}(\{s \in \mathbb{S} : \frac{d\tilde{\mu}^{(n)}}{d\mu^{(n)}}(s) \leq -K\}) \geq 0.$$

holds.

Proof. If $\nu \in \mathcal{M}(\mathbb{S})$, $\tilde{\nu}$ be a finite signed measure on \mathbb{S} , and $\tilde{\nu} \ll \nu$, then the Radon-Nikodym derivative $\frac{d\tilde{\nu}}{d\nu}$ is μ -integrable, that is, $\frac{d\tilde{\nu}}{d\nu} \in L^1(\mathbb{S}; \nu)$. This is true because $\int_{\mathbb{S}} |\frac{d\tilde{\nu}}{d\nu}| d\nu = \|\nu\| < \infty$. Set $f := \frac{d\tilde{\mu}}{d\mu}$, $f^{(n)} := \frac{d\tilde{\mu}^{(n)}}{d\mu^{(n)}}$, $n = 1, 2, \dots$. Then Theorem 2.1 yields Corollary 2.12. \square

We remark that, if $\{\tilde{\mu}^{(n)}\}_{n=1,2,\dots} \subset \mathcal{M}(\mathbb{S})$, then statement (ii) of Corollary 2.12 always holds because $\tilde{\mu}^{(n)}(\cdot) \geq 0$ for all $n = 1, 2, \dots$. Corollary 2.12 implies the following necessary and sufficient condition for the convergence in total variation of finite signed measures $\{\tilde{\mu}^{(n)}\}_{n=1,2,\dots}$.

Corollary 2.13. *Let (\mathbb{S}, Σ) be a measurable space, $\{\mu^{(n)}\}_{n=1,2,\dots} \subset \mathcal{M}(\mathbb{S})$, μ be a measure on \mathbb{S} , and $\{\tilde{\mu}, \tilde{\mu}^{(n)}\}_{n=1,2,\dots}$ be a sequence of finite signed measures on \mathbb{S} . Assume that $\tilde{\mu} \ll \mu$ and $\tilde{\mu}^{(n)} \ll \mu^{(n)}$ for each $n = 1, 2, \dots$. If the sequence $\{\mu^{(n)}\}_{n=1,2,\dots}$ converges in total variation to μ , then the sequence $\{\tilde{\mu}^{(n)}\}_{n=1,2,\dots}$ converges in total variation to $\tilde{\mu}$, that is,*

$$\lim_{n \rightarrow \infty} \sup_{S \in \Sigma} |\tilde{\mu}^{(n)}(S) - \tilde{\mu}(S)| = 0,$$

if and only if the following two statements hold:

(i) the sequence $\{\frac{d\tilde{\mu}^{(n)}}{d\mu^{(n)}}\}_{n=1,2,\dots}$ converges in measure μ to $\frac{d\tilde{\mu}}{d\mu}$, and, therefore, there exists a subsequence $\{\frac{d\tilde{\mu}^{(n_k)}}{d\mu^{(n_k)}}\}_{k=1,2,\dots} \subseteq \{\frac{d\tilde{\mu}^{(n)}}{d\mu^{(n)}}\}_{n=1,2,\dots}$ that converges μ -a.e. to $\frac{d\tilde{\mu}}{d\mu}$;

(ii) the following inequality holds:

$$\lim_{K \rightarrow +\infty} \sup_{n=1,2,\dots} |\tilde{\mu}^{(n)}|(\{s \in \mathbb{S} : |\frac{d\tilde{\mu}^{(n)}}{d\mu^{(n)}}(s)| \geq K\}) = 0,$$

where $|\tilde{\mu}^{(n)}|(S) = \int_S |\frac{d\tilde{\mu}^{(n)}}{d\mu^{(n)}}(s)| \mu^{(n)}(ds)$, $S \in \Sigma$.

Proof. Applying Corollary 2.12 to $\{\tilde{\mu}, \tilde{\mu}^{(n)}, \mu, \mu^{(n)}\}_{n=1,2,\dots}$ and $\{-\tilde{\mu}, -\tilde{\mu}^{(n)}, \mu, \mu^{(n)}\}_{n=1,2,\dots}$ yields this corollary. \square

3 Proof of Theorem 2.1

For a measurable function $g : \mathbb{S} \rightarrow \overline{\mathbb{R}}$, real number K , and set $S \in \Sigma$, we denote:

$$S_{g \geq K} := \{s \in S : g(s) \geq K\}, \quad S_{g > K} := \{s \in S : g(s) > K\},$$

$$S_{g \leq K} := \{s \in S : g(s) \leq K\}, \quad S_{g < K} := \{s \in S : g(s) < K\}.$$

The proof of Theorem 2.1 consists of four auxiliary lemmas.

Lemma 3.1. Let (\mathbb{S}, Σ) be a measurable space, $\{f^{(n)}, f\}_{n=1,2,\dots}$ be a sequence of measurable functions, μ be a measure on \mathbb{S} , and (2.2) hold for each $\varepsilon > 0$. Then there exists a subsequence $\{f^{(n_k)}\}_{k=1,2,\dots} \subseteq \{f^{(n)}\}_{n=1,2,\dots}$ such that (2.3) holds.

Proof. Fix an arbitrary $\varepsilon > 0$. According to (2.2), there exists a sequence $\{n_k\}_{k=1,2,\dots}$ such that $\mu(\mathbb{S}_{f-f^{(n_k)} \geq \varepsilon}) \leq 2^{-k}$, $k = 1, 2, \dots$. Thus,

$$\mu(\cup_{k=K}^{\infty} \mathbb{S}_{f-f^{(n_k)} \geq \varepsilon}) \leq \sum_{k=K}^{\infty} \mu(\mathbb{S}_{f-f^{(n_k)} \geq \varepsilon}) \leq \sum_{k=K}^{\infty} 2^{-k} \leq 2^{-K+1},$$

$K = 1, 2, \dots$. Therefore, $\mu(\cap_{K=1}^{\infty} \cup_{k=K}^{\infty} \mathbb{S}_{f-f^{(n_k)} \geq \varepsilon}) = 0$, that is, for each $\varepsilon > 0$

$$\mu(\{s \in \mathbb{S} : \liminf_{k \rightarrow \infty} f^{(n_k)}(s) \leq f(s) - \varepsilon\}) = 0.$$

Thus, if (2.2) holds for each $\varepsilon > 0$, then (2.3) holds. \square

Lemma 3.2. Let the assumptions of Theorem 2.1 hold. Then inequality (2.1) implies statement (i) from Theorem 2.1.

Remark 3.3. The proof is based on a contradiction. The starting observation is that, if statement (i) from Theorem 2.1 does not hold, then inequality (3.1) holds for some $\varepsilon^*, \delta^* > 0$ and an increasing sequence $\{n_k\}_{k=1,2,\dots} \subset \mathbb{N}$. The rest of the proof is divided into several steps. *Step 1* shows that inequality (3.5) holds for large k . For this purpose inequalities (3.3) and (3.4) are established. *Step 2* proves that for large k inequality (3.6) follows from (2.1). *Step 3* establishes (3.7) as a corollary from (3.5) and (3.6). *Step 4* concludes that the convergence $\{\mu^{(n)}\}_{n=1,2,\dots} \subset \mathcal{M}(\mathbb{S})$ in total variation to the measure μ contradicts (3.7).

Proof of Lemma 3.2. Let statement (i) from Theorem 2.1 does not hold, then there exist a sequence $\{n_k \uparrow \infty\}_{k=1,2,\dots}$ and positive constants ε^* and δ^* such that

$$\mu(\mathbb{S}_{f-f^{(n_k)} \geq \varepsilon^*}) \geq \delta^*, \quad k = 1, 2, \dots \quad (3.1)$$

Since the sequence of finite measures $\{\mu^{(n)}\}_{n=1,2,\dots}$ converges in total variation to the finite measure μ , there exists $K_1 = 1, 2, \dots$, such that

$$\sup_{S \in \Sigma} |\mu^{(n_k)}(S) - \mu(S)| \leq \frac{\delta^*}{4}, \quad k = K_1, K_1 + 1, \dots \quad (3.2)$$

Therefore, inequalities (3.1) and (3.2) yield that

$$\mu^{(n_k)}(\mathbb{S}_{f-f^{(n_k)} \geq \varepsilon^*}) \geq \frac{3\delta^*}{4}, \quad k = K_1, K_1 + 1, \dots \quad (3.3)$$

Let us set $C := \int_{\mathbb{S}} |f(s)| \mu(ds)$. Note that $C < \infty$, because $f \in L^1(\mathbb{S}; \mu)$. Chebyshev's inequality yields that $\mu(\mathbb{S}_{|f| \geq M}) \leq \frac{C}{M}$ for each $M > 0$. Thus, inequality (3.2) implies

$$\mu^{(n_k)}(\mathbb{S}_{|f| \geq \frac{4C}{\delta^*}}) \leq \frac{\delta^*}{2}, \quad k = K_1, K_1 + 1, \dots \quad (3.4)$$

Moreover, inequalities (3.3) and (3.4) yield

$$\mu^{(n_k)}(\mathbb{S}_{f-f^{(n_k)} \geq \varepsilon^*} \setminus \mathbb{S}_{|f| \geq \frac{4C}{\delta^*}}) \geq \frac{\delta^*}{4}, \quad k = K_1, K_1 + 1, \dots \quad (3.5)$$

Indeed, for $k = K_1, K_1 + 1, \dots$,

$$\begin{aligned} \frac{3\delta^*}{4} &\leq \mu^{(n_k)}(\mathbb{S}_{f-f^{(n_k)} \geq \varepsilon^*}) \leq \mu^{(n_k)}(\mathbb{S}_{|f| \geq \frac{4C}{\delta^*}}) + \mu^{(n_k)}(\mathbb{S}_{f-f^{(n_k)} \geq \varepsilon^*} \setminus \mathbb{S}_{|f| \geq \frac{4C}{\delta^*}}) \\ &\leq \frac{\delta^*}{2} + \mu^{(n_k)}(\mathbb{S}_{f-f^{(n_k)} \geq \varepsilon^*} \setminus \mathbb{S}_{|f| \geq \frac{4C}{\delta^*}}), \end{aligned}$$

where the first inequality follows from (3.3), the second inequality follows from subadditivity of the finite measure $\mu^{(n_k)}$, and the third inequality follows from (3.4). Inequality (2.1) implies the existence of $K_2 = K_1, K_1 + 1, \dots$ such that

$$\begin{aligned} &\int_{\mathbb{S}_{f-f^{(n_k)} \geq \varepsilon^*} \setminus \mathbb{S}_{|f| \geq \frac{4C}{\delta^*}}} f^{(n_k)}(s) \mu^{(n_k)}(ds) \\ &\quad - \int_{\mathbb{S}_{f-f^{(n_k)} \geq \varepsilon^*} \setminus \mathbb{S}_{|f| \geq \frac{4C}{\delta^*}}} f(s) \mu(ds) \geq -\frac{\varepsilon^* \delta^*}{8} \end{aligned} \quad (3.6)$$

for each $k = K_2, K_2 + 1, \dots$. The definition of $\mathbb{S}_{f-f^{(n_k)} \geq \varepsilon^*}$ and inequalities (3.5) and (3.6) yield that for each $k = K_2, K_2 + 1, \dots$

$$\begin{aligned} -\frac{\varepsilon^* \delta^*}{8} &\leq \int_{\mathbb{S}_{f-f^{(n_k)} \geq \varepsilon^*} \setminus \mathbb{S}_{|f| \geq \frac{4C}{\delta^*}}} f(s) \mu^{(n_k)}(ds) \\ &\quad - \int_{\mathbb{S}_{f-f^{(n_k)} \geq \varepsilon^*} \setminus \mathbb{S}_{|f| \geq \frac{4C}{\delta^*}}} f(s) \mu(ds) - \varepsilon^* \mu^{(n_k)}(\mathbb{S}_{f-f^{(n_k)} \geq \varepsilon^*} \setminus \mathbb{S}_{|f| \geq \frac{4C}{\delta^*}}) \\ &\leq \int_{\mathbb{S}_{f-f^{(n_k)} \geq \varepsilon^*} \setminus \mathbb{S}_{|f| \geq \frac{4C}{\delta^*}}} f(s) \mu^{(n_k)}(ds) - \int_{\mathbb{S}_{f-f^{(n_k)} \geq \varepsilon^*} \setminus \mathbb{S}_{|f| \geq \frac{4C}{\delta^*}}} f(s) \mu(ds) - \frac{\varepsilon^* \delta^*}{4}. \end{aligned}$$

Therefore, for each $k = K_2, K_2 + 1, \dots$,

$$\begin{aligned} &\int_{\mathbb{S}} f(s) \mathbf{I} \left\{ s \in \mathbb{S}_{f-f^{(n_k)} \geq \varepsilon^*} \setminus \mathbb{S}_{|f| \geq \frac{4C}{\delta^*}} \right\} \mu^{(n_k)}(ds) \\ &\quad - \int_{\mathbb{S}} f(s) \mathbf{I} \left\{ s \in \mathbb{S}_{f-f^{(n_k)} \geq \varepsilon^*} \setminus \mathbb{S}_{|f| \geq \frac{4C}{\delta^*}} \right\} \mu(ds) \geq \frac{\varepsilon^* \delta^*}{8}. \end{aligned} \quad (3.7)$$

Since each function $s \rightarrow f(s) \mathbf{I} \left\{ s \in \mathbb{S}_{f-f^{(n_k)} \geq \varepsilon^*} \setminus \mathbb{S}_{|f| \geq \frac{4C}{\delta^*}} \right\}$, $k = K_2, K_2 + 1, \dots$, is measurable and absolutely bounded by the constant $\frac{4C}{\delta^*}$ and the sequence of finite measures $\{\mu^{(n)}\}_{n=1,2,\dots}$ converges in total variation to $\mu \in \mathcal{M}(\mathbb{S})$,

$$\begin{aligned} &\int_{\mathbb{S}} f(s) \mathbf{I} \left\{ s \in \mathbb{S}_{f-f^{(n_k)} \geq \varepsilon^*} \setminus \mathbb{S}_{|f| \geq \frac{4C}{\delta^*}} \right\} \mu^{(n_k)}(ds) \\ &\quad - \int_{\mathbb{S}} f(s) \mathbf{I} \left\{ s \in \mathbb{S}_{f-f^{(n_k)} \geq \varepsilon^*} \setminus \mathbb{S}_{|f| \geq \frac{4C}{\delta^*}} \right\} \mu(ds) \rightarrow 0, \quad k \rightarrow \infty. \end{aligned}$$

This contradicts (3.7). Therefore, inequality (2.1) implies statement (i) of Theorem 2.1. \square

Lemma 3.4. Let (\mathbb{S}, Σ) be a measurable space, $\{\mu^{(n)}, \mu\}_{n=1,2,\dots} \subset \mathcal{M}(\mathbb{S})$, $f \in L^1(\mathbb{S}; \mu)$, and $f^{(n)} \in L^1(\mathbb{S}; \mu^{(n)})$, for each $n = 1, 2, \dots$. Then, inequality (2.1) and statement (i) from Theorem 2.1 imply statement (ii) from Theorem 2.1.

Remark 3.5. According to Lemmas 3.2 and 3.4, if the assumptions of Theorem 2.1 hold, then inequality (2.1) implies statements (i) and (ii) from Theorem 2.1.

Remark 3.6. The proof of Lemma 3.4 is divided into several steps. *Step 1* shows that it is sufficient to justify the existence of $N = 1, 2, \dots$ such that inequality (3.9) holds for each $\varepsilon > 0$. *Step 2* derives inequality (3.10) from (2.1). *Step 3* provides a lower estimate for the right hand-side of (3.10) that follows from (3.11). *Step 4* shows that each term in the right hand-side of (3.11) is small; see (3.12) and (3.13). *Step 5* establishes the required statement regarding inequality (3.9) as a corollary from (3.10) – (3.13).

Proof of Lemma 3.4. For each $Q \in \mathcal{M}(\mathbb{S})$ and $g \in L^1(\mathbb{S}; Q)$,

$$\int_{\mathbb{S}_{g \leq -K}} g(s)Q(ds) \rightarrow 0 \text{ as } K \rightarrow +\infty. \quad (3.8)$$

Therefore, statement (ii) of Theorem 2.1 is equivalent to the existence of a natural number N such that for each $\varepsilon > 0$

$$\liminf_{K \rightarrow +\infty} \inf_{n=N, N+1, \dots} \int_{\mathbb{S}_{f^{(n)} \leq -K}} f^{(n)}(s)\mu^{(n)}(ds) \geq -\varepsilon. \quad (3.9)$$

Let us fix an arbitrary $\varepsilon > 0$ and verify (3.9). According to inequality (2.1), there exists $N_1 = 1, 2, \dots$ such that for $n = N_1, N_1 + 1, \dots$

$$\inf_{S \in \Sigma} \left(\int_S f^{(n)}(s)\mu^{(n)}(ds) - \int_S f(s)\mu(ds) \right) \geq -\frac{\varepsilon}{2}.$$

Then, for $n = N_1, N_1 + 1, \dots$ and $K > 0$,

$$\int_{\mathbb{S}_{f^{(n)} \leq -K}} f^{(n)}(s)\mu^{(n)}(ds) \geq \int_{\mathbb{S}_{f \leq -K}} f(s)\mu(ds) - \frac{\varepsilon}{2}. \quad (3.10)$$

Direct calculations imply that, for $n = N_1, N_1 + 1, \dots$ and for $K > 0$,

$$\begin{aligned} \int_{\mathbb{S}_{f^{(n)} \leq -K}} f(s)\mu(ds) &= \int_{\mathbb{S}_{f-f^{(n)} < 1} \cap \mathbb{S}_{f^{(n)} \leq -K}} f(s)\mu(ds) \\ &+ \int_{\mathbb{S}_{f-f^{(n)} \geq 1} \cap \mathbb{S}_{f^{(n)} \leq -K}} f(s)\mu(ds) \geq - \int_{\mathbb{S}_{f \leq 1-K}} |f(s)|\mu(ds) \\ &- \int_{\mathbb{S}_{f^{(n)}-f \leq -1}} |f(s)|\mu(ds), \end{aligned} \quad (3.11)$$

where the inequality holds because $\mathbb{S}_{f-f^{(n)} < 1} \cap \mathbb{S}_{f^{(n)} \leq -K} \subseteq \mathbb{S}_{f \leq 1-K}$ and $\mathbb{S}_{f-f^{(n)} \geq 1} \cap \mathbb{S}_{f^{(n)} \leq -K} \subseteq \mathbb{S}_{f^{(n)}-f \leq -1}$. Due to (3.8)

$$\int_{\mathbb{S}_{f \leq -K+1}} |f(s)|\mu(ds) \rightarrow 0 \text{ as } K \rightarrow +\infty. \quad (3.12)$$

Statement (i) of Theorem 2.1 yields that $\mu(\mathbb{S}_{f^{(n)}-f \leq -1}) \rightarrow 0$ as $n \rightarrow \infty$. Therefore, since $f \in L^1(\mathbb{S}; \mu)$, there exists $N_2 = N_1, N_1 + 1, \dots$ such that

$$\int_{\mathbb{S}_{f^{(n)}-f \leq -1}} |f(s)| \mu(ds) \leq \frac{\varepsilon}{2}, \quad n = N_2, N_2 + 1, \dots \quad (3.13)$$

Thus (3.10) – (3.13) imply the existence of a natural number N such that for each $\varepsilon > 0$ (3.9) holds. Therefore, inequality (2.1) and statement (i) from Theorem 2.1 imply statement (ii) from Theorem 2.1. \square

Lemma 3.7. *Let the assumptions of Theorem 2.1 hold. Then statements (i) and (ii) from Theorem 2.1 yield inequality (2.1).*

Remark 3.8. The proof of Lemma 3.7 is divided into several steps. *Step 1* shows that it is sufficient to justify inequality (3.19). *Step 2* provides a lower estimate for $I(n, K)$; see inequality (3.20). *Step 3* estimates the asymptotic behavior of each term from the right hand-side of inequality (3.20) as $K \rightarrow +\infty$ and $n \rightarrow \infty$; see equality (3.21) and inequalities (3.22) and (3.23).

Proof of Lemma 3.7. The additivity of integrals and the property, that an infimum of a sum of two functions is greater than or equal to the sum of infimums, imply that, for $n = 1, 2, \dots$ and $K > 0$,

$$\begin{aligned} & \inf_{S \in \Sigma} \left(\int_S f^{(n)}(s) \mu^{(n)}(ds) - \int_S f(s) \mu(ds) \right) \\ & \geq \inf_{S \in \Sigma} \left(\int_{S_{f^{(n)} \leq -K}} f^{(n)}(s) \mu^{(n)}(ds) - \int_{S_{f^{(n)} \leq -K}} f(s) \mu(ds) \right) \\ & + \inf_{S \in \Sigma} \left(\int_{S_{f^{(n)} > -K}} f^{(n)}(s) \mu^{(n)}(ds) - \int_{S_{f^{(n)} > -K}} f(s) \mu(ds) \right), \end{aligned} \quad (3.14)$$

Note that, for $n = 1, 2, \dots$ and $K > 0$,

$$\begin{aligned} & \inf_{S \in \Sigma} \left(\int_{S_{f^{(n)} \leq -K}} f^{(n)}(s) \mu^{(n)}(ds) - \int_{S_{f^{(n)} \leq -K}} f(s) \mu(ds) \right) \\ & \geq \inf_{n=1,2,\dots} \int_{\mathbb{S}_{f^{(n)} \leq -K}} f^{(n)}(s) \mu^{(n)}(ds) - \int_{\mathbb{S}_{f^{(n)} \leq -K}} |f(s)| \mu(ds). \end{aligned} \quad (3.15)$$

Moreover, for $n = 1, 2, \dots$ and $K > 0$,

$$\int_{\mathbb{S}_{f^{(n)} \leq -K}} |f(s)| \mu(ds) \leq \int_{\mathbb{S}_{f \leq -K+1}} |f(s)| \mu(ds) + \int_{\mathbb{S}_{f-f^{(n)} \geq 1}} |f(s)| \mu(ds), \quad (3.16)$$

because, if $f^{(n)}(s) \leq -K$ and $f(s) > -K + 1$, then $f^{(n)}(s) < f(s) - 1$ and, thus, $f^{(n)}(s) \leq f(s) - 1$.

Since $f \in L^1(\mathbb{S}; \mu)$, then $\mu(\mathbb{S}_{f \leq -K+1}) \rightarrow 0$ as $K \rightarrow +\infty$. Therefore,

$$\int_{\mathbb{S}_{f \leq -K+1}} |f(s)| \mu(ds) \rightarrow 0 \text{ as } K \rightarrow +\infty. \quad (3.17)$$

Due to (2.2), $\mu(\mathbb{S}_{f-f^{(n)} \geq 1}) \rightarrow 0$ as $n \rightarrow \infty$. Similar to (3.17),

$$\int_{\mathbb{S}_{f-f^{(n)} \geq 1}} |f(s)| \mu(ds) \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (3.18)$$

According to (3.14) – (3.18), inequality (2.1) follows from statements (i) and (ii) of Theorem 2.1, if

$$\liminf_{K \rightarrow +\infty} \liminf_{n \rightarrow \infty} I(n, K) \geq 0, \quad (3.19)$$

where, for $n = 1, 2, \dots$ and $K > 0$,

$$I(n, K) := \inf_{S \in \Sigma} \left(\int_{S_{f^{(n)} > -K}} f^{(n)}(s) \mu^{(n)}(ds) - \int_{S_{f > -K}} f(s) \mu(ds) \right).$$

The rest of the proof establishes inequality (3.19). We observe that for each $K > 0$

$$I(n, K) \geq I_1(n, K) + I_2(n, K) + I_3(n, K), \quad n = 1, 2, \dots, \quad (3.20)$$

where

$$\begin{aligned} I_1(n, K) &= \inf_{S \in \Sigma} \left(\int_{S_{|f^{(n)}| < K}} f^{(n)}(s) \mu^{(n)}(ds) - \int_{S_{|f^{(n)}| < K}} f^{(n)}(s) \mu(ds) \right), \\ I_2(n, K) &= \inf_{S \in \Sigma} \left(\int_{S_{|f^{(n)}| < K}} f^{(n)}(s) \mu(ds) - \int_{S_{|f^{(n)}| < K}} f(s) \mu(ds) \right), \\ I_3(n, K) &= \inf_{S \in \Sigma} \left(\int_{S_{f^{(n)} \geq K}} f^{(n)}(s) \mu^{(n)}(ds) - \int_{S_{f^{(n)} \geq K}} f(s) \mu(ds) \right). \end{aligned}$$

Since $\{\mu^{(n)}\}_{n=1,2,\dots}$ converges in total variation to μ , then $I_1(n, K) \rightarrow 0$ as $n \rightarrow \infty$ for each $K > 0$. Therefore,

$$\liminf_{K \rightarrow +\infty} \liminf_{n \rightarrow \infty} I_1(n, K) = 0. \quad (3.21)$$

For $n = 1, 2, \dots$, $K > 0$, and $\varepsilon > 0$, the following inequalities hold:

$$\begin{aligned} I_2(n, K) &\geq \inf_{S \in \Sigma} \int_{S_{|f^{(n)}| < K} \cap S_{f^{(n)} - f > -\varepsilon}} (f^{(n)}(s) - f(s)) \mu(ds) \\ &\quad + \inf_{S \in \Sigma} \int_{S_{|f^{(n)}| < K} \cap S_{f - f^{(n)} > \varepsilon}} (f^{(n)}(s) - f(s)) \mu(ds) \\ &\geq -\varepsilon \mu(\mathbb{S}) - \int_{S_{|f^{(n)}| < K} \cap S_{f - f^{(n)} > \varepsilon}} |f^{(n)}(s)| \mu(ds) - \int_{S_{f - f^{(n)} > \varepsilon}} |f(s)| \mu(ds). \end{aligned}$$

and, therefore,

$$I_2(n, K) \geq -\varepsilon \mu(\mathbb{S}) - K \mu(S_{f - f^{(n)} > \varepsilon}) - \int_{S_{f - f^{(n)} > \varepsilon}} |f(s)| \mu(ds).$$

Thus, due to (2.2) and $f \in L^1(\mathbb{S}; \mu)$,

$$\liminf_{K \rightarrow +\infty} \liminf_{n \rightarrow \infty} I_2(n, K) \geq 0. \quad (3.22)$$

For $n = 1, 2, \dots$ and $K > 0$, the following inequalities hold:

$$\begin{aligned} I_3(n, K) &\geq K \left(\mu^{(n)}(S_{f^{(n)} \geq K} \cap S_{f(s) \leq K}) - \mu(S_{f^{(n)} \geq K} \cap S_{f(s) \leq K}) \right) \\ &\quad - \sup_{S \in \Sigma} \int_{S_{f^{(n)} \geq K} \cap S_{f(s) > K}} f(s) \mu(ds) \geq -K \sup_{S \in \Sigma} |\mu^{(n)}(S) - \mu(S)| - \int_{S_{f(s) > K}} f(s) \mu(ds). \end{aligned}$$

Therefore, since the sequence $\{\mu^{(n)}\}_{n=1,2,\dots}$ converges in total variation to μ and $f \in L^1(\mathbb{S})$,

$$\liminf_{K \rightarrow +\infty} \liminf_{n \rightarrow \infty} I_3(n, K) \geq 0. \quad (3.23)$$

Inequalities (3.20)–(3.23) yield (3.19). Therefore, statements (i) and (ii) of Theorem 2.1 imply inequality (2.1). \square

Proof of Theorem 2.1. Theorem 2.1 follows directly from Lemmas 3.1–3.7; see also Remark 3.5. \square

4 Counterexamples

Example 4.1 describes a probability space $(\mathbb{S}, \Sigma, \mu)$ and a sequence $\{f, f^{(n)}\}_{n=1,2,\dots}$ of uniformly bounded nonnegative measurable functions on it such that: (a) $\{f, f^{(n)}\}_{n=1,2,\dots}$ satisfy inequality (2.8); (b) inequality (2.2) takes place for each $\varepsilon > 0$; (c) inequality (2.3) does not hold for the function f and the entire sequence $\{f^{(n)}\}_{n=1,2,\dots}$. This example also demonstrates that Corollary 2.7 is essentially a more exact statement than the classic Fatou lemma.

Example 4.1. Let $\mathbb{S} = [0, 1]$, Σ be the Borel σ -field on \mathbb{S} , $\mu^{(n)} = \mu$ be the Lebesgue measure on \mathbb{S} , $f \equiv 1$, and $f^{(n)}(s) = 1 - \mathbf{I}\{s \in [\frac{j}{2^k}, \frac{j+1}{2^k}]\}$, where $k = \lfloor \log_2 n \rfloor$, $j = n - 2^k$, $s \in \mathbb{S}$, and $n = 1, 2, \dots$. Then

$$\lim_{n \rightarrow \infty} \int_{\mathbb{S}} (f^{(n)}(s) - f(s))^- \mu(ds) = \lim_{n \rightarrow \infty} \frac{1}{2^{\lfloor \log_2 n \rfloor}} = 0,$$

and, according to Remark 2.5, inequality (2.8) holds. Moreover, for each $\varepsilon > 0$

$$\mu(\{s \in \mathbb{S} : f^{(n)}(s) \leq f(s) - \varepsilon\}) = \frac{1}{2^{\lfloor \log_2 n \rfloor}} \rightarrow 0 \text{ as } n \rightarrow \infty,$$

that is, convergence in (2.2) takes place for each $\varepsilon > 0$. Moreover,

$$\liminf_{n \rightarrow \infty} f^{(n)}(s) = 0 < 1 = f(s) \text{ for } \mu\text{-a.e. } s \in \mathbb{S},$$

that is, inequality (2.3) does not hold for the function f and for the entire sequence $\{f^{(n)}\}_{n=1,2,\dots}$.

Corollary 2.7 yields

$$1 = \liminf_{n \rightarrow \infty} \int_{\mathbb{S}} f^{(n)}(s) \mu(ds) \geq \int_{\mathbb{S}} f(s) \mu(ds) = 1;$$

see equality (2.8) and Remark 2.5. But the classic Fatou lemma implies

$$1 = \liminf_{n \rightarrow \infty} \int_{\mathbb{S}} f^{(n)}(s) \mu(ds) \geq \int_{\mathbb{S}} \liminf_{n \rightarrow \infty} f^{(n)}(s) \mu(ds) = 0.$$

Therefore, Corollary 2.7 is a more exact statement than the classic Fatou lemma. \square

The following three examples demonstrate that the uniform Fatou lemma does not hold if convergence of measures in total variation is relaxed to setwise convergence. In particular, the necessary condition fails in Examples 4.2 and 4.3, and the sufficient condition fails in Example 4.4. As mentioned above, Fatou's lemma, which is a sufficient condition for inequality (2.5), which is weaker than inequality (2.1) in the

uniform Fatou lemma, holds for setwise converging measures and, if the notion of a limit of a function is appropriately modified, it also holds for weakly converging measures; see Royden [18, p. 231], Serfozo [21], Feinberg et al. [10], and references therein.

Example 4.2 demonstrates that, if convergence in total variation of finite measures $\{\mu^{(n)}\}_{n=1,2,\dots}$ to μ in Corollary 2.9 is relaxed to setwise convergence, equality (2.11) implies neither statement (i) nor statement (ii) from Theorem 2.1, and therefore neither statement (i) nor statement (ii) from Corollary 2.9 holds. Thus, inequality (2.1) does not yield either statement (i) or statement (ii) from Theorem 2.1, if the convergence in total variation of finite measures $\{\mu^{(n)}\}_{n=1,2,\dots}$ to μ in Theorem 2.1 is relaxed to setwise convergence.

Example 4.2. Let $\mathbb{S} = [0, 1]$, $\Sigma = \mathcal{B}(\mathbb{S})$ be a Borel σ -algebra on \mathbb{S} ,

$$g^{(n)}(s) := \begin{cases} \frac{1}{n}, & \text{if } 2k/2^n < s < (2k+1)/2^n \text{ for } k = 0, 1, \dots, 2^{n-1} - 1; \\ 2 - \frac{1}{n}, & \text{otherwise,} \end{cases}$$

$f^{(n)}(s) := -1/g^{(n)}(s)$, $s \in [0, 1]$, $n = 1, 2, \dots$, be the sequence of measurable functions, μ be the Lebesgue measure on $[0, 1]$, and $f \equiv -1$. Consider the sequence of probability measures $\mu^{(n)}$ on $[0, 1]$, $n = 1, 2, \dots$, defined as

$$\mu^{(n)}(S) := \int_S g^{(n)}(s) \mu(ds), \quad S \in \Sigma. \quad (4.1)$$

The sequence $\{\mu^{(n)}\}_{n=1,2,\dots}$ converges setwise to μ as $n \rightarrow \infty$. Indeed, according to Feinberg et al. [12, Theorem 2.3], measures $\mu^{(n)}$ converge setwise to the measure μ , if $\mu^{(n)}(C) \rightarrow \mu(C)$ for each open set C in $[0, 1]$. Since $\mu^{(n)}(\{0\}) = \mu(\{0\}) = \mu^{(n)}(\{1\}) = \mu(\{1\})$, $n = 1, 2, \dots$, then $\mu^{(n)}(C) \rightarrow \mu(C)$ for each open set C in $[0, 1]$ if and only if $\mu^{(n)}(C) \rightarrow \mu(C)$ for each open set C in $(0, 1)$. Choose an arbitrary open set C in $(0, 1)$. Then C is a union of a countable set of open disjoint intervals (a_i, b_i) . Therefore, for each $\varepsilon > 0$ there is a finite number n_ε of open intervals $\{(a_i, b_i) : i = 1, \dots, n_\varepsilon\}$ such that $\mu(C \setminus C_\varepsilon) \leq \varepsilon$, where $C_\varepsilon = \cup_{i=1}^{n_\varepsilon} (a_i, b_i)$. Due to $|g^{(n)}| \leq 2$, we obtain that $\mu^{(n)}(C \setminus C_\varepsilon) \leq 2\varepsilon$ for each $n = 1, 2, \dots$. Since $|\mu^{(n)}((a, b)) - \mu((a, b))| < 1/2^{n-1}$, $n = 1, 2, \dots$, for each interval $(a, b) \subseteq (0, 1)$, this implies that $|\mu(C_\varepsilon) - \mu^{(n)}(C_\varepsilon)| < \varepsilon$ if $n \geq N_\varepsilon$, where N_ε is a natural number satisfying $1/2^{N_\varepsilon-1} \leq \varepsilon$. Therefore, if $n \geq N_\varepsilon$ then $|\mu^{(n)}(C) - \mu(C)| \leq |\mu^{(n)}(C_\varepsilon) - \mu(C_\varepsilon)| + \mu(C \setminus C_\varepsilon) + \mu^{(n)}(C \setminus C_\varepsilon) < 4\varepsilon$. This implies that $\mu^{(n)}(C) \rightarrow \mu(C)$ as $n \rightarrow \infty$. Thus $\mu^{(n)}$ converge setwise to μ as $n \rightarrow \infty$.

Observe that for $S_n = \cup_{k=0}^{2^{n-1}-1} [2k/2^n, (2k+1)/2^n]$, $n = 1, 2, \dots$,

$$\mu^{(n)}(S_n) - \mu(S_n) = -\left(\frac{1}{2} - \frac{1}{2n}\right). \quad (4.2)$$

So, the sequence $\{\mu^{(n)}\}_{n=1,2,\dots}$ does not converge in total variation to μ because

$$\text{dist}(\mu^{(n)}, \mu) \geq \frac{1}{2} - \frac{1}{2n}, \quad n = 1, 2, \dots$$

Equality (2.11) holds since

$$\int_S f^{(n)}(s) \mu^{(n)}(ds) = \int_S f(s) \mu(ds) \quad \text{for all } S \in \Sigma, \quad n = 1, 2, \dots, \quad (4.3)$$

which is stronger than (2.11). Thus, inequality (2.1) also holds.

Statement (i) from Theorem 2.1 does not hold since

$$\mu(\{s \in \mathbb{S} : f^{(n)}(s) \leq f(s) - 1\}) = \frac{1}{2}, \quad n = 2, 3, \dots$$

Thus statement (i) from Corollary 2.9 does not hold either.

Statement (ii) from Theorem 2.1 does not hold since

$$\inf_{n=1,2,\dots} \int_{\mathbb{S}} f^{(n)}(s) \mathbf{I}_{\{s \in \mathbb{S} : f^{(n)}(s) \leq -K\}} \mu^{(n)}(ds) = \frac{1}{2}, \quad K > 1.$$

Thus statement (ii) from Corollary 2.9 does not hold either. \square

Example 4.3 demonstrates that, if convergence in total variation of finite measures $\{\mu^{(n)}\}_{n=1,2,\dots}$ to μ in Corollary 2.3, in which the functions $f^{(n)}$ are assumed to be nonnegative, is relaxed to setwise convergence, inequality (2.1) does not imply statement (i) from Theorem 2.1.

Example 4.3. Let $\mathbb{S} = [0, 1]$, $\Sigma = \mathcal{B}(\mathbb{S})$ be a Borel σ -algebra on \mathbb{S} ,

$$g^{(n)}(s) := \begin{cases} \frac{1}{2}, & \text{if } 2k/2^n < s < (2k+1)/2^n \text{ for } k = 0, 1, \dots, 2^{n-1} - 1; \\ \frac{3}{2}, & \text{otherwise,} \end{cases} \quad (4.4)$$

$f^{(n)}(s) := 1/g^{(n)}(s)$, $s \in [0, 1]$, $n = 1, 2, \dots$, be the sequence of measurable functions, μ be the Lebesgue measure on $[0, 1]$, and $f \equiv 1$. For the functions $g^{(n)}$ from (4.4), consider the sequence of probability measures $\mu^{(n)}$ on $[0, 1]$, $n = 1, 2, \dots$, defined in (4.1).

The sequence $\{\mu^{(n)}\}_{n=1,2,\dots}$ converges setwise to μ as $n \rightarrow \infty$, and (4.3) holds. These facts follow from the same arguments as in Example 4.2. In view of (4.3), inequality (2.1) holds. Statement (i) from Theorem 2.1 does not hold since $\mu(\{s \in \mathbb{S} : f^{(n)}(s) \leq f(s) - \frac{1}{3}\}) = \frac{1}{2}$, $n = 1, 2, \dots$. \square

Example 4.4 demonstrates that statements (i) and (ii) from Corollary 2.9 do not imply inequality (2.1) and therefore they do not imply equality (2.11), if convergence in total variation of finite measures $\{\mu^{(n)}\}_{n=1,2,\dots}$ to μ in Corollary 2.9 is relaxed to setwise convergence. Therefore, statements (i) and (ii) from Theorem 2.1 do not yield inequality (2.1), if the convergence in total variation of finite measures $\{\mu^{(n)}\}_{n=1,2,\dots}$ to μ in Theorem 2.1 is relaxed to setwise convergence.

Example 4.4. Consider a measurable space (\mathbb{S}, Σ) and a sequence $\{\mu^{(n)}\}_{n=1,2,\dots} \subset \mathcal{M}(\mathbb{S})$ that converges setwise to a measure μ on \mathbb{S} such that

$$\liminf_{n \rightarrow \infty} \inf_{S \in \Sigma} (\mu^{(n)}(S) - \mu(S)) < 0.$$

For example, in view of (4.2), the measurable spaces and measures defined in Example 4.2 can be considered for this example. Let $f = f^{(n)} \equiv 1$, $n = 1, 2, \dots$.

Note that, statements (i) and (ii) from Corollary 2.9 hold. Thus statements (i) and (ii) from Theorem 2.1 hold. Moreover, since

$$\liminf_{n \rightarrow \infty} \inf_{S \in \Sigma} \left(\int_{\mathbb{S}} f^{(n)}(s) \mu^{(n)}(ds) - \int_{\mathbb{S}} f(s) \mu(ds) \right) = \liminf_{n \rightarrow \infty} \inf_{S \in \Sigma} (\mu^{(n)}(S) - \mu(S)) < 0,$$

neither inequality (2.1) nor equality (2.11) holds. \square

We remark that the functions f and $f^{(n)}$, $n = 1, 2, \dots$, are nonnegative in Example 4.4. Therefore, unlike the case of measures converging in total variation described in Corollary 2.9, even for nonnegative functions f and $f^{(n)}$, the validity of statement (i) from Theorem 2.1 is not necessary for the validity of inequality (2.1) in the case of setwise converging measures.

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References

- [1] M. Aoki, Optimal control of partially observable Markovian systems, *J. Franklin Inst.* 280 (1965) 367-386.
- [2] Z. Artstein, A note on Fatou's lemma in several dimensions, *J. Math. Econ.* 6 (1979) 277-282.
- [3] K.J. Åström, Optimal control of Markov decision processes with incomplete state estimation, *J. Math. Anal. Appl.* 10 (1965) 174-205.
- [4] R.J. Aumann, Integrals of set-valued functions, *J. Math. Anal. Appl.* 12 (1965) 1-12.
- [5] E.J. Balder, A Fatou lemma for Gelfand integrals by means of Young measure theory, *Positivity* 6 (2002) 317-329.
- [6] J.P. Chen, B.E. Ugurcan, Entropic repulsion of Gaussian free field on high-dimensional Sierpinski carpet graphs, *Stoch. Proc. Appl.* 125 (2015) 4632-4673.
- [7] S. Chu, Y. Zhang, Markov decision processes with iterated coherent risk measures, *Int. J. Control* 87 (2014) 2286-2293.
- [8] E.B. Dynkin, Controlled random sequences, *Theory Probab. Appl.* 10 (1965) 1-14.
- [9] E.A. Feinberg, P.O. Kasyanov, N.V. Zadoianchuk, Average-cost Markov decision processes with weakly continuous transition probabilities, *Math. Oper. Res.* 37 (2012) 591-607.
- [10] E.A. Feinberg, P.O. Kasyanov, N.V. Zadoianchuk, Fatou's lemma for weakly converging probabilities, *Theor. Probab. Appl.* 58 (2014) 683-689.
- [11] E.A. Feinberg, P.O. Kasyanov, M.Z. Zgurovsky, Partially observable total-cost Markov decision processes with weakly continuous transition probabilities, *Math. Oper. Res.* (in press and published online), doi:10.1287/moor.2015.0746 .
- [12] E.A. Feinberg, P.O. Kasyanov, M.Z. Zgurovsky, Convergence of probability measures and Markov decision models with incomplete information, *P. Steklov Inst. Math.* 287 (2014) 96-117.
- [13] N.V. Gorban, O.V. Kapustyan, P.O. Kasyanov, Uniform trajectory attractor for non-autonomous reaction-diffusion equations with Caratheodory's nonlinearity, *Nonlinear Anal.-Theor.* 98 (2014) 13-26.

- [14] J. Jacod, A.N. Shiryaev, *Limit Theorems for Stochastic Processes*, Second edition, Springer, Berlin, 2003.
- [15] A. Jaskiewicz, A.S. Nowak, Zero-sum ergodic stochastic games with Feller transition probabilities, *SIAM J. Control Optim.* 45 (2006) 773-789.
- [16] R.S. Liptser, A.N. Shiryaev, *Statistics of Stochastic Processes*, Springer, Berlin, 1977.
- [17] D. Rhenius, Incomplete information in Markovian decision models, *Ann. Statist.* 2 (1974) 1327-1334.
- [18] H.L. Royden, *Real Analysis*, Second edition, Macmillan, New York, 1968.
- [19] M. Schäl, Average optimality in dynamic programming with general state space, *Math. Oper. Res.* 18 (1993) 163-172.
- [20] D. Schmeidler, Fatou's lemma in several dimensions, *Proc. Amer. Math. Soc.* 24 (1970) 300-306.
- [21] R. Serfozo, Convergence of Lebesgue integrals with varying measures, *Sankhya Ser. A* 44 (1982) 380-402.
- [22] A.N. Shiryaev, Some new results in the theory of controlled random processes, *Select. Transl. Math. Statist. Probab.* 8 (1969) 49-130.
- [23] M. Thorpe, F. Theil, A.M. Johansen, N. Cade, Convergence of the k -means minimization problem using Γ -convergence, *SIAM J. Appl. Math.* 75 (2015), 2444-2474.
- [24] N.C. Yannelis, Fatou's lemma in infinite-dimensional spaces, *Proc. Amer. Math. Soc.* 102 (1988), 303-310.
- [25] A. A. Yushkevich, Reduction of a controlled Markov model with incomplete data to a problem with complete information in the case of Borel state and control spaces, *Theory Probab. Appl.* 21 (1976) 153-158.
- [26] Zgurovsky M.Z., P.O. Kasyanov, O.V. Kapustyan, J. Valero, N.V. Zadoianchuk, *Evolution Inclusions and Variation Inequalities for Earth Data Processing III: Long-Time Behavior of Evolution Inclusions Solutions in Earth Data Analysis*, Springer, Berlin, 2012.
- [27] M.Z. Zgurovsky, V.S. Mel'nik, P.O. Kasyanov, *Evolution Inclusions and Variation Inequalities for Earth Data Processing I: Operator Inclusions and Variation Inequalities for Earth Data Processing*, Springer, Berlin, 2011.
- [28] M.Z. Zgurovsky, V.S. Mel'nik, P.O. Kasyanov, *Evolution Inclusions and Variation Inequalities for Earth Data Processing II: Differential-Operator Inclusions and Evolution Variation Inequalities for Earth Data Processing*, Springer, Berlin, 2011.
- [29] Y. Zhang, Average optimality for continuous-time Markov decision processes under weak continuity conditions, *J. Appl. Probab.* 51 (2014) 954-970.