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An accelerated exponential time integrator for semi-linear stochastic strongly damped wave equation with additive noise ☆

Ruisheng Qi [a], Xiaojie Wang [b,*]

[a] School of Mathematics and Statistics, Northeastern University at Qinhuangdao, Qinhuangdao, China
[b] School of Mathematics and Statistics, Central South University, Changsha, China

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ABSTRACT

This paper is concerned with the strong approximation of a semi-linear stochastic wave equation with strong damping, driven by additive noise. Based on a spatial discretization performed by a spectral Galerkin method, we introduce a kind of accelerated exponential time integrator involving linear functionals of the noise. Under appropriate assumptions, we provide error bounds for the proposed full-discrete scheme. It is shown that the scheme achieves higher strong order in time direction than the order of temporal regularity of the underlying problem, which allows for higher convergence rate than usual time-stepping schemes. Particularly for the space–time white noise case in two or three spatial dimensions, the scheme still exhibits a good convergence performance. Another striking finding is that, even for the velocity with low regularity, the scheme always promises first order strong convergence in time. Numerical examples are finally reported to confirm our theoretical findings.

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1. Introduction

Great attention has been devoted in the last decade to numerical approximations of hyperbolic stochastic partial differential equations (SPDEs) (see, e.g., [1,2,4,6,10,12–14,24,28,30] and references therein). In the present work, we concentrate on a class of semi-linear SPDEs of second order with damping, described by

$$\begin{cases} u_{tt} = \alpha L u_t + Lu + F(u) + \dot{W}(t), & \text{in } \mathcal{D} \times (0,\ T], \\ u(\cdot,0) = u_0,\ u_t(\cdot,0) = v_0, & \text{in } \mathcal{D}, \\ u = 0, & \text{on } \partial\mathcal{D} \times (0,\ T], \end{cases} \tag{1.1}$$

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* Corresponding author.
E-mail addresses: qirsh@neuq.edu.cn, qiruisheng123@sohu.com (R. Qi), x.j.wang7@csu.edu.cn, x.j.wang7@gmail.com (X. Wang).

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where $T \in (0,\infty)$, $\mathcal{D} \subset \mathbb{R}^d$, $d = 1,2,3$, is a bounded open domain with smooth boundary $\partial\mathcal{D}$, and where $L := \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\big(l_{ij}(x)\frac{\partial}{\partial x_j}\big), x \in \mathcal{D}$ is a linear second-order elliptic operator with smooth coefficients $\{l_{ij}\}_{i,j=1}^{d}$ being uniformly positive definite. Let $\alpha > 0$ be a fixed positive constant and let $\{W(t)\}_{t\in[0,T]}$ be a (possibly cylindrical) Q-Wiener process on $H := L_2(\mathcal{D})$, defined on a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\in[0,T]})$ with respect to the normal filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$. The initial data u_0, v_0 are assumed to be $\mathcal{F}_0$-measurable random variables.

The deterministic counterpart of (1.1), called strongly damped wave equation (SDWE), occurs in a wide range of applications such as modeling motion of viscoelastic materials [7,18,19]. From both the theoretical and numerical point of view, the deterministic problem has been extensively studied (e.g., [11,16,26]). However, the corresponding stochastic strongly damped wave equations are theoretically and numerically far from well-understood from existing literature [5,21]. In the classical monograph [5], a stochastic strongly damped wave equation with multiplicative noise was discussed and a unique mild solution was established. In our recent publication [21], we analyzed the regularity properties of the mild solution to (1.1) and examine error estimates of a full discretization, done by the finite element spatial approximation together with the well-known linear implicit Euler temporal discretization. It was shown there that, for a certain class of stochastic SDWEs, the convergence rates of the usual full discretization coincide with the space–time regularity properties of the mild solution (see Theorem 2.4 in [21]). In this article, we aim to introduce a so-called accelerated exponential time integrator for the problem (1.1), which, as we will show later, promises higher convergence order in time than the derived order of temporal regularity of the underlying problem.

Different from the usual Euler-type time-stepping schemes using the basic increments of the driving Wiener process [1,3,15,17,22,23,29,32], the accelerated exponential time integrators rely on suitable linear functionals of the Wiener process and usually attain higher approximation orders in time [8,9,30,31]. In 2009, such scheme was first constructed by Jentzen and Kloeden [8] for semi-linear parabolic SPDEs with additive space–time white noise. The order barrier in the numerical approximation of parabolic SPDEs was overcome and a strong convergence rate of order $1-\epsilon$ in time was obtained, for arbitrarily small $\epsilon > 0$, unfortunately with seriously restrictive commutativity condition imposed on the non-linear term (see [8, Assumption 2.4]). Afterwards, the accelerated schemes were extended to solve a larger class of parabolic SPDEs with more general noise and the error bounds were analyzed under relaxed conditions on the non-linear term [9,31]. Furthermore, the accelerated scheme was successfully adapted to solve semilinear stochastic wave equations and the order barrier $\frac{1}{2}$ was went beyond [30].

Following the idea of the acceleration technique, we discretize the considered problem (1.1) in space by a spectral Galerkin method and in time by an exponential integrator involving linear functionals of the noise. To analyze the resulting error bounds, we formulate mild assumptions on the nonlinear mapping F (see Assumption 2.3), to allow for a large class of nonlinear Nemytskii operators. Additionally, we assume the covariance operator $Q\colon L_2(\mathcal{D}) \to L_2(\mathcal{D})$ of the Wiener process obeys

$$\|A^{\frac{\gamma-1}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}} < \infty \quad \text{for some } \gamma \in (-1,2], \tag{1.2}$$

which covers both the space–time white noise case and the trace-class noise case. Here $A := -L$ with domain $D(A) = H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})$. Under these assumptions, the continuous problem (1.1) possesses a unique mild solution with the displacement $u(t), t \in [0,T]$ taking values in $L^p(\Omega; \dot{H}^{1+\min\{\gamma,1\}})$ for $\gamma \in (-1,2]$ and the velocity $v(t) := u_t(t), t \in [0,T]$ taking values in $L^p(\Omega; \dot{H}^{\gamma})$ for $\gamma \in [0,2]$. Here $\dot{H}^s = D(A^{\frac{s}{2}}), s \in \mathbb{R}$ will be specified later. For $M, N \in \mathbb{N}$, let $(u_m^{N,M}, v_m^{N,M})'$ be the numerical approximations of $(u(t_m), v(t_m))'$ at the temporal grid points $t_m = mk$ for $m \in \{0,1,...,M\}$, produced by the proposed full discrete scheme with time-step $k = T/M$. Under certain assumptions, our convergence analysis shows, there exists a constant C such that, for all $N, M \in \mathbb{N}$,

$$\sup_{m\in\{0,1,\cdots,M\}} \|u(t_m) - u_m^{N,M}\|_{L^2(\Omega;\dot{H}^0)} \le C(k^{\min\{1+\gamma,1\}} + \lambda_{N+1}^{-\frac{1+\min\{\gamma,1\}}{2}}), \quad \gamma \in (-1,2], \tag{1.3}$$

and

$$\sup_{m\in\{0,1,\cdots,M\}} \|v(t_m) - v_m^{N,M}\|_{L^2(\Omega;\dot{H}^0)} \leq C(k + \lambda_{N+1}^{-\frac{\gamma}{2}}), \quad \gamma \in [0,2], \tag{1.4}$$

where $\{\lambda_j\}_{j=1}^{\infty}$ are the eigenvalues of the linear operator A. It should be emphasized that, although the idea of the method construction comes from [8,9,30,31], the corresponding error analysis (section 4) is not easy and forces us to exploit a variety of regularity properties (Lemmas 4.2, 4.3) of the associated semigroup that are missing in existing works. As clearly indicated in (1.3) and (1.4), the convergence rate in space exactly agrees with the derived order of the spatial regularity of the mild solution in Theorem 2.1. However, the convergence rate in time behaves quite differently. When $\gamma \in (-1, 0]$, the temporal mean-square rate $\gamma+1$ is higher than the obtained temporal Hölder regularity order of the mild solution in Theorem 2.1. Particularly for the case of the additive space–time white noise ($Q = I$) in two and three dimensions when (1.2) is fulfilled with $\gamma < 0$ and $\gamma < -\frac{1}{2}$, respectively, the error estimate (1.3) implies that the introduced method solving the displacement has a good convergence performance. Also, we point out a surprising finding, when $\gamma \in [0, 2]$, the numerical solution $u_m^{N,M}$ and $v_m^{N,M}$ both enjoy a temporal convergence rate of order one that does not depend on the order of the temporal regularity of the mild solution. This means, even for the velocity v with very low regularity, the numerical approximation shows a first order strong convergence in time. Recall from [21] that, the finite element spatial discretization requires (1.2) to be satisfied with $\gamma \geq 0$ and this excludes the space–time white noise case in two or three dimensions. Furthermore, the convergence rates of numerical approximations obtained in [21] coincide exactly with the obtained orders of the regularity. Accordingly, the newly proposed scheme shows significantly improved convergence rates in time.

Finally, we would like to mention limitations of the presented numerical method. In order to easily implement the scheme (see section 3), the eigenfunctions of the dominating linear operator A and of the covariance operator Q of the Wiener process must coincide and must be known explicitly.

The remainder of this article is organized as follows. In the next section, some preliminaries are collected and assumptions are made for the noise and the nonlinearity. Further, a concrete example is presented to illustrate the abstract assumptions. In section 3 we propose the full-discretization scheme and state the main convergence result, together with some comments on the implementation of the scheme. The proof of the main result is elaborated in section 4. Finally, numerical experiments are performed to confirm the theoretical results.

2. Preliminaries and abstract framework

Given two separable $\mathbb{R}$-Hilbert spaces $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U)$ and $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$, we denote by $\mathcal{L}(U, H)$ the Banach space of all linear bounded operators from U into H and by $\mathcal{L}_2(U, H)$ we denote the Hilbert space of all Hilbert–Schmidt operators from U into H. For simplicity, we write $\mathcal{L}(U) = \mathcal{L}(U, U)$ and $\mathrm{HS} = \mathcal{L}_2(U, U)$. It is well known that $\|TS\|_{\mathcal{L}_2(U,H)} \leq \|T\|_{\mathcal{L}_2(U,H)}\|S\|_{\mathcal{L}(U)}$, for $T \in \mathcal{L}_2(U, H)$, $S \in \mathcal{L}(U)$. Let $Q \in \mathcal{L}(U)$ be a self-adjoint, positive semidefinite operator. We denote the space of the Hilbert–Schmidt operators from $Q^{\frac{1}{2}}(U)$ to H by $\mathcal{L}_2^0 := \mathcal{L}_2(Q^{\frac{1}{2}}(U), H)$ and the corresponding norm is given by $\|\Gamma\|_{\mathcal{L}_2^0} = \|\Gamma Q^{\frac{1}{2}}\|_{\mathcal{L}_2(U,H)}$.

For $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space and for $(H, \langle\cdot,\cdot\rangle)$ a Hilbert space we denote by $L^2(\Omega; H)$ the Hilbert space of equivalence classes of square-integrable random variables, equipped with the norm $\|v\|_{L^2(\Omega;H)} := \big(\mathbf{E}\big[\|v\|^2\big]\big)^{\frac{1}{2}}$. Next, we introduce a self-adjoint, positive definite, linear operator $A = -L$ with the domain $D(A) = H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})$, where $\mathcal{D} \subset \mathbb{R}^d$, $d = 1, 2, 3$, is a bounded open domain with smooth boundary $\partial\mathcal{D}$. Here $H^m(\mathcal{D})$ denotes the standard Sobolev spaces of integer order $m \geq 1$ and $H_0^1(\mathcal{D}) := \{\phi \in H^1(\mathcal{D})\colon \phi|_{\partial\mathcal{D}} = 0\}$, i.e., a space of functions ϕ with $\nabla\phi = \text{grad } \phi$ in $L_2(\mathcal{D})$ vanishing on ∂D. In order to define a mild solution of (1.1) in the semigroup framework as in [5], we introduce the time derivative of u as a new variable $v := u_t$ and rewrite (1.1) as a system of first order, given by

$$\begin{cases} \mathrm{d}X(t) = -\mathcal{A}X(t)\,\mathrm{d}t + \mathbf{F}(X(t))\,\mathrm{d}t + \mathbf{B}\,\mathrm{d}W(t), \quad t \in (0,T], \\ X(0) = X_0, \end{cases} \tag{2.1}$$

where $X(t) := (u(t), v(t))'$, $X_0 := (u_0, v_0)'$ and

$$\mathcal{A} := \begin{bmatrix} 0 & -I \\ A & \alpha A \end{bmatrix}, \quad \mathbf{F}(X) := \begin{bmatrix} 0 \\ F(u) \end{bmatrix} \text{ and } \mathbf{B} := \begin{bmatrix} 0 \\ I \end{bmatrix}. \tag{2.2}$$

To further get the existence, uniqueness and regularity properties of the mild solution of (2.1) and also for the purpose of the error analysis of the numerical scheme, we give some abstract assumptions on the linear operator, the nonlinear term, the noise process and the initial data as follows.

Assumption 2.1 *(Linear operator).* Let $(H, \langle\cdot,\cdot\rangle)$ be a real separable Hilbert space and let $A\colon D(A) \subset H \to H$ be a linear, densely-defined, positive self-adjoint unbounded operator with compact inverse.

We first mention that the above concrete setting with $A = -L, D(A) = H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})$ satisfies Assumption 2.1, where $\mathcal{D} \subset \mathbb{R}^d$, $d = 1, 2, 3$, is a bounded open domain with smooth boundary $\partial\mathcal{D}$. Under Assumption 2.1, one can find a sequence of real numbers $\{\lambda_j\}_{j=1}^{\infty}$ and an orthonormal basis $\{\phi_j\}_{j=1}^{\infty}$ such that $A\phi_j = \lambda_j\phi_j, j \in \mathbb{N}$. Since the operator A is self-adjoint, positive definite and unbounded, it holds that $\lambda_j \geq 0$ for all $j \in \mathbb{N}$ and $\sup_{j\in\mathbb{N}} \lambda_j = \infty$. Without loss of generality we assume $\lambda_{j+1} \geq \lambda_j$ for all $j \in \mathbb{N}$. Then the separable Hilbert space $\dot{H}^s = D(A^{\frac{s}{2}}), s \in \mathbb{R}$ can be defined [25, Chapter 3] using eigenfunction expansion, equipped with the corresponding norm

$$\|v\|_s := \|A^{\frac{s}{2}}v\| = \Big(\sum_{j=1}^{\infty} \lambda_j^s \langle v, \phi_j\rangle^2\Big)^{\frac{1}{2}}, \ s \in \mathbb{R}, \tag{2.3}$$

where $\{(\lambda_j, \phi_j)\}_{j=1}^{\infty}$ are the eigenpairs of A with orthonormal eigenvectors. It is well-known that $\dot{H}^0 = H := L_2(\mathcal{D})$, $\dot{H}^1 = H_0^1(\mathcal{D})$ and $\dot{H}^2 = H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})$, see, e.g., [27, Chapter 4] and [25, Chapter 3]. As clarified in [16, Lemma 2.1], $-\mathcal{A}$ generates an analytic semigroup $\mathcal{S}(t)$ in $\dot{H}^s \times \dot{H}^{s-\sigma}$, for $s \in \mathbb{R}, \sigma \in [0,2]$, given by

$$\mathcal{S}(t) = e^{-t\mathcal{A}} = \begin{bmatrix} \mathcal{S}_1(t) & \mathcal{S}_2(t) \\ \mathcal{S}_3(t) & \mathcal{S}_4(t) \end{bmatrix}. \tag{2.4}$$

Here, $\mathcal{S}_i(t)$, $i = 1, 2, 3, 4$, can be expressed in terms of the eigenfunction expansion, for $\varphi \in L_2(\mathcal{D})$,

$$\begin{cases} \mathcal{S}_1(t)\varphi = \sum_{j=1}^{\infty} \frac{\lambda_j^+ e^{-t\lambda_j^-} - \lambda_j^- e^{-t\lambda_j^+}}{\lambda_j^+ - \lambda_j^-} \langle\varphi, \phi_j\rangle\, \phi_j, \\ \mathcal{S}_2(t)\varphi = \sum_{j=1}^{\infty} \frac{e^{-t\lambda_j^-} - e^{-t\lambda_j^+}}{\lambda_j^+ - \lambda_j^-} \langle\varphi, \phi_j\rangle\, \phi_j, \\ \mathcal{S}_3(t)\varphi = \sum_{j=1}^{\infty} \frac{\lambda_j(e^{-t\lambda_j^+} - e^{-t\lambda_j^-})}{\lambda_j^+ - \lambda_j^-} \langle\varphi, \phi_j\rangle\, \phi_j, \\ \mathcal{S}_4(t)\varphi = \sum_{j=1}^{\infty} \frac{\lambda_j^+ e^{-t\lambda_j^+} - \lambda_j^- e^{-t\lambda_j^-}}{\lambda_j^+ - \lambda_j^-} \langle\varphi, \phi_j\rangle\, \phi_j, \end{cases} \tag{2.5}$$

where $\lambda_j^{\pm}$ are the solutions of the following equation

$$z^2 - \alpha\lambda_j z + \lambda_j = 0, \tag{2.6}$$

obeying

$$\lim_{j\to\infty} \lambda_j^- = \frac{4}{\alpha}, \quad \textit{and} \quad \lim_{j\to\infty} \frac{\lambda_j^+}{\lambda_j} = \alpha. \tag{2.7}$$

Assumption 2.2 *(Q-Wiener process).* Let $\{W(t)\}_{t\in[0,T]}$ be a (possibly cylindrical) Q-Wiener process, with the covariance operator $Q\colon H\to H$ being a symmetric nonnegative operator satisfying

$$\|A^{\frac{\gamma-1}{2}}Q^{\frac{1}{2}}\|_{\mathrm{HS}}<\infty \quad \text{for some } \gamma\in(-1,2]. \tag{2.8}$$

Assumption 2.3 *(Nonlinearity).* The deterministic mapping $F\colon H\to H$ is assumed to be twice Fréchet differentiable and there exists a positive constant K such that, for $\beta:=\min\{1,1+\gamma\}$,

$$\|F(x)-F(y)\|\le K\|x-y\|, \quad \forall x,\ y\in H, \tag{2.9}$$

$$\|A^{-\frac{\theta}{2}}F'(y)z\|\le K(\|y\|_\beta+1)\|z\|_{-\beta}, \quad \text{for some } \theta\in[1,2),\ \forall y\in\dot H^\beta,\ z\in H. \tag{2.10}$$

In view of (2.9) in Assumption 2.3, the mapping F also obeys the following linear growth condition,

$$\|F(x)\|\le K\|x\|+\|F(0)\|,\ \forall x\in H. \tag{2.11}$$

Also, we remark that such condition as (2.10) was also used in [31], where the condition is validated only for particular ranges of β such as $\beta<\frac{1}{2}$ and $\beta=1$. In Example 2.1, we give a class of concrete nonlinear Nemytskij operators to validate (2.10) for the whole range $\beta\in(0,1]$.

Assumption 2.4 *(Initial data).* Let u_0,v_0 be $\mathcal{F}_0$-measurable and $(u_0,v_0)'\in L^p(\Omega;\dot H^{\min\{2,\gamma+1\}}\times\dot H^\gamma)$, for any $p\ge 2$.

Under these assumptions, as shown in [21, Theorem 2.1], the equation (2.1) has a unique mild solution. Moreover, a slight modification of [21, Theorem 2.1] ensures the following regularity results.

Theorem 2.1. *Assume that Assumptions 2.1–2.4 hold. Then there exists a unique mild solution $X=(u,v)'$ of (2.1) given by*

$$X(t)=\mathcal{S}(t)X_0+\int_0^t\mathcal{S}(t-s)\mathbf{F}(X(s))\,\mathrm{d}s+\int_0^t\mathcal{S}(t-s)\mathbf{B}\,\mathrm{d}W(s). \tag{2.12}$$

Furthermore, there exists a constant such that, for all $p\ge 2$ and $t,s\in[0,T]$ it holds

$$\|u(t)\|_{L^p(\Omega;\dot H^\varrho)}+\|v(t)\|_{L^p(\Omega;\dot H^{\varrho-2})}\le C\big(1+\|u_0\|_{L^p(\Omega;\dot H^\varrho)}+\|v_0\|_{L^p(\Omega;\dot H^{\varrho-2})}\big), \tag{2.13}$$

$$\|u(t)-u(s)\|_{L^p(\Omega;\dot H^0)}\le C|t-s|^{\frac{\varrho}{2}}\big(1+\|u_0\|_{L^p(\Omega;\dot H^\varrho)}+\|v_0\|_{L^p(\Omega;\dot H^{\varrho-2})}\big), \tag{2.14}$$

where we write $\varrho:=\min\{1+\gamma,2\}$. For the particular case when Assumption 2.2 is fulfilled with $\gamma\in[0,2]$, it holds that, for $p\ge 2$ and $t,s\in[0,T]$,

$$\|u(t)\|_{L^p(\Omega;\dot H^\gamma)}+\|v(t)\|_{L^p(\Omega;\dot H^\gamma)}\le C\big(1+\|u_0\|_{L^p(\Omega;\dot H^\gamma)}+\|v_0\|_{L^p(\Omega;\dot H^\gamma)}\big), \tag{2.15}$$

$$\|v(t)-v(s)\|_{L^p(\Omega;\dot H^0)}\le C|t-s|^{\frac{\min\{\gamma,1\}}{2}}\big(1+\|u_0\|_{L^p(\Omega;\dot H^{\min\{\gamma,1\}})}+\|v_0\|_{L^p(\Omega;\dot H^{\min\{\gamma,1\}})}\big). \tag{2.16}$$

Here and below, C represents a generic constant that may change between occurrences but only depends on $K,T,\|F(0)\|,\|A^{\frac{\gamma-1}{2}}Q^{\frac{1}{2}}\|_{\mathrm{HS}}$ and the initial data. As an immediate consequence, Theorem 2.1 implies the following facts.

Lemma 2.1. *Under the assumptions of Theorem 2.1, there exists a constant C such that for all $p\ge 2$, $t,s\in[0,T]$ and $\varrho:=\min\{1+\gamma,2\}$ it holds*

$$\sup_{s\in[0,T]}\|F(u(s))\|_{L^p(\Omega;\dot{H}^0)}\le C\big(1+\|u_0\|_{L^p(\Omega;\dot{H}^0)}+\|v_0\|_{L^p(\Omega;\dot{H}^{-2})}\big),\tag{2.17}$$

$$\|F(u(t))-F(u(s))\|_{L^p(\Omega;\dot{H}^0)}\le C|t-s|^{\frac{\varrho}{2}}\big(1+\|u_0\|_{L^p(\Omega;\dot{H}^\varrho)}+\|v_0\|_{L^p(\Omega;\dot{H}^{\varrho-2})}\big).\tag{2.18}$$

We present a class of concrete nonlinear mappings F that fulfill Assumption 2.3.

Example 2.1. Let $H:=L_2(\mathcal{D})$, $A=-\Delta=-\sum_{i=1}^d\frac{\partial^2}{\partial x_i^2}$ with the domain $D(A)=H^2(\mathcal{D})\cap H_0^1(\mathcal{D})$ and let $F:H\to H$ be a Nemytskij operator associated to f, defined by

$$F(\varphi)(x)=f(x,\varphi(x)),\ x\in\mathcal{D},\ \varphi\in H,\tag{2.19}$$

where $\mathcal{D}\subset\mathbb{R}^d$, $d=1,2,3$, is a bounded open set with Lipschitz boundary and $f\in C^{1,2}(\mathcal{D}\times\mathbb{R},\mathbb{R})$ satisfies, with some constant $K>0$,

$$|f(x,z)|\le K(|z|+1),\tag{2.20}$$

$$\left|\frac{\partial f}{\partial z}(x,z)\right|\le K,\quad \left|\frac{\partial^2 f}{\partial x_i\partial z}(x,z)\right|\le K,\quad\text{and}\quad \left|\frac{\partial^2 f}{\partial z^2}(x,z)\right|\le K,\tag{2.21}$$

for all $x=(x_1,x_2,\cdots,x_d)'\in\mathcal{D}$, $z\in\mathbb{R}$. It is clear that F satisfies the following conditions

$$\|F(\varphi)\|\le\sqrt{2}K(\|\varphi\|+1),\tag{2.22}$$

$$\|F(\varphi_1)-F(\varphi_2)\|\le K\|\varphi_1-\varphi_2\|,\tag{2.23}$$

for $\varphi,\varphi_1,\varphi_2\in H$. The Fréchet derivative operator of F is given by

$$F'(\varphi)(\psi)(x)=\frac{\partial f}{\partial z}(x,\varphi(x))\cdot\psi(x),\ x\in\mathcal{D},\tag{2.24}$$

for all $\varphi,\psi\in H$. Note that the derivative operator $F'(\varphi),\varphi\in H$ defined in the above way is self-adjoint and by similar arguments used in the proof of [30, Lemma 4.4], we have, for $\beta=\min\{1,1+\gamma\}\in(0,1]$,

$$\|F'(\varphi)\psi\|_{W^{\beta,2}}\le C(\|\psi\|+\|\psi\|_{C(\mathcal{D},\mathbb{R})}+\|\varphi\|_{W^{\beta,2}}\|\psi\|_{C(\mathcal{D},\mathbb{R})})+C\|\psi\|_{W^{\beta,2}},\tag{2.25}$$

where $\varphi\in\dot{H}^\beta$, $\psi\in\dot{H}^\theta$ with $\theta\in[1,2)$ and $\theta>\frac{d}{2}$, $d=1,2,3$. Here $\|\cdot\|_{W^{\beta,2}}$ stands for the usual norm for the Sobolev space $W^{\beta,2}$. There exists a constant $C_\beta\in(0,\infty)$ such that [27, Chapter 4]

$$\frac{1}{C_\beta}\|v\|_{W^{\beta,2}}\le\|v\|_\beta\le C_\beta\|v\|_{W^{\beta,2}},\quad\text{for}\ \ v\in\dot{H}^\beta.\tag{2.26}$$

Furthermore, the Sobolev embedding theorem gives $\dot{H}^\theta\subset C(\mathcal{D},\mathbb{R})$ and $\dot{H}^\theta\subset\dot{H}^\beta$, for $\theta>\frac{d}{2}$ and $\theta\in[1,2)$. Due to (2.26), we infer that

$$\|F'(\varphi)\psi\|_{W^{\beta,2}}\le C(\|\varphi\|_\beta+1)\|\psi\|_\theta,\ \text{for}\ \varphi\in\dot{H}^\beta,\ \psi\in\dot{H}^\theta.\tag{2.27}$$

For $\varphi\in\dot{H}^\beta$ and $\psi\in\dot{H}^\theta\subset H_0^1(\mathcal{D})$, we have $F'(\varphi)\psi\in\dot{H}^\beta$ and

$$\|F'(\varphi)\psi\|_\beta\le\frac{1}{C_\beta}\|F'(\varphi)\psi\|_{W^{\beta,2}}\le C(\|\varphi\|_\beta+1)\|\psi\|_\theta.\tag{2.28}$$

Consequently, one can show that, for $\varphi\in\dot{H}^\beta$ and $\psi\in H$, $\beta=\min\{1,1+\gamma\}\in(0,1]$,

$$\begin{aligned}\|A^{-\frac{\theta}{2}}F'(\varphi)\psi\| &= \sup_{\chi\in\dot{H}^0}\frac{\langle A^{-\frac{\theta}{2}}F'(\varphi)\psi,\chi\rangle}{\|\chi\|} = \sup_{\chi\in\dot{H}^0}\frac{\langle A^{-\frac{\beta}{2}}\psi, A^{\frac{\beta}{2}}F'(\varphi)A^{-\frac{\theta}{2}}\chi\rangle}{\|\chi\|}\\ &\le \sup_{\chi\in\dot{H}^0}\frac{\|A^{-\frac{\beta}{2}}\psi\|\|A^{\frac{\beta}{2}}F'(\varphi)A^{-\frac{\theta}{2}}\chi\|}{\|\chi\|} \le \sup_{\chi\in\dot{H}^0}\frac{\|\psi\|_{-\beta}\|A^{\frac{\beta}{2}}F'(\varphi)A^{-\frac{\theta}{2}}\chi\|}{\|\chi\|}\\ &\le C\sup_{\chi\in\dot{H}^0}\frac{\|\psi\|_{-\beta}\left(1+\|\varphi\|_\beta\right)\|\chi\|}{\|\chi\|}\\ &\le C\|\psi\|_{-\beta}(1+\|\varphi\|_\beta). \end{aligned} \tag{2.29}$$

To conclude, the function F defined by (2.19) satisfies Assumption 2.3.

3. The proposed scheme and main result

In this section, we construct a full discrete scheme for (1.1), with the spatial approximation done by a spectral Galerkin method, along with temporal discretization by an accelerated exponential Euler scheme. To do this, we define a finite dimensional subspace of H by $H_N := \mathrm{span}\{\phi_1, \phi_2, \cdots, \phi_N\}$ and the projection operator $\mathcal{P}_N : \dot{H}^\kappa \to H_N$ by

$$\mathcal{P}_N v = \sum_{j=1}^{N}\langle v, \phi_j\rangle\,\phi_j,\ \forall v\in\dot{H}^\kappa,\ \kappa\in\mathbb{R}. \tag{3.1}$$

Then one can straightforwardly show that $\|\mathcal{P}_N v\| \le \|v\|$ and that, for $\forall\kappa\ge 0, v\in\dot{H}^\kappa$,

$$\|(I-\mathcal{P}_N)v\| = \Big(\sum_{j=N+1}^{\infty}\langle v,\phi_j\rangle^2\Big)^{\frac{1}{2}} \le \lambda_{N+1}^{-\frac{\kappa}{2}}\Big(\sum_{j=N+1}^{\infty}\lambda_j^\kappa\langle v,\phi_j\rangle^2\Big)^{\frac{1}{2}} \le \lambda_{N+1}^{-\frac{\kappa}{2}}\|v\|_\kappa. \tag{3.2}$$

Let $A_N : H_N \to H_N$ be defined as $A_N = \mathcal{P}_N A$. Then a spatial approximation of (2.1) leads to a stochastic differential equation in $H_N \times H_N$,

$$\begin{cases} \mathrm{d}X^N(t) = -\mathcal{A}_N X^N(t)\,\mathrm{d}t + \mathbf{F}_N(X^N(t))\,\mathrm{d}t + \mathbf{B}_N\,\mathrm{d}W(t), \quad t\in(0,T],\\ X^N(0) = X_0^N, \end{cases} \tag{3.3}$$

where $X^N(t) := (u^N(t), v^N(t))'$, $X_0^N := (\mathcal{P}_N u_0, \mathcal{P}_N v_0)'$ and

$$\mathcal{A}_N := \begin{bmatrix} 0 & -I\\ A_N & \alpha A_N\end{bmatrix},\quad \mathbf{F}_N(X^N) := \begin{bmatrix} 0\\ P_N F(u^N)\end{bmatrix}\ \text{ and }\ \mathbf{B}_N := \begin{bmatrix} 0\\ P_N\end{bmatrix}.$$

Similarly, $\mathcal{A}_N$ also generates an analytic semigroup $\mathcal{S}_N(t)$, $t\ge 0$, in $H_N\times H_N$ which is an analogue of (2.4), expressed by

$$\mathcal{S}_N(t) = e^{t\mathcal{A}_N} = \begin{bmatrix}\mathcal{S}_{1,N}(t) & \mathcal{S}_{2,N}(t)\\ \mathcal{S}_{3,N}(t) & \mathcal{S}_{4,N}(t)\end{bmatrix}, \tag{3.4}$$

where $\mathcal{S}_{i,N}(t)$, $i=1,2,3,4$ are defined as follows, for $\varphi\in H_N$,

$$\begin{cases} \mathcal{S}_{1,N}(t)\varphi = \sum_{j=1}^{N} \frac{\lambda_j^+ e^{-t\lambda_j^-} - \lambda_j^- e^{-t\lambda_j^+}}{\lambda_j^+ - \lambda_j^-} \langle \varphi, \phi_j \rangle \phi_j, \\ \mathcal{S}_{2,N}(t)\varphi = \sum_{j=1}^{N} \frac{e^{-t\lambda_j^-} - e^{-t\lambda_j^+}}{\lambda_j^+ - \lambda_j^-} \langle \varphi, \phi_j \rangle \phi_j, \\ \mathcal{S}_{3,N}(t)\varphi = \sum_{j=1}^{N} \frac{\lambda_j (e^{-t\lambda_j^+} - e^{-t\lambda_j^-})}{\lambda_j^+ - \lambda_j^-} \langle \varphi, \phi_j \rangle \phi_j, \\ \mathcal{S}_{4,N}(t)\varphi = \sum_{j=1}^{N} \frac{\lambda_j^+ e^{-t\lambda_j^+} - \lambda_j^- e^{-t\lambda_j^-}}{\lambda_j^+ - \lambda_j^-} \langle \varphi, \phi_j \rangle \phi_j. \end{cases} \tag{3.5}$$

Since A commutes with P_N, Assumptions 2.2–2.4 suffice to guarantee a unique solution $X^N(t) = (u^N(t), v^N(t))'$ for (3.3), given by

$$X^N(t) = \mathcal{S}_N(t)X_0^N + \int_0^t \mathcal{S}_N(t-s)\mathbf{F}_N(X^N(s))\,\mathrm{d}s + \int_0^t \mathcal{S}_N(t-s)\mathbf{B}_N\,\mathrm{d}W(s). \tag{3.6}$$

Next we turn our attention to the full discretization of (2.1). Given a time step-size $k = T/M$ for some $M \in \mathbb{N}^+$, we construct uniform mesh grids $t_m = mk$, for $m = 0, 1, 2, \cdots M$. On the gridpoints, we introduce the following full-discrete scheme $X^{N,M} = (u^{N,M}, v^{N,M})'$ based on the spatial discretization (3.6):

$$X_{m+1}^{N,M} = \mathcal{S}_N(k)X_m^{N,M} + k\mathcal{S}_N(k)\mathbf{F}_N(X_m^{N,M}) + \int_{t_m}^{t_{m+1}} \mathcal{S}_N(t_{m+1}-s)\mathbf{B}_N\,\mathrm{d}W(s). \tag{3.7}$$

With the notation in (3.5), one can rewrite (3.7) as

$$\begin{cases} u_{m+1}^{N,M} = \mathcal{S}_{1,N}(k)u_m^{N,M} + \mathcal{S}_{2,N}(k)v_m^{N,M} + k\mathcal{S}_{2,N}(k)P_N F(u_m^{N,M}) + \int_{t_m}^{t_{m+1}} \mathcal{S}_{2,N}(t_{m+1}-s)\,\mathrm{d}W(s), \\ v_{m+1}^{N,M} = \mathcal{S}_{3,N}(k)u_m^{N,M} + \mathcal{S}_{4,N}(k)v_m^{N,M} + k\mathcal{S}_{4,N}(k)P_N F(u_m^{N,M}) + \int_{t_m}^{t_{m+1}} \mathcal{S}_{4,N}(t_{m+1}-s)\,\mathrm{d}W(s), \end{cases} \tag{3.8}$$

for $m = 0, 1, 2, \cdots, M-1$, $M \in \mathbb{N}^+$. A natural question arises how to simulate this scheme and the key point apparently lies on how to simulate the two stochastic convolutions $\int_{t_m}^{t_{m+1}} \mathcal{S}_{2,N}(t_{m+1}-s)\,\mathrm{d}W(s)$ and $\int_{t_m}^{t_{m+1}} \mathcal{S}_{4,N}(t_{m+1}-s)\,\mathrm{d}W(s)$ in (3.8). It is not difficult to find that they are two correlated normally distributed random variables. If the operators A and Q own the same eigenfunctions, the correlation between the stochastic convolutions can be explicitly computed and the proposed scheme becomes rather easy to implement. To show this, from here and below in this section we assume that for all $j \in \mathbb{N}$ there exists a $\gamma_j \in [0, \infty)$ such that

$$Q\phi_j = \gamma_j \phi_j, \tag{3.9}$$

where $\{(\lambda_j, \phi_j)\}_{j=1}^{\infty}$ are the eigenpairs of A. Then the Wiener process $W(t)$ has the following orthogonal expansion [5, Chapter 4]:

$$W(t) = \sum_{j=1}^{\infty} \gamma_j^{\frac{1}{2}} \beta_j(t)\phi_j, \tag{3.10}$$

where $\{\beta_j(t)\}_{j=1}^{\infty}$ are a sequence of real-valued mutually independent standard Brownian motions. For $m = 0, 1, 2, \cdots, M-1$, $j = 1, 2, \cdots, N$, the following random variables

$$\eta_m^j := \left\langle \int_{t_m}^{t_{m+1}} \mathcal{S}_{2,N}(t_{m+1}-s)\,\mathrm{d}W(s), \phi_j \right\rangle = \int_{t_m}^{t_{m+1}} \frac{e^{-(t_{m+1}-s)\lambda_j^-} - e^{-(t_{m+1}-s)\lambda_j^+}}{\lambda_j^+ - \lambda_j^-} \gamma_j^{\frac{1}{2}}\,\mathrm{d}\beta_j(s),$$

and

$$\widehat{\eta}_m^j := \Big\langle \int_{t_m}^{t_{m+1}} \mathcal{S}_{4,N}(t_{m+1}-s)\mathrm{d}W(s), \phi_j \Big\rangle = \int_{t_m}^{t_{m+1}} \frac{\lambda_j^+ e^{-(t_{m+1}-s)\lambda_j^+} - \lambda_j^- e^{-(t_{m+1}-s)\lambda_j^-}}{\lambda_j^+ - \lambda_j^-} \gamma_j^{\frac{1}{2}} \mathrm{d}\beta_j(s),$$

are two series of mutually independent, normally distributed random variables, where λ_j^+ and λ_j^- are defined as in section 2. Furthermore, they satisfy $\mathbf{E}[\eta_m^j] = 0$, $\mathbf{E}[\widehat{\eta}_m^j] = 0$,

$$\mathrm{Var}(\eta_m^j) = \mathbf{E}[|\eta_m^j|^2] = \frac{\gamma_j}{(\lambda_j^+ - \lambda_j^-)^2}\Big[\frac{\alpha}{2} - \frac{e^{-2k\lambda_j^+}}{2\lambda_j^+} - \frac{e^{-2k\lambda_j^-}}{2\lambda_j^-} - \frac{2}{\alpha}(1 - e^{-2k\lambda_j})\Big], \tag{3.11}$$

$$\mathrm{Var}(\widehat{\eta}_m^j) = \mathbf{E}[|\widehat{\eta}_m^j|^2] = \frac{\gamma_j}{(\lambda_j^+ - \lambda_j^-)^2}\Big[\frac{\lambda_j}{2} - \frac{\lambda_j^+}{2}e^{-2k\lambda_j^+} - \frac{\lambda_j^-}{2}e^{-2k\lambda_j^-} - \frac{2}{\alpha\lambda_j}(1 - e^{-\alpha\lambda_j k})\Big], \tag{3.12}$$

and

$$\mathrm{Cov}(\eta_m^j, \widehat{\eta}_m^j) = \mathbf{E}[\eta_m^j \widehat{\eta}_m^j] = \frac{\gamma_j}{(\lambda_j^+ - \lambda_j^-)^2}\Big(\frac{e^{-2k\lambda_j^+} + e^{-2k\lambda_j^-}}{2} - e^{-\alpha k\lambda_j}\Big). \tag{3.13}$$

Let M_m^j be a family of 2×2 matrices with

$$C_m^j := \begin{bmatrix} \mathrm{Var}(\eta_m^j) & \mathrm{Cov}(\eta_m^j, \widehat{\eta}_m^j) \\ \mathrm{Cov}(\eta_m^j, \widehat{\eta}_m^j) & \mathrm{Var}(\widehat{\eta}_m^j) \end{bmatrix} = M_m^j (M_m^j)^T. \tag{3.14}$$

Hence the pair of random variables $(\eta_m^j, \widehat{\eta}_m^j)'$ can be fully characterized by

$$\begin{bmatrix} \eta_m^j \\ \widehat{\eta}_m^j \end{bmatrix} = M_m^j \begin{bmatrix} \xi_m^j \\ \widehat{\xi}_m^j \end{bmatrix}, \tag{3.15}$$

with ξ_m^j and $\widehat{\xi}_m^j$, for $j = 1, 2, \cdots, N$ and $m = 0, 1, 2, \cdots, M-1$ being independent, standard normally distributed random variables. Then for $j = 1, 2, \cdots, N$ and $m = 0, 1, 2, \cdots, M-1$ the coefficients of the expansion of $u_m^{N,M}$ and $v_m^{N,M}$ in the scheme (3.8) can be realized by the following recurrence equations:

$$\begin{aligned} \Big\langle u_{m+1}^{N,M}, \phi_j \Big\rangle &= \frac{\lambda_j^+ e^{-t\lambda_j^-} - \lambda_j^- e^{-k\lambda_j^+}}{\lambda_j^+ - \lambda_j^-} \big\langle u_m^{N,M}, \phi_j \big\rangle + \frac{e^{-k\lambda_j^-} - e^{-k\lambda_j^+}}{\lambda_j^+ - \lambda_j^-} \big\langle v_m^{N,M}, \phi_j \big\rangle \\ &\quad + k\frac{e^{-k\lambda_j^-} - e^{-k\lambda_j^+}}{\lambda_j^+ - \lambda_j^-} \big\langle F(u_m^{N,M}), \phi_j \big\rangle + \eta_m^j, \end{aligned} \tag{3.16}$$

$$\begin{aligned} \Big\langle v_{m+1}^{N,M}, \phi_j \Big\rangle &= \frac{\lambda_j(e^{-t\lambda_j^+} - e^{-k\lambda_j^-})}{\lambda_j^+ - \lambda_j^-} \big\langle u_m^{N,M}, \phi_j \big\rangle + \frac{\lambda_j^+ e^{-k\lambda_j^+} - \lambda_j^- e^{-k\lambda_j^-}}{\lambda_j^+ - \lambda_j^-} \big\langle v_m^{N,M}, \phi_j \big\rangle \\ &\quad + k\frac{\lambda_j^+ e^{-k\lambda_j^+} - \lambda_j^- e^{-k\lambda_j^-}}{\lambda_j^+ - \lambda_j^-} \big\langle F(u_m^{N,M}), \phi_j \big\rangle + \widehat{\eta}_m^j. \end{aligned} \tag{3.17}$$

Now we state our main result.

Theorem 3.1. *Let Assumptions 2.1–2.4 hold. Let $(u, v)'$ be the solution of (2.1) given by (2.12) and for $M, N \in \mathbb{N}$ let $(u^{N,M}, v^{N,M})'$ be the numerical solution produced by (3.8) with $k = T/M$. Then there exists a constant C such that, for all $N, M \in \mathbb{N}$ it holds that*

$$\sup_{m\in\{0,1,\cdots,M\}} \|u(t_m)-u_m^{N,M}\|_{L^2(\Omega;\dot H^0)} \le C(k^{\min\{1+\gamma,1\}}+\lambda_{N+1}^{-\frac{1+\min\{\gamma,1\}}{2}}), \quad \gamma\in(-1,2], \tag{3.18}$$

and

$$\sup_{m\in\{0,1,\cdots,M\}} \|v(t_m)-v_m^{N,M}\|_{L^2(\Omega;\dot H^0)} \le C(k+\lambda_{N+1}^{-\frac{\gamma}{2}}), \quad \gamma\in[0,2]. \tag{3.19}$$

Theorem 3.1 reveals that the obtained strong orders in space are optimal in the sense that the convergence orders coincide with the derived orders of the spatial regularity of $(u,v)'$ in Theorem 2.1. However, the strong order in time is twice as high as the obtained order of the time regularity of the mild solution for $\gamma\in(-1,0]$, especially for the case of the space–time white noise in two or three dimensions. For $\gamma\in[0,2]$, $v_m^{N,M}$ as well as $u_m^{N,M}$ strongly converges with a rate of order one, regardless of the order of regularity of the velocity v. As already discussed in the introduction, this scheme allows for higher strong order in time than the linear implicit Euler scheme investigated in [21]. The proof of Theorem 3.1 is provided in the following section.

4. Proof of the main result

This section is devoted to the proof of the main convergence result Theorem 3.1. In the first part, we present some useful preparatory results. In particular, the following proposition plays an essential role in the error analysis of our scheme.

Proposition 4.1. *Under the assumptions of Theorem 2.1, there exists a constant C such that, for all $0\le s<t\le T$,*

$$\|u(t)-u(s)\|_{L^p(\Omega;\dot H^{-\delta})} \le C(t-s)^{\frac{\varrho+\delta}{2}}\big(1+\|u_0\|_{L^p(\Omega;\dot H^{\varrho})}+\|v_0\|_{L^p(\Omega;\dot H^{\varrho-2})}\big), \tag{4.1}$$

where

$$\delta\in[0,1-\min\{1,|\gamma|\}], \quad \text{and} \quad \varrho:=\min\{\gamma+1,2\}, \ \gamma\in[1,2]. \tag{4.2}$$

Before showing Proposition 4.1, we quote some useful results from [16, Lemma 2.3] and [21, Lemma 2.9] and introduce two operators P_1 and P_2 defined by

$$P_i x = x_i, \ \forall x=(x_1,x_2)'\in H_1\times H_2$$

for two Hilbert spaces H_1, H_2. Then $P_1\mathcal{S}(t)x=\mathcal{S}_1(t)x_1+\mathcal{S}_2(t)x_2$ and $P_2\mathcal{S}(t)x=\mathcal{S}_3(t)x_1+\mathcal{S}_4(t)x_2$ for $x=(x_1,x_2)'$.

Lemma 4.1. *Denote $x=(x_1,x_2)'$ and let $\mathcal{S}(t)$ be the semigroup with four components $\mathcal{S}_i(t)$, $i=1,2,3,4$ as defined in (2.4) and (2.5). Then there exists a constant C such that, for all $0\le s<t\le T$,*

$$\|P_1\mathcal{S}(t)x\| \le C(\|x_1\|+\|x_2\|_{-\mu}), \qquad \mu\in[0,2], \tag{4.3}$$

$$\|P_1[\mathcal{S}(t)-\mathcal{S}(s)]x\| \le C(t-s)^{\frac{\mu}{2}}(\|x_1\|_{\mu}+\|x_2\|_{\mu-2}), \qquad \mu\in[0,2], \tag{4.4}$$

$$\|P_2[\mathcal{S}(t)-\mathcal{S}(s)]x\| \le C(t-s)^{\frac{\mu}{2}}(\|x_1\|_{\mu}+\|x_2\|_{\mu}), \qquad \mu\in[0,2], \tag{4.5}$$

$$\int_s^t \|P_2\mathcal{S}(t-r)x\|^2\,\mathrm{d}r \le C(t-s)^{\nu}(\|x_1\|_{\nu}^2+\|x_2\|_{\nu-1}^2), \qquad \nu\in[0,1]. \tag{4.6}$$

Proof of Lemma 4.1. The assertion (4.3) comes from [16, lemma 2.3] and (4.4)–(4.6) can be found in [21, Lemma 2.9]. We remark that the first assertion can be deduced directly from [20, Theorem 2.6.13 (c)] by recalling that $-\mathcal{A}$ generates an analytic semigroup in $\dot{H}^0 \times \dot{H}^{-\mu}$, for $\mu \in [0,2]$. The other three inequalities are, up to our judgments, not direct consequences of [20, Theorem 2.6.13]. □

Proof of Proposition 4.1. Noting that $\mathcal{S}(0) = I$, we employ (2.4) and (2.12) to obtain

$$u(t) - u(s) = P_1[\mathcal{S}(t-s) - \mathcal{S}(0)]X(s) + \int_s^t \mathcal{S}_2(t-r)F(u(r))\,\mathrm{d}r + \int_s^t \mathcal{S}_2(t-r)\,\mathrm{d}W(r), \tag{4.7}$$

which suggests that

$$\begin{aligned}\|u(t) - u(s)\|_{L^p(\Omega;\dot{H}^{-\delta})} &\le \|P_1[\mathcal{S}(t-s) - \mathcal{S}(0)]X(s)\|_{L^p(\Omega;\dot{H}^{-\delta})} \\ &\quad + \int_s^t \|\mathcal{S}_2(t-r)F(u(r))\|_{L^p(\Omega;\dot{H}^{-\delta})}\,\mathrm{d}r + \Big\|\int_s^t \mathcal{S}_2(t-r)\,\mathrm{d}W(r)\Big\|_{L^p(\Omega;\dot{H}^{-\delta})} \\ &:= \mathbb{I} + \mathbb{III} + \mathbb{IIII}. \end{aligned} \tag{4.8}$$

Since $\frac{\varrho+\delta}{2} \le 1$, for δ, ϱ as defined in (4.2), one can derive by (2.13) and (4.4), there exists a constant C such that, for all $0 \le s < t \le T$ it holds

$$\begin{aligned}\mathbb{I} &= \|P_1[\mathcal{S}(t-s) - \mathcal{S}(0)]\big(A^{-\frac{\delta}{2}}u(s), A^{-\frac{\delta}{2}}v(s)\big)'\|_{L^p(\Omega;\dot{H}^0)} \\ &\le C(t-s)^{\frac{\varrho+\delta}{2}}(\|u(s)\|_{L^p(\Omega;\dot{H}^\varrho)} + \|v(s)\|_{L^p(\Omega;\dot{H}^{\varrho-2})}) \\ &\le C(t-s)^{\frac{\varrho+\delta}{2}}(1 + \|u_0\|_{L^p(\Omega;\dot{H}^\varrho)} + \|v_0\|_{L^p(\Omega;\dot{H}^{\varrho-2})}). \end{aligned} \tag{4.9}$$

For $\mathbb{III}$, using (2.17) and applying (4.3) with $x = (0, F(u))'$ and $\mu = 0$ enable us to obtain that, there exists a constant C such that, for all $0 \le s < t \le T$ it holds

$$\mathbb{III} \le \int_s^t \|F(u(r))\|_{L^p(\Omega;\dot{H}^0)}\,\mathrm{d}r \le C(t-s)(1 + \|u_0\|_{L^p(\Omega;\dot{H}^0)} + \|v_0\|_{L^p(\Omega;\dot{H}^{-2})}). \tag{4.10}$$

To handle the estimate of $\mathbb{IIII}$, we utilize (4.4), the Burkholder–Davis–Gundy type inequality and the definition of the Hilbert–Schmidt norm to show, there exists a constant C such that, for all $0 \le s < t \le T$,

$$\begin{aligned}|\mathbb{IIII}|^2 &\le C_p^2 \int_s^t \big\|\mathcal{S}_2(t-r)A^{-\frac{\delta}{2}}Q^{\frac{1}{2}}\big\|_{\mathrm{HS}}^2\,\mathrm{d}r \\ &= C_p^2 \sum_{j=1}^{\infty} \int_s^t \big\|[\mathcal{S}_2(t-r) - \mathcal{S}_2(0)]A^{-\frac{\delta}{2}}Q^{\frac{1}{2}}\phi_j\big\|^2\,\mathrm{d}r \\ &\le C \sum_{j=1}^{\infty} \int_s^t (t-r)^{\varrho+\delta}\big\|A^{\frac{\varrho-2}{2}}Q^{\frac{1}{2}}\phi_j\big\|^2\,\mathrm{d}r \\ &\le C(t-s)^{1+\varrho+\delta}\big\|A^{\frac{\varrho-2}{2}}Q^{\frac{1}{2}}\big\|_{\mathrm{HS}}^2 \\ &\le C(t-s)^{1+\varrho+\delta}\big\|A^{\frac{\gamma-1}{2}}Q^{\frac{1}{2}}\big\|_{\mathrm{HS}}^2, \end{aligned} \tag{4.11}$$

where we also used the fact $\mathcal{S}_2(0)x = 0$. Inserting (4.9)–(4.11) into (4.8) finally finishes the proof of this proposition. □

To complete the proof of the main result, we additionally need to exploit further regularity results on the semigroup $\mathcal{S}(t)$, as stated in the following lemmas.

Lemma 4.2. *Assume that $\mathcal{S}(t)$ is the analytic semigroup with four components $\mathcal{S}_i(t)$, $i = 1, 2, 3, 4$ as defined in (2.5). Then for $x = (x_1, x_2)'$ and $0 < s < t \le T$ we have*

$$\|P_2\mathcal{S}(t)x\| \le C(\|x_1\| + t^{-\frac{\mu}{2}}\|x_2\|_{-\mu}), \qquad \mu \in [0, 2], \tag{4.12}$$

$$\|P_2[\mathcal{S}(t) - \mathcal{S}(s)]x\| \le C(t-s)^{\frac{\mu}{2}}(\|x_1\|_\mu + s^{-\frac{\mu-\nu}{2}}\|x_2\|_\nu), \qquad \mu \in [0, 2],\ \nu \in [-2, \mu], \tag{4.13}$$

$$\Big\| \int_s^t \mathcal{S}_4(t-r)x_2\,\mathrm{d}r \Big\| \le C(t-s)^{\frac{2-\mu}{2}}\|x_2\|_{-\mu}, \qquad \mu \in [0, 2]. \tag{4.14}$$

Proof of Lemma 4.2. To get (4.12), we first recall from [16, Lemma 2.3] that

$$\|\mathcal{S}_3(t)x_1\| + \|\mathcal{S}_4(t)x_2\|_j \le C(\|x_1\| + t^{-\frac{j}{2}}\|x_2\|),\ j \in \{0, 2\}. \tag{4.15}$$

Thanks to the interpolation theory and (4.15), there exists a constant C such that for all $\mu \in [0, 2]$ and $t > 0$ it holds

$$\begin{aligned} \|P_2\mathcal{S}(t)x\| &\le \|\mathcal{S}_3(t)x_1\| + \|A^{\frac{\mu}{2}}\mathcal{S}_4(t)A^{-\frac{\mu}{2}}x_2\| \\ &\le C(\|x_1\| + \|A\mathcal{S}_4(t)A^{-\frac{\mu}{2}}x_2\|^{\frac{\mu}{2}}\|\mathcal{S}_4(t)A^{-\frac{\mu}{2}}x_2\|^{\frac{2-\mu}{2}}) \\ &\le C(\|x_1\| + t^{-\frac{\mu}{2}}\|x_2\|_{-\mu}). \end{aligned} \tag{4.16}$$

Concerning (4.13), we first use (4.5) with the choice of $x = (x_1, 0)'$ to show, there exists a constant C such that, for all $0 \le s < t \le T$,

$$\|[\mathcal{S}_3(t) - \mathcal{S}_3(s)]x_1\| \le C(t-s)^{\frac{\mu}{2}}\|x_1\|_\mu,\ \forall x_1 \in \dot{H}^\mu,\ \mu \in [0,\ 2]. \tag{4.17}$$

Therefore, bearing the definitions of P_2 and $\mathcal{S}(t)$ in mind, it suffices to prove that, there exists a constant C such that, for all $0 \le s < t \le T$,

$$\|[\mathcal{S}_4(t) - \mathcal{S}_4(s)]x_2\| \le C(t-s)^{\frac{\mu}{2}}s^{-\frac{\mu-\nu}{2}}\|x_2\|_\nu,\ \forall x_2 \in \dot{H}^\nu,\ \mu \in [0, 2],\ \nu \in [-2, \mu]. \tag{4.18}$$

Employing (2.5) implies

$$\begin{aligned} \|[\mathcal{S}_4(t) - \mathcal{S}_4(s)]x_2\|^2 &= \sum_{j=0}^{\infty} \frac{[\lambda_j^+(e^{-t\lambda_j^+} - e^{-s\lambda_j^+}) - \lambda_j^-(e^{-t\lambda_j^-} - e^{-s\lambda_j^-})]^2}{[\lambda_j^+ - \lambda_j^-]^2}\langle x_2, \phi_j\rangle^2 \\ &\le 2\sum_{j=0}^{\infty} \frac{(\lambda_j^+)^2(e^{-t\lambda_j^+} - e^{-s\lambda_j^+})^2}{[\lambda_j^+ - \lambda_j^-]^2}\langle x_2, \phi_j\rangle^2 \\ &\quad + 2\sum_{j=0}^{\infty} \frac{(\lambda_j^-)^2(e^{-t\lambda_j^-} - e^{-s\lambda_j^-})^2}{[\lambda_j^+ - \lambda_j^-]^2}\langle x_2, \phi_j\rangle^2 \\ &:= B_1 + B_2. \end{aligned} \tag{4.19}$$

Before proceeding further, we recall (2.7), as a consequence of which, one also has

$$\sup_{j\in\mathbb{N}}\left|\frac{\lambda_j^+}{\lambda_j^+-\lambda_j^-}\right|<\infty,\quad \textit{and}\quad \sup_{j\in\mathbb{N}}\left|\frac{\lambda_j}{\lambda_j^+-\lambda_j^-}\right|<\infty. \tag{4.20}$$

In addition, there exists a constant C such that for all $j\in\mathbb{N}$, $(\lambda_j^+)^{-\frac{\mu}{2}}(1-e^{-(t-s)\lambda_j^+})\le C(t-s)^{\frac{\mu}{2}}$, $\mu\in[0,2]$. These facts together with the observation that $\sup_{s>0}s^{\mu-\nu}e^{-s}<\infty$ for $\nu\le\mu$ help us to find a constant C such that, for all $0\le s<t\le T$,

$$\begin{aligned}
B_1&\le C\sum_{j=0}^{\infty}(e^{-t\lambda_j^+}-e^{-s\lambda_j^+})^2\langle x_2,\phi_j\rangle^2\\
&\le C\sum_{j=0}^{\infty}\left[(\lambda_j^+)^{\frac{\mu-\nu}{2}}e^{-s\lambda_j^+}(\lambda_j^+)^{-\frac{\mu}{2}}(e^{-(t-s)\lambda_j^+}-1)\right]^2(\lambda_j^+)^{\nu}\langle x_2,\phi_j\rangle^2\\
&\le Cs^{-(\mu-\nu)}(t-s)^{\mu}\|x_2\|_{\nu}^2.
\end{aligned} \tag{4.21}$$

Similarly as above, there is a constant C such that, for all $j\in\mathbb{N}$, $(\lambda_j^-)^{-\frac{\mu}{2}}(1-e^{-(t-s)\lambda_j^-})\le C(t-s)^{\frac{\mu}{2}}$, $\mu\in[0,2]$. Owing to (2.7), (4.20) and the fact that $\sup_{s>0}s^{\mu-\nu}e^{-s}<\infty$ for $\nu\in[-2,\mu]$, one can find a constant C such that, for all $0\le s<t\le T$,

$$\begin{aligned}
B_2&\le C\sum_{j=0}^{\infty}(\lambda_j^-)^2(e^{-t\lambda_j^-}-e^{-s\lambda_j^-})^2\lambda_j^{-2}\langle x_2,\phi_j\rangle^2\\
&\le C\sum_{j=0}^{\infty}\left[(\lambda_j^-)^{\frac{\mu-\nu}{2}}e^{-s\lambda_j^-}(\lambda_j^-)^{-\frac{\mu}{2}}(e^{-(t-s)\lambda_j^-}-1)\right]^2(\lambda_j^-)^{\nu+2}\lambda_j^{-2}\langle x_2,\phi_j\rangle^2\\
&\le Cs^{-(\mu-\nu)}(t-s)^{\mu}\|x_2\|_{-2}^2\\
&\le Cs^{-(\mu-\nu)}(t-s)^{\mu}\|x_2\|_{\nu}^2.
\end{aligned} \tag{4.22}$$

Inserting this estimate and (4.21) into (4.19) ends the proof of (4.13). With regard to (4.14), the definition of $\mathcal{S}_4(t)$ enables us to deduce that

$$\left\|\int_s^t\mathcal{S}_4(t-r)x_2\,\mathrm{d}r\right\|=\left(\sum_{j=1}^{\infty}\left|\int_s^t\frac{\lambda_j^+e^{-\lambda_j^+(t-r)}-\lambda_j^-e^{-\lambda_j^-(t-r)}}{\lambda_j^+-\lambda_j^-}\,\mathrm{d}r\right|^2\langle x_2,\phi_j\rangle^2\right)^{\frac12}. \tag{4.23}$$

Moreover, there exists a constant C such that, for all $\lambda\in[0,\infty)$ and for all $0\le s<t\le T$ it holds

$$0\le\int_s^t\lambda^{\frac{\mu}{2}}e^{-\lambda(t-r)}\,\mathrm{d}r=\lambda^{\frac{\mu-2}{2}}(1-e^{-\lambda(t-s)})\le C(t-s)^{\frac{2-\mu}{2}},\quad \mu\in[0,2]. \tag{4.24}$$

Additionally, recalling (2.7) and (4.20) again leads us to get a constant C such that, for all $\mu\in[0,2]$ and for all $0\le s<t\le T$,

$$\begin{aligned}
\left\|\int_s^t\mathcal{S}_4(t-r)x_2\,\mathrm{d}r\right\|&\le 2C\left(\sum_{j=1}^{\infty}\left|\int_s^t(\lambda_j^+)^{\frac{\mu}{2}}e^{-\lambda_j^+(t-r)}\,\mathrm{d}r\right|^2(\lambda_j^+)^{-\mu}\langle x_2,\phi_j\rangle^2\right)^{\frac12}\\
&\quad+2C\left(\sum_{j=1}^{\infty}\left|\int_s^t(\lambda_j^-)^{\frac{\mu}{2}}e^{-\lambda_j^-(t-r)}\,\mathrm{d}r\right|^2(\lambda_j^-)^{2-\mu}\lambda_j^{-2}\langle x_2,\phi_j\rangle^2\right)^{\frac12}
\end{aligned}$$

$$\leq C(t-s)^{\frac{2-\mu}{2}}\Big(\sum_{j=1}^{\infty}\lambda_j^{-\mu}\langle x_2,\phi_j\rangle^2\Big)^{\frac{1}{2}}$$
$$\leq C(t-s)^{\frac{2-\mu}{2}}\|x_2\|_{-\mu}, \tag{4.25}$$

where we also used the fact $\sup_{j\in\mathbb{N}^+}(\lambda_j^-)^{2-\mu}<\infty$ for $\mu\in[0,2]$. This validates Lemma 4.2. □

Lemma 4.3. *Let $\mathcal{S}_4(t)$ be a component of the semigroup $\mathcal{S}(t)$ defined in (2.5). Then there exists a constant C such that for all $m\in\{0,1,\cdots,M\}, M\in\mathbb{N}$ it holds*

$$\mathbb{J}:=\Big\|\sum_{l=0}^{m-1}\int_{t_l}^{t_{l+1}}[\mathcal{S}_4(t_m-s)-\mathcal{S}_4(t_m-t_l)]x_2\,\mathrm{d}s\Big\|\leq Ck\|x_2\|,\ \forall\, x_2\in\dot{H}^0. \tag{4.26}$$

Proof of Lemma 4.3. We apply the same argument used in the proof of (4.14) to arrive at

$$\mathbb{J}\leq\Big(\sum_{j=1}^{\infty}\Big|\sum_{l=0}^{m-1}\int_{t_l}^{t_{l+1}}\frac{\lambda_j^+[e^{-\lambda_j^+(t_m-s)}-e^{-\lambda_j^+(t_m-t_l)}]}{\lambda_j^+-\lambda_j^-}\,\mathrm{d}s\Big|^2\langle x_2,\phi_j\rangle^2\Big)^{\frac{1}{2}}$$
$$+\Big(\sum_{j=1}^{\infty}\Big|\sum_{l=0}^{m-1}\int_{t_l}^{t_{l+1}}\frac{\lambda_j^-[e^{-\lambda_j^-(t_m-s)}-e^{-\lambda_j^-(t_m-t_l)}]}{\lambda_j^+-\lambda_j^-}\mathrm{d}s\Big|^2\langle x_2,\phi_j\rangle^2\Big)^{\frac{1}{2}}. \tag{4.27}$$

Note that there exists a constant C such that, for all $m\in\{0,1,\cdots,M\}, M\in\mathbb{N}$ and for all $\lambda>0$,

$$0\leq\sum_{l=0}^{m-1}\int_{t_l}^{t_{l+1}}[e^{-\lambda(t_m-s)}-e^{-\lambda(t_m-t_l)}]\,\mathrm{d}s=\sum_{l=0}^{m-1}\int_{t_l}^{t_{l+1}}\lambda e^{-\lambda(t_m-s)}\lambda^{-1}(1-e^{-\lambda(s-t_l)})\,\mathrm{d}s$$
$$\leq Ck\int_0^{t_m}\lambda e^{-\lambda(t_m-s)}\,\mathrm{d}s$$
$$\leq Ck. \tag{4.28}$$

Hence by inserting this result into (4.27) and applying (4.20), one finds a constant C such that, for all $m\in\{0,1,\cdots,M\}, M\in\mathbb{N}$ and for any $x_2\in\dot{H}^0$,

$$\mathbb{J}\leq C\Big(\sum_{j=1}^{\infty}\Big|\sum_{l=0}^{m-1}\int_{t_l}^{t_{l+1}}[e^{-\lambda_j^+(t_m-s)}-e^{-\lambda_j^+(t_m-t_l)}]\mathrm{d}s\Big|^2\langle x_2,\phi_j\rangle^2\Big)^{\frac{1}{2}}$$
$$+C\Big(\sum_{j=1}^{\infty}\Big|\sum_{l=0}^{m-1}\int_{t_l}^{t_{l+1}}[e^{-\lambda_j^-(t_m-s)}-e^{-\lambda_j^-(t_m-t_l)}]\mathrm{d}s\Big|^2\langle x_2,\phi_j\rangle^2\Big)^{\frac{1}{2}}$$
$$\leq Ck\|x_2\|. \tag{4.29}$$

This thus completes the proof of Lemma 4.3. □

Armed with the above preparatory results, we are now ready to prove the main convergence result.

Proof of Theorem 3.1. By recursion, the numerical solution $X_m^{N,M}$ can be written as

$$X_m^{N,M} = \mathcal{S}_N(t_m)X_0^{N,M} + \sum_{l=0}^{m-1}\int_{t_l}^{t_{l+1}} \mathcal{S}_N(t_m - t_l)\mathbf{F}_N(X_l^{N,M})\,\mathrm{d}s + \int_0^{t_m} \mathcal{S}_N(t_m - s)\mathbf{B}_N\,\mathrm{d}W(s). \tag{4.30}$$

Next we deal with the error $u(t_m) - u_m^{N,M}$. Owing to (2.12), (4.30), and the definitions of $\mathcal{S}(t)$ and $\mathcal{S}_N(t)$, one can easily do the following decompositions,

$$\begin{aligned} u(t_m) - u_m^{N,M} &= (I - \mathcal{P}_N)P_1\mathcal{S}(t_m)X_0 + \sum_{l=0}^{m-1}\int_{t_l}^{t_{l+1}} (I - \mathcal{P}_N)\mathcal{S}_2(t_m - s)F(u(s))\,\mathrm{d}s \\ &\quad + \sum_{l=0}^{m-1}\int_{t_l}^{t_{l+1}} [\mathcal{S}_2(t_m - s) - \mathcal{S}_2(t_m - t_l)]\mathcal{P}_N F(u(s))\,\mathrm{d}s \\ &\quad + \sum_{l=0}^{m-1}\int_{t_l}^{t_{l+1}} \mathcal{S}_2(t_m - t_l)\mathcal{P}_N[F(u(s)) - F(u(t_l))]\,\mathrm{d}s \\ &\quad + \sum_{l=0}^{m-1}\int_{t_l}^{t_{l+1}} \mathcal{S}_2(t_m - t_l)\mathcal{P}_N[F(u(t_l)) - F(u_l^{N,M})]\,\mathrm{d}s \\ &\quad + \int_0^{t_m} \mathcal{S}_2(t_m - s)(I - \mathcal{P}_N)\,\mathrm{d}W(s) \\ &:= I_1 + I_2 + I_3 + I_4 + I_5 + I_6. \end{aligned} \tag{4.31}$$

For the first term I_1, note that $(u_0, v_0)' \in L^2(\Omega; \dot{H}^\varrho \times \dot{H}^{\varrho-2})$ for $\varrho := \min\{1+\gamma, 2\}$, $\gamma \in (-1, 2]$. Then (4.3) and (3.2) help us to get a constant C such that, for all $N \in \mathbb{N}$ and $\varrho := \min\{1+\gamma, 2\}$, $\gamma \in (-1, 2]$,

$$\|I_1\|_{L^2(\Omega;\dot{H}^0)} \le \lambda_{N+1}^{-\frac{\varrho}{2}}\|P_1\mathcal{S}(t)X_0\|_{L^2(\Omega;\dot{H}^\varrho)} \le C\lambda_{N+1}^{-\frac{\varrho}{2}}(\|u_0\|_{L^2(\Omega;\dot{H}^\varrho)} + \|v_0\|_{L^2(\Omega;\dot{H}^{\varrho-2})}). \tag{4.32}$$

Similarly, using (3.2), (2.17) and (4.3) with $\mu = 2$ shows a constant C such that, for all $N \in \mathbb{N}$,

$$\begin{aligned} \|I_2\|_{L^2(\Omega;\dot{H}^0)} &\le \lambda_{N+1}^{-1}\int_0^{t_m} \|A\mathcal{S}_2(t_m - s)F(u(s))\|_{L^2(\Omega;\dot{H}^0)}\,\mathrm{d}s \\ &\le C\lambda_{N+1}^{-1}\|F(u)\|_{L^\infty([0,T],\ L^2(\Omega;\dot{H}^0))} \\ &\le C\lambda_{N+1}^{-1}(1 + \|u_0\|_{L^2(\Omega;\dot{H}^0)} + \|v_0\|_{L^2(\Omega;\dot{H}^{-2})}). \end{aligned} \tag{4.33}$$

Due to (2.17) and (4.4) with $\mu = 2$, we obtain a constant C such that, for all $k = T/M, M \in \mathbb{N}$,

$$\begin{aligned} \|I_3\|_{L^2(\Omega;\dot{H}^0)} &\le C\sum_{l=0}^{m-1}\int_{t_l}^{t_{l+1}} \big\|[\mathcal{S}_2(t_m - s) - \mathcal{S}_2(t_m - t_l)]P_N F(u(s))\big\|_{L^2(\Omega;\dot{H}^0)}\,\mathrm{d}s \\ &\le C\sum_{l=0}^{m-1}\int_{t_l}^{t_{l+1}} (s - t_l)\,\mathrm{d}s\,\|F(u)\|_{L^\infty([0,T],\ L^2(\Omega;\dot{H}^0))} \\ &\le Ck(1 + \|u_0\|_{L^2(\Omega;\dot{H}^0)} + \|v_0\|_{L^2(\Omega;\dot{H}^{-2})}). \end{aligned} \tag{4.34}$$

To treat the term I_4, we perform a linearization step and use (4.3) with $x_1 = 0$ to get a constant C such that, for all $m \in \{0, 1, \cdots, M\}, M \in \mathbb{N}$ and for $\theta \in [1, 2)$,

$$\begin{aligned}\|I_4\|_{L^2(\Omega;\dot{H}^0)} &\leq \sum_{l=0}^{m-1} \int_{t_l}^{t_{l+1}} \int_0^1 \|\mathcal{S}_2(t_m - t_l) P_N F'(\chi(t_l, s, r))[u(s) - u(t_l)]\|_{L^2(\Omega;\dot{H}^0)} \,\mathrm{d}r \,\mathrm{d}s \\ &\leq C \sum_{l=0}^{m-1} \int_{t_l}^{t_{l+1}} \int_0^1 \|A^{-\frac{\theta}{2}} F'(\chi(t_l, r, s))[u(s) - u(t_l)]\|_{L^2(\Omega;\dot{H}^0)} \,\mathrm{d}r \,\mathrm{d}s, \end{aligned} \tag{4.35}$$

where for simplicity of presentation we denote

$$\chi(t_l, s, r) := u(t_l) + r(u(s) - u(t_l)). \tag{4.36}$$

Recall that $1 - \min\{1, |\gamma|\} \leq \beta := \min\{1, 1 + \gamma\} \leq \varrho := \min\{1 + \gamma, 2\}$ for $\gamma \in (-1, 2]$. Hence employing (2.10), (2.13), (4.1), and Hölder's inequality yields a constant C such that, for all $k = T/M, M \in \mathbb{N}$,

$$\begin{aligned}\|A^{-\frac{\theta}{2}} F'(\chi(t_l, s, r))[u(s) - u(t_l)]\|_{L^2(\Omega;\dot{H}^0)} &\leq K(\|\chi(t_l, s, r)\|_{L^4(\Omega;\dot{H}^\beta)} + 1)\|u(s) - u(t_l)\|_{L^4(\Omega;\dot{H}^{-\beta})} \\ &\leq C(\|u\|_{L^\infty([0,T],\, L^4(\Omega;\dot{H}^\varrho))} + 1)\|u(s) - u(t_l)\|_{L^4(\Omega;\dot{H}^{-\beta})} \\ &\leq C k^{\min\{1+\gamma,1\}} \big(1 + \|u_0\|^2_{L^4(\Omega;\dot{H}^\varrho)} + \|v_0\|^2_{L^4(\Omega;\dot{H}^{\varrho-2})}\big). \end{aligned} \tag{4.37}$$

Plugging this into (4.35) gives a constant C such that, for all $k = T/M, M \in \mathbb{N}$,

$$\|I_4\|_{L^2(\Omega;\dot{H}^0)} \leq C k^{\min\{1+\gamma,1\}} (1 + \|u_0\|^2_{L^4(\Omega;\dot{H}^\varrho)} + \|v_0\|^2_{L^4(\Omega;\dot{H}^{\varrho-2})}). \tag{4.38}$$

Next, with the aid of (4.3) and the Lipschitz condition (2.9), we derive for the term I_5 that, for all $m \in \{0, 1, 2, \cdots, M\}, M \in \mathbb{N}$ it holds

$$\begin{aligned}\|I_5\|_{L^2(\Omega;\dot{H}^0)} &\leq \sum_{l=0}^{m-1} \int_{t_l}^{t_{l+1}} \|\mathcal{S}_2(t_m - t_l)\mathcal{P}_N\|_{\mathcal{L}(\dot{H}^0)} \|F(u(t_l)) - F(u_l^{N,M})\|_{L^2(\Omega;\dot{H}^0)} \,\mathrm{d}s \\ &\leq C k \sum_{l=0}^{m-1} \|u(t_l) - u_l^{N,M}\|_{L^2(\Omega;\dot{H}^0)}. \end{aligned} \tag{4.39}$$

For the last term I_6, we use (4.3), the Itô isometry and the property of P_N to show, for all $m \in \{0, 1, 2, \cdots, M\}, M \in \mathbb{N}$,

$$\begin{aligned}\|I_6\|_{L^2(\Omega;\dot{H}^0)} &= \Big(\int_0^{t_m} \|\mathcal{S}_2(t_m - s)(I - \mathcal{P}_N) Q^{\frac{1}{2}}\|^2_{\mathrm{HS}} \,\mathrm{d}s \Big)^{\frac{1}{2}} \\ &\leq C\sqrt{T} \|A^{-1}(I - \mathcal{P}_N) Q^{\frac{1}{2}}\|_{\mathrm{HS}} \\ &\leq C \lambda_{N+1}^{-\frac{\varrho}{2}} \|A^{\frac{\varrho-2}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}} \\ &\leq C \lambda_{N+1}^{-\frac{\varrho}{2}} \|A^{\frac{\gamma-1}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}}. \end{aligned} \tag{4.40}$$

Finally, inserting the above estimates into (4.31) and applying the discrete Gronwall inequality imply that, for $M, N \in \mathbb{N}$ and $\varrho = \min\{1 + \gamma, 2\}$

$$\sup_{m\in\{0,1,\cdots,M\}}\|u(t_m)-u_m^{N,M}\|_{L^2(\Omega;H)}\le C(\lambda_{N+1}^{-\frac{\varrho}{2}}+k^{\min\{1+\gamma,1\}})\big(1+\|u_0\|^2_{L^4(\Omega;\dot H^{\varrho})}+\|v_0\|^2_{L^4(\Omega;\dot H^{\varrho-2})}\big),\quad(4.41)$$

which verifies (3.18). Next we start to treat the error $\|v(t_m)-v_m^{N,M}\|_{L^2(\Omega;\dot H^0)}$:

$$\begin{aligned}v(t_m)-v_m^{N,M}&=(I-\mathcal{P}_N)P_2\mathcal{S}(t_m)X_0+\int_0^{t_m}\mathcal{S}_4(t_m-s)(I-\mathcal{P}_N)\,\mathrm{d}W(s)\\&\quad+\sum_{l=0}^{m-1}\int_{t_l}^{t_{l+1}}\big[\mathcal{S}_4(t_m-s)F(u(s))-\mathcal{S}_{4,N}(t_m-t_l)\mathcal{P}_NF(u_l^{N,M})\big]\,\mathrm{d}s\\&:=J_1+J_2+J_3.\end{aligned}\quad(4.42)$$

Analogously to the estimate of I_1, we use (4.12) with $\mu=0$ to bound the term J_1. There exists a constant C such that, for all $N\in\mathbb{N}$,

$$\|J_1\|_{L^2(\Omega;\dot H^0)}\le C\lambda_{N+1}^{-\frac{\gamma}{2}}\big(\|u_0\|_{L^2(\Omega;\dot H^{\gamma})}+\|v_0\|_{L^2(\Omega;\dot H^{\gamma})}\big),\quad\gamma\in[0,2].\quad(4.43)$$

Owing to (4.6) with $\nu=0$, Assumption 2.2 and the Itô isometry, we acquire that, there exists a constant C such that, for all $m\in\{0,1,2,\cdots,M\}, M\in\mathbb{N}$, $N\in\mathbb{N}$ and $\gamma\in[0,2]$,

$$\begin{aligned}\|J_2\|_{L^2(\Omega;\dot H^0)}&=\Big(\int_0^{t_m}\|\mathcal{S}_4(t_m-s)(I-\mathcal{P}_N)Q^{\frac12}\|^2_{\mathrm{HS}}\,\mathrm{d}s\Big)^{\frac12}\\&\le C\|A^{-\frac12}(I-\mathcal{P}_N)Q^{\frac12}\|_{\mathrm{HS}}\\&\le C\lambda_{N+1}^{-\frac{\gamma}{2}}\|A^{\frac{\gamma-1}{2}}Q^{\frac12}\|_{\mathrm{HS}}.\end{aligned}\quad(4.44)$$

Now it remains to estimate J_3. To this aim, we furthermore decompose J_3 into three terms

$$\begin{aligned}\|J_3\|_{L^2(\Omega;\dot H^0)}&\le\Big\|\sum_{l=0}^{m-1}\int_{t_l}^{t_{l+1}}(I-\mathcal{P}_N)\mathcal{S}_4(t_m-s)F(u(s))\,\mathrm{d}s\Big\|_{L^2(\Omega;\dot H^0)}\\&\quad+\Big\|\sum_{l=0}^{m-1}\int_{t_l}^{t_{l+1}}\mathcal{P}_N[\mathcal{S}_4(t_m-s)-\mathcal{S}_4(t_m-t_l)]F(u(s))\,\mathrm{d}s\Big\|_{L^2(\Omega;\dot H^0)}\\&\quad+\Big\|\sum_{l=0}^{m-1}\int_{t_l}^{t_{l+1}}\mathcal{P}_N\mathcal{S}_4(t_m-t_l)[F(u(s))-F(u_l^{N,M})]\,\mathrm{d}s\Big\|_{L^2(\Omega;\dot H^0)}\\&:=J_3^1+J_3^2+J_3^3.\end{aligned}\quad(4.45)$$

Below, we will estimate them separately. Concerning J_3^1, we employ (2.18), (4.12) with $\mu=2$, (4.14) with $\mu=2$ and (3.2) to show, for $\varrho=\min\{1+\gamma,2\}$, $\gamma\in[0,2]$ and for all $m\in\{0,1,2,\cdots,M\}, M\in\mathbb{N}$ and $N\in\mathbb{N}$ there exists a constant C such that

$$J_3^1\le\lambda_{N+1}^{-1}\Big[\int_0^{t_m}\|A\mathcal{S}_4(t_m-s)[F(u(s))-F(u(t_m))]\|_{L^2(\Omega;\dot H^0)}\,\mathrm{d}s$$

$$+\Big\|\int_0^{t_m} A\mathcal{S}_4(t_m-s)F(u(t_m))\,\mathrm{d}s\Big\|_{L^2(\Omega;\dot{H}^0)}\Big]$$

$$\le C\lambda_{N+1}^{-1}\Big[\int_0^{t_m}(t_m-s)^{-1}\|F(u(s))-F(u(t_m))\|_{L^2(\Omega;\dot{H}^0)}\,\mathrm{d}s+\|F(u(t_m))\|_{L^2(\Omega;\dot{H}^0)}\Big]$$

$$\le C\lambda_{N+1}^{-1}\Big[\int_0^{t_m}(t_m-s)^{-1}(t_m-s)^{\frac{\varrho}{2}}\,\mathrm{d}s+1\Big](1+\|u_0\|_{L^2(\Omega;\dot{H}^\varrho)}+\|v_0\|_{L^2(\Omega;\dot{H}^{\varrho-2})})$$

$$\le C\lambda_{N+1}^{-1}(1+\|u_0\|_{L^2(\Omega;\dot{H}^\varrho)}+\|v_0\|_{L^2(\Omega;\dot{H}^{\varrho-2})}). \tag{4.46}$$

Further, we utilize (2.18), (4.13) with $\mu=2$, $\nu=0$ and (4.26) with $x_2=F(u(t_m))$ to achieve a constant C such that, for all $m\in\{0,1,2,\cdots,M\}, M\in\mathbb{N}$,

$$J_3^2\le\sum_{l=0}^{m-1}\int_{t_l}^{t_{l+1}}\|[\mathcal{S}_4(t_m-s)-\mathcal{S}_4(t_m-t_l)]\,[F(u(s))-F(u(t_m))]\|_{L^2(\Omega;\dot{H}^0)}\,\mathrm{d}s$$

$$+\Big\|\sum_{l=0}^{m-1}\int_{t_l}^{t_{l+1}}[\mathcal{S}_4(t_m-s)-\mathcal{S}_4(t_m-t_l)]F(u(t_m))\,\mathrm{d}s\Big\|_{L^2(\Omega;\dot{H}^0)}$$

$$\le Ck\Big[\sum_{l=0}^{m-1}\int_{t_l}^{t_{l+1}}(t_m-s)^{-1}\|F(u(s))-F(u(t_m))\|_{L^2(\Omega;\dot{H}^0)}\,\mathrm{d}s+\|F(u(t_m))\|_{L^2(\Omega;\dot{H}^0)}\Big]$$

$$\le Ck\Big[\int_0^{t_m}(t_m-s)^{-1}(t_m-s)^{\frac{\varrho}{2}}\,\mathrm{d}s+1\Big](1+\|u_0\|_{L^2(\Omega;\dot{H}^\varrho)}+\|v_0\|_{L^2(\Omega;\dot{H}^{\varrho-2})})$$

$$\le Ck(1+\|u_0\|_{L^2(\Omega;\dot{H}^\varrho)}+\|v_0\|_{L^2(\Omega;\dot{H}^{\varrho-2})}). \tag{4.47}$$

At last, similarly to the proof of (4.38), one can employ (4.12) with $\mu=\theta$, (4.41), (2.10) and Hölder's inequality to show that, for $\theta\in[1,2)$, $\gamma\in[0,2]$ and for all $m\in\{0,1,2,\cdots,M\}, M\in\mathbb{N}, N\in\mathbb{N}$, there exists a constant C such that

$$J_3^3\le\sum_{l=0}^{m-1}\int_{t_l}^{t_{l+1}}\Big[\|\mathcal{S}_4(t_m-t_l)[F(u(s))-F(u(t_l))]\|_{L^2(\Omega;\dot{H}^0)}$$

$$+\|\mathcal{S}_4(t_m-t_l)[F(u(t_l))-F(u_l^{N,M})]\|_{L^2(\Omega;\dot{H}^0)}\Big]\,\mathrm{d}s$$

$$\le C\sum_{l=0}^{m-1}\Big[\int_{t_l}^{t_{l+1}}(t_m-t_l)^{-\frac{\theta}{2}}\|A^{-\frac{\theta}{2}}[F(u(s))-F(u(t_l))]\|_{L^2(\Omega;\dot{H}^0)}\,\mathrm{d}s+k\|u(t_l)-u_l^{N,M}\|_{L^2(\Omega;\dot{H}^0)}\Big]$$

$$\le Ck\sum_{l=0}^{m-1}\Big[k^{\min\{1+\gamma,1\}}(t_m-t_l)^{-\frac{\theta}{2}}+\lambda_{N+1}^{-\frac{1+\min\{\gamma,1\}}{2}}+k^{\min\{1+\gamma,1\}}\Big](1+\|u_0\|_{L^4(\Omega;\dot{H}^\varrho)}^2+\|v_0\|_{L^4(\Omega;\dot{H}^{\varrho-2})}^2)$$

$$\le C(k+\lambda_{N+1}^{-\frac{1+\min\{\gamma,1\}}{2}})(1+\|u_0\|_{L^4(\Omega;\dot{H}^\varrho)}^2+\|v_0\|_{L^4(\Omega;\dot{H}^{\varrho-2})}^2). \tag{4.48}$$

Gathering (4.46), (4.47) and (4.48) together shows, there exists a constant C such that, for all $m\in\{0,1,2,\cdots,M\}, M\in\mathbb{N}, N\in\mathbb{N}$,

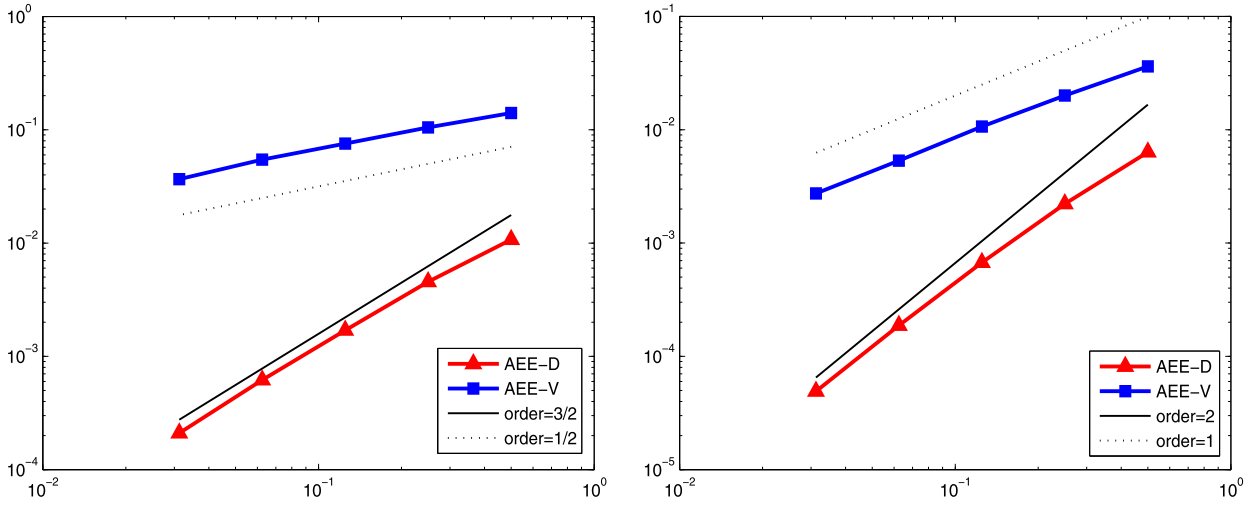


Fig. 1. Convergence rates for the spatial discretization (Left: $Q = I$; right: $Q = A^{-0.5005}$).

$$\|J_3\|_{L^2(\Omega;\dot{H}^0)} \leq C(k + \lambda_{N+1}^{-\frac{\gamma}{2}})(1 + \|u_0\|_{L^4(\Omega;\dot{H}^{1+\min\{\gamma,1\}})} + \|v_0\|_{L^4(\Omega;\dot{H}^{\gamma})})^2, \tag{4.49}$$

which together with (4.43) and (4.44) yields for all $N, M \in \mathbb{N}$

$$\sup_{m\in\{0,1,\cdots,M\}} \|v(t_m) - v_m^{N,M}\|_{L^2(\Omega;\dot{H}^0)} \leq C(k + \lambda_{N+1}^{-\frac{\gamma}{2}})(1 + \|u_0\|_{L^4(\Omega;\dot{H}^{1+\min\{\gamma,1\}})} + \|v_0\|_{L^4(\Omega;\dot{H}^{\gamma})})^2. \tag{4.50}$$

The proof of Theorem 3.1 is now complete. □

5. Numerical experiments

In this section, we test the previous theoretical findings by doing some numerical experiments. For simplicity, we consider the following stochastic strongly damped wave equation in one dimension,

$$\begin{cases} u_{tt} = \Delta u + \Delta u_t + \frac{1-u}{1+u^2} + \dot{W}(t), & t \in (0,1],\ x \in (0,1), \\ u(0,x) = \frac{\partial u}{\partial t}(0,x) = 0, & x \in (0,1), \\ u(t,0) = u(t,1) = 0, & t \in (0,1]. \end{cases} \tag{5.1}$$

Our task is to simulate the approximation errors at a fixed time $T = 1$ for the space–time white noise case ($Q = I$) and the trace-class noise case ($Q = A^{-0.5005}$). For these two cases, Assumption 2.2 is satisfied with $\gamma = \frac{1}{2} - \epsilon$ for arbitrarily small $\epsilon > 0$ and $\gamma = 1$. The numerical errors in the mean-square sense are achieved by the Monte-Carlo approach over 100 samples. Since no exact solution is available, we identify it as a numerical one with fine mesh step-sizes by choosing small k_{exact} and large N_{exact}.

At first, we perform the spatial discretization of the test equation with two noise cases mentioned earlier. Fig. 1 depicts the spatial numerical errors of the accelerated exponential Euler (AEE) scheme (3.8) with $N = 2^j$, $j = 1, 2, ..., 5$ and time step-size $k_{\text{exact}} = 2^{-14}$. Evidently, the resulting errors for both the displacement (AEE-D) and the velocity (AEE-V) decrease with slopes of orders as expected (see Theorem 3.1). Here note that the "exact" solution is computed by using $N_{\text{exact}} = 2^8$.

In what follows, we examine the temporal discretization errors using different time stepsizes $k = 2^{-i}, i = 3, 4, ..., 8$. The "exact" solution is approximated by AEE scheme (3.8) with small time step-size $k_{\text{exact}} = 2^{-12}$. For comparison, we present in Fig. 2 the convergence errors in time caused by the AEE scheme (3.8) and the linear implicit Euler (LIE) scheme [21]. From Fig. 2, one can observe that, the errors of (3.8) for the displacement (AEE-D) and for the velocity (AEE-V) both decrease with order 1 for the space–time white

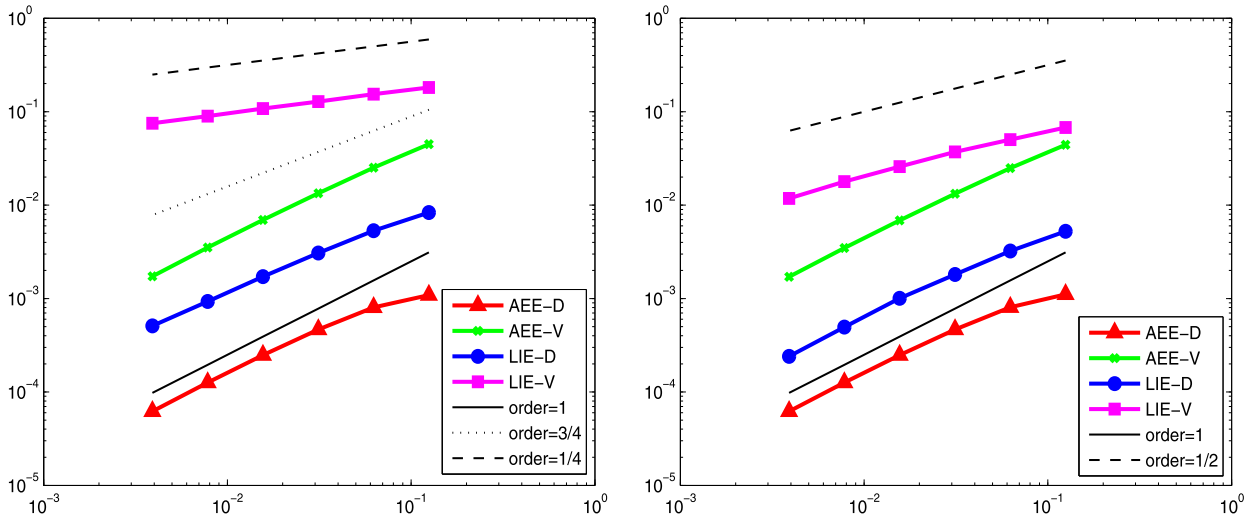


Fig. 2. Convergence rates for the temporal discretization (Left: $Q = I$; right: $Q = A^{-0.5005}$).

noise and the trace-class noise, which is consistent with assertions in Theorem 3.1. The corresponding errors for the LIE method (LIE-D and LIE-V), however, exhibit much worse performance. For example, the errors of LIE-D and LIE-V only show rates of order $\frac{3}{4}$ and order $\frac{1}{4}$, respectively, in the space–time white noise case. It turns out that the newly proposed scheme admits significantly improved convergence rates in time.

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