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# On the automorphism group of a certain infinite type domain in $\mathbb{C}^2$ ☆

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## ABSTRACT

In this study, we consider an infinite type domain  $\Omega_P$  in  $\mathbb{C}^2$ . The aim of this study is to investigate the holomorphic vector fields tangent to an infinite type model in  $\mathbb{C}^2$  vanishing at an infinite type point and to give an explicit description of the automorphism group of  $\Omega_P$ .

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## 1. Introduction

Let  $D$  be a domain in  $\mathbb{C}^n$ . An automorphism of  $D$  is a biholomorphic self-map. The set of all automorphisms of  $D$  makes a group under composition. We denote the automorphism group by  $\text{Aut}(D)$ . The topology based on  $\text{Aut}(D)$  has uniform convergence on compact sets (i.e., the compact-open topology).

Cartan provided a standard and classical result that if  $D$  is a bounded domain in  $\mathbb{C}^n$  and the automorphism group of  $D$  is noncompact, then a point  $x \in D$ , a point  $p \in \partial D$ , and the automorphisms  $\varphi_j \in \text{Aut}(D)$  exist such that  $\varphi_j(x) \rightarrow p$ . In this case, we call  $p$  a *boundary orbit accumulation point*.

In 1993, Greene and Krantz [14] conjectured that for a smoothly bounded pseudoconvex domain that admits a non-compact automorphism group, the point orbits can accumulate only at a point of finite type in the sense of Kohn, Catlin, and D'Angelo (see [11,25] for this concept). For details of this conjecture, we refer the reader to [18].

Evidence for the correctness of Greene–Krantz's conjecture was provided in [20]. Kang [20] proved that the automorphism group  $\text{Aut}(E_P)$  is compact, where  $E_P$  is a special type of Hartogs domain

$$E_P = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + P(z_2) < 1\} \Subset \mathbb{C}^2,$$

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where  $P$  is a real-valued,  $\mathcal{C}^\infty$ -smooth, subharmonic function that satisfies:

- (i)  $P(z_2) > 0$  if  $z_2 \neq 0$ ,
- (ii)  $P$  vanishes to infinite order only at the origin.

Note that  $E_P$  is of infinite type along the points  $(e^{i\theta}, 0) \in bE_P$ , and  $(e^{i\theta}, 0)$  are the only points of infinite type.

Using Pinchuk's scaling method, Kim and Krantz [23] proved that a boundary orbit accumulation point cannot be exponentially flat, which also confirms a version of Greene–Krantz's conjecture.

Recently, Krantz [26] showed that the domain

$$\Omega := \{z \in \mathbb{C}^n : |z_1|^{2m_1} + |z_2|^{2m_2} + \cdots + |z_{n-1}|^{2m_{n-1}} + \psi(|z_n|) < 1\},$$

where  $m_j$  are positive integers and where  $\psi$  is a real-valued, even, smooth, monotone and convex on  $[0, +\infty)$  function of a real variable with  $\psi(0) = 0$  that vanishes to infinite order at 0, has a compact automorphism group. In fact, the only automorphisms of  $\Omega$  are the separate rotations in each variable (cf. [14,19]).

We emphasize that the automorphism group of a domain in  $\mathbb{C}^n$  is not easy to describe explicitly and it is unknown in most cases. In the present study, we compute the automorphism group of an infinite type model

$$\Omega_P := \{(z_1, z_2) \in \mathbb{C}^2 : \rho(z_1, z_2) = \operatorname{Re} z_1 + P(z_2) < 0\},$$

where  $P : \mathbb{C} \rightarrow \mathbb{R}$  is a  $\mathcal{C}^\infty$ -smooth function that satisfies:

- (i)  $P(z) = q(|z|)$  for all  $z \in \mathbb{C}$ , where  $q : [0, +\infty) \rightarrow \mathbb{R}$  is a function with  $q(0) = 0$  such that it is strictly increasing and convex on  $[0, \epsilon_0)$  for some  $\epsilon_0 > 0$ , and
- (ii)  $P$  vanishes to infinite order at 0.

It is easy to see that  $(it, 0)$ ,  $t \in \mathbb{R}$  are points of infinite type in  $b\Omega_P$ , and hence  $\Omega_P$  is of infinite type.

In order to state the first main result, we recall the following terminology. A *holomorphic vector field* in  $\mathbb{C}^n$  takes the form

$$H = \sum_{k=1}^n h_k(z) \frac{\partial}{\partial z_k}$$

for some functions  $h_1, \dots, h_n$  holomorphic in  $z = (z_1, \dots, z_n)$ . A smooth real hypersurface germ  $M$  (of real codimension 1) at  $p$  in  $\mathbb{C}^n$  takes a defining function, say  $\rho$ , such that  $M$  is represented by the equation  $\rho(z) = 0$ . The holomorphic vector field  $H$  is said to be *tangent* to  $M$  if its real part  $\operatorname{Re} H$  is tangent to  $M$ , i.e.,  $H$  satisfies the equation

$$(\operatorname{Re} H)\rho(z) = 0 \text{ for all } z \in M. \quad (1)$$

Our first aim is to prove the following theorem, which is a characterization of tangential holomorphic vector fields.

**Theorem 1.** *Let  $P : \mathbb{C} \rightarrow \mathbb{R}$  be a  $\mathcal{C}^\infty$ -smooth function that satisfies*

- (i)  $P(z) = q(|z|)$  for all  $z \in \mathbb{C}$ , where  $q : [0, +\infty) \rightarrow \mathbb{R}$  is a function with  $q(0) = 0$  such that it is strictly increasing and convex on  $[0, \epsilon_0)$  for some  $\epsilon_0 > 0$ , and
- (ii)  $P$  vanishes to infinite order at 0.

If  $H = h_1(z_1, z_2)\frac{\partial}{\partial z_1} + h_2(z_1, z_2)\frac{\partial}{\partial z_2}$  with  $H(0, 0) = 0$  is holomorphic in  $\Omega_P \cap U$ ,  $\mathcal{C}^\infty$ -smooth in  $\overline{\Omega_P} \cap U$ , and tangent to  $b\Omega_P \cap U$ , where  $U$  is a neighborhood of  $(0, 0) \in \mathbb{C}^2$ , then  $H = i\beta z_2 \frac{\partial}{\partial z_2}$  for some  $\beta \in \mathbb{R}$ .

For the case where the tangential holomorphic vector field  $H$  is holomorphic in a neighborhood of the origin, [Theorem 1](#) was already proved by [\[7,15\]](#). The tangential holomorphic vector field  $H$  in [Theorem 1](#) is only holomorphic inside the domain, so it appears that some key techniques in [\[7\]](#) cannot be used for our problem. To avoid this difficulty, we first employ the Schwarz reflection principle to show that the holomorphic functions  $h_1, h_2$  must vanish to finite order at the origin. Then, Equation [\(1\)](#) implies that  $h_1 \equiv 0$ . Therefore, the proof follows from Chirka's curvilinear Hartogs' lemma (see the detailed proof in [Section 2](#)).

We note that  $\text{Aut}(\Omega_P)$  is noncompact because it contains biholomorphisms

$$(z_1, z_2) \mapsto (z_1 + is, e^{it}z_2), \quad s, t \in \mathbb{R}.$$

Let us denote  $\{R_t\}_{t \in \mathbb{R}}$  as the one-parameter subgroup of  $\text{Aut}(\Omega_P, 0)$  generated by the holomorphic vector field  $H_R(z_1, z_2) = iz_2 \frac{\partial}{\partial z_2}$ , i.e.,

$$R_t(z_1, z_2) = (z_1, e^{it}z_2), \quad \forall t \in \mathbb{R}.$$

In addition, we denote  $T_s(z_1, z_2) = (z_1 + is, z_2)$  for  $s \in \mathbb{R}$ .

To state the second main result, we need the following definitions. Recall that the Kobayashi metric  $K_D$  of  $D$  is defined by

$$K_D(\eta, X) := \inf \left\{ \frac{1}{R} \mid \exists f : \Delta \rightarrow D \text{ such that } f(0) = \eta, f'(0) = RX \right\},$$

where  $\eta \in D$  and  $X \in T_\eta^{1,0}\mathbb{C}^n$ , where  $\Delta_r$  is a disc with its center at the origin and radius  $r > 0$ , and  $\Delta := \Delta_1$ .

The following definition is derived from the study by Huang [\[17\]](#).

**Definition 1.** Let  $D$  be a domain in  $\mathbb{C}^n$  with  $\mathcal{C}^2$ -smooth boundary  $bD$  and  $z_0$  is a boundary point. For a  $\mathcal{C}^1$ -smooth, monotonic, increasing function  $g : [1, +\infty) \rightarrow [1, +\infty)$ , we say that  $D$  is  $g$ -admissible at  $z_0$  if a neighborhood  $V$  of  $z_0$  exists such that

$$K_D(z, X) \gtrsim g(\delta_D^{-1}(z))|X|$$

for any  $z \in V \cap D$  and  $X \in T_z^{1,0}\mathbb{C}^n$ , where  $\delta_D(z)$  is the distance from  $z$  to  $bD$ .

**Remark 1.**

- (i) In [\[6, p. 93\]](#) (see also in [\[28\]](#)), it was proved that if a plurisubharmonic peak function exists at  $z_0$ , then a neighborhood  $V$  of  $z_0$  exists such that

$$K_D(z, X) \leq K_{D \cap V}(z, X) \leq 2K_D(z, X),$$

for any  $z \in V \cap D$  and  $X \in T_z^{1,0}\mathbb{C}^n$ .

- (ii) If  $D$  is  $\mathcal{C}^\infty$ -smooth pseudoconvex of finite type, then  $D$  is  $t^\epsilon$ -admissible at any boundary point for some  $\epsilon > 0$  (cf. [\[10\]](#)). Recently, Khanh [\[21\]](#) (also see [\[22\]](#)) proved that a certain pseudoconvex domain of infinite type is also  $g$ -admissible for some function  $g$ .

**Definition 2** (see [21]). Let  $D \subset \mathbb{C}^n$  be a  $\mathcal{C}^2$ -smooth domain. Assume that  $D$  is pseudoconvex near  $z_0 \in bD$ . For a  $\mathcal{C}^1$ -smooth, monotonic, increasing function  $u : [1, +\infty) \rightarrow [1, +\infty)$  with decreasing  $u(t)/t^{1/2}$ , we say that a domain  $D$  has the  $u$ -property at the boundary point  $z_0$  if a neighborhood  $U$  of  $z_0$  and a family of  $\mathcal{C}^2$ -functions  $\{\phi_\eta\}$  exist such that

- (i)  $|\phi_\eta| < 1$ ,  $\mathcal{C}^2$ , and plurisubharmonic on  $D$ ;
- (ii)  $i\partial\bar{\partial}\phi_\eta \gtrsim u(\eta^{-1})^2 Id$  and  $|D\phi_\eta| \lesssim \eta^{-1}$  on  $U \cap \{z \in D : -\eta < r(z) < 0\}$ , where  $r$  is a  $\mathcal{C}^2$ -defining function for  $D$ .

In the following,  $\lesssim$  and  $\gtrsim$  denote inequalities up to a positive constant multiple. In addition, we use  $\approx$  for the combination of  $\lesssim$  and  $\gtrsim$ .

**Definition 3** (see [21]). We say that a domain  $D$  has the strong  $u$ -property at the boundary  $z_0$  if it has the  $u$ -property and  $u$  satisfies the following:

- (i)  $\int_t^{+\infty} \frac{da}{au(a)}$  for some  $t > 1$  and denote  $(g(t))^{-1}$  as this finite integral;
- (ii) The function  $\frac{1}{\delta g(1/\delta^n)}$  is decreasing and  $\int_0^d \frac{1}{\delta g(1/\delta^n)} d\delta < +\infty$  for a sufficiently small  $d > 0$  and for some  $0 < \eta < 1$ .

**Definition 4.** We say that  $\Omega_P$  satisfies the condition (T) at  $\infty$  if one of the following conditions hold:

- (i)  $\lim_{z \rightarrow \infty} P(z) = +\infty$ ;
- (ii) The function  $Q$  defined by setting  $Q(\zeta) := P(1/\zeta)$  can be extended to being  $\mathcal{C}^\infty$ -smooth in a neighborhood of  $\zeta = 0$ ,  $\Omega_Q$  and it has the strong  $\tilde{u}$ -property at  $(-r, 0)$  for some function  $\tilde{u}$ , where  $r = \lim_{z \rightarrow \infty} P(z)$ , and  $b\Omega_P$  and  $b\Omega_Q$  are not isomorphic as CR manifold germs at  $(0, 0)$  and  $(-r, 0)$  respectively.

The second aim of this study is to explain the following theorem.

**Theorem 2.** Let  $P : \mathbb{C} \rightarrow \mathbb{R}$  be a  $\mathcal{C}^\infty$ -smooth function that satisfies

- (i)  $P(z) = q(|z|)$  for all  $z \in \mathbb{C}$ , where  $q : [0, +\infty) \rightarrow \mathbb{R}$  is a function with  $q(0) = 0$  such that it is strictly increasing and convex on  $[0, \epsilon_0)$  for some  $\epsilon_0 > 0$ ;
- (ii)  $P$  vanishes to infinite order at 0; and
- (iii)  $P$  vanishes to finite order at any  $z \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ .

If we assume that  $\Omega_P$  has the strong  $u$ -property at  $(0, 0)$  and  $\Omega_P$  satisfies the property (T) at  $\infty$ , then

$$\text{Aut}(\Omega_P) = \{(z_1, z_2) \mapsto (z_1 + is, e^{it}z_2) : s, t \in \mathbb{R}\}.$$

**Remark 2.** Let  $\Omega_P$  be as described in Theorem 2 and let  $P_\infty(b\Omega_P)$  be the set of all points in  $b\Omega_P$  of D'Angelo infinite type. It is easy to see that  $P_\infty(b\Omega_P) = \{(it, 0) : t \in \mathbb{R}\}$ . Moreover, since  $\Omega_P$  is invariant under any translation  $(z_1, z_2) \mapsto (z_1 + it, z_2)$ ,  $t \in \mathbb{R}$ , then it satisfies the strong  $u$ -property at  $(it, 0)$  for any  $t \in \mathbb{R}$ .

**Remark 3.** Let  $P$  be a function defined by  $P(z_2) = \exp(-1/|z_2|^\alpha)$  if  $z_2 \neq 0$  and  $P(0) = 0$ , where  $0 < \alpha < 1$ . Then, by [21, Corollary 1.3],  $\Omega_P$  has the  $\log^{1/\alpha}$ -property at  $(it, 0)$ , and thus it is  $\log^{1/\alpha-1}$ -admissible at

$(it, 0)$  for any  $t \in \mathbb{R}$ . Furthermore, a computation shows that if  $0 < \alpha < 1/2$ , then  $\Omega_P$  has the strong  $\log^{1/\alpha}$ -property at  $(it, 0)$  for any  $t \in \mathbb{R}$ .

**Example 1.** Let  $E_j$ ,  $j = 1, \dots, 3$  be domains defined by

$$E_j := \{(z_1, z_2) \in \mathbb{C}^2 : \rho(z_1, z_2) = \operatorname{Re} z_1 + P_j(z_2) < 0\},$$

where  $P_j$  are defined by

$$\begin{aligned} P_1 &= \psi(|z_2|)e^{-1/|z_2|^\alpha} + (1 - \psi(|z_2|))\frac{1}{|z_2|^{2m}}, \\ P_2 &= \psi(|z_2|)e^{-1/|z_2|^\alpha} + (1 - \psi(|z_2|))e^{-|z_2|^\beta}, \\ P_3 &= \psi(|z_2|)e^{-1/|z_2|^\alpha} + (1 - \psi(|z_2|))|z_2|^2 \end{aligned}$$

if  $z_2 \neq 0$  and  $P(0) = 0$ , where  $0 < \alpha, \beta < 1/2$ ,  $m \in \mathbb{N}^*$  with  $\beta \neq \alpha$  and  $\psi(t)$  is a  $\mathcal{C}^\infty$ -smooth cut-off function such that  $\psi(t) = 1$  if  $|t| < a$  and  $\psi(t) = 0$  if  $|t| > b$  ( $0 < a < b$ ). From Remark 3 and a computation, it follows that  $E_j$ ,  $j = 1, \dots, 3$ , have the strong  $\log^{1/\alpha}$ -property and satisfy the property (T) at  $\infty$ . Therefore, by Theorem 2, we conclude that

$$\operatorname{Aut}(E_j) = \{(z_1, z_2) \mapsto (z_1 + is, e^{it}z_2) : s, t \in \mathbb{R}\}, \quad j = 1, \dots, 3.$$

Next, we explain the idea of the proof of Theorem 2. Let  $f \in \operatorname{Aut}(\Omega_P)$  be arbitrary. We show that  $t_1, t_2 \in \mathbb{R}$  exist such that  $f, f^{-1}$  extend smoothly to  $b\Omega_P$  near  $(it_1, 0)$  and  $(it_2, 0)$ , respectively and  $(it_2, 0) = f(it_1, 0)$  (cf. Lemma 6). After replacing  $f$  by  $T_{-t_2} \circ f \circ T_{t_1}$ , we may assume that  $f, f^{-1}$  extend smoothly to  $b\Omega_P$  near the origin and  $f(0, 0) = (0, 0)$ . Next, we consider the one-parameter subgroup  $\{F_t\}_{t \in \mathbb{R}}$  of  $\operatorname{Aut}(\Omega_P) \cap \mathcal{C}^\infty(\overline{\Omega_P} \cap U)$  defined by  $F_t = f \circ R_{-t} \circ f^{-1}$ . By employing Theorem 1, a real number  $\delta$  exists such that  $F_t = R_{\delta t}$  for all  $t \in \mathbb{R}$ . Using the property that  $P$  vanishes to infinite order at 0, we can prove that  $f = R_{t_0}$  for some  $t_0 \in \mathbb{R}$  (see the detailed proof in Section 4). This completes our proof.

The remainder of this paper is organized as follows. In Section 2, we prove Theorem 1. In Section 3, we prove several lemmas that are mainly used in the proof of Theorem 2. Section 4 gives the proof of Theorem 2. Finally, two lemmas are given in the Appendix.

## 2. Holomorphic vector fields tangent to an infinite type model

In this section, we provide the proof of Theorem 1. Assume that  $P : \mathbb{C} \rightarrow \mathbb{R}$  is a  $\mathcal{C}^\infty$ -smooth function that satisfies (i) and (ii) as stated in the Introduction.

Then, we consider a nontrivial holomorphic vector field  $H = h_1(z_1, z_2)\frac{\partial}{\partial z_1} + h_2(z_1, z_2)\frac{\partial}{\partial z_2}$  defined on  $\Omega_P \cap U$ , where  $U$  is a neighborhood of the origin. We only consider the case where  $H$  is tangent to  $b\Omega_P \cap U$ . Thus, they satisfy the identity

$$(\operatorname{Re} H)\rho(z_1, z_2) = 0, \quad \forall (z_1, z_2) \in b\Omega_P \cap U. \quad (2)$$

By a simple computation, we have

$$\begin{aligned} \rho_{z_1}(z_1, z_2) &= 1, \\ \rho_{z_2}(z_1, z_2) &= P'(z_2), \end{aligned}$$

and thus Equation (2) can be rewritten as

$$\operatorname{Re}\left[h_1(z_1, z_2) + P'(z_2)h_2(z_1, z_2)\right] = 0 \quad (3)$$

for all  $(z_1, z_2) \in b\Omega_P \cap U$ .

Since  $(it - P(z_2), z_2) \in b\Omega_P$  for any  $t \in \mathbb{R}$  with a sufficiently small  $t$ , then the equation above again admits a new form

$$\operatorname{Re}\left[h_1(it - P(z_2), z_2) + P'(z_2)h_2(it - P(z_2), z_2)\right] = 0 \quad (4)$$

for all  $z_2 \in \mathbb{C}$  and for all  $t \in \mathbb{R}$  with  $|z_2| < \epsilon_0$  and  $|t| < \delta_0$ , where  $\epsilon_0 > 0$  and  $\delta_0 > 0$  are sufficiently small.

**Lemma 1.** We find that  $\frac{\partial^{m+n}}{\partial z_1^m \partial z_2^n} h_1(z_1, 0)$  can be extended to being holomorphic in a neighborhood of  $z_1 = 0$  for every  $m, n \in \mathbb{N}$ .

**Proof.**  $\nu_0(P') = +\infty$ , so from (4) with  $t = 0$ , it follows that  $\operatorname{Re} h_1(it, 0) = 0$  for all  $t \in (-\delta_0, \delta_0)$ . By the Schwarz reflection principle,  $h_1(z_1, 0)$  can be extended to a holomorphic function on a neighborhood of  $z_1 = 0$ . For any  $m, n \in \mathbb{N}$ , by taking  $\frac{\partial^{m+n}}{\partial t^m \partial z_2^n} |_{z_2=0}$  on both sides of Equation (4), we have

$$\operatorname{Re}\left[i^m \frac{\partial^{m+n}}{\partial z_1^m \partial z_2^n} h_1(it, 0)\right] = 0$$

for all  $t \in (-\delta_0, \delta_0)$ . Again, by the Schwarz reflection principle,  $\frac{\partial^{m+n}}{\partial z_1^m \partial z_2^n} h_1(z_1, 0)$  can be extended to being holomorphic in a neighborhood of  $z_1 = 0$ , which completes the proof.  $\square$

**Corollary 1.** If  $h_1$  vanishes to infinite order at  $(0, 0)$ , then  $h_1 \equiv 0$ .

**Proof.**  $h_1$  vanishes to infinite order at  $(0, 0)$ , so  $\frac{\partial^{m+n}}{\partial z_1^m \partial z_2^n} h_1(z_1, 0)$  also vanishes to infinite order at  $z_1 = 0$  for all  $m, n \in \mathbb{N}$ . Moreover, by Lemma 1, these functions are holomorphic in a neighborhood of  $z_1 = 0$ . Therefore,  $\frac{\partial^{m+n}}{\partial z_1^m \partial z_2^n} h_1(z_1, 0) \equiv 0$  for every  $m, n \in \mathbb{N}$ .

Expand  $h_1$  into the Taylor series at  $(-\epsilon, 0)$  with a sufficiently small  $\epsilon > 0$  such that

$$h_1(z_1, z_2) = \sum_{m,n=0}^{\infty} \frac{1}{m!n!} \frac{\partial^{m+n}}{\partial z_1^m \partial z_2^n} h_1(-\epsilon, 0) (z_1 + \epsilon)^m z_2^n.$$

$\frac{\partial^{m+n}}{\partial z_1^m \partial z_2^n} h_1(-\epsilon, 0) = 0$  for all  $m, n \in \mathbb{N}$ ,  $h_1 \equiv 0$  on a neighborhood of  $(-\epsilon, 0)$ , and thus  $h_1 \equiv 0$  on  $\Omega_P$ .  $\square$

**Proof of Theorem 1.** Denote  $D_P(r) := \{z_2 \in \mathbb{C} : |z_2| < q^{-1}(r)\}$  ( $r > 0$ ). For each  $z_1$  with  $\operatorname{Re}(z_1) < 0$ , we have

$$h_1(z_1, z_2) = \sum_{n=0}^{\infty} a_n(z_1) z_2^n, \quad \forall z_2 \in D_P(-\operatorname{Re}(z_1)), \quad (5)$$

where  $a_n(z_1) = \frac{\partial^n}{\partial z_2^n} h_1(z_1, 0)$  for every  $n \in \mathbb{N}$ . Since  $h_1 \in \operatorname{Hol}(\Omega_P \cap U) \cap \mathcal{C}^\infty(\overline{\Omega_P} \cap U)$ ,  $a_n \in \operatorname{Hol}(\mathcal{H} \cap U_1) \cap \mathcal{C}^\infty(\overline{\mathcal{H}} \cap U_1)$  for every  $n = 0, 1, \dots$ , where  $\mathcal{H} := \{z_1 \in \mathbb{C} : \operatorname{Re}(z_1) < 0\}$  and  $U_1$  is a neighborhood of  $z_1 = 0$  in  $\mathbb{C}_{z_1}$ . Moreover, by expanding the function  $g_{z_1}(z_2) := h_1(z_1, z_2)$  into the Fourier series, we can see that (5) still holds for all  $z_2 \in \overline{D_P(-\operatorname{Re}(z_1))}$ . Therefore, the function  $h_1(it - P(z_2), z_2)$  can be rewritten as follows:

$$h_1(it - P(z_2), z_2) = \sum_{n=0}^{\infty} a_n(it - P(z_2)) z_2^n,$$

for all  $(t, z_2) \in (-\delta_0, \delta_0) \times \Delta_{\epsilon_0}$ , where  $\delta_0 > 0$ ,  $\epsilon_0 > 0$  are sufficiently small.

Similarly, we also have

$$h_2(it - P(z_2), z_2) = \sum_{n=0}^{\infty} b_n(it - P(z_2))z_2^n$$

for all  $(t, z_2) \in (-\delta_0, \delta_0) \times \Delta_{\epsilon_0}$ , where  $b_n \in \text{Hol}(\mathcal{H} \cap U_1) \cap \mathcal{C}^\infty(\overline{\mathcal{H}} \cap U_1)$  for every  $n = 0, 1, \dots$

Now, we prove that  $h_1 \equiv 0$ . Indeed, arguing by contradiction, we suppose that  $h_1 \not\equiv 0$ . If  $h_1$  vanishes to infinite order at  $(0, 0)$ , then by [Corollary 1](#), we obtain  $h_1 \equiv 0$ . Thus,  $h_1$  vanishes to finite order at  $(0, 0)$ . From [\(4\)](#), it follows that  $h_2$  also vanishes to finite order at  $(0, 0)$ ; otherwise,  $h_1$  vanishes to infinite order at  $(0, 0)$ .

Denote

$$\begin{aligned} m_0 &:= \min \left\{ m \in \mathbb{N} : \frac{\partial^{m+n}}{\partial^m z_1 \partial^n z_2} h_1(0, 0) \neq 0 \text{ for some } n \in \mathbb{N} \right\}, \\ n_0 &:= \min \left\{ n \in \mathbb{N} : \frac{\partial^{m_0+n}}{\partial^{m_0} z_1 \partial^n z_2} h_1(0, 0) \neq 0 \right\}, \\ k_0 &:= \min \left\{ m \in \mathbb{N} : \frac{\partial^{k+l}}{\partial^k z_1 \partial^l z_2} h_2(0, 0) \neq 0 \text{ for some } l \in \mathbb{N} \right\}, \\ l_0 &:= \min \left\{ l \in \mathbb{N} : \frac{\partial^{k_0+l}}{\partial^{k_0} z_1 \partial^l z_2} h_2(0, 0) \neq 0 \right\}. \end{aligned} \quad (6)$$

$\nu_0(P) = +\infty$ , and thus we obtain

$$\begin{aligned} h_1(i\alpha P(z_2) - P(z_2), z_2) &= a_{m_0, n_0} (i\alpha - 1)^{m_0} (P(z_2))^{m_0} (z_2^{n_0} + o(|z_2|^{n_0})), \\ h_2(i\alpha P(z_2) - P(z_2), z_2) &= b_{k_0, l_0} (i\alpha - 1)^{k_0} (P(z_2))^{k_0} (z_2^{l_0} + o(|z_2|^{l_0})), \end{aligned} \quad (7)$$

where  $a_{m_0, n_0} := \frac{\partial^{m_0+n_0}}{\partial^{m_0} z_1 \partial^{n_0} z_2} h_1(0, 0) \neq 0$ ,  $b_{k_0, l_0} := \frac{\partial^{k_0+l_0}}{\partial^{k_0} z_1 \partial^{l_0} z_2} h_2(0, 0) \neq 0$ , and  $\alpha \in \mathbb{R}$  is selected later.

Now, from [\(4\)](#) with  $t = \alpha P(z_2)$ , it follows that

$$\begin{aligned} &\text{Re} \left[ a_{m_0, n_0} (i\alpha - 1)^{m_0} (P(z_2))^{m_0} (z_2^{n_0} + o(|z_2|^{n_0})) + b_{k_0, l_0} (i\alpha - 1)^{k_0} (z_2^{l_0} + o(|z_2|^{l_0})) \right. \\ &\quad \left. \times (P(z_2))^{k_0} P'(z_2) \right] = 0 \end{aligned} \quad (8)$$

for all  $z_2 \in \Delta_{\epsilon_0}$  and for all sufficiently small  $\alpha \in \mathbb{R}$ . We note that in the case where  $n_0 = 0$  and  $\text{Re}(a_{m_0, 0}) = 0$ ,  $\alpha$  can be selected such that  $\text{Re}((i\alpha - 1)^{m_0} a_{m_0, 0}) \neq 0$ . Then, the equation above shows that  $k_0 > m_0$ . Furthermore,  $P$  is rotational, so it follows that  $\text{Re}(iz_2 P'(z_2)) \equiv 0$  (see [\[24, Lemma 4\]](#)), and hence we can assume that  $\text{Re}(b_{10}) \neq 0$  for the case that  $k_0 = 1$ ,  $l_0 = 0$ . However, [\(8\)](#) contradicts Lemma 3 in [\[24\]](#); therefore,  $h_1 \equiv 0$ .

It is given that  $h_1 \equiv 0$ , [\(4\)](#) is equivalent to

$$\text{Re} \left[ P'(z_2) h_2(it - P(z_2), z_2) \right] = 0 \quad (9)$$

for all  $(t, z_2) \in (-\delta_0, \delta_0) \times \Delta_{\epsilon_0}$ . Thus, for each  $z_2 \in \Delta_{\epsilon_0}^*$ , the function  $g_{z_2}$  defined by setting  $g_{z_2}(z_1) := h_2(z_1, z_2)$  is holomorphic in  $\{z_1 \in \mathbb{C} : \text{Re}(z_1) < -P(z_2)\}$  and  $\mathcal{C}^\infty$ -smooth up to the real line  $\{z_1 \in \mathbb{C} : \text{Re}(z_1) = -P(z_2)\}$ . Moreover,  $g_{z_2}$  maps this line onto the real line  $\text{Re}(P'(z_2)w) = 0$  in the complex plane  $\mathbb{C}_w$ . Thus, by the Schwarz reflection principle,  $g_w$  can be extended to being holomorphic in a neighborhood  $U$  of  $z_1 = 0$  in the plane  $\mathbb{C}_{z_1}$ . (The neighborhood  $U$  is independent of  $z_2$ .)

Now, our function  $h_2$  is holomorphic in  $z_1 \in U$  for each  $z_2 \in \Delta_{\epsilon_0}^*$  and holomorphic in  $(z_1, z_2)$  in the domain  $\{(z_1, z_2) \in \mathbb{C}^2 : \operatorname{Re}(z_1) < 0, |z_2| < q^{-1}(-\operatorname{Re}(z_1))\}$ . Therefore, from Chirka's curvilinear Hartogs' lemma (see [9]), it follows that  $h_2$  can be extended to being holomorphic in a neighborhood of  $(0, 0)$  in  $\mathbb{C}^2$ . Moreover, by (9) and by [15, Theorem 3], we conclude that  $h_2(z_1, z_2) \equiv i\beta z_2$  for some  $\beta \in \mathbb{R}^*$ . Thus, the proof is complete.  $\square$

### 3. Extension of automorphisms

If  $f : D \rightarrow \mathbb{C}^N$  is a continuous map on a domain  $D \subset \mathbb{C}^n$  and  $z_0 \in \partial D$ , we denote  $\mathcal{C}(f, z_0)$  as the cluster set of  $f$  at  $z_0$ :

$$\mathcal{C}(f, z_0) = \{w \in \mathbb{C}^N : w = \lim f(z_j), z_j \in D, \text{ and } \lim z_j = z_0\}.$$

**Definition 5** (see [1]). When  $\Gamma$  is an open subset of the boundary of a smooth domain  $D$ , we say that  $\Gamma$  satisfies *local condition R* if for each  $z \in \Gamma$ , an open set  $V$  in  $\mathbb{C}^n$  with  $z \in V$  exists such that for each  $s$ , an  $M$  exists such that

$$P(W^{s+M}(D \cap V)) \subset W^s(D \cap V),$$

where  $P$  is the Bergman projection and  $W^s(U)$  is the Sobolev space of order  $s$ . We say that  $D$  satisfies *local condition R* at  $z_0 \in bD$  if an open subset of the boundary  $bD$  containing  $z_0$  exists and it satisfies the local condition R.

**Definition 6.** Let  $D, G$  be domains in  $\mathbb{C}^n$  and let  $F : [0, +\infty) \rightarrow [0, +\infty)$  be an increasing function with  $F(0) = 0$ . Let  $z_0 \in bD$  and  $w_0 \in bG$ . We say that  $D, G$  satisfy the property  $(D, G)_{(z_0, w_0)}^F$  if for each proper holomorphic mapping  $f : D \rightarrow G$ , the neighborhoods  $U$  and  $V$  of  $z_0$  and  $w_0$  respectively, exist such that

$$\delta_G(f(z)) \leq F(\delta_D(z))$$

for any  $z \in U \cap D$  such that  $f(z) \in V \cap G$ , where  $\delta_D(z)$  is the Euclidean distance from  $z \in D$  to the boundary  $\partial D$ .

For the cases where  $D$  and  $G$  are bounded pseudoconvex domains with generic corners, Chakrabarti and Verma [8, Proposition 5.1] proved that an  $\epsilon \in (0, 1)$  exists such that

$$(\delta_D(z))^{1/\epsilon} \lesssim d_G(f(z)) \lesssim (\delta_D(z))^\epsilon$$

for all  $z \in D$ , which is a generalization of [12, 3]. Consequently,  $D, G$  satisfies the property  $(D, G)_{(z_0, w_0)}^F$ , where  $F(t) = t^\epsilon$ , for any  $z_0 \in bD$  and  $w_0 \in bG$ .

Now, we recall the general Hölder continuity (see [21]). Let  $f$  be an increasing function such that  $\lim_{t \rightarrow +\infty} f(t) = +\infty$ . For  $\Omega \subset \mathbb{C}^n$ , we define the  $f$ -Hölder space on  $\bar{\Omega}$  by

$$\Lambda^f(\bar{\Omega}) = \{u : \|u\|_\infty + \sup_{z, z+h \in \bar{\Omega}} f(|h|^{-1})|u(z+h) - u(z)| < +\infty\}.$$

Note that the  $f$ -Hölder space includes the standard Hölder space  $\Lambda_\alpha(\bar{\Omega})$  if we take  $f(t) = t^\alpha$  with  $0 < \alpha < 1$ .

The following lemma is a slight generalization of [21, Theorem 1.4].

**Lemma 2.** Let  $D$  and  $G$  be domains in  $\mathbb{C}^n$  with  $\mathcal{C}^2$ -smooth boundaries. Let  $g : [1, +\infty) \rightarrow [1, +\infty)$  and  $F : [0, +\infty) \rightarrow [0, +\infty)$  be nonnegative increasing functions with  $F(0) = 0$  such that the function  $\frac{1}{\delta g(1/F(\delta))}$



is decreasing and  $\int_0^d \frac{1}{\delta g(1/F(\delta))} d\delta < +\infty$  for sufficiently small  $d > 0$ . Assume that  $D$  and  $G$  satisfy the property  $(D, G)_{(z_0, w_0)}^F$ , and  $G$  is  $g$ -admissible at  $w_0$ . Let  $f : D \rightarrow G$  be a proper holomorphic map such that  $w_0 \in \mathcal{C}(f, z_0)$ . Then, the neighborhoods  $U$  and  $V$  of  $z_0$  and  $w_0$ , respectively, exist such that  $f$  can be extended as a general Höder continuous map  $\hat{f} : U \cap \overline{D} \rightarrow V \cap \overline{G}$  with a rate  $h(t)$  defined by

$$(h(t))^{-1} := \int_0^{t^{-1}} \frac{1}{\delta g(1/F(\delta))} d\delta.$$

**Proof.**  $G$  is  $g$ -admissible at  $w_0$ , so by using the Schwarz–Pick lemma for the Kobayashi metric and the upper bound of Kobayashi metric, we obtain the following estimate

$$g(\delta_G^{-1}(f(z)))|f'(z)X| \lesssim K_G(f(z), f'(z)X) \leq K_D(z, X) \lesssim \delta_D^{-1}(z)|X|$$

for any  $z \in D \cap U$  such that  $f(z) \in V \cap G$  and  $X \in T^{1,0}\mathbb{C}^n$ . Moreover, since the property  $(D, G)_{(z_0, w_0)}^F$  holds, we may assume that

$$\delta_G(f(z)) \leq F(\delta_D(z))$$

for any  $z \in D \cap U$  such that  $f(z) \in V \cap G$ . Therefore,

$$|f'(z)X| \lesssim \frac{1}{\delta_D(z)g(1/F(\delta_D(z)))}|X| \quad (10)$$

for any  $z \in D \cap U$  such that  $f(z) \in V \cap G$  and  $X \in T^{1,0}\mathbb{C}^n$ .

By using Henkin's technique (see [4, 29]), we prove that  $f$  extends continuously to  $z_0$ . Indeed, if we suppose that  $f$  does not extend continuously to  $z_0$ , there is an open ball  $B \subset V$  (with center at  $w_0$ ) and a neighborhoods basis  $U_j$  of  $z_0$  such that  $f(D \cap U_j)$  is connected and never contained in  $B$ . Then, since  $w_0 \in \mathcal{C}_\Omega(f, z_0)$ , a sequence  $\{z'_j\}$ ,  $z'_j \in U_j$  exists such that  $f(z'_j) \in \partial B$  and  $\lim f(z'_j) = w'_0 \in \partial B \cap \partial G$ .

Let  $\{z_j\} \subset \Omega \cap U$  such that  $\lim z_j = z_0$  and  $\lim f(z_j) = w_0$ . Let  $l_j := |z_j - z'_j|$  and  $\gamma_j : [0, 3l_j] \rightarrow \Omega \cap U$  be a  $\mathcal{C}^1$ -path such that:

- (a)  $\gamma_j(0) = z_j$  and  $\gamma_j(3l_j) = z'_j$ .
- (b)  $\delta_\Omega(\gamma(t)) \geq t$  on  $[0, l_j]$ ;  $\delta_\Omega(\gamma(t)) \geq l_j$  on  $[l_j, 2l_j]$ ;  $\delta_\Omega(\gamma(t)) \geq 3l_j - t$  on  $[2l_j, 3l_j]$ .
- (c)  $\|\frac{d\gamma_j(t)}{dt}\| \lesssim 1$ ,  $t \in [0, 3l_j]$ .

(See [16, Prop. 2, p. 203].)

Choose  $t_j \in [0, 3l_j]$  such that  $f \circ \gamma_j([0, t_j]) \subset \bar{B}$  and  $f \circ \gamma_j(t_j) \in \partial B$ . From (10) (b) and (c), it follows that  $|f(z_j) - f \circ \gamma_j(t_j)| \lesssim 1/h(1/l_j) + 1/g(1/F(l_j)) \rightarrow 0$  as  $j \rightarrow \infty$ , which is a contradiction. Hence,  $f$  extends continuously to  $z_0$ .

Now, we may assume that  $f(D \cap U) \subset G \cap V$  and we can apply [21, Lemma 1.4] to prove that  $f$  can be extended to a  $h$ -Höder continuous map  $\hat{f} : \overline{D} \cap U \rightarrow \overline{G} \cap V$  with the rate  $h(t)$  defined by

$$((h(t))^{-1} := \int_0^{t^{-1}} \frac{1}{\delta g(1/F(\delta))} d\delta. \quad \square$$

The following lemma is a local version of Fefferman's theorem (see [1]).

**Lemma 3.** *Suppose that  $D$  and  $G$  are  $\mathcal{C}^\infty$ -smooth domains in  $\mathbb{C}^n$  that satisfy the local condition R at  $z_0 \in bD$  and  $w_0 \in bG$ , respectively. Assume that  $D$  and  $G$  are pseudoconvex near  $z_0$  and  $w_0$ , respectively. Let  $g : [1, +\infty) \rightarrow [1, +\infty)$  and  $F : [0, +\infty) \rightarrow [0, +\infty)$  be nonnegative increasing functions with  $F(0) = 0$  such that the function  $\frac{1}{\delta g(1/F(\delta))}$  is decreasing and  $\int_0^d \frac{1}{\delta g(1/F(\delta))} d\delta < +\infty$  for sufficiently small  $d > 0$ . Suppose that  $D$  and  $G$  satisfy the property  $(D, G)_{(z_0, w_0)}^F$ . Let  $f$  be a biholomorphic mapping of  $D$  onto  $G$  such that  $w_0 \in \mathcal{C}(f, z_0)$ . Then,  $f$  extends smoothly to  $bD$  in some neighborhood of the point  $z_0$ .*

**Proof.** By Lemma 2, we may assume that neighborhoods  $U$  and  $V$  of  $z_0$  and  $w_0$ , respectively, exist such that  $f$  extends continuously to  $U \cap \bar{D}$ . Moreover, we may assume that  $f(U \cap D) = V \cap G$  and  $U \cap D$  and  $V \cap G$  are bounded  $\mathcal{C}^\infty$ -smooth pseudoconvex domains. Therefore, the proof follows from the theorem in [1, Section 7].  $\square$

**Lemma 4.** *Let  $D \subset \mathbb{C}^n$  be a  $\mathcal{C}^2$ -smooth domain and let  $0 < \eta < 1$ . Assume that  $D$  is pseudoconvex near  $z_0 \in bD$  and  $D$  has the  $u$ -property at  $z_0$ , where  $u : [1, +\infty) \rightarrow [1, +\infty)$  is a smooth monotonic increasing function with  $u(t)/t^{1/2}$  decreasing and  $\int_{t_0}^{+\infty} \frac{da}{au(a)} < +\infty$  for some  $t_0 > 1$ . Then,  $D$  is  $g$ -admissible at  $z_0$ , where  $g$  is a function defined by*

$$(g(t))^{-1} = \int_t^{+\infty} \frac{da}{au(a)}, \quad t_0 \leq t < +\infty.$$

Moreover, the property  $(D, G)_{(z_0, w_0)}^{F_2}$  holds for any  $\mathcal{C}^2$ -smooth domain  $G \subset \mathbb{C}^n$  and  $w_0 \in bG$ , where  $F_2(t) := c_2 t^\eta$ ,  $t > 0$ , for some  $c_2 > 0$ .

**Proof.** Let  $D \subset \mathbb{C}^n$  be a  $\mathcal{C}^2$ -smooth domain. Assume that  $D$  is pseudoconvex near  $z_0 \in bD$  and  $D$  has the  $u$ -property at  $z_0$ , where  $u : [1, +\infty) \rightarrow [1, +\infty)$  is a smooth monotonic increasing function with  $u(t)/t^{1/2}$  decreasing and  $\int_{t_0}^{+\infty} \frac{da}{au(a)} < +\infty$  for some  $t_0 > 1$ . From [21, Theorem 1.2], it follows that  $D$  is  $g$ -admissible at  $z_0$ , where  $g$  is a function defined by

$$(g(t))^{-1} = \int_t^{+\infty} \frac{da}{au(a)}, \quad t_0 \leq t < +\infty.$$

Denote  $\tilde{g}$  as the functions defined by

$$\tilde{g}(\delta) = \frac{1}{g^{-1}(1/(\gamma\delta))},$$

for any  $0 < \delta < \delta_0$ , where  $\gamma, \delta_0$  is sufficiently small. By [21, Theorem 3.1] and the proof of [21, Theorem 2.1], a family  $\psi_w(z)$  exists as described in Lemma 12 in the Appendix, where  $F_1 := c_1 \tilde{g}^\eta$  and  $F_2(t) := c_2 t^\eta$ ,  $t > 0$ , for some  $0 < \eta < 1$  and  $c_1, c_2 > 0$ . Therefore, from Lemma 12 in the Appendix, it follows that the property  $(D, G)_{(z_0, w_0)}^{F_2}$  holds for any  $\mathcal{C}^2$ -smooth domain  $G \subset \mathbb{C}^n$  and  $w_0 \in bG$ . This completes the proof.  $\square$

By the definition of the strong  $u$ -property, and Lemmas 3 and 4, we obtain the following corollary.

**Corollary 2.** *Suppose that  $D$  and  $G$  are  $\mathcal{C}^\infty$ -smooth domains in  $\mathbb{C}^n$  that satisfy the local condition R at  $z_0 \in bD$  and  $w_0 \in bG$ , respectively. Suppose that  $D$  and  $G$  are pseudoconvex near  $z_0$  and  $w_0$ , respectively. Assume*

that  $D$  (resp.  $G$ ) has the strong  $u$ -property at  $z_0$  (resp. strong  $\tilde{u}$ -property at  $w_0$ ). Let  $f$  be a biholomorphic mapping of  $D$  onto  $G$  such that  $w_0 \in \mathcal{C}(f, z_0)$ . Then,  $f$  and  $f^{-1}$  extend smoothly to  $bD$  in some neighborhoods of the points  $z_0$  and  $w_0$ , respectively.

**Remark 4.** Suppose that  $D$  is  $\mathcal{C}^\infty$ -smooth pseudoconvex of finite type near  $z_0 \in bD$ . Then,  $D$  has the  $t^\epsilon$ -property at  $z_0$  for some  $\epsilon > 0$  (cf. [10,21]). Moreover, a computation shows that the strong  $t^\epsilon$ -property exists at  $z_0$ . In addition,  $D$  satisfies the local condition R at  $z_0$  (cf. [2]).

By Corollary 2 and Remark 4, we obtain the following corollary, which was proved by Sukhov.

**Corollary 3** (see Corollary 1.4 in [29]). Suppose that  $D$  and  $G$  are  $\mathcal{C}^\infty$ -smooth domains in  $\mathbb{C}^n$ . Suppose that  $D$  and  $G$  are pseudoconvex of finite type near  $z_0 \in bD$  and  $w_0 \in bG$ , respectively. Let  $f$  be a biholomorphic mapping of  $D$  onto  $G$  such that  $w_0 \in \mathcal{C}(f, z_0)$ . Then,  $f$  and  $f^{-1}$  extend smoothly to  $bD$  in some neighborhoods of the points  $z_0$  and  $w_0$ , respectively.

It is well known that any boundary orbit accumulation point is pseudoconvex (cf. [13]). The following lemma states that the pseudoconvexity is invariant under any biholomorphism.

**Lemma 5.** Let  $D, G$  be  $\mathcal{C}^2$ -smooth domains in  $\mathbb{C}^n$  and let  $z_0 \in bD$  and  $w_0 \in bG$ . Let  $f : D \rightarrow G$  be a biholomorphism such that  $w_0 \in \mathcal{C}(f, z_0)$ . If  $D$  is pseudoconvex at  $z_0$ , then  $G$  is also pseudoconvex at  $w_0$ .

**Proof.** Since  $w_0 \in \mathcal{C}(f, z_0)$ , then we may assume that a sequence  $\{z_j\} \subset D$  exists such that  $z_j \rightarrow z_0$  and  $f(z_j) \rightarrow w_0$  as  $j \rightarrow \infty$ . By contrast, if we assume that  $G$  is not pseudoconvex at  $w_0$ , then there is a compact set  $K \Subset G$  such that the holomorphic hull  $\hat{K}$  of  $K$  contains  $V \cap G$ , where  $V$  is a small neighborhood of  $w_0$ . (Recall that  $\hat{K} := \{z \in G : |g(z)| \leq \max_K |g|, \forall g : G \rightarrow \mathbb{C} \text{ holomorphic}\}$ .) Consequently,  $f(z_j) \in \hat{K}$  for every  $j \geq j_0$ , where  $j_0$  is sufficiently large.

Denote  $L := f^{-1}(K)$ . Then,  $L$  is a compact subset in  $D$ . We prove that  $z_j \in \hat{L}$  for every  $j \geq j_0$  and hence the proof follows. Indeed, let  $g : D \rightarrow \mathbb{C}$  be any holomorphic function. Then, since  $f(z_j) \in \hat{K}$  for every  $j \geq j_0$ , we have

$$|g \circ f^{-1}(f(z_j))| \leq \max_K |g \circ f^{-1}|, \forall j \geq j_0.$$

This implies that

$$|g(z_j)| \leq \max_K |g \circ f^{-1}| = \max_{f^{-1}(K)} |g| = \max_L |g|, \forall j \geq j_0.$$

Therefore,  $z_j \in \hat{L}$  for every  $j \geq j_0$ , and thus the proof is complete.  $\square$

**Lemma 6.** Let  $\Omega_P$  be as described in Theorem 2 and let  $f \in \text{Aut}(\Omega_P)$  be arbitrary. Then,  $t_1, t_2 \in \mathbb{R}$  exist such that  $f$  and  $f^{-1}$  can extend to be locally  $\mathcal{C}^\infty$ -smooth up to the boundaries near  $(it_1, 0)$  and  $(it_2, 0)$ , respectively, and  $f(it_1, 0) = (it_2, 0)$ .

**Proof.** We next provide the proof of [5, Lemma 3.2]. Let  $\phi : \Omega_P \rightarrow \Delta$  be the function defined by

$$\phi(z_1, z_2) = \frac{z_1 + 1}{z_1 - 1}.$$

Then, we can see that  $\phi$  is continuous on  $\overline{\Omega_P}$  such that  $|\phi(z)| < 1$  for  $z = (z_1, z_2) \in \Omega_P$  and tends to 1 when  $z_1 \rightarrow \infty$ . Let  $f : \Omega_P \rightarrow \Omega_P$  be an automorphism. We claim that a  $t_1 \in \mathbb{R}$  exists such that  $\lim_{x \rightarrow 0^-} \inf |\pi_1 \circ$

$f(x + it_1, 0)| < +\infty$ , where  $\pi_1, \pi_2$  are the projections of  $\mathbb{C}^2$  onto  $\mathbb{C}_{z_1}$  and  $\mathbb{C}_{z_2}$ , respectively, i.e.,  $\pi_1(z) = z_1$  and  $\pi_2(z) = z_2$ . Indeed, if this is not the case, the function  $\phi \circ f$  would be equal to 1 on the half plane  $\{(z_1, z_2) \in \mathbb{C}^2 : \operatorname{Re} z_1 < 0, z_2 = 0\}$ , but this is impossible because  $|\phi(z)| < 1$  for every  $z \in \Omega_P$ . Therefore, we may assume that a sequence  $x_k < 0$  exists such that  $\lim_{k \rightarrow \infty} x_k = 0$  and  $\lim_{k \rightarrow \infty} \pi_1 \circ f(x_k + it_1, 0) = w_1^0 \in \overline{\mathcal{H}}$ , where  $\mathcal{H} := \{z_1 \in \mathbb{C} : \operatorname{Re}(z_1) < 1\}$ .

After taking some subsequence, if necessary, we prove that  $\lim_{k \rightarrow \infty} \pi_2 \circ f(x_k + it_1, 0) = w_2^0$  for some  $w_2^0 \in \mathbb{C}$ . Indeed, arguing by contradiction, we assume that  $\pi_2 \circ f(x_k + it_1, 0) \rightarrow \infty$  as  $k \rightarrow \infty$ . Due to the convergence of  $\{\pi_1 \circ f(x_k + it_1, 0)\}$ , the sequence  $\{P(\pi_2 \circ f(x_k + it_1, 0))\}$  is bounded, which is a contradiction if  $\lim_{z_2 \rightarrow \infty} P(z_2) = +\infty$ . Therefore, after taking some subsequence, if necessary, we may assume that

$$\lim_{k \rightarrow \infty} P(\pi_2 \circ f(x_k + it_1, 0)) = r \geq 0.$$

Define  $\psi(w_1, w_2) = (w_1, 1/w_2)$ . Then, the map  $\psi \circ f$  is well defined near  $(it_1, 0)$  and

$$\lim_{k \rightarrow \infty} \psi \circ f(x_k + it_1, 0) = (w_1^0, 0).$$

Moreover, the defining function for  $\psi \circ f(\Omega_P \cap U)$  near  $(w_1^0, 0)$ , where  $U$  is a small neighborhood of  $(it_1, 0)$ , is

$$\operatorname{Re} w_1 + Q(w_2) < 0,$$

where

$$Q(w_2) = \begin{cases} P(1/w_2) & \text{if } w_2 \neq 0 \\ r & \text{if } w_2 = 0. \end{cases}$$

Note that  $\psi \circ f$  is a local biholomorphism on  $\Omega_P \cap U$ .  $\Omega_P \cap U$  is pseudoconvex near  $(0, 0)$ , so  $\psi \circ f(\Omega_P \cap U)$  is pseudoconvex near  $(-r, 0)$ . Moreover, the domain

$$\Omega_Q = \{(w_1, w_2) \in \mathbb{C}^2 : \operatorname{Re} w_1 + Q(w_2) < 0\}$$

has the strong  $\tilde{u}$ -property at  $(w_1^0, 0)$ . Therefore, from [Corollary 2](#), it follows that the local biholomorphisms  $\psi \circ f$  and  $(\psi \circ f)^{-1}$  can be extended to being  $\mathcal{C}^\infty$ -smooth up to the boundaries in neighborhoods of  $(it_1, 0)$  and  $(w_1^0, 0)$ , respectively. However,  $b\Omega_P$  and  $b\Omega_Q$  are not isomorphic as CR manifold germs at  $(0, 0)$  and  $(-r, 0)$ , respectively, which is a contradiction.

Given that  $\lim_{k \rightarrow \infty} f(x_k + it_1, 0) = w^0 := (w_1^0, w_2^0) \in b\Omega_P$ , from [Lemma 5](#), it follows that  $\Omega_P$  is pseudoconvex near  $w^0$ . Moreover, [Corollary 2](#) also ensures that  $f$  and  $f^{-1}$  extend to being locally  $\mathcal{C}^\infty$ -smooth up to the boundaries. Hence,  $\tau_{w^0}(b\Omega_P) = \tau_{(it_1, 0)}(b\Omega_P) = +\infty$ , which means that  $w^0 = (it_2, 0)$  for some  $t_2 \in \mathbb{R}$  by virtue of [Remark 2](#). Thus, the lemma is proved.  $\square$

#### 4. Automorphism group of $\Omega_P$

In this section, we prove [Theorem 2](#). First, let  $P$  be as describe in [Theorem 2](#). Let  $p(r)$  be a  $\mathcal{C}^\infty$ -smooth function on  $(0, \epsilon_0)$  ( $\epsilon_0 > 0$ ) such that the function

$$P(z) = \begin{cases} e^{p(|z|)} & \text{if } z \in \Delta_{\epsilon_0}^* \\ 0 & \text{if } z = 0. \end{cases}$$

**Remark 5.** Since  $\nu_0(P) = +\infty$ ,  $\lim_{r \rightarrow 0^+} p(r) = -\infty$ . Moreover, we observe that  $\limsup_{r \rightarrow 0^+} |rp'(r)| = +\infty$ ; otherwise, we have  $|p(r)| \lesssim |\log(r)|$  for every  $0 < r < \epsilon_0$ , and thus  $P$  does not vanish to infinite order at 0. In addition, from [27, Corollary 1], it follows that the function  $P(r)p'(r)$  also vanishes to infinite order at  $r = 0$ .

To prove Theorem 2, we need the following lemmas.

**Lemma 7** (see Lemma 2 in [27]). Suppose that  $0 < \alpha \leq 1$  and  $\beta > 0$  exist such that

$$\lim_{z \rightarrow 0} \frac{P(\alpha z)}{P(z)} = \beta.$$

Then,  $\alpha = \beta = 1$ .

**Lemma 8** (see Lemma 3 in [27]). Let  $\beta \in \mathcal{C}^\infty(\Delta_{\epsilon_0})$  with  $\beta(0) = 0$ . Then,

$$P(z + z\beta(z)) - P(z) = P(z) \left( |z|p'(|z|)(\operatorname{Re}(\beta(z) + o(\beta(z)))) \right) + o((\beta(z))^2)$$

for any  $z \in \Delta_{\epsilon_0}^*$  satisfying  $z + z\beta(z) \in \Delta_{\epsilon_0}$ .

In the following, we denote  $\mathcal{H} := \{z_1 \in \mathbb{C} : \operatorname{Re}(z_1) < 0\}$  as the left half-plane.

**Lemma 9.** If  $f \in \operatorname{Aut}(\Omega_P \cap U) \cap \mathcal{C}^\infty(\overline{\Omega_P} \cap U)$  that satisfies  $f_1(z_1, z_2) = a_{01}z_1 + \tilde{a}_0(z_1)$  and  $f_2(z_1, z_2) = b_{10}z_2 + z_2\tilde{b}_1(z_1)$ , where  $a_{01}, b_{10} \in \mathbb{C}^*$  with  $b_{10} > 0$  and  $\tilde{a}_0, \tilde{b}_1 \in \operatorname{Hol}(\mathcal{H} \cap U_1) \cap \mathcal{C}^\infty(\overline{\mathcal{H}} \cap U_1)$  with  $\nu_0(\tilde{a}_0) \geq 2$  and  $\nu_0(\tilde{b}_1) \geq 1$ , where  $U$  and  $U_1$  are neighborhoods of the origins in  $\mathbb{C}^2$  and  $\mathbb{C}_{z_1}$ , respectively, then  $a_{01} = b_{10} = 1$ .

**Proof.**  $f(b\Omega_P \cap U) \subset b\Omega_P$ , so we have

$$\operatorname{Re} \left( a_{01}(it - P(z_2)) + \tilde{a}_0(it - P(z_2)) \right) + P \left( b_{10}z_2 + z_2\tilde{b}_1(it - P(z_2)) \right) \equiv 0 \quad (11)$$

on  $\Delta_{\epsilon_0} \times (-\delta_0, \delta_0)$  for some  $\epsilon_0, \delta_0 > 0$ . Thus, from (11) with  $z_2 = 0$ , it follows that

$$\operatorname{Re}(a_{01}it) + o(t) = 0$$

for every sufficiently small  $t \in \mathbb{R}$ , which yields  $\operatorname{Im}(a_{01}) = 0$ .

However, if we let  $t = 0$  in (11), then we have

$$P \left( b_{10}z_2 + z_2O(P(z_2)) \right) - \operatorname{Re}(a_{01})P(z_2) + o(P(z_2)) \equiv 0 \quad (12)$$

on  $\Delta_{\epsilon_0}$ , which implies that  $\lim_{z_2 \rightarrow 0} P(b_{10}z_2 + z_2O(P(z_2)))/P(z_2) = \operatorname{Re}(a_{01}) = a_{01} > 0$ .

By assumption, we can write  $P(z_2) = e^{p(|z_2|)}$  for all  $z_2 \in \Delta_{\epsilon_0}^*$  for some function  $p \in \mathcal{C}^\infty(0, \epsilon_0)$  with  $\lim_{t \rightarrow 0^+} p(t) = -\infty$  such that  $P$  vanishes to infinite order at  $z_2 = 0$ . Therefore, by Lemma 8 and the fact that  $P(z_2)p'(|z_2|)$  vanishes to infinite order at  $z_2 = 0$  (cf. Remark 5), we obtain

$$\lim_{z_2 \rightarrow 0} \frac{P(b_{10}z_2)}{P(z_2)} = \lim_{z_2 \rightarrow 0} \frac{P(b_{10}z_2 + z_2O(P(z_2)))}{P(z_2)} = a_{01} > 0.$$

Hence, Lemma 7 ensures that  $a_{01} = b_{10} = 1$ , which completes the proof.  $\square$

**Lemma 10.** *If  $f \in \text{Aut}(\Omega_P \cap U) \cap \mathcal{C}^\infty(\overline{\Omega_P} \cap U)$  that satisfies  $f_1(z_1, z_2) = z_1 + \tilde{a}_0(z_1)$  and  $f_2(z_1, z_2) = z_2 + z_2 \tilde{b}_1(z_1)$ , where  $\tilde{a}_0 \in \text{Hol}(U_1)$  and  $\tilde{b}_1 \in \text{Hol}(\mathcal{H} \cap U_1) \cap \mathcal{C}^\infty(\overline{\mathcal{H}} \cap U_1)$  with  $\nu_0(\tilde{a}_0) \geq 2$  and  $\nu_0(\tilde{b}_1) \geq 1$ , where  $U$  and  $U_1$  are neighborhoods of the origins in  $\mathbb{C}^2$  and  $\mathbb{C}_{z_1}$ , respectively, then  $f = \text{id}$ .*

**Proof.** By expanding  $\tilde{a}_0$  into the Taylor series at 0, we have

$$\tilde{a}_0(z_1) = \sum_{k=2}^{\infty} a_{0k} z_1^k,$$

where  $a_{0k} \in \mathbb{C}$  for every  $k \geq 2$ .

$f$  preserves  $b\Omega_P \cap U$ , so it follows that

$$\text{Re}\left((it - P(z_2)) + \sum_{k=2}^{\infty} a_{0k} (it - P(z_2))^k\right) + P\left(z_2 + \tilde{b}_1(it - P(z_2))\right) \equiv 0, \quad (13)$$

or equivalently that

$$P\left(z_2 + z_2 \tilde{b}_1(it - P(z_2))\right) - P(z_2) + \text{Re}\left(\sum_{k=2}^{\infty} a_{0k} (it - P(z_2))^k\right) \equiv 0 \quad (14)$$

on  $\Delta_{\epsilon_0} \times (-\delta_0, \delta_0)$  for some  $\epsilon_0, \delta_0 > 0$ .

If  $f_1(z_1, z_2) \equiv z_1$ , then let  $k_1 = +\infty$ . By contrast, let  $k_1 \geq 2$  be the smallest integer  $k$  such that  $a_{0k} \neq 0$ . Similarly, if  $\tilde{b}_1(z_1)$  vanishes to infinite order at  $z_1 = 0$ , then denote  $k_2 = +\infty$ . Otherwise, let  $k_2 \geq 1$  be the smallest integer  $k$  such that  $b_{1k} := \frac{\partial^k}{\partial z_1^k} \tilde{b}_1(0) \neq 0$ .

Note that we may choose  $t = \alpha P(z_2)$  in (14) (where  $\alpha \in \mathbb{R}$  is selected later), and thus we obtain

$$\begin{aligned} & P\left(z_2 + z_2 b_{1k_2} P^{k_2}(z_2)(\alpha i - 1)^{k_2} + z_2 o(P^{k_2}(z_2))\right) - P(z_2) \\ & + \text{Re}\left(a_{0k_1} P^{k_1}(z_2)(\alpha i - 1)^{k_1} + o(P^{k_1}(z_2))\right) \equiv 0 \end{aligned} \quad (15)$$

on  $\Delta_{\epsilon_0} \times (-\delta_0, \delta_0)$ . Moreover, by Lemma 8, we obtain

$$\begin{aligned} & P^{k_2+1}(z_2)|z_2|p'(|z_2|)\left(\text{Re}(b_{1k_2}(\alpha i - 1)^{k_2} + g_2(z_2))\right) \\ & + P^{k_1}(z_2)\text{Re}\left(a_{0k_1}(\alpha i - 1)^{k_1} + g_1(z_2)\right) \equiv 0 \end{aligned} \quad (16)$$

on  $\Delta_{\epsilon_0}$ , where  $g_1, g_2 \in \mathcal{C}^\infty(\Delta_{\epsilon_0})$  with  $g_1(0) = g_2(0) = 0$ .

We remark that  $\alpha$  can be chosen such that  $\text{Re}(b_{1k_2}(\alpha i - 1)^{k_2}) \neq 0$  and  $\text{Re}(a_{0k_1}(\alpha i - 1)^{k_1}) \neq 0$ . Furthermore, since  $\limsup_{r \rightarrow 0^+} |rp'(r)| = +\infty$  (cf. Remark 5), (16) yields that  $k_2 + 1 > k_1$ . However, by the fact that  $P(z_2)p'(|z_2|)$  vanishes to infinite order at  $z_2 = 0$  (see Remark 5) and by (16), we have  $k_1 > k_2$ . Hence, we conclude that  $k_1 = k_2 = +\infty$ .

$k_1 = k_2 = +\infty$ , so it follows that  $f_1(z_1, z_2) \equiv z_1$ , and (14) is equivalent to

$$P\left(z_2 + \tilde{b}_1(it - P(z_2))\right) \equiv P(z_2), \quad (17)$$

on  $\Delta_{\epsilon_0} \times (-\delta_0, \delta_0)$ . The level sets of  $P$  are circles, so (17) implies that  $\tilde{b}_1(z_1) \equiv 0$ . Thus, the proof is complete.  $\square$

**Proof of Theorem 2.** Let  $f = (f_1, f_2) \in \text{Aut}(\Omega_P)$ . By Lemma 6,  $t_1, t_2 \in \mathbb{R}$  exist such that  $f$  and  $f^{-1}$  extend smoothly to the boundaries near  $(it_1, 0)$  and  $(it_2, 0)$ , respectively, and  $f(it_1, 0) = (it_2, 0)$ . After replacing  $f$  with  $T_{-t_2} \circ f \circ T_{t_1}$ , we may assume that  $f(0, 0) = (0, 0)$ , and neighborhoods  $U$  and  $V$  of  $(0, 0)$  exist such that  $f$  is a local CR diffeomorphism between  $V \cap b\Omega_P$  and  $V \cap b\Omega_P$ .

For each  $t \in \mathbb{R}$ , let us define  $F_t$  by setting  $F_t := f \circ R_{-t} \circ f^{-1}$ . Then,  $\{F_t\}_{t \in \mathbb{R}}$  is a one-parameter subgroup of  $\text{Aut}(\Omega_P) \cap \mathcal{C}^\infty(\overline{\Omega_P} \cap U)$ .

By Theorem 1, a real number  $\delta$  exists such that  $F_t = R_{\delta t}$  for all  $t \in \mathbb{R}$ , which implies that

$$f = R_{\delta t} \circ f \circ R_t, \quad \forall t \in \mathbb{R}. \quad (18)$$

We note that if  $\delta = 0$ , then  $f = f \circ R_t$ , and thus  $R_t = id$  for any  $t \in \mathbb{R}$ , which is a contradiction. Hence, we can assume that  $\delta \neq 0$ .

We prove that  $\delta = -1$ . Indeed, by (18), we have

$$f_2(z_1, z_2) \equiv e^{i\delta t} f_2(z_1, z_2 e^{it}) \quad (19)$$

on a neighborhood  $U$  of  $(0, 0) \in \mathbb{C}^2$  and for all  $t \in \mathbb{R}$ .

By expanding  $f_2$  into the Taylor series, we obtain

$$f_2(z_1, z_2) = \sum_{n=0}^{\infty} b_n(z_1) z_2^n,$$

where  $b_n$ ,  $n = 0, 2, \dots$  are in  $Hol(\mathcal{H}) \cap \mathcal{C}^\infty(\overline{\mathcal{H}})$  and  $b_0(0) = f_2(0, 0) = 0$ . Hence, Eq. (19) is equivalent to

$$\sum_{n=0}^{\infty} b_n(z_1) z_2^n \equiv \sum_{n=0}^{\infty} b_n(z_1) z_2^n e^{i\delta t + int} \quad (20)$$

on  $U$  for all  $t \in \mathbb{R}$ , which immediately implies that  $b_0(z_1) \equiv 0$ . Since  $f$  is biholomorphism,  $b_1(z_1) \not\equiv 0$ . Therefore, (20) gives that  $\delta = -1$  and  $b_n = 0$  for every  $n \in \mathbb{N} \setminus \{1\}$ , which means that  $f_2(z_1, z_2) \equiv z_2 b_1(z_1)$ .

We conclude that  $F_t = R_{-t}$  for all  $t \in \mathbb{R}$ , which implies that

$$f = R_{-t} \circ f \circ R_t, \quad \forall t \in \mathbb{R}, \quad (21)$$

and this implies that

$$f_1(z_1, z_2) \equiv f_1(z_1, z_2 e^{it})$$

on a neighborhood  $U$  of  $(0, 0)$  in  $\mathbb{C}^2$  for all  $t \in \mathbb{R}$ . Thus, we have  $f_1(z_1, z_2) = a_0(z_1)$ .

$f$  preserves the boundary  $b\Omega_P \cap U$ , so we have

$$\text{Re}(a_0(is - P(z_2))) + P(z_2 b_1(is - P(z_2))) = 0 \quad (22)$$

for all  $(z_2, s) \in \Delta_{\epsilon_0} \times (-\delta_0, +\delta_0)$ . If we let  $z_2 = 0$  in (22), we obtain

$$\text{Re}(a_0(is)) = 0 \quad (23)$$

for all  $s \in (-\delta_0, +\delta_0)$ . Hence, by the Schwarz reflection principle,  $a_0$  extends to being holomorphic in a neighborhood of the origin  $z_1 = 0$ , where we also denote the extension by  $a_0$  and the Taylor expansion of  $a_0$  at  $z_1 = 0$  is given by

$$a_0(z_1) = \sum_{m=1}^{\infty} a_{0m} z_1^m.$$

Moreover,  $f \in \text{Aut}(\Omega_P)$ , so it follows that  $a_{01} \neq 0$ . From (23), we have

$$\text{Im}(a_{01}) = 0.$$

Next, we show that  $b_1(0) \neq 0$ . By contrast, if we suppose that  $\nu_0(b_1) \geq 1$ , then from (22) with  $s = 0$ , it follows that

$$\lim_{z_2 \rightarrow 0} \frac{P(z_2 b_1(P(z_2)))}{P(z_2)} = \text{Re}(a_{01}) = a_{01} > 0,$$

which is impossible because

$$\lim_{z_2 \rightarrow 0} \frac{P(z_2 b_1(P(z_2)))}{P(z_2)} = \lim_{z_2 \rightarrow 0} \frac{P(z_2 b_1(P(z_2)))}{z_2 b_1(P(z_2))} \lim_{z_2 \rightarrow 0} \frac{z_2 b_1(P(z_2))}{P(z_2)} = 0.$$

Hence, we conclude that

$$f_2(z_1, z_2) = b_{10} z_2 + z_2 \tilde{b}_1(z_1),$$

where  $b_{10} \in \mathbb{C}^*$  and  $\tilde{b}_1 \in \text{Hol}(\mathcal{H}) \cap \mathcal{C}^\infty(\overline{\mathcal{H}})$  with  $\tilde{b}_1(0) = 0$ . In addition, by replacing  $f$  with  $f \circ R_\theta$  for some  $\theta \in \mathbb{R}$ , we can assume that  $b_{10}$  is a positive real number.

Now, we apply Lemma 9 to find that  $a_{01} = b_{10} = 1$ . Furthermore, by Lemma 10, we conclude that  $f = \text{id}$ . Hence, the proof is complete.  $\square$

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## Appendix

We recall the following lemma, which is a version of the Hopf lemma.

**Lemma 11** (see Lemma 2.3 in [29]). *Let  $\Omega \subset \mathbb{C}^n$  be a bounded domain with  $\mathcal{C}^2$  boundary. Let  $K \Subset \Omega$  be a compact set nonempty interior and select  $L > 0$ . Then,  $C = C(K, L) > 0$  exists such that for any negative plurisubharmonic function  $u$  in  $\Omega$  that satisfies the condition  $u(z) < -L$  on  $K$ , the following bound holds:*

$$|u(z)| \geq C \delta_\Omega(z) \text{ for } z \in \Omega.$$

The following lemma is a slight generalization of [29, Lemma 2.4].

**Lemma 12.** *Let  $D$  and  $G$  be domains in  $\mathbb{C}^n$  with  $\mathcal{C}^2$ -smooth boundaries,  $z_0 \in bD$ ,  $w_0 \in bG$ , and  $F_1, F_2 : [0, +\infty) \rightarrow [0, +\infty)$  are nonnegative functions with  $F_1(0) = F_2(0) = 0$  such that  $F_1$  is increasing. Assume that there is a neighborhood  $U$  of  $z_0$  such that for each  $w \in U \cap bD$ , a plurisubharmonic function  $\psi_w$  exists such that*

$$(i) \quad \lim_{D \times bD \ni (z, w) \rightarrow (z_0, w_0)} \psi_w(z) = 0,$$



- (ii)  $\psi_w(z) \leq -F_1(|z - w|)$ ,
- (iii)  $\psi_{\pi(z)}(z) \geq -F_2(\delta_D(z))$

for  $z \in U \cap D$ . Let  $f : D \rightarrow G$  be a proper holomorphic map such that  $w_0 \in \mathcal{C}(f, z_0)$ . Then, neighborhoods  $\tilde{U} \subset U$  and  $V$  of  $z_0$  and  $w_0$ , respectively, exist such that  $\delta_G(f(z)) \lesssim F_2(\delta_D(z))$  for any  $z \in \tilde{U} \cap D$  such that  $f(z) \in V \cap G$ .

**Proof.** The proof proceeds along the same lines described in [29, Section 2], but we give the detailed proof for convenience.

For  $\epsilon > 0$ , we consider the open set

$$D^\epsilon = \{z \in U \cap D : \psi_{z_0} > -\epsilon\}.$$

By virtue of (ii),  $\epsilon_0 > 0$  exists such that for any  $\epsilon \in (0, \epsilon_0]$ , we have  $\bar{D}^\epsilon \subset \bar{U}$ . Hence, the boundary  $bD^\epsilon \subset (\bar{U} \cap bD) \cup S^\epsilon$ , where  $S^\epsilon = \{z \in \bar{U} \cap D : \psi_{z_0}(z) = -\epsilon\}$ .

We fix  $\epsilon \in (0, \epsilon_0/2]$  and choose  $\epsilon_1 > 0$  such that  $D \cap (z_0 + \epsilon_1\mathbb{B}) \subset D^\epsilon$ , where  $\mathbb{B}$  is the open unit ball in  $\mathbb{C}^n$ . Without any loss of generality, we may assume that the neighborhood  $U$  is sufficiently small such that  $\delta_D(z) = |z - \pi(z)|$  for  $z \in U \cap D$ . We fix a positive number  $\delta$  with the properties  $\epsilon_1/50 < \delta < 2\delta < \epsilon_1/10$  and consider the compact set  $K = \bar{D} \cap (z_0 + 2\delta\bar{\mathbb{B}}) \setminus (z_0 + \delta\bar{\mathbb{B}})$ . For  $\epsilon_2 < \epsilon_1/100$ , by (ii), we have

$$\begin{aligned} \max\{\psi_\zeta(z) : z \in K, \zeta \in bD \cap (z_0 + \epsilon_2\bar{\mathbb{B}})\} &\leq -F_1\left(d(K, bD \cap (z_0 + \epsilon_2\bar{\mathbb{B}}))\right) \\ &\leq -F_1(\delta - \epsilon_2). \end{aligned}$$

However, by (i), we can choose  $\epsilon_2$  such that

$$-F_1(\delta - \epsilon_2) < \gamma := \min\{\psi_\zeta(z) : z \in D \cap (z_0 + \epsilon_2\bar{\mathbb{B}}), \zeta \in bD \cap (z_0 + \epsilon_2\bar{\mathbb{B}})\}.$$

We fix  $\epsilon_2 > 0$ . Let  $\tau > 0$  be such that

$$-F_1(\delta - \epsilon_2) < -\tau < -\tau/2 < \gamma < 0.$$

We consider a smooth nondecreasing convex function  $\phi(t)$  with the properties  $\phi(t) = -\tau$  for  $t \leq -\tau$  and  $\phi(t) = t$  for  $t > -\tau/2$ . We set  $\rho_\zeta(z) = \tau^{-1}\phi \circ \psi_\zeta(z)$ . Then,  $\rho_\zeta(z)|_K = -1$  for  $\zeta \in bD \cap (z_0 + \epsilon_2\bar{\mathbb{B}})$ , and we can extend  $\rho_\zeta(z)$  to  $D$  by setting  $\rho_\zeta(z) = -1$  for  $z \in D \setminus (z_0 + 2\delta\bar{\mathbb{B}})$ . We obtain a function  $\rho_\zeta(z)$ , which is a negative continuous plurisubharmonic function on  $D$  that satisfies  $\rho_\zeta(z) = -1$  on  $D \setminus (z_0 + \delta\bar{\mathbb{B}})$  and  $\rho_\zeta(z) = \tau^{-1}\psi_\zeta(z)$  on  $D \cap (z_0 + \epsilon_2\bar{\mathbb{B}})$  for  $\zeta \in bD \cap (z_0 + \epsilon_2\bar{\mathbb{B}})$ .

$\epsilon_3 \in (0, \epsilon_2/2)$  exist such that  $\pi(z) \in bD \cap (z_0 + \epsilon_2\bar{\mathbb{B}})$  for any  $z \in D \cap (z_0 + \epsilon_3\bar{\mathbb{B}})$ . We also fix a point  $p \in D \cap (z_0 + \epsilon_3\bar{\mathbb{B}})$  and define the function

$$\varphi_p(w) = \begin{cases} \sup\{\rho_{\pi(p)}(z) : z \in f^{-1}(w)\} & \text{for } w \in f(D^\epsilon), \\ -1 & \text{for } w \in G \setminus f(D^\epsilon). \end{cases}$$

$f$  is proper, so the function  $\varphi_p(w)$  is a continuous negative plurisubharmonic function on  $G$  (see [29, Lemma 2.2]).

Let  $V$  be a neighborhood of the point  $w_0$  such that the surface  $V \cap bG$  is smooth. We fix a compact set  $K \Subset f(D^{\epsilon_2}) \cap V$  with nonempty interior (this is possible because  $w_0 \in \mathcal{C}(f, z_0)$  and  $f(D^{\epsilon_2})$  is an open set). Assume that  $2\max_{w \in K} \varphi_p(w) \leq -L = -L(p)$ . Then, by Lemma 11, we have  $|\varphi_p(w)| \geq C(L)\delta_G(w)$  for

$w \in G \cap V$ , where  $C = C(L) > 0$  depends only on  $L = L(p)$ . Now, we show that  $L$  (and hence  $C$ ) can be chosen independently of  $p$ .

We have

$$\begin{aligned}\max_{w \in K} \varphi_p(w) &= \max\{\rho_{\pi(p)}(z) : z \in f^{-1}(w) \cap D^{\epsilon_2}, w \in K\} \\ &= \max\{\rho_{\pi(p)}(z) : z \in f^{-1}(K) \cap D^{\epsilon_2}\}.\end{aligned}$$

Since  $f$  is proper,  $\mathcal{C}(f, z) \subset bG$  for  $z \in U \cap bD$ , and thus the set  $f^{-1}(K)$  has no limit points on  $U \cap bD$ . Therefore, the set  $K' = \overline{f^{-1}(K) \cap D^{\epsilon_2}}$  is relatively compact in  $U \cap D$ . If  $z \in K'$ , then by (iii), we have

$$\begin{aligned}\max\{\psi_{\pi(p)}(z) : z \in K'\} &\leq -F_1\left(\min\{|z - \zeta| : z \in K', \zeta \in bD \cap (z_0 + \epsilon_2\mathbb{B})\}\right) \\ &= -F_1\left(d(K', bD \cap (z_0 + \epsilon_2\mathbb{B}))\right).\end{aligned}$$

The last quantity does not exceed some constant  $-N < 0$ , so we may set

$$2 \max_{z \in K'} \rho_{\pi(p)}(z) \leq 2\tau^{-1}N := L,$$

and  $L$  is independent of  $p$ .

Now, if  $f(p) \in V \cap G$ , then by (iii), we have

$$\begin{aligned}\delta_G(f(p)) &\leq C|\varphi_p(f(p))| \leq C|\rho_{\pi(p)}(p)| \\ &\leq CF_2(\delta_D(p)).\end{aligned}$$

In this case,  $C > 0$  does not depend on  $p$ , so we have

$$\delta_G(f(z)) \lesssim F_2(\delta_D(z))$$

for any  $z \in D \cap (z_0 + \epsilon_3\mathbb{B})$  such that  $f(z) \in G \cap V$ . This completes the proof.  $\square$

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