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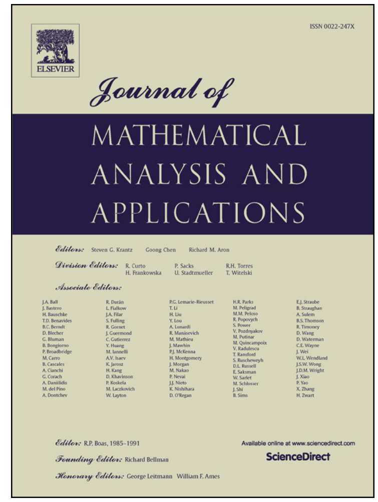

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Polynomial inequalities on certain algebraic hypersurfaces

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Abstract

We prove that any Markov set in $\mathbb{C}^N$ satisfies a Schur type inequality for polynomials and we give a generalization for polynomial matrices. As a consequence, we obtain polynomial inequalities on compact subsets of algebraic hypersurfaces of the form $V = \{z_{N+1}^k = s(z_1, \ldots, z_N)\} \subset \mathbb{C}^{N+1}$, where s is a non constant polynomial of N variables. We also give a condition equivalent to the Markov inequality on compact subsets of V.

Keywords: Markov inequality, Schur inequality, division inequality, algebraic hypersurfaces

2010 MSC: 41A17, 32E30, 14Q10

1. Introduction

In the constellation of polynomial inequalities, the Markov inequality – which relates, on a given compact set in $\mathbb{C}^N$, the growth of a polynomial to that of its derivatives – certainly plays one of the most important roles. Far from being an isolated result, it has become a classical tool in numerical analysis, especially in problems involving discretizations (see the construction of admissible meshes in

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[8] for a recent example), it is essentially equivalent to quite a few other useful inequalities and it also has a deep theoretical significance. For instance, it has been connected to fundamental concepts of (pluri)potential theory and shown to be equivalent to the existence of extension operators for C^∞ functions (see [19]). Since polynomials are essential objects on algebraic sets, it is natural to look for an analogue to the Markov inequality for compact subsets of algebraic sets.

The first major result in this direction was obtained in [5] where it was shown that the Markov inequality – in which standard derivation is replaced by tangential derivation – characterizes smooth algebraic submanifolds in $\mathbb{R}^N$. For further related works, we refer to [6], [3], [11], [7], [16] and the references therein.

In this paper, we follow a different path. Our generalization of the Markov inequality on algebraic hypersurfaces uses ordinary higher derivatives. We are interested in polynomial inequalities for compact subsets of algebraic hypersurfaces of the form $V = \{z_{N+1}^k = s(z_1, \ldots, z_N)\}$, $N \geq 1$, $k \in \{1, 2, \ldots\}$ and s is a non constant polynomial of N variables. Namely, we propose two versions of the Markov inequality on a compact set $E \subset V$ and we prove that one of them is equivalent to the fact that the natural projection of E onto $\mathbb{C}^N$ is a Markov set (see below). Moreover, a division inequality (also called Schur type estimate) on the set E is proved under the assumption that its projection $\pi(E)$ satisfies the Markov inequality. The properties mentioned above have been the subject of many works in recent years, e.g. by Baran, Brudnyi, Levenberg, Pleśniak in view of their many applications to numerical analysis, constructive function theory and approximation. However, the case of compact subsets of algebraic hypersurfaces in complex space requires different techniques and seems to be of independent interest.

To derive our results, we need to prove, see section 3, an explicit division inequality in the ordinary case which has relevant applications in other topics. The main motivation for our study was an open problem in [2] where the authors asked about a generalized Markov property for compact subsets of $\{x^3+y^3 = 1\}$

and $\{x^4+y^4=1\}$ (see [2, Open problem 1]). Our approach relies on elementary arithmetic and algebraic tools that one cannot expect to use in more general situations. However, our results provide an incentive for looking for similar results on general algebraic sets. More involved methods are certainly required to obtain such generalizations, probably with less precise estimates.

We usually denote by $\mathcal{P}(\mathbb{C}^N)$ (resp. $\mathcal{P}_d(\mathbb{C}^N)$) the space of all polynomials of N complex variables with coefficients in $\mathbb{C}$ (resp. of total degree at most d). Sometimes, however, it is more convenient to write $\mathcal{P}(z_1,\ldots,z_N)$ or $\mathcal{P}(z)$, $z=(z_1,\ldots,z_N)$ for $\mathcal{P}(\mathbb{C}^N)$ and likewise for the subspaces of polynomials of a given degree. We use a standard multinomial notation. In particular, for $\alpha=(\alpha_1,\ldots,\alpha_N)\in\mathbb{N}^N$, we have $|\alpha|=\alpha_1+\cdots+\alpha_N$, $z^\alpha=z_1^{\alpha_1}\ldots z_N^{\alpha_N}$ and $D^\alpha=\partial^{|\alpha|}/(\partial z_1^{\alpha_1}\ldots\partial z_N^{\alpha_N})$. Given $v\in\mathbb{C}^N$, the n-th directional derivative of a holomorphic function f at a along v is

$$D_v^n(f)(a)=df^n(a)(v,v,\ldots,v)=\frac{d^n}{dt^n}f(a+tv)\Big|_{t=0}$$

where $d^nf(a)$ denotes the complete n-th Frechet derivative of f at a.

We will work with the group $\mathbb{U}_k$ of the k-th roots of unity in $\mathbb{C}$. Any generator of $\mathbb{U}_k$ is called a primitive k-th root of unity.

Finally, given a compact set E and a continuous function f on E, as usual, we set $\|f\|_E=\max\{|f(z)|,z\in E\}$.

2. Basic facts on Markov inequality and companion inequalities

Definition 1 (Markov set and Markov inequality)**.** A compact set $E\subset\mathbb{C}^N$ is said to be *a Markov set* if there exist constants $M,m>0$ such that

$$\|D^\alpha p\|_E\le M^{|\alpha|}(\deg p)^{m|\alpha|}\|p\|_E,\quad p\in\mathcal{P}(\mathbb{C}^N),\quad \alpha\in\mathbb{N}^N. \tag{1}$$

Such inequality is called a *Markov inequality* for E.

By iteration, inequality (1) is satisfied for all α once it is satisfied for all α of length one. The following properties immediately follow from the definition.

1. A compact set $E \subset \mathbb{C}^N$ is a Markov set if and only if so is $A(E)$ where A is any affine automorphism of $\mathbb{C}^N$.

2. A finite union of Markov sets is a Markov set.

3. The Cartesian product of two Markov sets E_i in $\mathbb{C}^{N_i}$, $i = 1, 2$ is a Markov set in $\mathbb{C}^{N_1+N_2}$.

4. A Markov set E in $\mathbb{C}^N$ is $\mathcal{P}(\mathbb{C}^N)$-*determining* (*determining* for short) that is, $p \in \mathcal{P}(\mathbb{C}^N)$ and $\|p\|_E = 0$ implies $p = 0$. (Otherwise, (1) cannot hold for a polynomial p of minimal positive degree which vanishes on E.)

Considerable work has been done in the last decades about the problem of finding (geometrical) conditions ensuring that a given compact is a Markov set and that of finding (near) optimal constants in Markov inequalities for a given compact set. The survey paper [20] and the references therein provide an account of the current knowledge. For theoretical applications, the main references are [18], [19]. For connections to (pluri-) potential theory, the reader may for instance consult [1].

Observe that if E is a $\mathcal{P}(\mathbb{C}^N)$-determining compact set, the function $p \to \|p\|_E$ defines a norm on $\mathcal{P}(\mathbb{C}^N)$. Since the map $p \in \mathcal{P}_n(\mathbb{C}^N) \to D^\alpha p$ is linear continuous on $(\mathcal{P}_n(\mathbb{C}^N), \| \dots \|_E)$, its (operator) norm $M(\alpha, n)$ is well defined and

$$\|D^\alpha p\|_E \leq M(\alpha, n)\|p\|_E, \quad p \in \mathcal{P}_n(\mathbb{C}^N).$$

Thus, to say that a determining compact set E is a Markov set is equivalent to say that the norm $M(\alpha, n)$ grows polynomially in n, i.e $M(\alpha, n) = O(n^m)$, with $m \in \mathbb{R}^+$, for all α of length 1. We will set

$$M(\ell, n) = \max_{|\alpha|=\ell} M(\alpha, n). \tag{2}$$

Let E be a $\mathcal{P}(\mathbb{C}^N)$-determining compact set and q be a non constant polynomial.

The map $\phi : h \in q\mathcal{P}_n(\mathbb{C}^N) \to h/q \in \mathcal{P}_n(\mathbb{C}^N)$ is linear, hence continuous (with the norm $\| \cdot \|_E$) and its operator norm $D(E, q, n)$ is well defined in $\mathbb{R}^+$.

Taking $h = pq$ so that $\phi(h) = p$, we have

$$\|p\|_E \le D(E, q, n)\|pq\|_E, \quad p \in \mathcal{P}_n(\mathbb{C}^N). \tag{3}$$

Definition 2 (Division set and division inequality)**.** A determining compact set E in $\mathbb{C}^N$ is said to be a *division set* if for all non zero polynomial q, the norm $D(E, q, n)$ grows polynomially in n. Equation (3) in which $D(E, q, n)$ is replaced by any explicit bound is called a *division inequality*.

The most classical division inequality, which holds for $E = [-1, 1]$, is due to Schur [23] and states that

$$\|p\|_{[-1,1]} \le (1 + \deg p)\|pq\|_{[-1,1]}, \quad q(x) = x, \quad p \in \mathcal{P}(\mathbb{R}).$$

For this reason division inequalities are sometimes called Schur inequalities. Such kind of estimate has been investigated by e.g. Stein [24] and Goetgheluck [12], [13], [14]. It is shown in [4] that, in the complex one-dimensional case, Markov sets and division sets coincide. Let us point out that, of course, one may define Markov sets and division sets in $\mathbb{R}^N$ by restricting the above definitions to $\mathcal{P}(\mathbb{R}^N)$. It is readily seen that, for $E \subset \mathbb{R}^N$, to be a Markov (resp. division) set as a subset of $\mathbb{R}^N$ is equivalent to be a Markov (division) set as a subset of $\mathbb{C}^N$ but the involved constant may (slightly) differ.

In the next section we prove a specific division inequality for compact sets in $\mathbb{C}^N$ that will be used later in the paper.

3. A Polynomial division inequality for Markov compact sets in $\mathbb{C}^N$

We prove that a Markov set E in $\mathbb{C}^N$ is a division set and we give a simple polynomial bound for $D(E, q, n)$ in terms of the constants involved in the Markov inequality for E.

Theorem 3. *Let E be a compact set in $\mathbb{C}^N$ satisfying Markov inequality (1) and $q \in \mathcal{P}(\mathbb{C}^N)$ a non zero polynomial of degree d. There exists a positive constant C depending only on q and E such that*

$$\|p\|_E \le C(d + n)^{dm}\|pq\|_E, \quad p \in \mathcal{P}_n(\mathbb{C}^N) \tag{4}$$

An explicit value for C is given in the proof. Of course we may assume $d \geq 1$, the case $d = 0$ being obvious.

Lemma 4. *Let E be a determining compact set in $\mathbb{C}^N$. We have*

$$\left\|D_v^k p\right\|_E \leq \|v\|^k M(k,n)\|p\|_E, \quad p \in \mathcal{P}_n(\mathbb{C}^N),\ k \in \mathbb{N},\ v = (v_1, \ldots, v_N) \in \mathbb{C}^N$$

where $M(k,n)$ is defined in (2) *and* $\|v\| = |v_1| + \cdots + |v_N|$.

In particular, if (1) holds true, for all $k \in \mathbb{N}$ and $v = (v_1, \ldots, v_N) \in \mathbb{C}^N$ we have

$$\left\|D_v^k p\right\|_E \leq \|v\|^k M^k (\deg p)^{mk} \|p\|_E, \quad p \in \mathcal{P}(\mathbb{C}^N). \tag{5}$$

Proof. By the multivariate Taylor formula, for $p \in \mathcal{P}_n(\mathbb{C}^N)$ and $a \in E$, we have

$$p(a+h) = \sum_{j=0}^{n} \sum_{|\alpha|=j}^{n} \binom{j}{\alpha} \frac{h^\alpha}{j!} D^\alpha p(a), \quad h \in \mathbb{C}^N.$$

Taking $h = tv$ and differentiating k times with respect to t at $t = 0$, we get

$$D_v^k p(a) = \sum_{|\alpha|=k} \binom{k}{\alpha} v^\alpha D^\alpha p(a). \tag{6}$$

Hence, using the definition of $M(k,n)$,

$$\begin{aligned} \left|D_v^k p(a)\right| \leq & M(k,n)\|p\|_E \sum_{|\alpha|=k} \binom{k}{\alpha} (|v_1|, \ldots, |v_N|)^\alpha \\ = & (|v_1| + \cdots + |v_N|)^k M(k,n)\|p\|_E \end{aligned}$$

where the last equality uses the multinomial formula. The lemma follows. □

Proof of Theorem 3. It is similar in spirit to that of [13, Lemma 3]. Let $\hat{q}$ be the homogeneous part of degree d in q. Choose $v = (v_1, \ldots, v_N) \in \mathbb{C}^N$ such that $\|v\| = 1$ and $\hat{q}(v) \neq 0$. We will prove that (4) holds with

$$C = C(q, E) = \frac{\gamma^d M^d}{|\hat{q}(v)|}, \quad \gamma = 5/2 \tag{7}$$

where M is the constant from (1) or (5) and the value of γ arises from technical reasons that we will explain below. To optimize our estimate, we should therefore choose v on the unit ball which maximizes $|\hat{q}(v)|$.

Let p be a polynomial of degree $n \geq 1$. Take a point $z_0 \in E$ with $|p(z_0)| = \|p\|_E$. We define $d+1$ positive numbers c_i as

$$c_i = i! \frac{(\gamma M n^m)^i}{C n^{md}}, \quad i = 0, \dots, d. \tag{8}$$

If $|q(z_0)| \geq c_0$ then

$$\|p\|_E = |p(z_0)| \leq \frac{|p(z_0)q(z_0)|}{c_0} \leq C n^{md} \|pq\|_E \leq C(d+n)^{md} \|pq\|_E,$$

and inequality (4) is satisfied. We will now deal with the more complicated case where $|q(z_0)| < c_0$. Using (6) and taking into account that $D^\alpha \hat{q}$ is constant for $|\alpha| = d$, we have

$$\begin{aligned} D_v^d q(a) &= \sum_{|\alpha|=d} \binom{d}{\alpha} v^\alpha D^\alpha q(a) = \sum_{|\alpha|=d} \binom{d}{\alpha} v^\alpha D^\alpha \hat{q}(a) \\ &= \sum_{|\alpha|=d} \binom{d}{\alpha} v^\alpha D^\alpha \hat{q}(0) = d! \hat{q}(v) \end{aligned}$$

for any $a \in \mathbb{C}$ where the last equality is just a Taylor formula. Hence, since, in view of (8) and (7), $d!|\hat{q}(v)| = c_d$, we have $|D_v^d q(z_0)| = c_d$. Thus if $|q(z_0)| < c_0$, we have, in particular, that $|D_v q(z_0)| \geq c_1$ in the case of $d = 1$. Hence there exists $l \in \{1, \dots, d\}$ such that

$$\left| D_v^i q(z_0) \right| < c_i \text{ for } 0 \leq i \leq l-1 \text{ while } \left| D_v^l q(z_0) \right| \geq c_l. \tag{9}$$

The idea of the proof is that $\|p\|_E$ is comparable to $|p(z_0) D_v^l q(z_0)|$ which, due to the definition of l, is the dominant term in $D_v^l (pq)(z_0)$ and, in view of the Markov inequality, this derivative can be estimated in terms of $\|pq\|_E$. The details are as follows. We start from

$$\|p\|_E = |p(z_0)| \leq \frac{1}{c_l} \left| p(z_0) D_v^l q(z_0) \right|, \tag{10}$$

which uses the definition of l. By the Leibniz rule, we have

$$p D_v^l q = D_v^l (pq) - \sum_{i=0}^{l-1} \binom{l}{i} D_v^{l-i} p D_v^i q,$$

hence, from (10),

$$\begin{aligned}\|p\|_E &\le \frac{1}{c_l}\left[\left|D_v^l(pq)(z_0)\right| + \sum_{i=0}^{l-1}\binom{l}{i}\left|D_v^{l-i}p(z_0)D_v^i q(z_0)\right|\right]\\ &\le \frac{1}{c_l}\left[\left\|D_v^l(pq)\right\|_E + \sum_{i=0}^{l-1}\binom{l}{i}\left\|D_v^{l-i}p\right\|_E\ \left|D_v^i q(z_0)\right|\right]\end{aligned}$$

and, in view of the definition of l in (9),

$$\|p\|_E \le \frac{1}{c_l}\left[\left\|D_v^l(pq)\right\|_E + \sum_{i=0}^{l-1} c_i\binom{l}{i}\left\|D_v^{l-i}p\right\|_E\right].$$

We may now use the Markov inequality in the form (5), taking into account that $\|v\| = 1$, to get

$$\|p\|_E \le \frac{1}{c_l}\left[M^l(d+n)^{lm}\|pq\|_E + \sum_{i=0}^{l-1} c_i\binom{l}{i}M^{l-i}n^{(l-i)m}\|p\|_E\right].$$

Inserting the values of the coefficients c_i and C (see (8), (7)), we obtain,

$$\|p\|_E \le \gamma^{d-l}\frac{M^d(d+n)^{lm}n^{m(d-l)}}{l!|\hat{q}(v)|}\|pq\|_E + \|p\|_E\sum_{i=0}^{l-1}\binom{l}{i}\frac{i!}{l!}\gamma^{i-l}$$

Now, we bound the first term on the right hand side using $l \ge 1$ and $n \le d+n$ and observe that the sum in the second one are the first l terms in the series expansion for $\exp(1/\gamma) - 1$ to finally obtain

$$\|p\|_E \le \frac{\gamma^{d-1}M^d}{|\hat{q}(v)|}(d+n)^{dm}\|pq\|_E + (\exp(1/\gamma) - 1)\|p\|_E.$$

Hence,

$$(2 - \exp(1/\gamma))\,\|p\|_E \le \frac{\gamma^{d-1}M^d}{|\hat{q}(v)|}(d+n)^{dm}\|pq\|_E.$$

We obtain

$$\|p\|_E \le \frac{\gamma^d M^d}{|\hat{q}(v)|}(d+n)^{dm}\|pq\|_E,$$

provided than $1 \le \gamma(2 - \exp(1/\gamma))$. The value $\gamma(2 - \exp(1/\gamma))$ equals 1 for $\gamma \approx 2.258$ and any greater γ works, for instance $\gamma = 5/2$. $\square$

Applying the inequality to p^k rather than p, we obtain the following corollary.

Corollary 5. *Under the assumption of Theorem 3, for all $k \in \mathbb{N}$, we have*

$$\|p\|_E \leq C^{1/k}(d+nk)^{dm/k}\|p\ |q|^{1/k}\|_E, \quad p \in \mathcal{P}_n(\mathbb{C}^N), \quad n \in \mathbb{N},$$

with the same constant C as in (4).

We will need a slight extension of Theorem 3 to the case of polynomial vectors.

Let $\mathbf{P} = (p_1, \ldots, p_l)^T$ is a column vector whose entries p_i are polynomials in $\mathcal{P}(\mathbb{C}^N)$ and $\mathbf{A} = (q_{ij})$ is a $l \times l$ matrix whose entries q_{ij} are elements of $\mathcal{P}(\mathbb{C}^N)$. The matrix product $\mathbf{AP}$ is again a column vector of polynomials. Observe that $\det \mathbf{A}$ is itself a polynomial. We will write $\|\mathbf{P}\|_E = \max\{\|p_i\|_E : i = 1, \ldots, l\}$ and $\|\mathbf{A}\|_E = \sum_{j=1}^n \|\mathrm{Col}_j(\mathbf{A})\|_E$ where $\mathrm{Col}_j(\mathbf{A})$ denotes the j-th column of $\mathbf{A}$ so that, just as in the ordinary case, we have $\|\mathbf{AP}\|_E \leq \|\mathbf{A}\|_E\|\mathbf{P}\|_E$.

Corollary 6. *Let $E \subset \mathbb{C}^N$ be a compact set in $\mathbb{C}^N$ satisfying Markov inequality (1) and $\mathbf{A}$ be a fixed polynomial matrix as above whose determinant is a non zero polynomial of degree r. Then there exists a positive constant c depending only on $\mathbf{A}$ and E such that*

$$\|\mathbf{P}\|_E \leq c(r+n)^{rm}\|\mathbf{AP}\|_E, \quad \mathbf{P} = (p_1, \ldots, p_l)^T, \quad p_i \in \mathcal{P}_n(\mathbb{C}^N). \tag{11}$$

Proof. Since $\det \mathbf{A} \not\equiv 0$, we may apply Theorem 3 with $q = \det \mathbf{A}$ to get,

$$\|p_j\|_E \leq C(r+n)^{rm}\|(\det \mathbf{A})p_j\|_E, \quad j = 1, \ldots, l,$$

with a constant $C = C(E, \det \mathbf{A})$. Hence

$$\|\mathbf{P}\|_E \leq C(r+n)^{rm}\|(\det \mathbf{A})\mathbf{P}\|_E.$$

Let now $\mathbf{B}$ denote the transpose of the comatrix of $\mathbf{A}$; i.e. $\mathbf{B}$ is a polynomial matrix satisfying $\mathbf{BA} = (\det \mathbf{A})I$ where I is the identity matrix. Replacing $(\det \mathbf{A})\mathbf{P}$ by $\mathbf{BAP}$ in the above estimate, we obtain

$$\|\mathbf{P}\|_E \leq C(r+n)^{rm}\|\mathbf{BAP}\|_E \leq C(r+n)^{rm}\|\mathbf{B}\|_E\|\mathbf{AP}\|_E$$

so that inequality (11) holds with $c = C \cdot \|\mathbf{B}\|_E$ which depends only on E and $\mathbf{A}$. ∎

4. Algebraic preliminaries on the hypersurfaces $\{y^k = s(z_1, \dots, z_N)\} \subset \mathbb{C}^{N+1}$

We want to study an extension of the Markov inequality to compact subsets of an hypersurface of the form

$$V(f) = \{f(z, y) = 0, (z, y) = (z_1, \dots, z_N, y) \in \mathbb{C}^{N+1}\}, \quad f(z, y) = y^k - s(z)$$

where $k \geq 1$ and s is a non constant polynomial in $\mathcal{P}(\mathbb{C}^N)$ and we use y instead of z_{N+1} to emphasize the particular role played by this variable. A basic but fundamental observation is that f is invariant under the group $\mathbb{U}_k$ (of the k-th roots of unity in $\mathbb{C}$), that is, $f(z, wy) = f(z, y)$ for any $w \in \mathbb{U}_k$. In particular $(z, y) \in V \implies (z, wy) \in V$.

We now establish the algebraic tools that will be used for our purpose.

4.1. Arithmetic properties of the polynomial f

Lemma 7. *The polynomial f is always square free. It is reducible if and only if $s(z) = r(z)^n$ for some polynomial $r \in \mathcal{P}(\mathbb{C}^N)$ and power $n > 1$ such that n divides k.*

The first property enables us to use the Nullstellensatz ([9, Theorem 2, p. 172]): if a polynomial p vanishes on $V(f)$ then f divides p. The second part is a classical result due to Cappelli a more involved version of which is given in the following result.

Theorem 8 (Capelli's Theorem, see [22])**.** *A binomial $y^n - a$ is reducible over a field K (of characteristic 0) if and only if either $a = b^p$, p prime, $p|n$, $b \in K$ or $a = -4b^4$, $4|n$, $b \in K$.*

Proof of Lemma 7. For the irreducibility of f, we may use the above theorem with $K = \mathcal{F}(z)$ the field of fractions of $\mathcal{P}(z)$, taking into account that the irreducibility of f as an element of the ring of polynomials in y with coefficients in $\mathcal{F}(z)$ implies its irreducibility as an element of $\mathcal{P}(z, y)$ and that s is a power in $\mathcal{F}(z)$ if and only if it is a power in $\mathcal{P}(z)$.

To prove that f is square free, we proceed by contradiction as follows. Assume that $p^2(z,y)$ divides $f(z,y)$ in $\mathcal{P}(z,y)$ with p irreducible. It is readily seen that the degree of p in y must be positive. Next, take (z_0, y_0) so that $p(z_0, y_0) = 0$. It follows that y_0 is a multiple root of the univariate polynomial $y^k - s(z_0)$ which forces $s(z_0) = 0$. We proved that $s = 0$ on $\{p(z,y) = 0\}$, and, since p is irreducible, p divides s which forces $s = 0$ since y appears in p but not in s. We get a contradiction since s is not constant. □

Since f is invariant under $\mathbb{U}_k$, the uniqueness of decomposition in irreducible factors gives that, if $w \in \mathbb{U}_k$, for any irreducible divisor p of f we have either $p(z, wy) = p(z,y)$ or $p(z, wy)$ is another irreducible divisor of f. In fact, it is not difficult to see that the decomposition of f in irreducible factors is of the form

$$f(z,y) = y^k - (r(z))^n = \prod_{i=1}^{n} p(z, \xi_i y) \tag{12}$$

where p is an irreducible divisor of f invariant under $\mathbb{U}_m$ for $m \cdot n = k$ and the ξ_i are representative of the n elements of the factor group $\mathbb{U}_k/\mathbb{U}_m$.

4.2. The ring of polynomials on V

Recall that the ring of polynomials on $V = V(f)$ is

$$\mathcal{P}(V) = \left\{ p_{|V}, p \in \mathcal{P}(z,y) \right\}.$$

We have a very simple algebraic structure for $\mathcal{P}(V)$ as shown by the following lemma and this is one of two key technical points used in the sequel.

Lemma 9. *We have*

$$\mathcal{P}(z) \otimes \mathcal{P}_{k-1}(y) \simeq \mathcal{P}(V).$$

Here, as usual, $\mathcal{P}(z) \otimes \mathcal{P}_{k-1}(y)$ denotes the subspace of $\mathcal{P}(z,y)$ formed of all polynomials of the form $\sum_{i=0}^{k-1} c_i(z) y^i$ with $c_i \in \mathcal{P}(z)$. A specific isomorphism

$$\Phi : \mathcal{P}(z) \otimes \mathcal{P}_{k-1}(y) \longrightarrow \mathcal{P}(V)$$

is merely the restriction to V, that is $\Phi(p) = p_{|V}$ while Φ^{-1} is the unique linear map on $\mathcal{P}(V)$ obtained by substituting $s(z)$ for y^k, that is

$$\Phi^{-1}\left((z^\alpha y^m)_{|V}\right) = z^\alpha s^q(z) y^r$$

where $m = qk + r$, $r \in \{0, \ldots, k-1\}$.

Proof. We first prove that the linear map Φ above is one-to-one. If $p \in \ker \Phi$ then $p = 0$ on V and, in view of Lemma 7, the Nullstellensatz implies that $p = qf$. Comparing the degrees in y of both sides, one sees that one must have $q = 0$ hence $p = 0$. The fact that the map is onto is obvious because on V we have $y^m = y^r s^q(z)$ when $m = qk + r$, $r \in \{0, \ldots, k-1\}$. Thus, on V, any polynomial coincides with a polynomial from $\mathcal{P}(z) \otimes \mathcal{P}_{k-1}(y)$. $\square$

4.3. The degree of a polynomial on V

Since it is a basic element in the Markov inequality we need to suitably define the degree of a polynomial on V. The natural definition (which works for any algebraic set) is as follows.

Definition 10. The degree $\deg_V p$ of a polynomial $p \in \mathcal{P}(V)$ is defined as

$$\deg_V p = \min\left\{\deg P : P_{|V} = p\right\}.$$

In particular, for any $P \in \mathcal{P}(z, y)$, we have $\deg_V P_{|V} \le \deg P$. In many cases, equality occurs, but it is not difficult to see that inequality may be strict.

Lemma 11. *Let $P \in \mathcal{P}(z, y)$. We have $\deg_V P_{|V} < \deg P$ if and only if the leading homogeneous component $\hat{f}$ of f divides the leading homogeneous component $\hat{P}$ of P.*

Proof. Assume that there exists $Q \in \mathcal{P}(z, y)$ such that $\deg Q < \deg P$ but $P = Q$ on V. The latter implies that f divides $P - Q$ or $Q = P - fT$ with $T \in \mathcal{P}(z, y)$. Since the degree of Q is smaller than the degree of P, the leading homogeneous component of P and fT must be cancelled. Hence, $\hat{P} = \widehat{fT} = \hat{f}\hat{T}$ which shows that the condition is necessary. Conversely, if $\hat{P} = \hat{f}t$, take $Q = P - ft$ we have $Q = P$ on V and $\deg Q < \deg P$. $\square$

In general, a polynomial P such that $\deg P = \deg_V p$ is not unique. For instance, if $\deg s = k$ and $p = y^k_{|V}$ then both $P(z,y) = y^k$ and $Q(z,y) = s(z)$ furnish $\deg p$. Yet, the above proof gives an easy algorithm to compute $\deg_V p$ and find a representative of minimal degree. Note that the lemma shows that if $k > \deg s$ then $\Phi^{-1}(p)$ always provides a polynomial of minimal degree for p (because $\hat{f}$ which is of degree d in y cannot divide $\Phi^{-1}(p)$ which is of degree at most $d-1$ in y). This observation also follows from the next lemma.

Lemma 12. *For any $p \in \mathcal{P}(V)$ we have*

$$\deg_V p \le \deg \Phi^{-1}(p) \le \max\left\{1, \frac{\deg s}{k}\right\} \deg_V p$$

Proof. The lower bound is obvious. Take P such that $\deg P = \deg_V p$ and $P = p$ on V. Put $\delta = \deg P$. So P is a sum of monomials $z^\alpha y^m$ with $|\alpha| + m \le \delta$ and $\Phi^{-1}(p)$ is a sum of terms $z^\alpha s^q(z) y^r$ when $m = qk + r$ as above. The degree of $\Phi^{-1}(p)$ is therefore not bigger than the maximum of the degree of such terms which are of the form $|\alpha| + qd + r$ where $d = \deg s$. So we are left with the problem of estimating $\delta' = |\alpha| + qd + r$ subject to $|\alpha| + m = |\alpha| + qk + r \le \delta$. If $d \le k$ we obviously have $\delta' \le \delta$ (and hence $\delta' = \delta$). For $d > k$,

$$\delta' = |\alpha| + qd + r \le \frac{d}{k}\big(|\alpha| + qk + r\big) \le \frac{d}{k}\delta.$$

This readily implies the lemma. $\square$

4.4. More on the space $\mathcal{P}(z) \otimes \mathcal{P}_{k-1}(y)$

We need a classical lemma on the way of recapturing the coefficients of a polynomial in terms of its values on $\mathbb{U}_n$.

Lemma 13. *Let $g(t) = \sum_{i=0}^{n-1} a_i t^i \in \mathcal{P}(\mathbb{C})$ and w is a primitive n-th root of unity then*

$$a_m = \frac{1}{n}\sum_{k=0}^{n-1} \frac{g(w^k)}{w^{mk}}, \quad m = 0, \dots, n-1. \tag{13}$$

In particular,

$$|a_m| \le \|g\|_{\mathbb{U}_n}, \quad m = 0, \dots, n-1.$$

Proof. Look at the factor of a_j in the sum on the right hand side of (13) and observe that $\sum_{k=0}^{n-1} w^{k(j-m)} = n\delta_{jm}$. We refer to [15, Lemma 2.2g, p. 85] for details. □

Applying Lemma 13 to $g(t) = p(z, ty)$ (with $n = k$) we get

Lemma 14. *If $p(z,y) = \sum_{i=0}^{k-1} p_i(z) y^i \in \mathcal{P}(z) \otimes \mathcal{P}_{k-1}(y)$ then*

$$|p_i(z)y^i| \leq \max_{w \in \mathbb{U}_k} |p(z, wy)|, \quad i = 0, \ldots, k-1, \quad (z, y) \in \mathbb{C}^{N+1}.$$

To explain the way we will use this result we first need the following definition.

Definition 15. A compact set E in V is said to be $\mathbb{U}_k$-invariant if $(z, y) \in E$ implies $(z, wy) \in E$ for any $w \in \mathbb{U}_k$.

Lemma 16. *Let E be a $\mathbb{U}_k$-invariant compact subset of V. If*

$$p(z,y) = \sum_{i=0}^{k-1} p_i(z) y^i \in \mathcal{P}(z) \otimes \mathcal{P}_{k-1}(y)$$

then

$$\|p_i(z)y^i\|_E \leq \|p\|_E, \quad i = 0, \ldots, k-1.$$

4.5. The multiplication homomorphism on $\mathcal{P}(z) \otimes \mathcal{P}_{k-1}(y)$

The isomorphism Φ endows $\mathcal{P}(z) \otimes \mathcal{P}_{k-1}(y)$ with a structure of $\mathcal{P}(z)$-module for which the external product is

$$T(z) \cdot L(z,y) = \Phi^{-1}\big((T(z) \cdot L(z,y))_{|V}\big).$$

Now, given $Q \in \mathcal{P}(z) \otimes \mathcal{P}_{k-1}(y)$, the map

$$\mathcal{M}_Q : L \in \mathcal{P}(z) \otimes \mathcal{P}_{k-1}(y) \longrightarrow \Phi^{-1}\left((LQ)_{|V}\right) \in \mathcal{P}(z) \otimes \mathcal{P}_{k-1}(y)$$

is a $\mathcal{P}(z)$-homomorphism which we may compute as follows. If $Q(z,y) = \sum_{i=0}^{k-1} q_i(z) y^i$ then

$$\mathcal{M}_Q(y^j) = \sum_{i=0}^{k-1} q_i(z) y^{i+j}, \quad j = 0, \ldots, k-1.$$

Substituting $s(z)y^{i+j-k}$ for y^{i+j} when $i+j \geq k$ we find

$$\mathcal{M}_Q(y^j) = \sum_{i=0}^{k-1-j} q_i(z)y^{i+j} + \sum_{i=k-j}^{k-1} q_i(z)s(z)y^{i+j-k}, \tag{14}$$

and, writing $l = i+j$ in the first sum and $l = i+j-k$ in the second one,

$$\mathcal{M}_Q(y^j) = \sum_{l=j}^{k-1} q_{l-j}(z)y^l + \sum_{l=0}^{j-1} q_{k-j+l}(z)s(z)y^l. \tag{15}$$

Thus, if $L = \sum_{i=0}^{k-1} L_i(x)y^i$ and $\mathcal{M}_Q(L_i) = \sum_{i=0}^{k-1} L_i'(z)y^i$ then the components L_i' are given in terms of the L_i via the relation

$$\begin{pmatrix} q_0 & q_{k-1}s & q_{k-2}s & \dots & q_2 s & q_1 s \\ q_1 & q_0 & q_{k-1}s & \dots & q_3 s & q_2 s \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ q_{k-3} & q_{k-4} & q_{k-5} & \dots & q_{k-1}s & q_{k-2}s \\ q_{k-2} & q_{k-3} & q_{k-4} & \dots & q_0 & q_{k-1}s \\ q_{k-1} & q_{k-2} & q_{k-3} & \dots & q_1 & q_0 \end{pmatrix} \begin{pmatrix} L_0 \\ L_1 \\ \vdots \\ L_{k-3} \\ L_{k-2} \\ L_{k-1} \end{pmatrix} = \begin{pmatrix} L_0' \\ L_1' \\ \vdots \\ L_{k-3}' \\ L_{k-2}' \\ L_{k-1}' \end{pmatrix}. \tag{16}$$

The matrix above, whose coefficients are elements of $\mathcal{P}(z)$, will be denoted $M_Q(z)$ and the corresponding matrix in which s is replaced by y^k, will be denoted by $M_Q^V(z,y)$. We therefore have

$$M_Q^V(z,y) = M_Q(z), \quad (z,y) \in V.$$

In fact, up to conjugation by diagonal matrices, $M_Q^V(z,y)$ is a classical *circulant matrix* built on the sequence $(q_0(z), q_1(z)y, q_2(z)y^2, \dots, q_{k-1}(z)y^{k-1})$, namely

$$M_Q^V(z,y) = \mathrm{Diag}\left(1, \frac{1}{y}, \dots, \frac{1}{y^{k-1}}\right) \times \begin{pmatrix} q_0 & q_{k-1}y^{k-1} & q_{k-2}y^{k-2} & \dots & q_1 y \\ q_1 y & q_0 & q_{k-1}y^{k-1} & \dots & q_2 y^2 \\ \vdots & \vdots & \vdots & \vdots & \\ q_{k-1}y^{k-1} & q_{k-2}y^{k-2} & q_{k-3}y^{k-3} & \dots & q_0 \end{pmatrix} \times \mathrm{Diag}\left(1, y, \dots, y^{k-1}\right) \tag{17}$$

The classical result on the determinant of circulant matrices, see e.g. [10, Exercise 300] gives that

$$\det M_Q^V(z,y) = \prod_{w\in\mathbb{U}_k}\left(\sum_{i=0}^{k-1} q_i(z)w^i y^i\right).$$

We proved the following lemma.

Lemma 17. *With the notation above,*

$$\det M_Q^V(z,y) = \prod_{w\in\mathbb{U}_k} Q(z,wy). \tag{18}$$

Lemma 18. *Let Q be as above. The polynomial $\det M_Q(z)$ is the zero polynomial if and only if f and Q have a non trivial common factor, that is,* $\deg(\gcd(Q,f)) > 0$.

In particular, if f is irreducible then $\det M_Q(z) = 0$ if and only if $Q = 0$ (for we must have $f|Q$ and $Q \in \mathcal{P}(z)\otimes\mathcal{P}_{k-1}(y)$ forces $Q = 0$).

Proof. The determinant of $M_Q(z)$ is the zero polynomial if and only if the determinant of $M_Q^V(z,y)$ is the zero polynomial on V. As above, by the Nullstellensatz, f divides $\det M_Q^V(z,y)$. Let p be an irreducible divisor of f. The polynomial p divides $\det M_Q^V(z,y)$ hence, it divides at least one of the factors in the right hand side of (18). Thus, for some $w \in \mathbb{U}_k$, $p(z,y)$ divides $Q(z,wy)$. It follows that $p(z,y/w)$ divides $Q(z,y)$ but, as we have seen, either $p(z,y/w) = p(z,y)$ or it is another irreducible factor of f. Thus, in any case, f and Q have a common non trivial divisor and this prove that the condition $\deg(\gcd(Q,f)) > 0$ is necessary.

We may proceed similarly to prove that the condition is sufficient with the help of (12) but a direct proof can be easily derived as follows.

If $\deg(\gcd(Q,f)) > 0$ then one can define the polynomial $r = f/\gcd(Q,f)$ which satisfies $rQ = 0$ on V (for f divides rQ) but $r_{|V}$ is not the zero polynomial and the same is true of its pre-image under the isomorphism Φ. Hence, if

$$\Phi^{-1}(r)(y,z) = \sum_{i=0}^{k-1} r_i(z)y^i,$$

at least one of the r_i, say r_{i_0}, is not the zero polynomial. Now, the equality $\mathcal{M}_Q(r) = 0$ gives, for each z such that $r_{i_0}(z) \neq 0$, a linear dependency relation on the columns of $M_Q(z)$. Therefore $\det M_Q(z) = 0$ on $\mathbb{C}^N \setminus \{r_{i_0} = 0\}$, and thus on the whole of $\mathbb{C}^N$, because it is a polynomial. □

The lemma can be summarized as follows $\det M_Q(z)$ is not the zero polynomial as soon as Q and f are relatively prime.

5. Polynomial inequalities on V

5.1. Markov inequalities on V

We use the notation introduced in the previous section. In particular, $V = \{f = 0\} \subset \mathbb{C}^{N+1}$, $f(z,y) = y^k - s(z)$. To measure the "'derivatives"' of polynomials on V, it seems natural to use the following semi-norm. Given a compact set E in $V = \{f = 0\}$ as in the previous section, we set

$$|p|^V_{\alpha,E} := \inf\left\{\|D^\alpha P\|_E : P_{|V} = p, P \in \mathcal{P}(z,y)\right\}, \quad p \in \mathcal{P}(V).$$

Definition 19 (Markov set and Markov inequality on V)**.** A compact set $E \subset V$ is said to be a *V-Markov set* if there exist constants $M, m > 0$ such that

$$|p|^V_{\alpha,E} \leq M^{|\alpha|}(\deg_V p)^{m|\alpha|}\|p\|_E, \quad p \in \mathcal{P}(V), \quad \alpha \in \mathbb{N}^N. \tag{19}$$

This inequality is called a *Markov inequality* for E in V or a *V-Markov inequality*.

This definition raises evident difficulties as it seems complicated to estimate $|p|^V_{\alpha,E}$. In fact, we will only prove a much stronger inequality in which $|p|^V_{\alpha,E}$ is replaced by its upper bound $\|D^\alpha\Phi^{-1}(p)\|_E$. Another obvious difficultly is that, in contrast with the ordinary case, it is not possible to simply iterate inequality (19) from the case $|\alpha| = n$ to $|\alpha| = n+1$.

The isomorphism $\mathcal{P}(z) \otimes \mathcal{P}_{k-1}(y) \simeq \mathcal{P}(V)$ next suggests the following definition.

Definition 20 (Markov set and Markov inequality on $\mathbf{W}$)**.** Let $\mathbf{W}$ be an infinite dimensional subspace of $\mathcal{P}(\mathbb{C}^{N+1})$ which is invariant under derivation. A compact set $E \subset \mathbb{C}^{N+1}$ is said to be a $\mathbf{W}$-*Markov set* if there exist $M, m > 0$ such that

$$\|D^\alpha p\|_E \le M^{|\alpha|}(\deg p)^{m|\alpha|}\|p\|_E, \quad p \in \mathbf{W}, \quad \alpha \in \mathbb{N}^N. \tag{20}$$

This inequality is called a $\mathbf{W}$-*Markov inequality* for E.

To say that $\mathbf{W}$ is invariant under derivation simply means that $p \in \mathbf{W}$ implies $D^\alpha p \in \mathbf{W}$ for all α and, of course, it suffices to check the property for $|\alpha| = 1$. The space $\mathcal{P}(z) \otimes \mathcal{P}_{k-1}(y)$ is obviously invariant by derivation.

The constant m in (19) and in (20) is called the *exponent* of the respective inequalities.

Lemma 21. *Let E be a compact subset of V and $\mathbf{W} = \mathcal{P}(z) \otimes \mathcal{P}_{k-1}(y)$. If E is a $\mathbf{W}$-Markov set then E is also a V-Markov set. The exponent m in the $\mathbf{W}$-Markov inequality may be used in the V-Markov inequality as well.*

Proof. Take $p \in \mathcal{P}(V)$ and set $P = \Phi^{-1}(p)$. Since $P \in \mathbf{W}$, we may apply (20) to get

$$\|D^\alpha P\|_E \le M^{|\alpha|}(\deg P)^{m|\alpha|}\|P\|_E,$$

and, using the bound for $\deg P$ given in Lemma 12,

$$\|D^\alpha P\|_E \le \left(\max\left\{1, \frac{\deg s}{k}\right\}\right)^{m|\alpha|} M^{|\alpha|}(\deg_V p)^{m|\alpha|}\|p\|_E.$$

The lemma follows since $|p|^V_{\alpha,E} \le \|D^\alpha P\|_E$. $\square$

From now on, we denote by π the projection from $V \subset \mathbb{C}^{N+1}$ onto the space $\mathbb{C}^N$, i.e. $\pi(z, y) = z$ for $(z, y) \in V$. In particular, if E is a compact subset of V then

$$\pi(E) = \{z \in \mathbb{C}^N : (z, y) \in E \text{ for some } y \in \mathbb{C}\}.$$

Theorem 22. *Let E be a $\mathbb{U}_K$-invariant compact set in V and $\mathbf{W} = \mathcal{P}(z) \otimes \mathcal{P}_{k-1}(y)$. Then E is a $\mathbf{W}$-Markov set if and only if $\pi(E)$ is a Markov set in*

$\mathbb{C}^N$. In particular, E is a V-Markov set with exponent $m\left(1+\frac{(k-1)d}{k}\right)$ as soon as $\pi(E)$ is a Markov set with exponent m in $\mathbb{C}^N$.

The definition of a $\mathbb{U}_k$-invariant set is given in Definition 15 above.

Proof. The second statement is a consequence of the first one via Lemma 21 and the fact that $\pi(E)$ is a Markov set when E is a $\mathbf{W}$-Markov set is obvious since $\mathcal{P}(z) \subset \mathbf{W}$. To prove the remaining claim, we assume that $\pi(E)$ is a Markov set and prove that E is a $\mathbf{W}$-Markov set. Fix a polynomial $P \in \mathcal{P}(z) \otimes \mathcal{P}_{k-1}(y)$,

$$P(z,y) = \sum_{i=0}^{k-1} p_i(z) y^i,$$

and an $(N+1)$-index α. We write $\beta = (\alpha_1, \dots, \alpha_N, 0)$ so that $D^\alpha = D^\beta D^{\alpha_{N+1}}$ where $D^{\alpha_{N+1}}$ indicates α_{N+1} derivations with respect to the last variable, i.e, with respect to y. Thus, we have

$$D^\alpha P(z,y) = \sum_{i=0}^{k-1} D^\beta p_i(z) D^{\alpha_{N+1}} y^i. \tag{21}$$

Markov inequality (1) for $\pi(E)$ yields

$$\|D^\beta p_i\|_{\pi(E)} \le M^{|\beta|} (\deg p_i)^{m|\beta|} \|p_i\|_{\pi(E)}, \quad i = 1, \dots, k-1. \tag{22}$$

Now, in view of Corollary 5 which we apply with $q(z) = s^i(z)$ of degree id, $d = \deg s$, we have

$$\|p_i\|_{\pi(E)} \le C_0^{1/k} \big(id + k \deg p_i\big)^{idm/k} \big\|p_i |s|^{i/k}\big\|_{\pi(E)} \tag{23}$$

with a constant C_0 depending only on E, k and s. Yet, since $E \subset V$,

$$\|p_i(z)|s(z)|^{i/k}\|_{\pi(E)} = \|p_i(z) y^i\|_E. \tag{24}$$

Since E is $\mathbb{U}_k$-invariant, we may next use Lemma 16 to get

$$\big\|p_i(z) y^i\big\|_E \le \|P\|_E. \tag{25}$$

Now using (23), (24) and (25) in (22), together with the fact that $\deg p_i \le \deg P$ and $|\beta| \le |\alpha|$, we obtain

$$\begin{aligned} \|D^\beta p_i\|_{\pi(E)} &\le M^{|\beta|} (\deg P)^{m|\beta|} C_0^{\frac{1}{k}} \left(id + k \deg P\right)^{m\frac{id}{k}} \|P\|_E & (26)\\ &\le C' (\deg P)^{m|\alpha| + m\frac{(k-1)d}{k}} \|P\|_E & (27) \end{aligned}$$

where C' depends only on E, k and s. Therefore

$$\|D^{\beta}p_i\|_{\pi(E)} \le C'(\deg p)^{m(1+\frac{(k-1)d}{k})|\alpha|}\|P\|_E \tag{28}$$

where we used $|\alpha| \ge 1$. This implies the required inequality, as follows from (21),

$$\left\|D^{\alpha}P\right\|_E \le C'' \max_{0\le i\le k-1}\|D^{\beta}p_i\|_{\pi(E)}$$

where $C'' = \max_{0\le\alpha_{N+1}\le k-1}\left\|\sum_{i=0}^{k-1}|D^{\alpha_{N+1}}y^i|\right\|_{\pi(E)}$. □

Example 23. Let $V = \{y^3 = z^2 - 1\} \subset \mathbb{C}^2$ and $\mathbb{D}$ be the (closed) unit disc in $\mathbb{C}$. The compact set $E = \{(z, y) \in V : y \in \mathbb{D}\}$ is a V-Markov set.

Proof. We have $\pi(E) = \{z \in \mathbb{C} : z^2 - 1 \in \mathbb{D}\}$ which is the lemniscate of Bernoulli (with its interior) and E is $\mathbb{U}_3$-invariant. By a result of Szegö, see e.g. [21, Th.15.3.5], and Bernstein's pointwise estimate, see e.g. [21, Th.15.1.1], we can show that $\pi(E)$ is a Markov set and satisfies the Markov inequality (1) with $m = 1$. Therefore, the set E is a **W**-Markov set and the conclusion follows from the theorem. If follows from the proof, see (28), that the exponent can be taken as $1 + 2 \cdot (2/3) = 7/3$. □

5.2. Division inequality on V

In contrast with the case of Markov inequality, the notion of division set immediately extends to the case of an algebraic set.

A compact set E in V is $\mathcal{P}_V$*-determining* if for all $p \in \mathcal{P}(z, y)$, $p = 0$ on E implies $p = 0$ on V.

Definition 24 (Division set and division inequality on V)**.** A $\mathcal{P}_V$-determining compact subset E in V is said to be a V-division set if, for any non constant polynomial q on V, there exists a sequence $D_V(E, q, n)$ in $\mathbb{R}^+$ which grows polynomially in n such that

$$\|p\|_E \le D_V(E, q, n)\|pq\|_E, \quad \deg_V p \le n.$$

Any explicit bound for $D_V(E, q, n)$ is called a V*-division inequality.*

A large class of V-division compact sets is given by the following theorem.

Theorem 25. *Assume that the polynomial f defining V is irreducible and let E be a $\mathbb{U}_k$-invariant compact set in V. If $\pi(E)$ is a Markov set then E is a V-division set.*

This is a particular case of the following result which does not require f to be irreducible.

Theorem 26. *Let E be a $\mathbb{U}_k$-invariant, $\mathcal{P}_V$-determining compact set in V such that $\pi(E)$ is a Markov set in $\mathbb{C}^N$ and q be a non constant polynomial in $\mathcal{P}(z,y)$. There exists a sequence $D_V(E,q,n)$ that grows polynomially in n such that*

$$\|p\|_E \leq D_V(E,q,n)\|pq\|_E \text{ for } \deg_V p \leq n$$

if and only if q and f are relatively prime.

It is readily seen that any $\mathbb{U}_k$-invariant compact set in V is $\mathcal{P}_V$-determining excepted when the y-projection of E reduces to $\{0\}$.

Proof. The condition is obviously necessary for, if q and f are not relatively prime then $w = f/\gcd(q,f)$ is a polynomial which is not zero on V, and the inequality

$$\|p\|_E \leq D_V(E,q,n)\|pq\|_E$$

cannot hold for $p = w^n$. Indeed the right hand side is zero while the left hand side is not (since, E is $\mathcal{P}_V$-determining, $\|w\|_E = 0$ would imply $\|w\|_V = 0$ which is not true).

We now assume that q and f are relatively prime. Let $p \in \mathcal{P}(V)$. Put $P = \Phi^{-1}(p)$ and $P = \sum_{i=1}^{k-1} p_i y^i$. In view of Lemma 18, the determinant of the polynomial matrix $M_q(z)$ is not zero. We may therefore apply Corollary 6 with $\mathbf{P} = (p_1, \ldots, p_l)^T$ and $\mathbf{A} = M_q(z)$ on the compact set $\pi(E)$ in $\mathbb{C}^N$ which is assumed to be a Markov set. We obtain the inequality

$$\|\mathbf{P}\|_{\pi(E)} \leq C_1(r+n)^{rm}\|M_q(z)\mathbf{P}\|_{\pi(E)}$$

where $\mathbf{P} = (p_1, \ldots, p_l)^T$, $n = \max_{i=0,\ldots,k-1} \deg p_i$, $r = \deg \det M_q(z)$, m is the Markov exponent for $\pi(E)$ and C_1 depends only on q, s and $\pi(E)$. In fact, using Lemma 12,

$$\deg p_i \leq \deg P \leq \lambda \deg_V p, \quad \lambda = \max \left\{1, \frac{\deg s}{k}\right\}.$$

Hence, the above two estimates yield

$$\|\mathbf{P}\|_{\pi(E)} \leq C_1 (r + \lambda \deg_V p)^{rm} \|M_q(z)\mathbf{P}\|_{\pi(E)}. \tag{29}$$

Now, on one side, we have

$$\|p\|_E = \|P\|_E \leq C_2 \|\mathbf{P}\|_{\pi(E)}, \quad C_2 = \left\| \sum_{i=0}^{k-1} |y^i| \right\|_E. \tag{30}$$

On the other side, see (16), $M_q(z)\mathbf{P}$ gives the components of $\Phi^{-1}(Pq)$, that is

$$M_q(z)\mathbf{P} = \widetilde{\mathbf{P}} = (\widetilde{p}_0, \ldots, \widetilde{p}_{k-1}), \quad \mathcal{M}_q(P) = \Phi^{-1}(Pq) = \sum_{i=0}^{k-1} \widetilde{p}_i y^i.$$

Using again the division inequality for $\pi(E)$ from Corollary 5,

$$\|\widetilde{p}_i\|_{\pi(E)} \leq C_3 (i \deg s + k \deg \widetilde{p}_i)^{(mi \deg s)/k} \|\widetilde{p}_i |s|^{i/k}\|_{\pi(E)} \tag{31}$$

where C_3 depends only on s, k and $\pi(E)$. In view of Lemma 12,

$$\deg \widetilde{p}_i \leq \deg(Pq) = \deg P + \deg q \leq \lambda \deg_V p + \deg q.$$

For $d = \deg s$, taking into account that $i \leq k-1$, (31) gives

$$\|\widetilde{p}_i\|_{\pi(E)} \leq C_3 \left((k-1)d + k\lambda \deg_V p + k \deg q\right)^{md(k-1)/k} \|\widetilde{p}_i |s|^{i/k}\|_{\pi(E)}$$

Now, since $\|\widetilde{p}_i |s|^{i/k}\|_{\pi(E)} = \|\widetilde{p}_i(z) y^i\|_E$, a use of Lemma 16 gives

$$\|\widetilde{p}_i\|_{\pi(E)} \leq C_3 \left((k-1)d + k\lambda \deg_V p + k \deg q\right)^{md(k-1)/k} \|pq\|_E.$$

Observe that, in order to use Lemma 16, we needed E to be $\mathbb{U}_k$-invariant. Summing up,

$$\max_{0 \leq i \leq k-1} \|\widetilde{p}_i\|_{\pi(E)} \leq C_3 \left((k-1)d + k\lambda \deg_V p + k \deg q\right)^{\frac{md(k-1)}{k}} \|pq\|_E. \tag{32}$$

From (29), (30), (32) and $\|M_q(z)\mathbf{P}\|_{\pi(E)} = \max_{0\le i\le k-1} \|\widetilde{p}_i\|_{\pi(E)}$, we obtain

$$\|p\|_E \le C_4(\deg_V p)^{m(d(k-1)/k+r)}\|pq\|_E \tag{33}$$

which gives a bound with the required properties. □

From (16), we deduce that the integer $r = \deg\det M_q(z)$ used in the proof above satisfies $r \le k\deg q + (k-1)\deg s$ so that the exponent in (33) is bounded by $mk(\deg q + \deg s)$.

5.3. Conclusions and final remarks

Our results concern algebraic hypersurfaces of the form

$$V = \{z_{N+1}^k = s(z_1,\dots,z_N)\} \subset \mathbb{C}^{N+1}, \tag{34}$$

where s is a non constant polynomial of N variables. It is not clear to what extent our results can be generalized by using similar approach to other classes of algebraic varieties. We will now briefly discuss some possible generalizations of our results.

If for the variety $U = \{(u,v,w)\in\mathbb{C}^3 : 2v^2+w^2 = 2uv+2vw-2uw\}$ we take the linear transformation $(u,v,w) = (x,y,z-x+y)$, we obtain the algebraic manifold $\{(x,y,z)\in\mathbb{C}^3 : z^2 = x^2-y^2\}$ that is of the form (34). Nondegenerate linear mappings do not change properties of sets related to polynomial inequalities, so the presented methods work on compact subsets of U. Therefore, our results can be used also for images of algebraic hypersurfaces of the form (34) under linear transformations. It seems likely this observation can be extended to certain polynomial mappings. A characterization of such polynomial mappings and algebraic sets obtained in this way would be interesting.

In [17] (Exercise on p. 107) one can find

Lemma 27. *Let $d\in\mathbb{N}$ and $E\subset\mathbb{C}^N$ be a compact, circled set, i.e. $(z_1,\dots,z_N)\in E$ implies $(e^{it}z_1,\dots,e^{it}z_N)\in E$ for all real t. Then for any polynomial $p_d = h_d + h_{d-1} + \ldots + h_0$ of degree d written as a sum of homogeneous polynomials, we have $\|h_j\|_E \le \|p_d\|_E$ for $j = 0,\dots,d$.*

It appears that for compact subsets of varieties of the form (34), the $\mathbb{U}_k$-invariance is a sufficient assumption to obtain the analogous estimate, see Lemma 16 in Section 4.4. The $\mathbb{U}_k$-invariance is closely related to the structure of the algebraic set V. This suggests the following question: what other classes of algebraic varieties with symmetries could be analyzed in a similar fashion.

In the general case, when Z is the set of zeros of a polynomial g in $N+1$ complex variables, one can ask about a condition which implies a similar estimate as in Lemma 16 for compact subsets of Z. For instance, can one obtain analogous results studying compact subsets of toric varieties? Toric varieties is an important class frequently considered in algebraic geometry. As an example, we can take the cuspidal cubic $C = \{(x,y) \in \mathbb{C}^2 \,:\, x^3 = y^2\}$ which is a toric algebraic set of the form (34).

Finally, we point out that our results allow us to construct an admissible mesh on V (see [8] for the definition). Namely, if E is a $\mathbb{U}_k$-invariant compact set in V and $(\mathcal{A}_n)_{n\in\mathbb{N}}$ is an admissible mesh for $\pi(E)$ then $\pi^{-1}\left((\mathcal{A}_n)_{n\in\mathbb{N}}\right)$ is an admissible mesh for E.

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References

[1] M. Baran and L. Bialas-Ciez, Hölder continuity of the Green function and Markov brothers' inequality, Constr. Approx. **40**(1) (2014), 121–140.

[2] M. Baran and A. Kowalska, Sets with the Bernstein and generalized Markov properties, Ann. Polon. Math. **111**(3) (2014), 259–270.

[3] M. Baran and W. Pleśniak, Characterization of compact subsets of alge-

braic varieties in terms of Bernstein type inequalities, Studia Math. **141**(3) (2000), 221–234.

[4] L. Bialas-Ciez, Equivalence of Markov's and Schur's inequalities on compact subsets of the complex plane, J. Inequal. Appl. **3**(1) (1999), 45–49.

[5] L. P. Bos, N. Levenberg, P. Milman and B. A. Taylor, Tangential Markov inequalities characterize algebraic submanifolds of $\mathbb{R}^N$, Indiana Univ. Math. J. **44** (1995), 115–138.

[6] L. P. Bos, N. Levenberg, P. Milman and B. A. Taylor Tangential Markov inequalities on real algebraic varieties, Indiana Univ. Math. J. **47** (1998), 1257–1272.

[7] L. P. Bos and P. D. Milman, Tangential Markov inequalities on singular varieties, Indiana Univ. Math. J., **55**(1) (2006), 65–73.

[8] J-P. Calvi and N. Levenberg, Uniform approximation by discrete least squares polynomials, J. Approx. Theory **152** (2008), 82–100.

[9] D. Cox, J. Litte and D. O'Shea, Ideas, varieties, and algorithms, Undergraduate Texts in Mathematics Springer-Verlag, New York, 1992.

[10] D. K. Faddeev and I. S. Sominskii, Problems in higher algebra, Translated by J. L. Brenner W. H. Freeman and Co., San Francisco-London, 1965.

[11] L. Gendre, Inégalités de Markov tangentielles locales sur les algébriques singuliéres de $\mathbb{R}^n$, Ann. Pol. Math. **86**(1) (2005), 59–77.

[12] P. Goetgheluck, Un résultant de division obtenu par les méthodes de l'approximation, J. Approx. Theory **40**(4) (1984), 355–363.

[13] P. Goetgheluck, Polynomial inequalities on general subsets of $\mathbb{R}^N$, Colloq. Math. **57**(1) (1989), 127–136.

[14] P. Goetgheluck, On the problem of sharp exponents in multivariate Nikolskii-type inequalities, J. Approx. Theory **77** (1994), 167–178.

[15] P. Henrici, Applied and computational complex analysis. Vol. 1., Wiley Classics Library. John Wiley & Sons, Inc., New York, 1988.

[16] A. Kowalska, Tangential Markov inequalites on semialgebraic curves and some semialgebraic surfaces, Ann. Polon. Math. **106** (2012), 215–222.

[17] N. Levenberg, Approximation in C^N, Surveys in Approximation Theory **2** (2006), 92–140.

[18] W. Pawłucki and W. Pleśniak, Markov's inequality and $\mathcal{C}^\infty$ function on sets with polynomial cusps, Math. Ann., **275**(3) (1986), 467–480.

[19] W. Pleśniak, Markov's inequality and the existence of an extension operator for $\mathcal{C}^\infty$ function, J. Approx. Theory **61** (1990), 106–117.

[20] W. Pleśniak, Inégalité de Markov en plusieurs variables, Int. J. Math. Sci. **12** (2006), Art. ID 24549.

[21] Q. I. Rahman and G. Schmeisser, Analytic Theory of Polynomials, Oxford Sci. Publ., Oxford, 2002.

[22] A. Schinzel, Solved and unsolved problems on polynomials, Banach Center Publ. **34** (1995), 149–159.

[23] I. Schur, Über das Maximumm des absoluten Betrages eines Polynoms in einem gegebenen Intervall, Math. Z. **4**(3-4) (1919), 271–287.

[24] E. M. Stein, Interpolation in polynomial classes and Markoff's inequality, Duke Math. J. **24** (1957), 467–476.