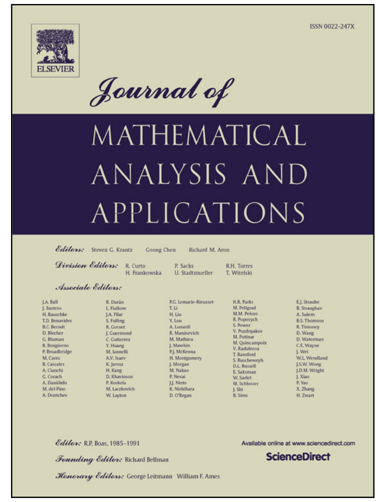

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SPECTRALITY OF MORAN MEASURES WITH FOUR-ELEMENT DIGIT SETS

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ABSTRACT. Let $\delta_E = \frac{1}{\#E}\sum_{a\in E}\delta_a$ denote the uniformly discrete probability measure on a finite set E. We prove that the infinite convolution (Moran measure)

$$\mu_{b,\{\mathcal{D}_k\}} = \delta_{b^{-1}\mathcal{D}_1} * \delta_{b^{-2}\mathcal{D}_2} * \cdots$$

admits an orthonormal basis of exponential provided that $\{\mathcal{D}_k\}_{k=1}^{\infty}$ is a uniformly bounded sequence of 4-digit spectral sets, $b = 2^{l+1}q$ with $q > 1$ an odd integer, and l sufficiently large (depends on $\mathcal{D}_k$). We also give some examples to illustrate the result.

1. Introduction

Let μ be a compactly supported Borel probability measure on $\mathbb{R}^d$. μ is called a *spectral measure* if there exists a countable set $\Lambda \subset \mathbb{R}^d$ such that $E(\Lambda) := \left\{e^{2\pi i<\lambda,x>} : \lambda \in \Lambda\right\}$ forms an orthonormal basis for $L^2(\mu)$. In this case, Λ is called a *spectrum* of μ and (μ, Λ) is called a *spectral pair*. If the normalized Lebesgue measure restricting on a Borel set Ω is a spectral measure, then Ω is called a *spectral set*. The study of spectral measures was first initiated by B.Fuglede in 1974 [8], who conjectured that $\Omega \subset \mathbb{R}^d$ *is a spectral set if and only if* Ω *is a translational tile*. The conjecture has been studied by many authors, e.g., Iosevich, Jorgensen, Kolountzakis, Laba, Lagarias, Matolcsi, Pedersen, Tao, Wang and many others ([17–23, 27, 28, 30, 34]), and it had baffled experts for 30 years until Tao [34] constructed the first counterexample, a spectral set which is not a tile on $\mathbb{R}^d, d \geq 5$. The example and technique were refined later to disprove the conjecture in both directions on $\mathbb{R}^d$ for $d \geq 3$. It is still open in dimensions $d = 1$ and $d = 2$. Despite the counterexamples, the exact relationship between spectral measures and tiling is still mysterious.

For non-atomic singular measures, a class of spectral measures was first found by Jorgensen and Pedersen (i.e., the $1/q$-Cantor measure where q is even) [19], and Strichartz supplemented their result with a simplified proof [31]. The result was

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further extended to other singular measures, and brought in a lot of interesting developments in the topic. There are two classes which have been studied in literature.

I. Self-similar/Self-affine measure

Let $\{f_i(x)\}_{i=1}^n$ be an *iterated function system* (IFS) [9], that is, all $f_i(x)$ are contractive in $\mathbb{R}^d$. Then it determines a unique nonempty compact set T, called an *attractor*, and a Borel probability measure μ supported on T satisfying

$$T = \bigcup_{i=1}^{n} f_i(T), \qquad \mu(\cdot) = \sum_{i=1}^{n} p_i \mu \circ f_i^{-1}(\cdot),$$

where $\{p_i\}_{i=1}^n$ is a probability weight, that is, $p_i > 0$ and $\sum_{i=1}^n p_i = 1$. Moreover, if the IFS $\{f_i(x)\}_{i=1}^n$ are similarity transformations, then the Borel probability measure μ is called a *self-similar measure* and T a *self-similar set*. In the same way the measure μ generated by the IFS $\{f_i(x)\}_{i=1}^n$ of affine transformations is called a *self-affine measure*.

There is a considerable number of articles on the spectrality of self-similar /self-affine measures and the construction of their spectrums [4–7, 14, 16, 24, 29, 33]. In particular, Fu, He and Lau [10] studied the spectrality of a self-similar measure $\mu_{4,\mathcal{D}}$ with equal weight probability generated by $\{f_d(x) = 4^{-1}(x+d)\}_{d\in\mathcal{D}}$ and obtained the following theorem.

Theorem 1.1. *Let $\mathcal{D} \subset \mathbb{Z}^+$ be a digit set with $\#\mathcal{D} = 4$($\#\mathcal{D}$ is the cardinality of $\mathcal{D}$) and $gcd(\mathcal{D}) = 1$. The following are equivalent*
(i) $\mu_{4,\mathcal{D}}$ is a spectral measure;
(ii) T is a tile;
(iii) $\mathcal{D} = \{0, a, 2^t l, a + 2^t l'\}$,where a, t, l, l' are odd integers. In this case $\mu_{4,\mathcal{D}}$ is the normalized Lebesgue measure on T.

The digit set $\mathcal{D}$ can be expressed as $\mathcal{D} = (\{0, 1\} \pmod 2) + 2^t\{0, 2\}) \pmod{2^{2(t+1)}}$, a modulo product-form studied in detail in [25, 26] in connection with the tilings.

Definition 1.2. The set $\mathcal{D} = \{0, a, 2^t l, a + 2^t l'\}$ with $\gcd(\mathcal{D}) = 1$ called a 4-*digit spectral set*, where a, t, l, l' are odd integers.

II. Moran measure

Let $\{b_k\}_{k=1}^\infty$ be a sequence of integer numbers with all $b_k \geq 2$ and let $\{\mathcal{D}_k\}_{k=1}^\infty$ be a sequence of digit sets with $0 \in \mathcal{D}_k \subset \mathbb{N}$ for each $k \geq 1$. We call the function system $\{f_{k,d}(x) = b_k^{-1}(x + d) : d \in \mathcal{D}_k\}_{k=1}^\infty$ a *Moran IFS*, which is a generalization of an IFS. If $\sup\{x : x \in b_k^{-1}\mathcal{D}_k, k \geq 1\} < \infty$, then associated to the *Moran IFS*, there exists a

Borel probability measure with compact support defined by the convolution

$$\mu_{\{b_k\},\{\mathcal{D}_k\}} = \delta_{b_1^{-1}\mathcal{D}_1} * \delta_{(b_1 b_2)^{-1}\mathcal{D}_2} * \cdots ,$$

where $\delta_{rE} = \frac{1}{\#E}\sum_{a\in E}\delta_{ra}$ and δ_{ra} is the Dirac measure at ra, the sign $*$ means the convolution and the convergence is in the weak sense. In this case $\mu_{\{b_k\},\{\mathcal{D}_k\}}$ is called a *Moran measure*, and its support is the *Moran set*

$$T(\{b_k\},\{\mathcal{D}_k\}) = \left\{\sum_{k=1}^{\infty}(b_1 b_2\cdots b_k)^{-1}d_k : d_k \in \mathcal{D}_k, k \geq 1\right\} := \sum_{k=1}^{\infty}(b_1 b_2\cdots b_k)^{-1}\mathcal{D}_k.$$

To simplify notations, we write $\mu_{b,\{\mathcal{D}_k\}} = \mu_{\{b_k\},\{\mathcal{D}_k\}}$ if all b_k are equal to b.

In fact, the Moran measure is a non-self-similar extension of the Cantor measure through the infinite convolution. Moran sets and Moran measures appear frequently in dynamic systems, multifractal analysis and geometry number theory (see [12]) etc. Until now, there are only a few results on the spectrality of Moran measures [1–3, 11, 15].

Obverse that $\delta_{b^{-1}\mathcal{D}}$ is a spectral measure if and only if there exists a set $\mathcal{C}$ such that

$$N^{-\frac{1}{2}}[e^{-2\pi i b^{-1}dc}]_{d\in\mathcal{D},c\in\mathcal{C}}$$

is a unitary matrix, where N is the cardinality of $\mathcal{D}$ and also $\mathcal{C}$. In the case that $\mathcal{D},\mathcal{C} \subset \mathbb{Z}$, we call $(b^{-1}\mathcal{D},\mathcal{C})$ a *compatible pair*, or just say $(b,\mathcal{D})$ is *admissible* for short.

To study the spectrality of the Moran measure $\mu_{\{b_k\},\{\mathcal{D}_k\}}$, the natural assumption is that all $(b_k,\mathcal{D}_k)$ are admissible. However, this assumption is not sufficient for the Moran measure being a spectral measure (see Example 5.2 [3]). In [3], An, He and Lau proved that *the Moran measure* $\mu_{b,\{\mathcal{D}_k\}} = \delta_{b^{-1}\mathcal{D}_1} * \delta_{b^{-2}\mathcal{D}_2} * \cdots$ *is a spectral measure provided that there is a common* $\mathcal{C} \subset \mathbb{Z}^+$ *such that all the* $(b^{-1}\mathcal{D}_k,\mathcal{C})$ *are compatible pairs and* $\mathcal{C}+\mathcal{C} \subset \{0,1,2,\cdots,b-1\}$. Without the condition of common compatible pairs, An, He and Li [2] proved that $\mu_{b,\{\mathcal{D}_k\}} = \delta_{b^{-1}\mathcal{D}_1} * \delta_{b^{-2}\mathcal{D}_2} * \cdots$ *is a spectral measure provided that* $b = 2^{l+1}q$ *is an integer so that* $l > L$ *if* $q = 1$ *, and* $l \geq L$ *if* $q > 1$ *is odd* where $\mathcal{D}_k = \{a_k, b_k\}$ and L is the maximal integer number such that $2^L|(a_k - b_k)$ for some $k \geq 1$.

Motivated by their ideas and results, in this paper, we focus on the Moran measure $\mu_{b,\{\mathcal{D}_k\}} = \delta_{b^{-1}\mathcal{D}_1} * \delta_{b^{-2}\mathcal{D}_2} * \cdots$ with $\#\mathcal{D}_k = 4$. We always assume that $\{\mathcal{D}_k\}_{k=1}^{\infty}$ are uniformly bounded and $\mathcal{D}_k$ are 4-digit spectral sets of the form $\{0, a_k, 2^{t_k}l_k, a_k + 2^{t_k}l'_k\}$ and denote $t_{\max} = \max_{k\geq 1} t_k$.

Our main result is the following theorem.

Theorem 1.3. *Let $b = 2^{l+1}q$ with $q > 1$ an odd integer, and let $\{\mathcal{D}_k\}_{k=1}^{\infty}$ be a uniformly bounded sequence of 4-digit spectral sets. Then the Moran measure $\mu_{b,\{\mathcal{D}_k\}}$ is a spectral measure if $l \geq t_{\max}$.*

The main idea of proof is to use Strichartz's technique of finite approximation [32], and is similar to the one used in the proof of Theorem 1.3 in [3]. The crucial part in the finite approximation is to analyze the set of zeros of the Fourier transform $\mu_{b,\{\mathcal{D}_k\}}$, and to construct a candidate spectrum (see (3.9)) for D_k.

The case digit sets with three elements has been studied in [11, 13, 35], and for the prime case, some difficulty on determining the zeros, and one side of the Fuglede problem "self-similar spectral set implies tile" is still not known.

The paper is organized as follows. In Section 2, we introduce preliminary results. In Section 3, we prove Theorem 1.3 and give some examples to illustrate our conclusion.

2. Preliminaries

Let μ be a probability measure with compact support on $\mathbb{R}$. The Fourier transform of μ is defined as usual,

$$\hat{\mu}(\xi) = \int e^{-2\pi i \xi x} \mathrm{d}\mu(x).$$

It is clear that a set Λ such that $E(\Lambda)$ is an orthonormal family for $L^2(\mu)$ if and only if

$$(\Lambda - \Lambda)\backslash\{0\} \subset \mathcal{Z}(\hat{\mu}),$$

where $\mathcal{Z}(f) := \{\xi : f(\xi) = 0\}$ is the set of the roots of f. In this case, we just call the set Λ an *orthonormal* set of μ for convenience. Without loss of generality, we will assume that 0 is in the orthonormal set Λ, and thus $\Lambda \subset \Lambda - \Lambda$. Firstly, we recall the fundamental criterion for spectral measures [19], which is a directed application of Parseval's identity.

Proposition 2.1. *Let μ be a Borel probability measure with compact support and let $Q_\Lambda(\xi) = \sum_{\lambda \in \Lambda} |\hat{\mu}(\xi + \lambda)|^2$ for $\Lambda \subset \mathbb{R}$. Then*
(i) Λ is an orthonormal set of μ if and only if $Q_\Lambda(\xi) \leq 1$ for $\xi \in \mathbb{R}$;
(ii) Λ is a spectrum of μ if and only if $Q_\Lambda(\xi) \equiv 1$ for $\xi \in \mathbb{R}$.
Moreover, if Λ is an orthonormal set, then $Q_\Lambda(z)$ is an entire function.

Next, we give some results that are needed to prove Theorem1.3. The following properties of compatible pairs can be found in [19] or to be checked directly.

Proposition 2.2. *Let $\mathcal{D}, \mathcal{C} \subset \mathbb{Z}$ and let $b \geq 2$ be an integer such that $(b^{-1}\mathcal{D}, \mathcal{C})$ is a compatible pair. Then*
(i) $b^{-1}\mathcal{C}$ is a spectrum of the measure δ_D;
(ii) the elements in $\mathcal{D}$ are in different cosets of $\mathbb{Z}/b\mathbb{Z}$;
(iii) $(b^{-1}\mathcal{D} + c, \mathcal{C} + a)$ for $a, c \in \mathbb{R}$ and $(b^{-1}\mathcal{D}, -\mathcal{C})$ are compatible pairs;
(iv) suppose that $\widetilde{\mathcal{C}} \subset \mathbb{Z}$ such that $\widetilde{\mathcal{C}} \equiv \mathcal{C} \pmod b$, then $(b^{-1}\mathcal{D}, \widetilde{\mathcal{C}})$ is a compatible pair;
(v) if all $(b^{-1}\mathcal{D}_k, \mathcal{C}_k)$ are compatible pairs, then $(b^{-1}\mathcal{D}_1 + \cdots + b^{-k}\mathcal{D}_k, \mathcal{C}_1 + \cdots + b^{k-1}\mathcal{C}_k)$ is a compatible pair for each $k > 1$.

Remark. By (iii) of Proposition 2.2, we will always assume $0 \in \mathcal{D}$.

We give two lemmas that Fu, He and Lau [10] used in the characterization of a 4-digit set $\mathcal{D} = \{0, a, b, c\}$ to be spectral (Theorem1.1).

Lemma 2.3. *$\mathcal{Z}(\hat{\delta}_{\mathcal{D}})$ is a nonempty set if and only if two of a, b, c are odd and one is even. Without loss of generality, we assume $a < c$ and a, c are odd , b is even. The elements of $\mathcal{Z}(\hat{\delta}_{\mathcal{D}})$ are*

$$\frac{2\mathbb{Z}+1}{2gcd(a, c-b)} \bigcup \frac{2\mathbb{Z}+1}{2gcd(c, b-a)} \bigcup \frac{2\mathbb{Z}+1}{2gcd(b, c-a)}.$$

Lemma 2.4. *$\delta_{\mathcal{D}}$ is a spectral measure if $\mathcal{D} = \{0, a, 2^t l, a + 2^t l'\}$ where a, t, l, l' are odd integers. In such case, it admits a spectrum $\mathcal{C} = \frac{1}{2^{t+1}}\{0, 1, 2^t, 1 + 2^t\}$.*

For any integer $b > 1$ and finite digit set $\mathcal{C} \subset \mathbb{Z}$, we define

$$T(b, \mathcal{C}) := \left\{ \sum_{k=1}^{\infty} c_k b^{-k} : c_k \in \mathcal{C} \right\} := \sum_{k=1}^{\infty} b^{-k}\mathcal{C}. \tag{2.1}$$

It is a compact set generated by the iterated function system $\{f_c = b^{-1}(x+c)\}_{c \in \mathcal{C}}$ [9]. For any $x \in T(b, \mathcal{C})$, x can be expressed (not uniquely in general) as

$$x = \sum_{k=1}^{\infty} c_k b^{-k}, \forall c_k \in \mathcal{C}.$$

We say that x has an *infinite expansion* in $T(b, \mathcal{C})$ if there are infinitely many nonzero c_k's, and *eventually periodic expansion* if the sequence $\{c_k\}_{k=1}^{\infty}$ is eventually periodic. In this case, we say a period of $\{c_k\}_{k=l+1}$ is also a period of x. The following lemma can be found in [2].

Lemma 2.5. *Let $\mathcal{C}$ be a digit set in $\mathbb{Z}$ and let $b > 1$ be an integer. Then $x \in T(b, \mathcal{C})$ is rational if and only if x has an eventually periodic expansion in $T(b, \mathcal{C})$.*

3. Proof of Theorem 1.3

Recall that $\{\mathcal{D}_k\}_{k=1}^{\infty}$ is a uniformly bounded sequence of 4-digit spectral sets. Then $t_{\max} := \max_{k\geq 1} t_k < \infty$ and there are only finitely many distinct $\mathcal{D}_k$'s. We rearrange these distinct $\mathcal{D}_k$'s as $\mathcal{D}_{\varepsilon_1}, \mathcal{D}_{\varepsilon_2}, \cdots, \mathcal{D}_{\varepsilon_N}$, and let $\mathcal{E} = \{\mathcal{D}_{\varepsilon_1}, \cdots, \mathcal{D}_{\varepsilon_N}\}$. Clearly,

$$\hat{\mu}_{b,\{\mathcal{D}_k\}}(\xi) = \prod_{k=1}^{\infty} M_{\mathcal{D}_k}(b^{-k}\xi), \quad \hat{\mu}_{b,\mathcal{D}_{\varepsilon_n}}(\xi) = \prod_{k=1}^{\infty} M_{\mathcal{D}_{\varepsilon_n}}(b^{-k}\xi), \tag{3.1}$$

where $M_{\mathcal{D}_k}(\xi) = \hat{\delta}_{\mathcal{D}_k}(\xi) = \frac{1}{\#\mathcal{D}_k}\Sigma_{a\in\mathcal{D}_k} e^{-2\pi i a\xi}$. Therefore we have $\mathcal{Z}(M_{\mathcal{D}_k}) \subseteq \bigcup_{n=1}^{N} \mathcal{Z}(M_{\mathcal{D}_{\varepsilon_n}})$ and $\mathcal{Z}(\hat{\mu}_{b,\mathcal{D}_k}) \subseteq \bigcup_{n=1}^{N} \mathcal{Z}(\hat{\mu}_{b,\mathcal{D}_{\varepsilon_n}})$. To simplify notations, we write

$$\mathcal{Z}_{q\mathcal{E}} := \bigcup_{n=1}^{N} \mathcal{Z}(M_{q\mathcal{D}_{\varepsilon_n}}), \quad \mathcal{Z}_{b,q\mathcal{E}} := \bigcup_{n=1}^{N} \mathcal{Z}(\hat{\mu}_{b,q\mathcal{D}_{\varepsilon_n}}) \tag{3.2}$$

where $q\mathcal{D}_{\varepsilon_n} = \{qx : x \in \mathcal{D}_{\varepsilon_n}\}$, it follows from (3.1) that

$$\mathcal{Z}_{b,q\mathcal{E}} = \bigcup_{k=1}^{\infty} b^k \mathcal{Z}_{q\mathcal{E}}\ . \tag{3.3}$$

Let $l \geq t_{\max}$ be fixed, and let $b = 2^{l+1}q$ where $q > 1$ is an odd integer, Lemma 2.4 implies that $q\mathcal{C}_k = q\{0, 2^{l-t_k}, 2^l, 2^l + 2^{l-t_k}\}$ is an integral spectrum of $\delta_{b^{-1}\mathcal{D}_k}$. According to Proposition 2.1, Λ is a spectrum of $\mu_{b,\{\mathcal{D}_k\}}$ if and only if $\frac{1}{q}\Lambda$ is a spectrum of $\mu_{b,q\{\mathcal{D}_k\}}$. In the rest of the section, we consider $\mu_{b,q\{\mathcal{D}_k\}}$ instead of $\mu_{b,\{\mathcal{D}_k\}}$. Since all $1 \leq t_k \leq t_{\max} \leq l$, we have

$$\bigcup_{k=1}^{\infty} \mathcal{C}_k \subset \mathcal{G} := \{0, 1, 2, \cdots, 2^l + 2^{l-1}\}. \tag{3.4}$$

Lemma 3.1. *With the above notations, each element in $T(b, \pm\mathcal{G})$ has a unique expansion.*

Proof. Suppose that x in $T(b, \pm\mathcal{G})$ has two different expansions, then

$$x = \sum_{k=1}^{\infty} c_k b^{-k} = \sum_{k=1}^{\infty} c'_k b^{-k}, c_k, c'_k \in \pm\mathcal{G}.$$

Without loss of generality, we assume that $c_1 \neq c'_1$. Since $q > 1$ and

$$| c_k - c'_k | \leq 2(2^l + 2^{l-1}) \leq b - 2, \forall k \geq 1,$$

then we have

$$| c_1 - c'_1 | = \sum_{k=2}^{\infty} | c_k - c'_k | b^{-k+1} \leq \sum_{k=1}^{\infty} (b-2)b^{-k} < 1,$$

which contradicts the assumption. Hence, the result follows. □

Lemma 3.2. *Suppose $x \in \mathcal{Z}_{q\mathcal{E}} \bigcap b^{-1}T(b, \pm\mathcal{G})$, then x has an infinite expansion.*

Proof. Suppose that x_0 in $\mathcal{Z}_{q\mathcal{E}} \bigcap b^{-1}T(b, \pm\mathcal{G})$ has finite expansion. Then there is a $\mathcal{D}_k = \{0, a_k, 2^{t_k}l_k, a + 2^{t_k}l'_k\}$ such that x_0 in $\mathcal{Z}(M_{q\mathcal{D}_k}) \bigcap b^{-1}T(b, \pm\mathcal{G})$ has finite expansion. To simplify notations, we write $\mathcal{D}_k = \{0, a, 2^t l, a + 2^t l'\}$. In this case, let

$$p_1 = \gcd(a, a + 2^t(l' - l)), p_2 = \gcd(a + 2^t l', 2^t l - a) \text{ and } p_3 = \gcd(l, l').$$

It is easy to see that $p_i \in 2\mathbb{Z} + 1$ and $\gcd(p_i, p_j) = 1$ for $i \neq j$. By Lemma 2.3, we know that

$$\mathcal{Z}(M_{q\mathcal{D}_k}) = \frac{2\mathbb{Z} + 1}{2p_1 q} \bigcup \frac{2\mathbb{Z} + 1}{2p_2 q} \bigcup \frac{2\mathbb{Z} + 1}{2^{t+1}p_3 q}.$$

We prove that this assumption is impossible by considering the following three cases:

Case 1. There is an integer u such that

$$\frac{2u + 1}{2p_1 q} = \frac{1}{b^i} \sum_{j=1}^{n} \frac{c_j}{b^j} = \frac{1}{b^{i+n}}(c_n + \cdots + b^{n-1}c_1),$$

where $i \geq 1, c_j \in \pm\mathcal{G}$ and $c_1, c_n \neq 0$. The above identity can be rewritten as

$$(2u + 1)b^{i+n} = 2p_1 q(c_n + bc_{n-1} + b^2 c_{n-2} + \cdots + b^{n-1}c_1). \tag{3.5}$$

When c_n is odd , then $1 \leq c_n < 2^l + 2^{l-1}$. The above identity is equivalent to

$$(2u + 1)2^{(l+1)(i+n)}q^{i+n-1} = 2p_1(\pm c_n + 2M)$$

for some integer M. It is impossible because the power of 2 on the two sides can not be the same.

When c_n is even, then $1 < c_n \leq 2^l + 2^{l-1}$. (3.5) is equivalent to

$$(2u + 1)2^{(l+1)(i+n)}q^{i+n-1} = 2p_1 2^{i_n}(\pm 1 + 2M)$$

for some integer M, where $1 \leq i_n \leq l$. Similarly, the identity is impossible.

Case 2. There is an integer u such that

$$\frac{2u + 1}{2p_2 q} = \frac{1}{b^i} \sum_{j=1}^{n} \frac{c_j}{b^j} = \frac{1}{b^{i+n}}(c_n + \cdots + b^{n-1}c_1).$$

Similar to **Case 1**, it is impossible.

Case 3. There is an integer u such that

$$\frac{2u+1}{2^{t+1}p_3q} = \frac{1}{b^i}\sum_{j=1}^{n}\frac{c_j}{b^j} = \frac{1}{b^{i+n}}(c_n + \cdots + b^{n-1}c_1),$$

where $i \geq 1, c_j \in \pm\mathcal{G}$ and $c_1, c_n \neq 0$. The above identity can be rewritten as

$$(2u+1)b^{i+n} = 2^{t+1}p_3q(c_n + bc_{n-1} + b^2c_{n-2} + \cdots + b^{n-1}c_1). \tag{3.6}$$

When c_n is odd, then $1 \leq c_n < 2^l + 2^{l-1}$. (3.6) is equivalent to

$$(2u+1)2^{(l+1)(i+n)}q^{i+n} = p_3q2^{t+1}(\pm c_n + 2M)$$

for some integer M. It is impossible because the power of 2 on the two sides can not be the same.

When c_n is even, then $1 < c_n \leq 2^l + 2^{l-1}$. (3.6) is equivalent to

$$(2u+1)2^{(l+1)(i+n)}q^{i+n-1} = p_32^{t+1}2^{i_n}(\pm 1 + 2M)$$

for some integer M, where $1 \leq i_n \leq l, 1 \leq t \leq l$. Similarly, the identity is impossible. □

We define

$$\mathcal{G}_{p,b} = \mathcal{G} + b\mathcal{G} + \cdots + b^{p-1}\mathcal{G} - b^p\mathcal{G} - \cdots - b^{2p-1}\mathcal{G}.$$

By (2.1), we know that

$$T_{p,b} := T(b^{2p}, \mathcal{G}_{p,b}) = \sum_{k=1}^{\infty} b^{-2pk}\mathcal{G}_{p,b}.$$

It is easy to see that

$$T_{p,b} \subset T(b, \pm\mathcal{G}).$$

Lemma 3.3. *There exists p_0 so that $\hat{\mu}_{b,q\mathcal{D}_{\varepsilon_n}}(\xi)$ has no roots on $T_{p,b}$ for any $1 \leq n \leq N$ and $p \geq p_0$. Consequently, there exists $c, \eta > 0$ depending on p such that*

$$\prod_{n=1}^{N}\left|\hat{\mu}_{b,q\mathcal{D}_{\varepsilon_n}}(\xi)\right|^2 \geq c > 0, \quad \xi \in (T_{p,b})_\eta.$$

Where $(E)_\eta = \{y : d(y, E) \leq \eta\}$.

Proof. Note that

$$T_{p,b} = \sum_{k=0}^{\infty}\frac{1}{b^{2pk}}\left(-\frac{\mathcal{G}}{b} - \cdots - \frac{\mathcal{G}}{b^p} + \frac{\mathcal{G}}{b^{p+1}} + \cdots + \frac{\mathcal{G}}{b^{2p}}\right). \tag{3.7}$$

By Lemma 3.1 and Lemma 3.2, there exists a unique infinite expansion for each $x \in \Omega := \mathcal{Z}_{q\mathcal{E}} \bigcap b^{-1}T(b, \pm\mathcal{G})$. Since $\mathcal{Z}_{q\mathcal{E}}$ is a subset of rational number(by Lemma2.3), the expansion of x is eventually periodic by Lemma 2.5 and its minimal period P_x

is divided by $2p$ according to the equation (3.7). Since $\prod_{n=1}^{N} \hat{\mu}_{b,q\mathcal{D}_{\mathcal{E}_n}}(\xi)$ is an entire function, there are at most finitely many elements in Ω. Choose p_0 so that $2p_0 > \max_{x\in\Omega} P_x$. Next we claim that

$$\Phi := \mathcal{Z}_{q\mathcal{E}} \bigcap \bigcup_{i=1}^{2p} b^{-i} T_{p,b} = \phi, \text{ for } p \geq p_0. \tag{3.8}$$

As $b^{-i}T_{p,b} \subseteq b^{-1}T(b, \pm\mathcal{G})$ for $1 \leq i \leq 2p$, then $\Phi \subseteq \Omega$. If there exists $x \in \Phi$, we have $P_x \geq 2p \geq 2p_0$. But all P_x are less than $2p_0$ for $x \in \Omega$, which yields a contradiction and thus the claim follows.

Suppose that the first assertion of the lemma is false. Recall that

$$\mathcal{Z}_{b,q\mathcal{E}} := \bigcup_{n=1}^{N} \mathcal{Z}(\hat{\mu}_{b,q\mathcal{D}_{\mathcal{E}_n}}) \text{ and } \mathcal{Z}_{b,q\mathcal{E}} = \bigcup_{k=1}^{\infty} b^k \mathcal{Z}_{q\mathcal{E}}.$$

Then there exist $k \geq 1$ so that $b^k \mathcal{Z}_{q\mathcal{E}} \bigcap T_{p,b} \neq \phi$. Observing that $b^{-2pn}T_{p,b} \subseteq T_{p,b}$ for $n \geq 1$, we have $\mathcal{Z}_{q\mathcal{E}} \bigcap b^{-j}T_{p,b} \neq \phi$ for some $1 \leq j \leq 2p$, which contradicts (3.8). Hence the first assertion follows. The last inequality is immediate by compactness of $T_{p,b}$. $\square$

We define $\widetilde{\mathcal{D}}_k = \sum_{i=1}^{2p} b^{2p-i} q\mathcal{D}_{i+2p(k-1)}$ for $k \geq 1$ and denote

$$\mu_m = \delta_{b^{-2p}\widetilde{\mathcal{D}}_1} * \delta_{b^{-4p}\widetilde{\mathcal{D}}_2} * \cdots * \delta_{b^{-2pm}\widetilde{\mathcal{D}}_m}.$$

Now we construct a set

$$\Lambda_m(p,b) = \sum_{k=1}^{2pm} (-1)^{\tau(k)} b^{k-1} \mathcal{C}_k,$$

where $\mathcal{C}_k = \{0, 2^{l-t_k}, 2^l, 2^l + 2^{l-t_k}\}$ is a spectrum of $\delta_{b^{-1}q\mathcal{D}_k}$, τ is a periodic function on $\mathbb{Z}$ with period $2p$ and takes values 0 for $1 \leq k \leq p$, and 1 for $p+1 \leq k \leq 2p$. It is not difficult to check that

$$b^{-2pm}\Lambda_m(p,b) \subseteq T_{p,b}.$$

Let

$$\Lambda(p,b) = \bigcup_{m=1}^{\infty} \Lambda_m(p,b). \tag{3.9}$$

Lemma 3.4. *With the above notations, $\Lambda_m(p,b)$ is a spectrum of the measure μ_m and $\Lambda(p,b)$ is an orthogonal set of $\mu_{b,\{q\mathcal{D}_k\}}$.*

Proof. Since all $(b^{-1}q\mathcal{D}_k, \mathcal{C}_k)$ are compatible pairs, so is $(\sum_{k=1}^{2pm} b^{-k}q\mathcal{D}_k, \Lambda_m(p,b))$ by Proposition 2.2. Hence $\Lambda_m(p,b)$ is a spectrum of the measure μ_m. Since

$$\Lambda_1(p,b) \subseteq \Lambda_2(p,b) \subseteq \cdots \subseteq \Lambda_m(p,b) \subseteq \Lambda_{m+1}(p,b) \subseteq \cdots ,$$

and

$$\mathcal{Z}(\hat{\mu}_{b,\{q\mathcal{D}_k\}}) = \bigcup_{m=1}^{\infty} \mathcal{Z}(\hat{\mu}_m).$$

the second assertion follows. □

Now, Theorem1.3 is rewritten as the following theorem.

Theorem 3.5. *Let $b = 2^{l+1}q$ be an integer with $q > 1$ an odd number, and let $\{\mathcal{D}_k\}_{k=1}^{\infty}$ be a uniformly bounded sequence of 4-digit spectral sets. Suppose $l \geq t_{\max}$, then the Moran measure $\mu_{b,\{q\mathcal{D}_k\}}$ is a spectral measure with a spectrum $\Lambda(p,b)$ for $p \geq p_0$ where p_0 is given in Lemma 3.3.*

Proof. By Lemma 3.4, $\Lambda_m(p,b)$ is a spectrum of the measure μ_m and $\Lambda(p,b)$ is an orthogonal set of $\mu_{b,\{q\mathcal{D}_k\}}$. According to Proposition 2.1, we have

$$Q_m(\xi) = \sum_{\lambda\in\Lambda_m} \left|\hat{\mu}_m(\xi+\lambda)\right|^2 \equiv 1 \quad \text{and} \quad Q_\Lambda(\xi) = \sum_{\lambda\in\Lambda} \left|\hat{\mu}_{b,\{q\mathcal{D}_k\}}(\xi+\lambda)\right|^2 \leq 1. \tag{3.10}$$

Since Λ_m increases to Λ, and μ_m converges to $\mu_{b,\{q\mathcal{D}_k\}}$ weakly, we will use the dominated convergence theorem to justify $Q_\Lambda(\xi) \equiv 1$. Define

$$f_m(\lambda) = \begin{cases} \left|\hat{\mu}_m(\xi+\lambda)\right|^2, & \text{if } \lambda \in \Lambda_m; \\ 0, & \text{otherwise,} \end{cases}$$

and

$$f(\lambda) = \begin{cases} \left|\hat{\mu}_{b,\{q\mathcal{D}_k\}}(\xi+\lambda)\right|^2, & \text{if } \lambda \in \Lambda; \\ 0, & \text{otherwise.} \end{cases}$$

Then $\lim_{n\to\infty} f_m(\lambda) = f(\lambda)$ for $\lambda \in \Lambda$. By (3.10) we have $\sum_{\lambda\in\Lambda} \left|\hat{\mu}_{b,\{q\mathcal{D}_k\}}(\xi+\lambda)\right|^2 = \sum_{\lambda\in\Lambda} f(\lambda) \leq 1$ for any $\xi \in \mathbb{R}$. Now we construct a dominated function by means of $f(\lambda)$. For any $\lambda \in \Lambda_m$, we have $b^{-2pm}\lambda \in T_{p,b}$ and

$$\begin{aligned} f(\lambda) &= |\hat{\mu}_{b,\{q\mathcal{D}_k\}}(\xi+\lambda)|^2 = \prod_{k=1}^{\infty} \left|M_{q\mathcal{D}_k}(b^{-k}(\xi+\lambda))\right|^2 = \prod_{k=1}^{\infty} \left|M_{\widetilde{\mathcal{D}}_k}(b^{-2pk}(\xi+\lambda))\right|^2 \\ &= |\hat{\mu}_m(\xi+\lambda)|^2 \prod_{k=1}^{\infty} \left|M_{\widetilde{\mathcal{D}}_{k+m}}(b^{-2pk}(b^{-2pm}\xi + b^{-2pm}\lambda))\right|^2 \\ &= f_m(\lambda) \prod_{k=1}^{\infty} \left|M_{q\mathcal{D}_{k+2pm}}(b^{-k}(b^{-2pm}\xi + b^{-2pm}\lambda))\right|^2 \\ &\geq f_m(\lambda) \prod_{n=1}^{N} \left|\hat{\mu}_{b,q\mathcal{D}_{\varepsilon_n}}(b^{-2pm}\xi + b^{-2pm}\lambda)\right|^2 . \end{aligned}$$

When $\xi \in (0,1)$ and $b^{-2pm} < \eta$, where η is given in Lemma 3.3, we have $f(\lambda) \geq cf_m(\lambda)$ for $\lambda \in \Lambda_m$. Hence, $c^{-1}f$ is a dominated function and so $Q_\Lambda(\xi) \equiv 1$ for $\xi \in (0,1)$. The assertion follows by Proposition 2.1. $\square$

As an illustration of Theorem1.3, we give the following example.

Example 3.6. Let $\{0, a_k, b_k, c_k\}$ be a sequence of digit sets that is uniformly bounded, $a_k = 1, b_k = 2 + 4m_k, c_k = 3 + 4m_k$ for $k \geq 1$, and $b = 4q$ for some odd number $q > 1$, m_k be positive integers with bounded M. Then the Moran measure $\mu_{4q,\{0,a_k,b_k,c_k\}}$ is a spectral measure.

Proof. In this example, $l = 1$ and $t_{\max} = 1$, By Theorem 1.3, the assertion follows. $\square$

Clearly, the condition $q > 1$ in Theorem 1.3 is not necessary. The following example will illustrate this fact.

Example 3.7. Let $\mathcal{D}_1 = \{0, 1, 6, 7\}$ and $\mathcal{D}_k = \{0, 1, 2, 3\}$ for $k \geq 2$. Then the Moran measure $\mu_{4,\{\mathcal{D}_k\}}$ is a spectral measure.

Proof. It is easy to check that $(\frac{1}{4}\mathcal{D}_1, \mathcal{C})$ is compatible pair, where $\mathcal{C} = \{0, 1, 2, 3\}$. Observe that

$$\begin{aligned} \mu_{4,\{\mathcal{D}_k\}}(\cdot) &= \delta_{4^{-1}\mathcal{D}_1}(\cdot) * \delta_{4^{-2}\mathcal{D}_2}(\cdot) * \cdots \\ &= \delta_{4^{-1}\mathcal{D}_1}(\cdot) * \mathcal{L}|_{[0,1]}(4\cdot) \end{aligned}$$

where $\mathcal{L}|_{[0,1]}$ is the Lebesgue measure restricted on the interval $[0,1]$, then $\mu_{4,\{\mathcal{D}_k\}}$ is a spectral measure (see Theorem 5.3 [16]). $\square$

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