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Global strong solutions to 1-D vacuum free boundary problem for compressible Navier–Stokes equations with variable viscosity and thermal conductivity

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ABSTRACT

In this paper, we investigate the vacuum free boundary problem of one-dimensional heat-conducting compressible Navier–Stokes equations where the viscosity coefficient depends on the density, and the heat conductivity coefficient depends on the temperature, satisfying a physical assumption from the Chapman–Enskog expansion of the Boltzmann equation. The fluid connects to the vacuum continuously, thus the system is degenerate near the free boundary. The global existence and uniqueness of strong solutions for the free boundary problem are established when the initial data are large. The result is proved by using both the Lagrangian mass coordinate and the Lagrangian trajectory coordinate. An key observation is that the Jacobian between these coordinates are bounded from above and below by positive constants.

Contents

1. Introduction

The free boundary problem of one-dimensional heat-conductive compressible Navier–Stokes equations reads as follows:

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$$\begin{cases} \varrho_t + (\varrho u)_x = 0, & \text{in } \mathcal{I}(t) \times [0,T], \\ \varrho(u_t + uu_x) + (p - \mu u_x)_x = 0, & \text{in } \mathcal{I}(t) \times [0,T], \\ \varrho(\vartheta_t + u\vartheta_x) + (p - \mu u_x)u_x = (\kappa(\vartheta)\vartheta_x)_x, & \text{in } \mathcal{I}(t) \times [0,T], \\ \varrho > 0, & \text{in } \mathcal{I}(t) \times [0,T], \\ u(\Gamma_1(t),t) = 0,\ (p - \mu u_x)(\Gamma_2(t),t) = 0,\ \ \vartheta_x(\Gamma_i(t),t) = 0, & \text{on } [0,T], \\ (\varrho, u, \vartheta)(x,0) = (\varrho_0, u_0, \vartheta_0)(x), & x \in \mathcal{I}(0) =: I, \end{cases} \tag{1.1}$$

where ϱ, u, ϑ and p denote the density, velocity, absolute temperature and pressure, respectively. The notations μ and κ are viscous coefficient and heat conductivity coefficient, respectively. $\mathcal{I}(t) = (\Gamma_1(t), \Gamma_2(t))$ is the free interval occupied by the fluids with $\mathcal{I}(0) = (0,1)$, where $\Gamma_i(t)$ represents the moving interface between the fluid and vacuum defined by

$$\begin{cases} \frac{d\Gamma_1(t)}{dt} = u(\Gamma_1(t),t), \quad t > 0 \\ \Gamma_1(0) = 0, \end{cases} \tag{1.2}$$

and

$$\begin{cases} \frac{d\Gamma_2(t)}{dt} = u(\Gamma_2(t),t), \quad t > 0 \\ \Gamma_2(0) = 1. \end{cases} \tag{1.3}$$

In addition, we consider the degenerate case that $\rho_0(1) = 0$, which indeed gives $\varrho(\Gamma_2(t),t) = 0$. In this paper, we study the situation of ideal polytropic gas, where the pressure law reads

$$p = R\varrho\vartheta,$$

with R being a positive constant.

In the classical literature, many famous mathematical results on the well-posedness of heat-conductive compressible Navier–Stokes equations focused on the cases without vacuum, which can describe the movement of heat-conductive viscous liquid. When μ and κ are both positive constants, the well-posedness of strong solutions for the initial–boundary value problem has been investigated successfully, either for the local-in-time theory (see [20]) or for the global-in-time theory (see [13,19]). These results are extended to the study of viscous heat-conductive “real gases” in [10,11], for instance, where the heat conductivity depends on the temperature. The global existence of strong or classical solutions for the one-dimensional free boundary problem was achieved by Wang [25], Qin and Yao [23], and other authors.

In the presence of vacuum, the global well-posedness theory for the full compressible Navier–Stokes equations is far from completed. Feireisl [7] studied the global existence of so-called variational solutions in a multi-dimensional bounded domain, where the temperature satisfied only an inequality from thermal equation. Later, Bresch and Desjardins [1] proposed some different assumptions on the viscosity, thermal conductivity and equation of state, and established the global existence of weak solutions in T^3 or R^3. Recently, Wen and Zhu [27,28] obtained the unique global classical solution for the initial–boundary problem with large initial data and vacuum in one dimension, or three dimensions (with symmetric structure). The global existence of weak solution for the free boundary problem with radial symmetry was obtained by Chen and Kratka [3], while the global strong solution was investigated by Li [14] for constant viscosity and heat-conductivity, for instance.

When one derives the full Navier–Stokes equations (1.1) from the Boltzmann equation by using the Chapman–Enskog expansion (see [2]), the viscosity coefficient μ and the heat conductivity coefficient κ are indeed functions of temperature in the form

$$\mu = \overline{\mu}\theta^b, \quad \kappa = \overline{\kappa}\theta^b, \qquad b > \frac{1}{2}, \tag{1.4}$$

where $\overline{\mu}$ and $\overline{\kappa}$ are positive constants. This physical case leads to strong nonlinearity and possible degeneracy, and produces additional difficulty. If the viscosity coefficient and heat conduction coefficient satisfy $\mu = \widetilde{\mu}h(u)\theta^{\alpha}$ and $\kappa = \widetilde{\kappa}h(u)\theta^{\alpha}$, Liu, Yang, Zhao and Zou [16], Wang and Zhao [26] obtained the global non-vacuum classical solutions under some smallness assumptions (the adiabatic component γ is close to 1, or $|\alpha|$ is small, respectively). However, under the assumption (1.4), the global well-posedness of large solutions to (1.1) is still open. Note that, when the viscosity is a constant and the heat conductivity depends on temperature, some of the ideas by Kazhikhov [12] still work. For instance, Jenssen and Karper [9], Pan and Zhang [22] proved the global existence of non-vacuum weak solutions or strong solutions, respectively, to the 1-D initial–boundary value problem under the assumption

$$\mu = \overline{\mu}, \quad \kappa = \overline{\kappa}\theta^{b}.$$

The main purpose of this paper is to establish the global existence of vacuum strong and classical solutions to the free boundary problem (1.1) with the assumptions

$$\mu = \overline{\mu}(1 + \rho^{\beta}), \quad \kappa = \overline{\kappa}\theta^{q}, \quad \beta \in [0, +\infty),\ q \in (0, \infty), \tag{1.5}$$

where $\overline{\mu}$ and $\overline{\kappa}$ are positive constants. We remark that the assumption $\mu = \overline{\mu}(1 + \rho^{\beta})$ is valid in some sense when the fluid is in the low Mach number regime that the pressure $p = R\rho\theta \approx$ constant (see [5]). We generalize the results of [9,22] to the case of free boundary problems, where additional nonlinearity and singularity are involved, thus the classical methods for the initial–boundary problem cannot be applied directly. Moreover, we also extend the works in [14] for constant viscosity and heat conductivity to the variable ones satisfying (1.5). In contrast with [14], the non-constant viscosity coefficient and heat conductivity coefficient will create some troubles in establishing the global estimates for the solution, and the method for setting up the energy estimates are in a different way.

Since it is important to trace the free boundary in this problem, we need to prove that the velocity field is Lipschitz continuous in spatial variable. Motivated by [4,17], we take the method of Lagrangian trajectory to establish global energy estimates. However, the system is degenerate near free boundary and strongly nonlinear, thus the classical theory cannot be applied directly. To overcome the trouble, we first derive the point-wise upper and lower bounds for the Jacobian J, which governs the evolution of the free boundary. Then we derive the estimates for the spatial and time derivatives of the solution by some weighted inequalities. It is also required to show the lower bound of the temperature, which seems to be difficult to derive with the Lagrangian trajectory coordinate, due to the degeneracy of the coefficient for the time derivative of the temperature. Thus we take the advantage of Lagrangian mass coordinate to show lower bound of the temperature, which is equivalent to the one in the Lagrangian trajectory coordinate.

We remark that there are much progress on the free boundary problems of Navier–Stokes equations and related models in the isentropic regime, for instance, [6,8,15,18,21,29] and the references therein.

The rest of the paper is organized as follows. In Section 2, we present the main theorems of the paper. In Section 3, some useful lemmas are stated, which will be used to prove the global existence of strong solutions. In Section 4, we give the proof of Theorem 2.1.

Notations.

(1) $I = (0, 1)$, $\partial I = \{0, 1\}$, $Q_T = I \times [0, T]$ for $T > 0$.
(2) For $1 \leq p \leq \infty$ and positive integer k, we use $L^p = L^p(I)$ and $W^{k,p} = W^{k,p}(I)$ to denote the standard Lebesgue and Sobolev spaces, respectively, and in the case that $k = 2$, we use H^p instead of $W^{2,p}$. The norm of Sobolev space $W^{k,p}$ is denoted as $\|\cdot\|_{W^{k,p}}$ or $\|\cdot\|_{H^k}$ for $p = 2$.
(3) Throughout this paper, the same letter C (sometimes used as $C(X)$ to emphasize the dependence of C on X) denote various generic positive constant.

2. The Lagrangian coordinates and main result

Let y be the Lagrangian trajectory coordinate, and define the coordinate transform between the Lagrangian trajectory coordinate y and the Euler coordinate x as

$$x = \eta(y,t),$$

where $\eta(y,t)$ is the flow map determined by u, that is

$$\begin{cases} \eta_t(y,t) = u(\eta(y,t),t), \text{ for } t > 0 \\ \eta(y,0) = y, \; y \in I. \end{cases}$$

For simplicity of presentation, the reference domain is denoted as

$$I := (0,1).$$

Denote by ρ, v, θ and π the density, velocity, temperature, and pressure, respectively, in the Lagrangian trajectory coordinate. That is, we define

$$\begin{aligned} \rho(y,t) &:= \varrho(\eta(y,t),t), \quad v(y,t) := u(\eta(y,t),t), \\ \theta(y,t) &:= \vartheta(\eta(y,t),t), \quad \pi(y,t) := p(\eta(y,t),t). \end{aligned} \tag{2.6}$$

Define the function $J = J(y,t)$ as

$$J = \eta_y(y,t),$$

then it follows

$$J_t = v_y(y,t), \tag{2.7}$$

$$\begin{cases} \rho_t + \rho\frac{v_y}{J} = 0, & \text{in } I \times [0,T], \\ \rho v_t + \frac{\pi_y}{J} = \frac{1}{J}\left(\frac{\mu v_y}{J}\right)_y, & \text{in } I \times [0,T], \\ \rho\theta_t + \pi\frac{v_y}{J} = \mu\frac{v_y^2}{J^2} + \frac{1}{J}\left(\frac{\kappa\theta_y}{J}\right)_y, & \text{in } I \times [0,T], \\ \rho > 0, & \text{in } I \times [0,T], \\ \rho|_{y=1} = 0, \quad v|_{y=0} = 0, \quad v_y|_{y=1} = 0, \quad \theta_y|_{y=0,1} = 0, & \\ (\rho, v, \theta)|_{t=0} = (\rho_0, v_0, \theta_0) & \text{on } I \times \{t = 0\}, \end{cases} \tag{2.8}$$

where $\pi = R\rho\theta$, $\rho_0(y)$ satisfies that $\rho_0 > 0$ in $I \cup \{y = 0\}$ and $\rho_0 = 0$ on $\{y = 1\}$. Due to (2.7) and $(2.8)_1$, we find that

$$(\rho J)_t = \rho_t J + \rho J_t = \rho_t J + \rho v_y = J\left(\rho_t + \rho\frac{v_y}{J}\right) = 0, \quad (y,t) \in I \times [0,T], \tag{2.9}$$

which together with $\rho(y,t)|_{t=0} = \rho_0$ and $J(y,t)|_{t=0} = \eta_y|_{t=0} = 1$ yields

$$J\rho = \rho_0.$$

We should note that $\eta_y > 0$ for all $(y,t) \in I \times [0,T]$, which can be verified later in the *a priori* estimates. One can replace $(2.8)_1$ with (2.7), by setting $\rho = \frac{\rho_0}{J}$, and rewrite (2.8) as

$$\begin{cases} J_t = v_y, & \text{in } I \times \{t=0\} \\ \rho_0 v_t + \pi_y = \left(\frac{\mu v_y}{J}\right)_y, & \text{in } I \times \{t=0\} \\ \rho_0 \theta_t + \pi v_y = \mu \frac{v_y^2}{J} + \left(\frac{\kappa \theta_y}{J}\right)_y, & \text{in } I \times \{t=0\} \\ v|_{y=0} = 0, \quad v_y|_{y=1} = 0, \quad \theta_y|_{y=0,1} = 0, & \\ (J, v, \theta) = (1, v_0, \theta_0) & \text{on } I \times \{t=0\}, \end{cases} \tag{2.10}$$

where $\pi = R\frac{\rho_0}{J}\theta$.

Theorem 2.1. *(i) Suppose that μ and κ satisfy (1.5) for some positive constants $\overline{\mu}$ and $\overline{\kappa}$. If the initial data $(\rho_0, v_0, \theta_0)(x) \in H^1 \times H^2 \times H^2$, $\rho_0 > 0$ in $[0,1)$, $\rho_0|_{y=1} = v_{0y}|_{y=1} = 0$, $v|_{y=0} = 0$, $\theta_{0y}|_{y=0,1} = 0$ and there is constant $\underline{\theta}$, such that*

$$0 < \underline{\theta} \le \theta_0(y) \text{ in } I \cup \partial I. \tag{2.11}$$

Moreover, the following compatibility conditions are imposed:

$$\|\sqrt{\rho_0} v_t(\cdot, 0)\|_{L^2(I)} + \|\sqrt{\rho_0}\theta_t(\cdot, 0)\|_{L^2(I)} \le C. \tag{2.12}$$

Furthermore, we suppose that, if $\beta \in (0,1)$,

$$\|\rho_0^\beta\|_{H^2(I)} \le C. \tag{2.13}$$

Then for any $T > 0$, there exists a unique global strong solution (J, v, θ) to (2.10) satisfying $\theta \ge C_1 > 0$ in $I \times (0, T]$ with

$$\begin{aligned} &(v, \theta) \in L^\infty(0, T; H^2(I)), \quad \theta_{ty}, v_{ty} \in L^2(0, T; L^2(I)), \\ &\theta_t \in L^2(0, T; L^\infty(I)), \ (\sqrt{\rho_0} v_t, \sqrt{\rho_0}\theta_t) \in L^\infty(0, T; L^2(I)). \end{aligned} \tag{2.14}$$

(ii) The expanding rate of the free interval $(\Gamma_1(t), \Gamma_2(t))$ is

$$\underline{C} \le \Gamma_2(t) - \Gamma_1(t) \le C_q(1+t), t \in [0, +\infty), \tag{2.15}$$

for some positive constants $\underline{C}$ and C_q.

Remark 2.1. It should be noted that our Theorem 2.1 holds for large initial data, when heat conductivity depends on temperature in the power law of the Chapman–Enskog expansion. The assumption (2.13) means that the initial viscosity belongs to $H^2(I)$, for instance, $\mu(\rho_0) = 1 + \rho_0^\beta$, where $\rho_0^\beta = (1-x)^2$.

Remark 2.2. We can also establish the global classical or smooth solution to (2.10), provided that the initial data are smooth enough, since the uniform upper and low bounds for J and the lower bound for θ are already obtained.

The existence and uniqueness of local-in-time solution can be obtained by a standard fixed point argument, cf. [20,19]. The following lemma gives the local existence of strong solutions, however, the proof is omitted here.

Lemma 2.1. *If (1.5) holds, and the initial data $(\rho_0, v_0, \theta_0)(x) \in H^1 \times H^2 \times H^2$, $\rho_0 > 0$ in $[0,1)$, $\rho_0|_{y=1} = v_{0y}|_{y=1} = 0$, $v|_{y=0} = 0$, $\theta_{0y}|_{y=0,1} = 0$ and $0 < \underline{\theta} \le \theta_0(y)$ in $I \cup \partial I$. Moreover, the compatibility conditions*

(2.12) are satisfied. Then there exist positive constants T^* and C_*, such that the system (2.10) admits a unique strong solution (J, v, θ) in $I \times (0, T]$, satisfying $\theta \geq C_* > 0$ in $I \times (0, T^*]$ with

$$\begin{aligned} &(v, \theta) \in L^\infty(0, T^*; H^2(I)), \quad \theta_{ty}, v_{ty} \in L^2(0, T^*; L^2(I)), \\ &\theta_t \in L^2(0, T^*; L^\infty(I)), \ (\sqrt{\rho_0} v_t, \sqrt{\rho_0}\theta_t) \in L^\infty(0, T^*; L^2(I)). \end{aligned} \tag{2.16}$$

3. Preliminaries

Lemma 3.1 *([28]). Assume that ρ be an on-negative function such that, $0 < M \leq |\int_I \rho dy| \leq L$ for constants $M, L > 0$. Then for any $\theta \in H^1(I)$.*

$$\|\theta\|_{L^\infty(I)} \leq \frac{L}{M}\|\theta_y\|_{L^1(I)} + \frac{1}{M}\int\limits_I \rho\theta dy.$$

In addition, assume $\|\rho\theta\|_{L^1(I)} \leq C$. Then for any $q > 0$, there exists a positive constant $C = C(L, M, K, q)$ such that

$$\|\theta^q\|_{L^\infty(I)} \leq C\|(\theta^q)_y\|_{L^1(I)} + C,$$

for any $\theta^q \in H^1(I)$.

Remark 3.1. The multi-dimensional version of Lemma 3.1 can be found in [7].

Lemma 3.2 *(Aubin–Lions–Simon Lemma [24]). Assume $X \subset E \subset Y$ are Banach spaces and $X \hookrightarrow\hookrightarrow E$. Then the following imbeddings are compact:*

(i) $\Big\{\varphi(y, t) | \varphi \in L^p(0, T; X), \frac{\partial}{\partial_t}\varphi \in L^1(0, T; Y)\Big\} \hookrightarrow\hookrightarrow L^p(0, T; E)$ *if* $1 \leq p \leq \infty$*;*
(ii) $\Big\{\varphi(y, t) | \varphi \in L^\infty(0, T; X), \frac{\partial}{\partial_t}\varphi \in L^r(0, T; Y)\Big\} \hookrightarrow\hookrightarrow C(0, T; E)$ *if* $1 < r \leq \infty$*.*

4. A priori estimates and global existence

In this section, we will perform the energy estimates which are stated in the following lemmas to prove Theorem 2.1. Furthermore, we get a unique global strong solution of (2.10) by using the *a priori* estimates of the solutions based on the local existence. We now assume that $(u, v, \theta)(x, t)$ is the unique strong solution of (2.10) defined on $[0, 1] \times [0, T]$, satisfying that

$$J > 0, \quad \theta > 0. \tag{4.17}$$

The above assumptions will be recovered in the energy estimates. For simplicity of presentation, we will denote $R = c_v = \bar{\mu} = \bar{\kappa} = 1$.

Lemma 4.1. *It holds that*

$$\int\limits_I \left(\rho_0\theta + \frac{1}{2}\rho_0 v^2\right) dy(t) = E_0, \tag{4.18}$$

for $t \in (0, \infty)$, where $E_0 := \int_I \left(\rho_0\theta_0 + \frac{1}{2}\rho_0 v_0^2\right) dy$.

Proof. Multiplying equation $(2.10)_2$ by v, adding it to $(2.10)_3$ and integrating the result over I, one gets from integrating by parts that

$$\frac{d}{dt}\int\limits_I \left(\rho_0\theta + \frac{1}{2}\rho_0 v^2\right) dy = 0,$$

where we have used $(2.10)_4$ and $\rho_0|_{y=1} = 0$. □

Lemma 4.2. *For any $(y,t) \in I \times [0,T]$, one has*

$$\underline{C} \le J(y,t) \le C\left\{1 + \int\limits_0^t \rho_0\theta ds\right\}, \tag{4.19}$$

$$\int\limits_I J dy(t) \le C(1+t), \tag{4.20}$$

where $\underline{C}$ is a positive constant independent of T and q, and $\bar{C}(q)$ is positive constant depending on q but not on T.

Proof. *Step 1. Point-wise lower bound of J.*

Due to $(2.10)_1$, it follows from $(2.10)_2$ that

$$-\frac{1}{\beta}\left(\frac{\rho_0^\beta}{J^\beta}\right)_{ty} + (\log J)_{ty} = \rho_0 v_t + \pi_y.$$

Integrating the above equation with respect to y over $(y,1)$ and observing $\pi = \rho_0\theta/J$, $\rho_0|_{y=1} = 0$ and $J_t|_{y=1} = v_y|_{y=1} = 0$, yield that

$$-\frac{1}{\beta}\left(\frac{\rho_0^\beta}{J^\beta}\right)_t + (\log J)_t = -\int\limits_y^1 \rho_0 v_t dz + \pi.$$

Next, we integrate this equation with respect to t and use $J(y,0) = 1$ to obtain

$$-\frac{1}{\beta}\frac{\rho_0^\beta}{J^\beta} + \log J = -\int\limits_y^1 (\rho_0 v - \rho_0 v_0)dy + \int\limits_0^t \pi(y,s)ds - \frac{1}{\beta}\rho_0^\beta,$$

from which, recalling $\pi = \frac{\rho_0\theta}{J}$, one gets

$$J = \exp\left\{-\int\limits_y^1 (\rho_0 v - \rho_0 v_0)dy\right\}\exp\left\{\int\limits_0^t \frac{\rho_0\theta}{J}(y,s)ds\right\}\exp\left\{\frac{\rho_0^\beta}{\beta}\left(\frac{1}{J^\beta} - 1\right)\right\}. \tag{4.21}$$

Multiplying (4.21) by $\rho_0\theta$, we obtain

$$\frac{\partial}{\partial t}\mathfrak{B}(y,t) = \rho_0\theta \exp\{\mathfrak{M}(y,t)\}\exp\{\mathfrak{N}(y,t)\}, \tag{4.22}$$

where

$$\mathfrak{M}(y,t) := \int_y^1 (\rho_0 v - \rho_0 v_0)dz, \quad \mathfrak{N}(y,t) := \frac{\rho_0^\beta}{\beta}\left(\frac{1}{J^\beta} - 1\right),$$

$$\mathfrak{B}(y,t) := \exp\left\{\int_0^t \frac{\rho_0\theta}{J}(y,s)ds\right\}.$$

Integrating (4.22) with respect to t over $(0,t)$, one has

$$\mathfrak{B}(y,t) = 1 + \int_0^t \rho_0\theta \exp\{\mathfrak{M}(y,s)\}\exp\{\mathfrak{N}(y,s)\}\,ds,$$

which together with (4.21) yields

$$J = \exp\{-\mathfrak{M}(y,t)\}\exp\{\mathfrak{N}(y,t)\}\left\{1 + \int_0^t \rho_0\theta \exp\{\mathfrak{M}(y,s)\}\,ds\right\}. \tag{4.23}$$

Note that

$$\begin{aligned}|\mathfrak{M}(y,t)| &\le \left(\int_I \rho_0 dy\right)^{\frac{1}{2}}\left[\left(\int_I \rho_0 v^2 dy\right)^{\frac{1}{2}} + \left(\int_I \rho_0 v_0^2 dy\right)^{\frac{1}{2}}\right]\\ &\le C,\end{aligned}$$

which yields

$$\exp\{-C\} \le \exp\{\mathfrak{M}(y,t)\} \le \exp\{C\}. \tag{4.24}$$

This combines with (4.23) and the fact that $\rho_0\theta \ge 0$ to give

$$J \ge \exp\{-\mathfrak{M}(y,t)\}\exp\left\{-\frac{\rho_0^\beta}{\beta}\right\} \ge \underline{C}. \tag{4.25}$$

Step 2. Point-wise upper bound of J.
It follows from (4.23), (4.24), (4.25) and $\rho_0\theta \ge 0$ that

$$\begin{aligned}J &= \exp\{-\mathfrak{M}(y,t) + \mathfrak{N}(y,t)\}\left\{1 + \int_0^t \rho_0\theta\exp\{\mathfrak{M}(y,s)\}\,ds\right\}\\ &\le C\left\{1 + \int_0^t \rho_0\theta ds\right\}\end{aligned} \tag{4.26}$$

and

$$\int_I J dy \le C\int_I \left(1 + \int_0^t \rho_0(y)\theta(y,s)ds\right)dy$$

$$
\begin{aligned}
&\leq C\left(1 + t \max_{s\in[0,t]} \int\limits_I \rho_0 \theta dy\right) \\
&\leq C(1+t).
\end{aligned} \tag{4.27}
$$

Thus, (4.25)–(4.27) give the proof of Lemma 4.2. □

To obtain the upper bound of J, it needs to get the lower bound of θ. Consequently, it is convenient to deal with the problem (1.1) in the Lagrangian mass coordinates. From $(1.1)_1$, (1.2) and (1.3), we have

$$
\int\limits_{\Gamma_1(t)}^{\Gamma_2(t)} \varrho(\xi,t)d\xi = \int\limits_{\Gamma_1(0)}^{\Gamma_2(0)} \varrho_0(\xi)d\xi = \int\limits_0^1 \varrho_0(\xi)d\xi,
$$

for simplicity, we assume that $\int_0^1 \varrho_0(\xi)d\xi = 1$. For any $x \in [\Gamma_1(t), \Gamma_2(t)]$, $t \in [0,T]$, we define the Lagrangian mass coordinates transformation

$$
z(x,t) = \int\limits_{\Gamma_1(t)}^{x} \varrho(\xi,t)d\xi \text{ and } \tau = t,
$$

which translates the domain $[0,T] \times [\Gamma_1(t), \Gamma_2(t)]$ into $[0,T] \times [0,1]$ and satisfies

$$
\frac{\partial z}{\partial x} = \varrho, \ \frac{\partial z}{\partial t} = -\varrho u, \ \frac{\partial \tau}{\partial t} = 1, \text{ and } \frac{\partial \tau}{\partial x} = 0.
$$

The free boundary problem (1.1) are changed to be

$$
\begin{cases}
\varrho_\tau + \varrho^2 u_z = 0, & (\tau,z) \in [0,T] \times (0,1), \\
u_\tau + P_z = (\mu\varrho u_z)_z, & (\tau,z) \in [0,T] \times (0,1), \\
\vartheta_\tau + P u_z = \mu\varrho u_z^2 + (\kappa(\vartheta)\rho\vartheta_z)_z, & (\tau,z) \in [0,T] \times (0,1), \\
u(\tau,0) = 0, \ (\mu\varrho u_z - P)(\tau,1) = 0, \ \vartheta_z(\tau,0) = \vartheta_z(\tau,1) = 0, & \\
(\varrho,u,\vartheta)|_{\tau=0} = (\varrho_0,u_0,\vartheta_0), & z \in (0,1).
\end{cases} \tag{4.28}
$$

Lemma 4.3. *There exists a constant $C > 0$ such that*

$$
\int\limits_I \left(\frac{1}{2}u^2 + \vartheta\right) dz(\tau) \leq C, \quad \tau \geq 0. \tag{4.29}
$$

Proof. Multiplying equation $(4.28)_2$ by u, adding it to $(4.28)_3$ and integrating the result over I, one gets from integrating by parts that

$$
\frac{d}{d\tau} \int\limits_I \left(\frac{1}{2}u^2 + \vartheta\right) dz = 0,
$$

where we have used $(4.28)_4$. □

Lemma 4.4. *There exists a constant $C > 0$ such that*

$$
\varrho \leq C. \tag{4.30}
$$

Proof. Integrating the $(4.28)_2$ with respect to z over $(z, 1)$ and noticing $(\mu\varrho u_z - P)|_{z=1} = 0$ yields

$$-\mu\varrho u_z + P = \int_z^1 u_\tau dz.$$

It combines with $(4.27)_1$ to yield that

$$(\ln\varrho)_\tau + \frac{1}{\beta}\left(\varrho^\beta\right)_\tau + P = \int_z^1 u_\tau dz,$$

from which, integrating with respect to t over $(0, t)$, one obtains

$$\ln\varrho + \frac{1}{\beta}\varrho^\beta = \ln\varrho_0 + \frac{1}{\beta}\varrho_0^\beta + \int_z^1 (u - u_0)dy - \int_0^t P ds.$$

By using Lemma 4.3 and the nonnegativity of ϱ and P, we find

$$\begin{aligned}\varrho &= \varrho_0 \exp\left\{\int_z^1 (u-u_0)dy\right\}\exp\left\{\frac{1}{\beta}(\varrho_0^\beta - \varrho^\beta)\right\}\exp\left\{-\int_0^t P ds\right\}\\ &\le \varrho_0\exp\left\{\left(\int_0^1 u^2 dy\right)^{\frac12} + \left(\int_0^1 u_0^2 dy\right)^{\frac12}\right\}\exp\left\{\frac{1}{\beta}\varrho_0^\beta\right\}\\ &\le C. \quad \square\end{aligned}$$

The following estimate will indeed leads to the positive lower bound of θ.

Lemma 4.5. *There exists a constant $C > 0$ such that, for any $p > 2$,*

$$\operatorname*{ess\ sup}_{\tau\in[0,T]}\left\|\frac{1}{\vartheta}\right\|_{L^{p-1}} \le C, \quad \textit{and, in particular,} \quad \left\|\frac{1}{\vartheta}\right\|_{L^\infty(0,T;L^\infty(0,1))} \le C. \tag{4.31}$$

Proof. Multiplying $(4.28)_3$ by ϑ^{-p} and integrating with respect to z over $[0, 1]$, gives that

$$\begin{aligned}\frac{1}{p-1}\frac{d}{d\tau}\int_I \left(\frac{1}{\vartheta}\right)^{p-1} dz + \int_I \frac{\mu\varrho u_z^2}{\vartheta^p}dz + p\int_I \frac{\kappa(\vartheta)\rho\vartheta_z^2}{\vartheta^{p+1}}dz &= \int_I \frac{\varrho\vartheta u_z}{\vartheta^p}dz\\ &\le \frac12\int_I \frac{\mu\varrho u_z^2}{\vartheta^p}dz + C\int_I \frac{\varrho\vartheta^2}{\vartheta^p}dz\\ &\le \frac12\int_I \frac{\mu\varrho u_z^2}{\vartheta^p}dz + C\left(\int_I\left(\frac{1}{\vartheta}\right)^{p-1}dz\right)^{\frac{p-2}{p-1}},\end{aligned} \tag{4.32}$$

where we have used the lower bound of J and Hölder's inequality. (4.32) implies that

$$\frac{d}{d\tau}\left\|\frac{1}{\vartheta}\right\|_{L^{p-1}}\left\|\frac{1}{\vartheta}\right\|_{L^{p-1}}^{p-2} \le C\left\|\frac{1}{\vartheta}\right\|_{L^{p-1}}^{p-2}$$

which gives

$$\left\| \frac{1}{\vartheta} \right\|_{L^{p-1}} \leq C(T). \tag{4.33}$$

The constant C in (4.33) depends only on T and initial data. Letting $p \to \infty$, we have proven Lemma 4.5. □

To estimate the upper bound of J, now it suffices to evaluate $\int_0^t \|\theta(\cdot, s)\|_{L^\infty} ds$, which is discussed in two cases as follows.

Lemma 4.6. *For any $(y,t) \in I \times [0,T]$, one has*

$$\underline{C} \leq J(y,t) \leq C(1+t)^2, \quad \int_0^t \|\theta(\cdot, s)\|_{L^\infty} ds \leq C(q)(1+t)^2. \tag{4.34}$$

Proof. *Case (i): $q \geq 1$.* Multiplying $(2.10)_3$ by $\frac{1}{\theta}$, one achieves

$$\rho_0 (\log\theta)_t + \frac{\rho_0 \theta v_y}{\theta J} = \left(\frac{\kappa(\theta)\theta_y}{J\theta} \right)_y + \frac{\kappa(\theta)|\theta_y|^2}{J\theta^2} + \frac{\mu v_y^2}{J\theta}.$$

Integrating the above equation in $I \times (0,t)$ and using (4.19) and (4.18), we can obtain

$$\begin{aligned}
&\int_0^t \int_I \left(\frac{\theta^q |\theta_y|^2}{J\theta^2} + \mu \frac{v_y^2}{J\theta} \right) dyds \\
&= \int_I \rho_0 \log\theta dy - \int_I \rho_0 \log\theta_0 dy + \int_0^t \int_I \frac{\rho_0 \theta v_y}{\theta J} dyds \\
&\leq \int_{\{\theta \geq 1\}} \rho_0 \log\theta dy - \int_{\{\underline{\theta} \leq \theta_0 < 1\}} \rho_0 \log\theta_0 dy + \frac{1}{2} \int_0^t \int_I \frac{\mu v_y^2}{J\theta} dyds + C\|J^{-1}\|_{L^\infty_{y,t}} \int_0^t \int_I \rho_0 \theta dyds \\
&\leq \int_{\{\theta \geq 1\}} \rho_0 \theta dy + (-\log\underline{\theta}) \int_{\{\underline{\theta} \leq \theta_0 < 1\}} \rho_0 dy + \frac{1}{2} \int_0^t \int_I \frac{\mu v_y^2}{J\theta} dyds + Ct \sup_{t \in [0,T]} \int_I \rho_0 \theta dy(t) \\
&\leq C(1+t) + \frac{1}{2} \int_0^t \int_I \frac{\mu v_y^2}{J\theta} dyds,
\end{aligned}$$

which gives that

$$\int_0^t \int_I \left(\frac{\theta^q |\theta_y|^2}{J\theta^2} + \frac{\mu v_y^2}{J\theta} \right) dyds \leq C(1+t). \tag{4.35}$$

From Young's inequality, Lemma 3.1, (4.20) and (4.35), for $q \geq 1$, we have

$$
\begin{aligned}
\int_0^t \|\theta(\cdot,s)\|_{L^\infty} ds &\le \int_0^t \|\theta(\cdot,s)\|_{L^\infty}^q ds + Ct \\
&\le \int_0^t \|\theta^{\frac{q}{2}}(\cdot,s)\|_{L^\infty}^2 ds + Ct, \\
&\le \int_0^t \|(\theta^{\frac{q}{2}})_y(\cdot,s)\|_{L^1}^2 ds + C(t+1) \\
&\le \sup_{s\in[0,t]} \int_I J(y,s)dy \int_0^t \int_I \frac{\theta^q|\theta_y|^2}{J\theta^2} dyds + C(t+1) \\
&\le C(1+t)^2 + C(t+1) \le C(1+t)^2.
\end{aligned}
\tag{4.36}
$$

Case (ii): $q \in (0,1)$. Multiplying $(2.10)_3$ by $\theta^{-\alpha}(0<\alpha<q<1)$, we obtain

$$
\frac{1}{1-\alpha}(\rho_0\theta^{1-\alpha})_t + \frac{\rho_0\theta v_y}{\theta^\alpha J} = \left(\frac{\kappa(\theta)\theta_y}{J\theta^\alpha}\right)_y + \frac{\alpha\kappa(\theta)|\theta_y|^2}{J\theta^{1+\alpha}} + \frac{\mu v_y^2}{J\theta^\alpha}.
$$

Then we integrate the above equation over $I\times(0,t)$, and use (4.18) to get

$$
\begin{aligned}
&\int_0^t \int_I \left(\frac{\alpha\kappa(\theta)|\theta_y|^2}{J\theta^{1+\alpha}} + \frac{\mu v_y^2}{J\theta^\alpha}\right) dyds \\
=&\frac{1}{1-\alpha}\int_I \rho_0\theta^{1-\alpha}dy - \frac{1}{1-\alpha}\int_I \rho_0\theta_0^{1-\alpha}dy + \int_0^t\int_I \frac{\rho_0\theta v_y}{J\theta^\alpha}dyds \\
\le& C(\alpha) + \frac{1}{2}\int_0^t\int_I \frac{\mu v_y^2}{J\theta^\alpha}dyds + C\sup_{s\in[0,t]}\int_I \rho_0\theta dy \int_0^t \|\theta(\cdot,s)\|_{L^\infty}^{1-\alpha}ds,
\end{aligned}
$$

which gives

$$
\int_0^t \int_I \left(\frac{\alpha\kappa(\theta)|\theta_y|^2}{J\theta^{1+\alpha}} + \frac{v_y^2}{J\theta^\alpha}\right) dyds \le C\left(1 + \int_0^t \|\theta(\cdot,s)\|_{L^\infty}^{1-\alpha}ds\right). \tag{4.37}
$$

Thus by Lemmas 3.1, (4.20) and (4.37), we have

$$
\begin{aligned}
&\int_0^t \|\theta(\cdot,s)\|_{L^\infty} ds \le C\left(t + \int_0^t\int_I |\theta_y|dyds\right) \\
=&C\left(t + \int_0^t\int_I \frac{\theta^{\frac{q}{2}}|\theta_y|}{\sqrt{J}\theta^{\frac{1+\alpha}{2}}}\cdot\sqrt{J}\theta^{\frac{1+\alpha-q}{2}}dyds\right) \\
\le& Ct + \epsilon\max_{s\in[0,t]}\int_I J(y,s)dy\int_0^t \|\theta(\cdot,s)\|_{L^\infty}^{1+\alpha-q}ds + C(\epsilon)\int_0^t\int_I \frac{\theta^q|\theta_y|^2}{J\theta^{1+\alpha}}dyds \\
\le& C(\alpha)(1+t) + C\epsilon(1+t)\int_0^t \|\theta(\cdot,s)\|_{L^\infty}^{1+\alpha-q}ds + C(\alpha,\epsilon)\int_0^t \|\theta(\cdot,s)\|_{L^\infty}^{1-\alpha}ds.
\end{aligned}
$$

Note that $0 < 1-\alpha < 1$ and $0 < 1+\alpha-q < 1$. Then we choose ϵ, which depends on t, to be small enough in the above inequality, then use the Young inequality to get, for $q \in (0,1)$,

$$\int_0^t \|\theta(\cdot,s)\|_{L^\infty} ds \le C(q)(1+t)^2,\ t \in [0,T]. \tag{4.38}$$

It follows from (4.36) and (4.38), we obtain that for any $q > 0$,

$$\int_0^t \|\theta(\cdot,s)\|_{L^\infty} ds \le C(q)(1+t)^2,\ t \in [0,T], \tag{4.39}$$

which combine with (4.19) gives

$$J(y,t) \le C(q)(1+t)^2 \quad \text{for} \quad (y,t) \in I \times [0,T]. \tag{4.40}$$

Thus, (4.39) and (4.40) give the proof of Lemma 4.6. □

Remark 4.1. Lemma 4.6 verifies the assumption that $J > 0$ for any $(y,t) \in I \times [0,T]$. Moreover, from now on, we always use the conclusion that $\underline{C} \le J \le C(q,T)$ without additional claim.

Lemma 4.7. *For any $t \in [0,T]$, we have*

$$\int_0^t \int_I \mu v_y^2 dxdt \le C, \tag{4.41}$$

$$\int_0^t \int_I \theta^{q-1-\alpha}\theta_y^2 dydt + \int_0^t \|\theta\|_{L^\infty}^{1+q-\alpha} dt \le C, \quad 0 < \alpha < \min(1,q). \tag{4.42}$$

Proof. Multiplying $(2.10)_2$ by v and integrating over $I \times (0,t)$ and using (4.18) and (4.34), we can obtain

$$\begin{aligned}
&\frac{1}{2}\int_I \rho_0 v^2 dy + \int_0^t \int_I \frac{\mu v_y^2}{J} dydt \\
&\le \frac{1}{2}\int_I \rho_0 v_0^2 dy + \int_0^t \int_I \frac{\rho_0 \theta |v_y|}{J} dydt \\
&\le C + \frac{1}{2}\int_0^t \int_I \frac{\mu v_y^2}{J} dydt + C \sup_{s\in[0,t]} \int_I \rho_0 \theta dy \int_0^t \|\theta(\cdot,s)\|_{L^\infty} ds \\
&\le C + \frac{1}{2}\int_0^t \int_I \frac{\mu v_y^2}{J} dydt,
\end{aligned}$$

which gives (4.41) immediately. Moreover, (4.42) is derived from (4.37) and (4.34), and using Lemma 3.1. □

Lemma 4.8. *For any $t \in [0,T]$, one has*

$$\int_I J_y^2 dy(t) \le C. \tag{4.43}$$

Proof. We rewrite $(2.10)_2$ as

$$\sigma_t = \pi_y, \tag{4.44}$$

where

$$\sigma := (\log J - \frac{1}{\beta}\rho_0^\beta J^{-\beta})_y = J^{-1}J_y(1+\rho_0^\beta J^{-\beta}).$$

Then we multiply (4.44) by σ and integrate to get

$$\begin{aligned}\frac{1}{2}\int_I \sigma^2 dy(t) &= \frac{1}{2}\int_I \sigma^2(y,0)dy + \int_0^t\int_I \sigma\pi_y dydt \\ &\le C + \int_0^t\int_I \frac{\sigma}{J}\left[\rho_{0y}\theta + \rho_0\theta_y + \sigma(1+\rho_0^\beta J^{-\beta})^{-1})\right] dydt \\ &\le C + \delta\int_0^t\int_I (\theta^{1+q-\alpha} + \theta^{q-1-\alpha}\theta_y^2)dydt + \frac{C}{\delta}\int_0^t\int_I (1+\theta^{1+\alpha-q})\sigma^2 dydt,\end{aligned} \tag{4.45}$$

which gives

$$\begin{aligned}\frac{1}{2}\int_I \sigma^2 dy(t) &\le C + C\int_0^t (1+\|\theta\|_{L^\infty}^{1+q-\alpha}\|\theta\|_{L^\infty}^{-2(q-\alpha)})\int_I \sigma^2 dydt \\ &C + C\int_0^t (1+\|\theta\|_{L^\infty}^{1+q-\alpha})\int_I \sigma^2 dydt,\end{aligned} \tag{4.46}$$

by employing (4.42) and the lower bound of θ established in Lemma 4.5. Finally, we apply the Gronwall inequality to (4.46) and using (4.42) to achieve

$$\int_I \sigma^2 dy(t) \le C, \quad t \in [0,T],$$

which yields (4.43). □

Lemma 4.9. *For any $t \in [0,T]$, we have*

$$\int_I \left(v_y^2 + (\rho_0\theta)^2 + \rho_0\theta^{q+2}\right) dy(t) + \int_0^t\int_I \left(\rho_0 v_t^2 + \theta^{2q}\theta_y^2\right) dydt \le C, \tag{4.47}$$

and

$$\int_0^T \|v_y\|_{L^\infty} dt \le C. \tag{4.48}$$

Proof. Observe that

$$1 \le |\mu| = 1 + \rho_0^\beta J^{-\beta} \le C, \quad |\mu_t| = \beta \rho_0^\beta J^{-\beta-1}|v_y| \le C|v_y|, \tag{4.49}$$

and

$$|\mu_y| = \beta \rho_0^{\beta-1} J^{-\beta-1}|\rho_{0y}J^{-1} - \rho_0 J^{-2}J_y| \le C + C|J_y|, \tag{4.50}$$

where the assumption (2.13) is used. Then the rest of the proof will be divided into two steps.

Step 1. Multiplying $(2.10)_2$ by v_t and integrating with respect to y over $[0,1]$ give that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_I \frac{\mu v_y^2}{J} dy + \int_I \rho_0 v_t^2 dy &= \frac{1}{2}\int_I \left(\frac{\mu_t v_y^2}{J} - \frac{\mu v_y^3}{J^2}\right) dy + \int_I \frac{\rho_0 \theta v_{ty}}{J} dy \\ &:= I_1 + I_2. \end{aligned} \tag{4.51}$$

We estimate I_1 and I_2 respectively as follows:

$$|I_1| \le C\|v_y\|_{L^\infty} \int_I \frac{v_y^2}{J} dy \le C\|v_y\|_{L^\infty} \int_I \frac{\mu v_y^2}{J} dy, \tag{4.52}$$

$$\begin{aligned} I_2 &= \frac{d}{dt}\int_I J^{-1}\rho_0\theta v_y dy - \int_I v_y (J^{-1}\rho_0\theta)_t dy \\ &= \frac{d}{dt}\int_I J^{-1}\rho_0\theta v_y dy - \int_I (v_y - \mu^{-1}\rho_0\theta)(J^{-1}\rho_0\theta)_t dy \\ &\quad - \frac{1}{2}\frac{d}{dt}\int_I (\mu J)^{-1}(\rho_0\theta)^2 dy + \frac{1-\beta}{2}\int_I (\mu J)^{-2}\rho_0^\beta J^{-\beta} v_y \rho_0^2\theta^2 dy \\ &:= I_{21} + I_{22} + I_{23} + I_{24}, \end{aligned} \tag{4.53}$$

where

$$|I_{24}| \le C\|J^{-1}v_y\|_{L^\infty} \int_I (\rho_0\theta)^2 dy, \tag{4.54}$$

and

$$\begin{aligned} |I_{22}| &= \left| \int_I \frac{1}{\mu}\left(\mu\frac{v_y}{J} - \frac{\rho_0\theta}{J}\right)\left(\frac{\rho_0\theta}{J}v_y - \rho_0\theta_t\right) dy \right| \\ &\le \left| \int_I \left[\frac{1}{\mu}\left(\mu\frac{v_y}{J} - \frac{\rho_0\theta}{J}\right)\left(2\frac{\rho_0\theta}{J}v_y - \mu\frac{v_y^2}{J}\right) + \rho_0 v_t \frac{\kappa(\theta)\theta_y}{J}\right] dy \right. \end{aligned}$$

$$
\begin{aligned}
&\left.-\int\limits_I \frac{\mu_y}{\mu^2}\left(\mu\frac{v_y}{J}-\frac{\rho_0\theta}{J}\right)\frac{\kappa(\theta)\theta_y}{J}dy\right| \\
&\leq C\left(\|v_y\|_{L^\infty}+\|\theta\|_{L^\infty}+1+\|\mu_y\|_{L^\infty}^2\right)\int\limits_I\left(\frac{\mu v_y^2}{J}+(\rho_0\theta)^2\right)dy \\
&+\delta\int\limits_I \rho_0 v_t^2 dy + C(\delta)\int\limits_I \kappa^2(\theta)\theta_y^2 dy,
\end{aligned}
\tag{4.55}
$$

by using $(2.10)_3$, $(2.10)_2$. Therefore, submitting (4.52)–(4.55) into (4.51), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\limits_I \frac{1}{J}\left(\sqrt{\mu}v_y-\frac{\rho_0\theta}{\sqrt{\mu}}\right)^2 dy+\int\limits_I \rho_0 v_t^2 dy \\
&\leq \left(\|v_y\|_{L^\infty}+\|\theta\|_{L^\infty}+1+\|J_y\|_{L^2}^2\right)\int\limits_I\left(\frac{\mu v_y^2}{J}+\frac{(\rho_0\theta)^2}{\mu J}\right)dy+\int\limits_I \kappa^2(\theta)\theta_y^2 dy \\
&\leq C(\delta)\left(1+\|\theta\|_{L^\infty}+\int\limits_I\left(\frac{\mu v_y^2}{J}+\frac{(\rho_0\theta)^2}{\mu J}\right)dy\right)\int\limits_I\left(\frac{\mu v_y^2}{J}+\frac{(\rho_0\theta)^2}{\mu J}\right)dy \\
&+\delta\int\limits_I \rho_0 v_t^2 dy + C\int\limits_I \kappa^2(\theta)\theta_y^2 dy.
\end{aligned}
\tag{4.56}
$$

In fact, integrating $(2.10)_2$ over $(y,1)$, we get

$$
-\mu\frac{v_y}{J}=-\frac{\rho_0\theta}{J}+\int\limits_y^1 \rho_0 v_t dz.
$$

Thus

$$
\|v_y\|_{L^\infty}\leq\|\theta\|_{L^\infty}+C\left(\int\limits_I \rho_0 v_t^2 dy\right)^{\frac{1}{2}}. \tag{4.57}
$$

Step 2. To deal with the last term in the right hand of (4.56), we multiply $(2.10)_3$ by $\frac{1}{q+2}\theta^{q+1}$ and integrate it over I to get

$$
\begin{aligned}
&\frac{d}{dt}\int\limits_I \rho_0\theta^{q+2}dy+\frac{q+1}{q+2}\int\limits_I \frac{\theta^{2q}\theta_y^2}{J}dy \\
&=-\frac{1}{q+2}\int\limits_I \frac{\rho_0\theta}{J}v_y\theta^{q+1}dy+\frac{1}{q+2}\int\limits_I \frac{\mu v_y^2\theta^{q+1}}{J}dy \\
&\leq C\|\theta^{q+1}\|_{L^\infty}\left(\int\limits_I \rho_0\theta|v_y|dy+\int\limits_I v_y^2 dy\right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\left(\left(\left(\int_I \theta^{2q}\theta_y^2 dy\right)^{\frac{1}{2}}+1\right)\left(\int_I (\rho_0\theta)^2 dy+\int_I v_y^2 dy\right)\right. \\
&\leq \delta\int_I \theta^{2q}\theta_y^2 dy + C(\delta)\left[\left(1+\int_I (v_y^2+(\rho_0\theta)^2)dy\right)\int_I (v_y^2+(\rho_0\theta)^2)dy\right],
\end{aligned}
$$

which gives

$$
\frac{d}{dt}\int_I \rho_0\theta^{q+2}dy+\int_I \theta^{2q}\theta_y^2 dy \leq C(\delta)\left[\left(1+\int_I (v_y^2+(\rho_0\theta)^2)dy\right)\int_I (v_y^2+(\rho_0\theta)^2)dy\right]. \tag{4.58}
$$

Adding (C_1+1) times (4.58) to (4.56), for sufficiently large C_1, and choosing δ suitably small, we have

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\int_I \frac{1}{J}\left(\sqrt{\mu}v_y-\frac{\rho_0\theta}{\sqrt{\mu}}\right)^2 dy+\int_I \rho_0\theta^{q+2}dy\right)+\int_I \rho_0 v_t^2 dy+\int_I \theta^{2q}\theta_y^2 dy \\
&\leq C\left(1+\|\theta\|_{L^\infty}+\int_I\left(\frac{\mu v_y^2}{J}+\frac{(\rho_0\theta)^2}{\mu J}\right)dy\right)\int_I\left(\frac{\mu v_y^2}{J}+\frac{(\rho_0\theta)^2}{\mu J}\right)dy.
\end{aligned} \tag{4.59}
$$

Next, we integrate (4.59) with respect to the time variable over $[0,t]$ to find

$$
\begin{aligned}
&\int_I \left(v_y^2+(\rho_0\theta)^2+\rho_0\theta^{q+2}\right)dy+\int_0^t\int_I \rho_0 v_t^2 dydt+\int_0^t\int_I \theta^{2q}\theta_y^2 dydt \\
&\leq C+C\int_I |v_y|\rho_0\theta dy + C\int_0^t\left(1+\|\theta\|_{L^\infty}+\int_I\left(v_y^2+\rho_0\theta^2\right)dy\right)\int_I\left(v_y^2+\rho_0\theta^2\right)dydt \\
&\leq C+\frac{1}{2}\int_I |v_y|^2 dy+\frac{1}{2}\int_I \rho_0\theta^{q+2}dy+C\int_I \rho_0^2\theta dy \\
&\quad +C\int_0^t\left(1+\|\theta\|_{L^\infty}+\int_I\left(v_y^2+(\rho_0\theta)^2\right)dy\right)\int_I\left(v_y^2+(\rho_0\theta)^2\right)dydt,
\end{aligned} \tag{4.60}
$$

which together with the Gronwall inequality, (4.34) and (4.41), gives

$$
\begin{aligned}
&\int_I \left[v_y^2+(\rho_0\theta)^2+\rho_0\theta^{q+2})\right]dy+\int_0^T\int_I \left[\rho_0 v_t^2+\theta^{2q}\theta_y^2\right]dydt \\
&\leq C\exp\left\{C(T+\int_0^T\int_I v_y^2 dydt+\int_0^T\|\theta\|_{L^\infty}\int_I \rho_0\theta dydt)\right\}\leq C.
\end{aligned} \tag{4.61}
$$

It follows from (4.57), (4.34) and (4.61) that

$$\int_0^T \|v_y\|_{L^\infty} dt \le \int_0^T \|\theta\|_{L^\infty} dt + \int_0^T \left(\int_I \rho_0 v_t^2 dy \right)^{\frac{1}{2}} dt \le C. \tag{4.62}$$

Thus, we finish the proof of Lemma 4.9. □

Lemma 4.10. *For any $t \in [0, T]$, we have*

$$\int_I \left(\rho_0 v_t^2 + \kappa^2(\theta)\theta_y^2\right) dy(t) + \int_0^t \int_I \left(v_{ty}^2 + \rho_0 \kappa(\theta)\theta_t^2\right) dydt \le C, \tag{4.63}$$

and

$$\|\theta(\cdot, t)\|_{L^\infty} + \|v_y(\cdot, t)\|_{L^\infty} \le C. \tag{4.64}$$

Proof. We show this lemma in two steps.

Step 1. Differentiating $(2.10)_2$ with respect to t, we obtain

$$\rho_0 v_{tt} - \left(\frac{\mu v_{ty}}{J}\right)_y = -\left(\frac{\mu v_y^2}{J^2}\right)_y + \left(\frac{\mu_t v_y}{J}\right)_y + \left(\frac{\rho_0 \theta v_y}{J^2} - \frac{\rho_0 \theta_t}{J}\right)_y.$$

Then we multiply the resulting equation by v_t, integrate over I and use (4.47) to get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_I \rho_0 v_t^2 dy + \int_I \mu \frac{v_{ty}^2}{J} dy \\
\le & \delta \int_I v_{ty}^2 dy + C(\delta) \int_I \left(\frac{v_y^4}{J^4} + \rho_0^2 \frac{\theta_t^2}{J^2} + \frac{\rho_0^2 \theta^2 v_y^2}{J^4} \right) dy \\
\le & \delta \int_I v_{ty}^2 dy + C(\delta) \left(\|v_y\|_{L^\infty}^2 \int_I (v_y^2 + (\rho_0\theta)^2) dy + \int_I \rho_0^2 \theta_t^2 dy \right) \\
\le & \delta \int_I v_{ty}^2 dy + C(\delta) \left(\|v_y\|_{L^\infty}^2 + \int_I \rho_0 \kappa(\theta) \theta_t^2 dy \right),
\end{aligned} \tag{4.65}$$

since $|\mu_t| \le C|v_y|$. From (4.57), Lemma 3.1 and (4.31), one has

$$\|v_y\|_{L^\infty}^2 \le C \left(\|\theta\|_{L^\infty}^2 + \int_I \rho_0 v_t^2 dy \right), \tag{4.66}$$

and

$$\|\theta\|_{L^\infty}^2 \le \int_I \theta_y^2 dy + \int_I \rho_0 \theta dy \le C \int_I \theta^{2q} \theta_y^2 dy + C. \tag{4.67}$$

Therefore, submitting (4.66) and (4.67) into (4.65), integrating the result in $(0, t)$ and using (4.47), we have

$$
\begin{aligned}
&\int_I \rho_0 v_t^2 dy + \int_0^t \int_I v_{ty}^2 dydt \\
&\leq \int_0^t \int_I \rho_0 v_t^2 dydt + C_1 \int_0^t \int_I \rho_0 \kappa(\theta)\theta_t^2 dydt + \int_0^t \int_I \theta^{2q}\theta_y^2 dydt + C \\
&\leq C_2 \int_0^t \int_I \rho_0 \kappa(\theta)\theta_t^2 dydt + C.
\end{aligned}
\tag{4.68}
$$

Step 2. To deal with the term $\int_0^t \int_I \rho_0\kappa(\theta)\theta_t^2 dydt$ in above inequality (4.68), we multiply the $(2.10)_3$ by $\kappa(\theta)\theta_t$, integrate over I to get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_I \frac{\kappa^2(\theta)\theta_y^2}{J} dy + \int_I \rho_0\kappa(\theta)\theta_t^2 dy \\
=& -\frac{1}{2}\int_I \frac{v_y\kappa^2(\theta)\theta_y^2}{J^2} dy - \int_I \frac{\rho_0\theta}{J} v_y \kappa(\theta)\theta_t dy + \int_I \frac{\mu v_y^2}{J}\kappa(\theta)\theta_t dy \\
:=& A_1 + A_2 + A_3.
\end{aligned}
\tag{4.69}
$$

We estimate each A_i $(i = 1, 2, 3)$ as follows.

$$
|A_1| \leq C\|v_y\|_{L^\infty} \int_I \kappa^2(\theta)\theta_y^2 dy. \tag{4.70}
$$

It follows from Cauchy's inequality, Lemma 3.1 and (4.47) that

$$
\begin{aligned}
|A_2| &\leq \delta \int_I \rho_0\kappa(\theta)\theta_t^2 dy + C(\delta)\|\theta\|_{L^\infty}\|\kappa(\theta)\theta\|_{L^\infty} \int_I v_y^2 dy \\
&\leq \delta \int_I \rho_0\kappa(\theta)\theta_t^2 dy + C(\delta)\|\theta\|_{L^\infty}\left(\int_I \kappa^2(\theta)\theta_y^2 dy + 1\right).
\end{aligned}
\tag{4.71}
$$

Notice that

$$
\begin{aligned}
A_3 =& \frac{1}{q+1}\frac{d}{dt}\int_I \frac{\mu v_y^2}{J}\theta^{q+1} dy - \frac{1}{q+1}\int_I \theta^{q+1}\left[\mu\left(\frac{2v_y v_{ty}}{J} - \frac{v_y^3}{J^2}\right) + \frac{\mu_t v_y^2}{J}\right] dx \\
:=& A_{31} + A_{32}.
\end{aligned}
\tag{4.72}
$$

Using Cauchy's inequality, (4.47) and Lemma 3.1, we have

$$
\int_0^t |A_{31}| dt \leq \|\theta\kappa(\theta)\|_{L^\infty} \int_I v_y^2 dy + \|\theta_0\kappa(\theta_0)\|_{L^\infty} \int_I v_{0y}^2 dy \leq \delta \int_I \kappa^2(\theta)\theta_y^2 dy + C(\delta), \tag{4.73}
$$

and

$$
\begin{aligned}
|A_{32}| &\le \delta \int_I v_{ty}^2 dy + C(\delta)\|\theta\kappa(\theta)\|_{L^\infty}^2 \int_I v_y^2 dy + \|\theta\kappa(\theta)\|_{L^\infty}\|v_y\|_{L^\infty} \int_I v_y^2 dy \\
&\le \delta \int_I v_{ty}^2 dy + C(\delta)(1+\|v_y\|_{L^\infty}) \int_I \kappa^2(\theta)\theta_y^2 dy + C(\delta)(\|v_y\|_{L^\infty}+1).
\end{aligned} \tag{4.74}
$$

Submitting (4.70)–(4.74) into (4.69), integrating the result in $(0,t)$ and using (4.34) and (4.48), we have

$$
\begin{aligned}
&\int_I \kappa^2(\theta)\theta_y^2 dy + \int_0^t \int_I \rho_0\kappa(\theta)\theta_t^2 dydt \\
&\le \delta \int_0^t \int_I v_{ty}^2 dydt + C(\delta) \int_0^t \left[(1+\|\theta\|_{L^\infty}+\|v_y\|_{L^\infty}) \int_I \kappa^2(\theta)\theta_y^2 dy \right] + C(\delta).
\end{aligned} \tag{4.75}
$$

Adding (C_2+1) times (4.75) to (4.68) for large C_2, then choosing δ suitably small, we have

$$
\begin{aligned}
&\int_I \kappa^2(\theta)\theta_y^2 dy + \int_I \rho_0 v_t^2 dy + \int_0^t \int_I \rho_0\kappa(\theta)\theta_t^2 dydt + \int_0^t \int_I v_{ty}^2 dydt \\
&\le C \int_0^t \left[(1+\|\theta\|_{L^\infty}+\|v_y\|_{L^\infty}) \int_I \kappa^2(\theta)\theta_y^2 dy \right] + C.
\end{aligned} \tag{4.76}
$$

Then we apply Gronwall's inequality to (4.76) and use (4.34) to get

$$
\int_I \kappa^2(\theta)\theta_y^2 dy + \int_I \rho_0 v_t^2 dy + \int_0^t \int_I \rho_0\kappa(\theta)\theta_t^2 dydt + \int_0^t \int_I v_{ty}^2 dydt \le C, \tag{4.77}
$$

which together with Lemma 3.1 gives

$$
\|\theta\|_{L^\infty}^{q+1} = \|\theta^{q+1}\|_{L^\infty} \le \int_I \theta^{2q}\theta_y^2 dy + C \le C. \tag{4.78}
$$

Thus, using (4.57), we derive

$$
\|v_y\|_{L^\infty}^2 \le \|\theta\|_{L^\infty}^2 + \int_I \rho_0 v_t^2 dy \le C. \tag{4.79}
$$

Finally, (4.77)–(4.79) give the proof of Lemma 4.10. □

Lemma 4.11. *For any $t \in [0,T]$, we have*

$$
\int_I \rho_0\kappa(\theta)\theta_t^2 dy(t) + \int_0^t \int_I \kappa^2(\theta)\theta_{ty}^2 dxdt \le C, \tag{4.80}
$$

$$
\|\theta_y(\cdot,t)\|_{L^\infty} + \int_0^t \|\theta_t\|_{L^\infty}^2 dt \le C. \tag{4.81}
$$

Proof. Differentiating $(2.10)_3$ with respect to t, we obtain

$$\rho_0\theta_{tt} - \left(\frac{\kappa(\theta)\theta_y}{J}\right)_{ty} = -\left(\frac{\rho_0\theta}{J}v_y\right)_t + \left(\frac{\mu v_y^2}{J}\right)_t$$

Multiplying the above equation by $\kappa(\theta)\theta_t$ and integrating the resulting equation over I yield

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_I \rho_0\kappa(\theta)\theta_t^2 dy + \int_I \frac{1}{J}|(\kappa(\theta)\theta_t)_y|^2 dy \\
&=\frac{1}{2}\int_I \rho_0\kappa'(\theta)\theta_t^3 dy + \int_I \frac{v_y}{J^2}\kappa(\theta)\theta_y\left(\kappa(\theta)\theta_t\right)_y dx \\
&\quad - \int_I \rho_0\kappa(\theta)\theta_t\left(\frac{\theta_t}{J}v_y + \frac{\theta}{J}v_{ty} - \theta\frac{v_y^2}{J^2}\right)dy \\
&\quad + \int_I \kappa(\theta)\theta_t\left[\mu\left(\frac{2v_yv_{ty}}{J} - \frac{v_y^3}{J^2}\right) + \frac{\mu_t v_y^2}{J}\right]dy \\
&:=B_1 + B_2 + B_3 + B_4.
\end{aligned} \tag{4.82}$$

We estimate B_1 through B_4 respectively as follows:

$$\begin{aligned}
|B_1| &\le C\left\|\theta^{-1}\right\|_{L^\infty}^{q+1}\|\kappa(\theta)\theta_t\|_{L^\infty}\int_I \rho_0\kappa(\theta)\theta_t^2 dy \\
&\le C\left(\|(\kappa(\theta)\theta_t)_y\|_{L^2} + \int_I \rho_0\kappa(\theta)|\theta_t|dy\right)\int_I \rho_0\kappa(\theta)\theta_t^2 dy \\
&\le C\left(\|(\kappa(\theta)\theta_t)_y\|_{L^2} + \left(\int_I \rho_0\kappa(\theta)|\theta_t|^2 dy\right)^{\frac{1}{2}}\left(\int_I \rho_0\kappa(\theta)dy\right)^{\frac{1}{2}}\right)\int_I \rho_0\kappa(\theta)\theta_t^2 dy \\
&\le \delta\int_I |(\kappa(\theta)\theta_t)_y|^2 dy + C(\delta)\left(\int_I \rho_0\kappa(\theta)\theta_t^2 dy\right)^2 + C\left(\int_I \rho_0\kappa(\theta)dy\right)^2 \\
&\le \delta\int_I |(\kappa(\theta)\theta_t)_y|^2 dy + C(\delta)\left(\int_I \rho_0\kappa(\theta)\theta_t^2 dy\right)^2 + C,
\end{aligned} \tag{4.83}$$

$$\begin{aligned}
|B_2| &\le \delta\int_I |(\kappa(\theta)\theta_t)_y|^2 dy + C(\delta)\|v_y\|_{L^\infty}^2\int_I \kappa(\theta)^2\theta_y^2 dy \\
&\le \delta\int_I |(\kappa(\theta)\theta_t)_y|^2 dy + C(\delta),
\end{aligned} \tag{4.84}$$

$$|B_3| \le \|v_y\|_{L^\infty}\int_I \rho_0\kappa(\theta)\theta_t^2 dy + \int_I v_{ty}^2 dy + \|\kappa(\theta)\theta^2\|_{L^\infty}\int_I \rho_0\kappa(\theta)\theta_t^2 dy$$

$$
\begin{aligned}
&+\left(\int_I \rho_0\kappa(\theta)\theta_t^2 dx\right)^{\frac{1}{2}}\left(\int_I \rho_0\kappa(\theta)\theta^2 v_y^4 dy\right)^{\frac{1}{2}} \\
&\le \int_I v_{ty}^2 dy + C\int_I \rho_0\kappa(\theta)\theta_t^2 dy + C,
\end{aligned}
\tag{4.85}
$$

and, noting that $|\mu_t| \le C|v_y|$,

$$
\begin{aligned}
|B_4| &\le \|\kappa(\theta)\theta_t\|_{L^\infty}(\|v_y\|_{L^2}\|v_{ty}\|_{L^2} + \|v_y\|_{L^\infty}\|v_y\|_{L^2}^2) \\
&\le \|\kappa(\theta)\theta_t\|_{L^\infty}(\|v_{ty}\|_{L^2} + C) \\
&\le \delta\|\kappa(\theta)\theta_t\|_{L^\infty}^2 + C(\delta)\|v_{ty}\|_{L^2}^2 + C(\delta) \\
&\le \delta\int_I |(\kappa(\theta)\theta_t)_y|^2 dy + \delta\int_I \rho_0\kappa(\theta)\theta_t^2 dy + C(\delta)\int_I v_{ty}^2 dy + C(\delta).
\end{aligned}
\tag{4.86}
$$

Therefore, submitting (4.83)–(4.86) and into (4.82), we have

$$
\begin{aligned}
&\frac{d}{dt}\int_I \rho_0\kappa(\theta)\theta_t^2 dy + \int_I ((\kappa(\theta)\theta_t)_y)^2 dydt \\
&\le \left(\int_I \rho_0\kappa(\theta)\theta_t^2 dy + 1\right)\int_I \rho_0\kappa(\theta)\theta_t^2 dx + \int_I |v_{ty}|^2 dy + C.
\end{aligned}
\tag{4.87}
$$

Applying the Gronwall inequality and (4.63), one gets

$$
\int_I \rho_0\kappa(\theta)\theta_t^2 dy + \int_0^t\int_I |(\kappa(\theta)\theta_t)_y|^2 dydt \le C, \quad t\in[0,T],
\tag{4.88}
$$

which gives, by using Lemma 3.1 again,

$$
\int_0^T \|\kappa(\theta)\theta_t\|_{L^\infty}^2 dt \le \int_0^T\int_I |(\kappa(\theta)\theta_t)_y|^2 dydt + \int_0^T\int_I |\rho_0\kappa(\theta)\theta_t| dydt \le C.
\tag{4.89}
$$

Note that from (4.88), (4.31), (4.63), and (4.89)

$$
\begin{aligned}
\int_0^t\int_I \kappa^2(\theta)\theta_{ty}^2 dydt &\le \int_0^t\int_I |(\kappa(\theta)\theta_t)_y|^2 dydt + \int_0^t\int_I |\left(\kappa(\theta)\right)_y\theta_t|^2 dydt \\
&\le C + \left\|\frac{1}{\theta^{q+1}}\right\|_{L^\infty}^2 \sup_t \|\theta^q\theta_y(\cdot,t)\|_{L^2}^2 \int_0^t \|\theta^q\theta_t\|_{L^\infty}^2 dt \\
&\le C.
\end{aligned}
\tag{4.90}
$$

Next, we integrate $(2.10)_2$ from 0 to y to get

$$
\frac{\kappa(\theta)\theta_y}{J} = \int_0^y \rho_0\theta_t ddz + \int_0^y \frac{\rho_0\theta}{J}v_z dz - \int_0^y \frac{\mu v_z^2}{J}dz
\tag{4.91}
$$

Thus (4.88), (4.64) and (4.47) imply that

$$\|\kappa(\theta)\theta_y\|_{L^\infty} \le C\left(\int_I \rho_0\theta^q\theta_t^2 dy\right)^{\frac{1}{2}} + C\|v_y\|_{L^\infty}\int_I \rho_0\theta dy + C\int_I v_y^2 dy \le C. \tag{4.92}$$

Thus, it follows from (4.92), (4.88) and (4.89) that

$$\|\theta_y\|_{L^\infty} + \int_0^t\int_I \theta_{ty}^2 dydt + \int_0^t \|\theta_t\|_{L^\infty}^2 dt \le C.$$

Thus, (4.88), (4.90) and (4.92) show Lemma 4.11. □

Lemma 4.12. *For any $t \in [0,T]$, we have*

$$\int_I (v_{yy}^2 + \theta_{yy}^2)dy(t) \le C. \tag{4.93}$$

Proof. We rewrite $(2.10)_2$ and $(2.10)_3$ as follows:

$$\frac{\mu v_{yy}}{J} = \rho_0 v_t + \frac{\rho_0\theta}{J} + \frac{\mu v_y^2}{J^2} - \frac{\mu_y v_y}{J},$$

and

$$\frac{\kappa(\theta)\theta_{yy}}{J} = \rho_0\theta_t + \frac{\rho_0\theta}{J}v_y - \mu\frac{v_y^2}{J} - \frac{q\theta^{q-1}\theta_y^2}{J} + \frac{\theta^q\theta_y v_y}{J^2},$$

which combine with (4.43), (4.63), (4.64), (4.80), (4.81), and the fact that $|\mu_y| \le C|J_y| + C$, give the following estimates

$$\|v_{yy}\|_{L^\infty(0,T;L^2)} + \|\theta_{yy}\|_{L^\infty(0,T;L^2)} \le C. \tag{4.94}$$

The proof of Lemma 4.12 is complete. □

Proof of Theorem 2.1. Then the proof of Theorem 2.1 follows from Lemma 2.1 which signifying the local existence of the strong solution and the global (in time) a priori estimates in Section 4. In fact, by Lemma 2.1, there exists a local strong solution (J, v, θ) on the time interval $(0, T_*]$ with $T_* > 0$. Now let with $T^* > 0$ be the maximal existing time of the strong solution (J, v, θ) in Lemma 2.1. Then obviously one has $T^* \ge T_*$. Now we claim that $T^* \ge T$ with $T > 0$ being any fixed positive constant given in Theorem 2.1. Otherwise, if $T^* < T$, then all the a priori estimates in Section 4 hold with T being replaced by T^*. Therefore, it follows from a priori estimates in Section 4 that $(J, v, \theta)(x, T^*)$ satisfy assumptions in Theorem 2.1. By using Lemma 2.1 again, there exists a $T_1^* > 0$ such that the strong solution (u, v, θ) in Lemma 2.1 exists on $(0, T^* + T_1^*]$, which contradicts with T^* being the maximal existing time of the strong solution (J, v, θ). Thus it holds that $T^* > T$.

Finally, the expanding rate of the free interval $(\Gamma_1(t), \Gamma_2(t))$ comes from the estimate in (4.34). The proof of Theorem 2.1 is completed. □

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